Inverse Numerator Relationship Matrix

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Problem 1: Inverse Numerator Relationship Matrix

During the lecture the method of computing the inverse numerator relationship matrix A^{-1} directly was introduced. The computation is based on the LDL-decomposition. As a result, we can write

$$A^{-1} = (L^T)^{-1} \cdot D^{-1} \cdot L^{-1}$$

where $L^{-1} = I - P$, and D^{-1} is a diagonal matrix with $(D^{-1})_{ii} * \sigma_u^{-2} = var(m_i)^{-1}$.

Tasks

- Use the example pedigree given below and compute the matrices L^{-1} and D^{-1} to compute A^{-1}
- Verify your result using the function `getAinv()` from package pedigreemm.

Pedigree

```
nr_animal <- 6
tbl_pedigree <- tibble::tibble(Calf = c(1:nr_animal),
                               Sire = c(NA, NA, NA, 1 ,3, 4),
                               Dam = c(NA, NA, NA, 2, 2, 5))
tbl_pedigree
```

```
## # A tibble: 6 x 3
##    Calf  Sire   Dam
##   <int> <dbl> <dbl>
## 1     1    NA    NA
## 2     2    NA    NA
## 3     3    NA    NA
## 4     4     1     2
## 5     5     3     2
## 6     6     4     5
```

Solution

The matrix P comes from the simple decomposition and can be constructed using the pedigree.

```
P = matrix(0, nrow = nr_animal, ncol = nr_animal)
for (i in 1:nr_animal){
  s <- tbl_pedigree$Sire[i]
  d <- tbl_pedigree$Dam[i]
  if (!is.na(s)){
    P[i,s] <- 0.5
  }
```

```r
  if(!is.na(d)){
    P[i,d] <- 0.5
  }
}
P
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]  0.0  0.0  0.0  0.0  0.0    0
## [2,]  0.0  0.0  0.0  0.0  0.0    0
## [3,]  0.0  0.0  0.0  0.0  0.0    0
## [4,]  0.5  0.5  0.0  0.0  0.0    0
## [5,]  0.0  0.5  0.5  0.0  0.0    0
## [6,]  0.0  0.0  0.0  0.5  0.5    0
```

With that the matrix L^{-1} is

```r
I <- diag(1, nrow = nr_animal, ncol = nr_animal)
Linv <- I - P
Linv
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]  1.0  0.0  0.0  0.0  0.0    0
## [2,]  0.0  1.0  0.0  0.0  0.0    0
## [3,]  0.0  0.0  1.0  0.0  0.0    0
## [4,] -0.5 -0.5  0.0  1.0  0.0    0
## [5,]  0.0 -0.5 -0.5  0.0  1.0    0
## [6,]  0.0  0.0  0.0 -0.5 -0.5    1
```

The matrix D is obtained from package pedigreemm

```r
ped <- pedigreemm::pedigree(sire = tbl_pedigree$Sire,
                            dam  = tbl_pedigree$Dam,
                            label = as.character(1:nr_animal))
D <- pedigreemm::Dmat(ped = ped)
Dinv <- diag(1/D, nrow = nr_animal, ncol = nr_animal)
Dinv
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    1    0    0    0    0    0
## [2,]    0    1    0    0    0    0
## [3,]    0    0    1    0    0    0
## [4,]    0    0    0    2    0    0
## [5,]    0    0    0    0    2    0
## [6,]    0    0    0    0    0    2
```

The inverse numerator relationship matrix is

```r
Ainv <- t(Linv) %*% Dinv %*% Linv
Ainv
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]  1.5  0.5  0.0 -1.0  0.0    0
## [2,]  0.5  2.0  0.5 -1.0 -1.0    0
## [3,]  0.0  0.5  1.5  0.0 -1.0    0
## [4,] -1.0 -1.0  0.0  2.5  0.5   -1
## [5,]  0.0 -1.0 -1.0  0.5  2.5   -1
## [6,]  0.0  0.0  0.0 -1.0 -1.0    2
```

