

OHP Picture 1

- Recap:
- Problem of correcting systematic environment with
 - regression
 - selection index

→ Solution: Use BLUP together with a mixed linear effect model to estimate systematic environment (herd, sex, season) effects and to predict breeding values as random effects simultaneously from the same data set.

1. part

$$y = X\beta + Zu + e \rightarrow \begin{cases} E(u) = \\ E(e) = \\ E(y) = \end{cases} \quad \begin{cases} \text{var}(u) = \dots \\ \text{var}(e) = \dots \\ \text{var}(y) = \dots \end{cases}$$

with β , u and e being unknown

Fit data, obtain $\hat{\beta}$ as estimates for fixed effects and \hat{u} as predictions of breeding values

Problem with \hat{u} and $\hat{\beta}$: depend on V^{-1} where $V = \text{var}(y)$ [Practical evaluations of y can have the length of 10]

Solution: Mixed model equations

OHP Picture 2

Mixed Model Equations (MME)

$$\begin{bmatrix} X^T R^{-1} X \\ Z^T R^{-1} X \end{bmatrix} \begin{bmatrix} X^T R^{-1} Z \\ Z^T R^{-1} Z + G^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X^T R^{-1} y \\ Z^T R^{-1} y \end{bmatrix}$$

- Remember: $y = X\beta + Zu + e$; $\text{var}(e) = R_j$; variance-covariance matrix of residuals

in MME we need R^{-1}

1. We assume that residual terms e_1, e_2, \dots, e_N , they have the same variance $\Rightarrow \text{var}(e_i) = \sigma_e^2$
 $\text{var}(e_1) = \text{var}(e_2) = \dots = \text{var}(e_i) = \dots = \text{var}(e_N) = \sigma_e^2$

$R = \begin{bmatrix} \text{var}(e_1) & \text{cov}(e_1, e_2) & \dots & \text{cov}(e_1, e_N) \\ \vdots & \ddots & \ddots & \vdots \\ \text{cov}(e_2, e_1) & \text{var}(e_2) & \dots & \text{cov}(e_2, e_N) \\ \vdots & \ddots & \ddots & \vdots \\ \text{cov}(e_N, e_1) & \text{cov}(e_N, e_2) & \dots & \text{var}(e_N) \end{bmatrix}$

vector $e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix}$ where N is the number of observations in data set.

2. Covariance between two residual effects is \emptyset : $\text{cov}(e_i, e_j) = \emptyset$ for $i \neq j$

OHP Picture 3

Inverse R^{-1} of matrix R :

$$R = \begin{bmatrix} G_c^2 & 0 & \dots & 0 \\ 0 & B_c^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & - & B_c^2 \end{bmatrix} = I \cdot B_c^2$$

Identity matrix: $I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & - & 0 & \dots & 1 \end{bmatrix}$

$$R^{-1} = \begin{bmatrix} \frac{1}{B_c^2} & 0 & \dots & 0 \\ 0 & \frac{1}{B_c^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & - & \frac{1}{B_c^2} \end{bmatrix} = I \cdot B_c^{-2}$$

$$R^{-1} \cdot R = I$$

Diagram illustrating the inverse operation:

Matrix R (bottom left) is multiplied by R^{-1} (top right) to yield the identity matrix I (bottom right).

The multiplication is shown as follows:

- Matrix R is partitioned into four quadrants:
 - Top-left: $\begin{bmatrix} G_c^2 & 0 \\ 0 & B_c^2 \end{bmatrix}$
 - Top-right: $\begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_c^2 \end{bmatrix}$
 - Bottom-left: $\begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_c^2 \end{bmatrix}$
 - Bottom-right: $\begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_c^2 \end{bmatrix}$
- Matrix R^{-1} is partitioned into four quadrants:
 - Top-left: $\begin{bmatrix} \frac{1}{B_c^2} & 0 \\ 0 & \frac{1}{B_c^2} \end{bmatrix}$
 - Top-right: $\begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{1}{B_c^2} \end{bmatrix}$
 - Bottom-left: $\begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{1}{B_c^2} \end{bmatrix}$
 - Bottom-right: $\begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{1}{B_c^2} \end{bmatrix}$
- The product $R^{-1} \cdot R$ is calculated by multiplying corresponding quadrants:
 - Top-left: $\begin{bmatrix} G_c^2 & 0 \\ 0 & B_c^2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{B_c^2} & 0 \\ 0 & \frac{1}{B_c^2} \end{bmatrix} = \begin{bmatrix} G_c^2 \cdot \frac{1}{B_c^2} & 0 \\ 0 & B_c^2 \cdot \frac{1}{B_c^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
 - Top-right: $\begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_c^2 \end{bmatrix} \cdot \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{1}{B_c^2} \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$
 - Bottom-left: $\begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_c^2 \end{bmatrix} \cdot \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{1}{B_c^2} \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$
 - Bottom-right: $\begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_c^2 \end{bmatrix} \cdot \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{1}{B_c^2} \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$

OHP Picture 4

Matrix G:

$$\square \text{ Model definition } \text{var}(u) = G = \begin{bmatrix} \text{var}(u_1) & \text{cov}(u_1 u_2) & \cdots \\ \text{cov}(u_2 u_1) & \text{var}(u_2) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

□ Depends on relationship between breeding values

□ Fish example: Sire model

Breeding values of sires are random effects S

~~Female animals and~~ ^{male} animals without offspring do not get breeding values.

□ Model: $y = X\beta + Zs + e ; E[e] = 0$

□ Data: $y = \begin{bmatrix} 2.61 \\ 2.31 \\ \vdots \\ 3.16 \end{bmatrix} \quad s = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_5 \end{bmatrix} \quad E[s] = 0$

$$\beta = \begin{bmatrix} \text{herd 1} \\ \text{herd 2} \\ \vdots \\ \text{herd 6} \end{bmatrix} \quad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_6 \end{bmatrix} \quad E[e] = R = I \cdot \text{Fe}^2$$

$$\text{var}[e] = R = G$$

$$\text{var}[y] = ZGZ^T + R$$

OHP Picture 5

Design matrices for sire model:

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

Solutions using Mixed Model Equations

$$\begin{bmatrix} X^T X & X^T Z \\ I^T X & I^T Z + G^{-1} \end{bmatrix} \begin{bmatrix} \beta \\ \hat{\sigma}^2 \end{bmatrix} = \begin{bmatrix} X^T y \\ Z^T y \end{bmatrix}$$

Simplify general MME, assuming $R^{-1} = I \cdot \bar{\sigma}_e^{-2}$

$$\begin{bmatrix} X^T I \bar{\sigma}_e^{-2} X & X^T I \bar{\sigma}_e^{-2} Z \\ I^T I \bar{\sigma}_e^{-2} X & I^T I \bar{\sigma}_e^{-2} Z + G^{-1} \end{bmatrix} \begin{bmatrix} \beta \\ \hat{\sigma}^2 \end{bmatrix} = \begin{bmatrix} X^T I \bar{\sigma}_e^{-2} y \\ I^T I \bar{\sigma}_e^{-2} y \end{bmatrix}$$

$$X^T I \bar{\sigma}_e^{-2} X - \underbrace{X^T I \cdot I \cdot X \cdot \bar{\sigma}_e^{-2}}_{\text{scalar}} - X^T X \cdot \bar{\sigma}_e^{-2}$$