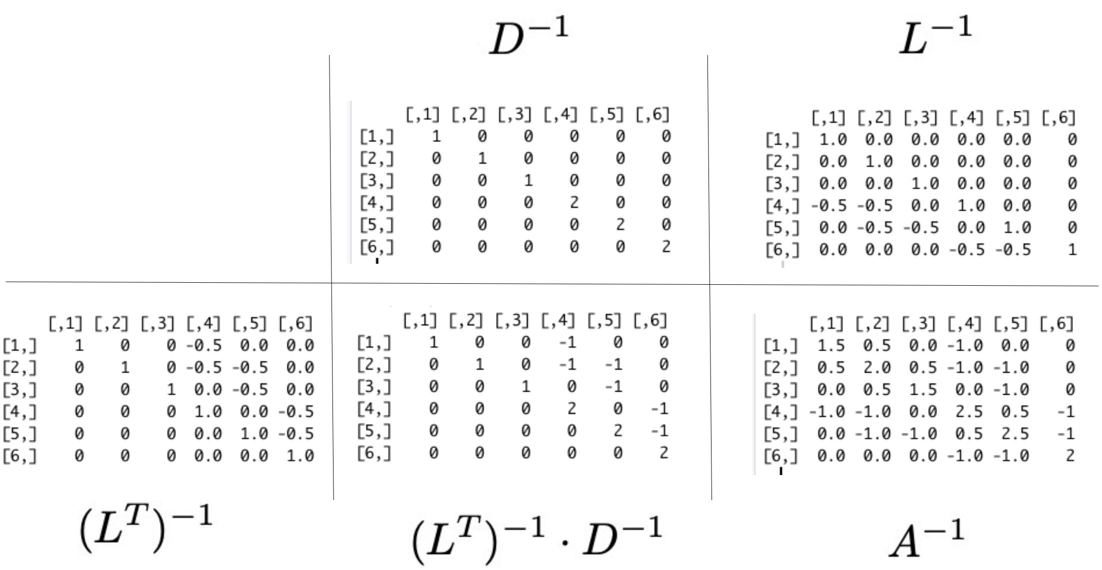
Verification

```r
pedigreemm::getAInv(ped = ped)
```

```
## 6 x 6 Matrix of class "dgeMatrix"
##      1    2    3    4    5  6
## 1  1.5  0.5  0.0 -1.0  0.0  0
## 2  0.5  2.0  0.5 -1.0 -1.0  0
## 3  0.0  0.5  1.5  0.0 -1.0  0
## 4 -1.0 -1.0  0.0  2.5  0.5 -1
## 5  0.0 -1.0 -1.0  0.5  2.5 -1
## 6  0.0  0.0  0.0 -1.0 -1.0  2
```

Problem 2: Rules

The following diagram helps to illustrate the rules for constructing A^{-1}



D^{-1}

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	1	0	0	0	0	0
[2,]	0	1	0	0	0	0
[3,]	0	0	1	0	0	0
[4,]	0	0	0	2	0	0
[5,]	0	0	0	0	2	0
[6,]	0	0	0	0	0	2

L^{-1}

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	1.0	0.0	0.0	0.0	0.0	0
[2,]	0.0	1.0	0.0	0.0	0.0	0
[3,]	0.0	0.0	1.0	0.0	0.0	0
[4,]	-0.5	-0.5	0.0	1.0	0.0	0
[5,]	0.0	-0.5	-0.5	0.0	1.0	0
[6,]	0.0	0.0	0.0	-0.5	-0.5	1

$(L^T)^{-1}$

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	1	0	0	-0.5	0.0	0.0
[2,]	0	1	0	-0.5	-0.5	0.0
[3,]	0	0	1	0.0	-0.5	0.0
[4,]	0	0	0	1.0	0.0	-0.5
[5,]	0	0	0	0.0	1.0	-0.5
[6,]	0	0	0	0.0	0.0	1.0

$(L^T)^{-1} \cdot D^{-1}$

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	1	0	0	-1	0	0
[2,]	0	1	0	-1	-1	0
[3,]	0	0	1	0	-1	0
[4,]	0	0	0	2	0	-1
[5,]	0	0	0	0	2	-1
[6,]	0	0	0	0	0	2

A^{-1}

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]
[1,]	1.5	0.5	0.0	-1.0	0.0	0
[2,]	0.5	2.0	0.5	-1.0	-1.0	0
[3,]	0.0	0.5	1.5	0.0	-1.0	0
[4,]	-1.0	-1.0	0.0	2.5	0.5	-1
[5,]	0.0	-1.0	-1.0	0.5	2.5	-1
[6,]	0.0	0.0	0.0	-1.0	-1.0	2