OHP Picture 6

o MNE:

$$\begin{bmatrix} X^T X + \bar{\beta}_e^{-2} \\ I^T X + \bar{\beta}_e^{-2} \end{bmatrix}$$

$$\begin{bmatrix} X^T \bar{\beta}_e^{-2} \\ I^T \bar{\beta}_e^{-2} + G_e^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{s} \end{bmatrix} = \begin{bmatrix} X^T y + \bar{\beta}_e^{-2} \\ I^T y + \bar{\beta}_e^{-2} \end{bmatrix}$$

$$/\bar{\beta}_e^{-2}$$

$$\begin{bmatrix} X^T X & X^T \bar{\beta}_e^{-2} \\ I^T X & I^T \bar{\beta}_e^{-2} + G_e^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{s} \end{bmatrix} = \begin{bmatrix} X^T y \\ I^T y \end{bmatrix}$$

$$\xrightarrow{I^T \bar{\beta}_e^{-2}} I^T \lambda \text{ with } \lambda = \frac{\bar{\beta}_e^{-2}}{G_e^{-2}}$$

o $G = \text{var}(s) = \begin{bmatrix} \text{var}(s_1) & \text{cov}(s_1, s_2) & \text{cov}(s_1, s_3) \\ \text{cov}(s_2, s_1) & \text{var}(s_2) & \text{cov}(s_2, s_3) \\ \text{cov}(s_3, s_1) & \text{cov}(s_3, s_2) & \text{var}(s_3) \end{bmatrix}$

o From data set (pedigree) we can see that sires 1-3 do not have any known parents

$\Rightarrow \text{cov}$ between their effects is 0: $\text{cov}(s_1, s_2)$

$$= \text{cov}(s_1, s_3)$$

$$= \text{cov}(s_2, s_3) = 0$$

$$\text{var}(s_i) = \bar{\beta}_s^{-2}$$

$$\Rightarrow G = I * \bar{\beta}_s^{-2} \Rightarrow G^{-1} = I * \bar{\beta}_s^{-2}$$

OHP Picture 7

a MME:

$$\begin{bmatrix} X^T X + \bar{\sigma}_e^{-2} & X^T Y + \bar{\sigma}_e^{-2} \\ Z^T X + \bar{\sigma}_e^{-2} & Z^T Y + \bar{\sigma}_e^{-2} + G^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{s} \end{bmatrix} = \begin{bmatrix} X^T Y + \bar{\sigma}_e^{-2} \\ Z^T Y + \bar{\sigma}_e^{-2} \end{bmatrix}$$

$$+ \bar{\sigma}_e^2$$

$$\begin{bmatrix} X^T X & X^T Z \\ Z^T X & Z^T Z + G^{-1} + \bar{\sigma}_e^{-2} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{s} \end{bmatrix} = \begin{bmatrix} X^T Y \\ Z^T Y \end{bmatrix}$$

$$I(\lambda) \text{ with } \lambda = \frac{\bar{\sigma}_e^2}{\bar{\sigma}_B^2}$$

a $G = \text{var}(s) =$

$$\begin{bmatrix} \text{var}(s_1) & \text{cov}(s_1, s_2) & \text{cov}(s_1, s_3) \\ \text{cov}(s_2, s_1) & \text{var}(s_2) & \text{cov}(s_2, s_3) \\ \text{cov}(s_3, s_1) & \text{cov}(s_3, s_2) & \text{var}(s_3) \end{bmatrix}$$

a From data set (pedigree) we can see that sires 1-3 do not have any known parents

$\Rightarrow \text{cov between their effects is } 0 : \text{cov}(s_1, s_2)$

$$= \text{cov}(s_1, s_3)$$

$$= \text{cov}(s_2, s_3) = 0$$

$$\text{var}(s_1) = \bar{\sigma}_s^2$$

$$\Rightarrow G = I * \bar{\sigma}_s^2 \Rightarrow G^{-1} = I * \bar{\sigma}_s^{-2}$$

OHP Picture 8

Exercise 4:

Fixed Linear effect model :

$$y = X\beta + e \quad \text{where } \beta = \begin{bmatrix} \text{head}_1 \\ \text{head}_2 \end{bmatrix}$$

In R: lm()

