

## OHP Picture 1

Recap

Predictions of breeding values using  
Linear Mixed Effect Models (LME).

□ Model:

$$\underline{y} = \underline{X}\beta + \underline{Z}\underline{u} + \underline{e}$$

$\underline{y}$ : Observations

$\beta$ : fixed effects

$\underline{u}$ : random effects  $\rightarrow$  breeding values

$\underline{e}$ : random residuals

□ Example : Sire Model  $\rightarrow$  Dairy Cattle

- Replaced  $\underline{u}$  by  $\underline{s}$  ( $\underline{y} = \underline{X}\beta + \underline{Z}\underline{s} + \underline{e}$ )

$$\underline{y} = \underline{X}\beta + \underline{Z}\underline{u}_s + \underline{e}$$

vector of breeding  
values for sires

① Estimate fixed effects and  
predict breeding values for sires, simultaneously

② Dams do not get breeding values.

## OHP Picture 2

### Animal Model

- The problem with the sire model that only sires get predicted breeding values is solved with the animal model

- LME:

$$y = X\beta + Z_u + e$$

breeding values for all animals

- Expectations and Variance-Covariances

$$E(y) = \underline{0} ; E(e) = \underline{0} ; E(y) = X\beta$$

$$\text{var}(e) = R = I \cdot \sigma_e^2$$

$$\text{var}(u) = G = A \cdot \sigma_u^2$$

$$\text{var}(y) = \Sigma G \Sigma^T + R$$

numerical relationship  
matrix

OHP Picture 3

Animal	Herd	Sire
3	1 → herd <sub>1</sub>	1
4	2 → herd <sub>2</sub>	NA
5	2	4
6	1	5
	2 herds	

$$\begin{array}{l}
 \begin{array}{c}
 \begin{array}{c} y \\ \rightarrow 4.5 \\ \rightarrow 2.9 \\ \rightarrow 3.9 \\ \rightarrow 5.5 \end{array} = \left[ \begin{array}{cc} \text{herd}_1 & \text{herd}_2 \\ \begin{array}{|c|c|} \hline 1 & \emptyset \\ \emptyset & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline \beta \\ \text{herd}_1 \\ \text{herd}_2 \\ \hline \end{array} \right] + \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \begin{array}{c} u_s \\ u_1 \\ u_4 \\ u_5 \end{array} \\
 \end{array} \\
 \text{Solutions for estimate } \hat{\beta} \text{ of fixed herd effect } \beta \text{ and} \\
 \text{predictions } u_s \text{ for sire breedability values by} \\
 \text{Mixed Model Equations.}
 \end{array}$$

## OHP Picture 4

Mixed Model Equations (MME) :

$$\begin{array}{c}
 \text{Matrix - notation} \\
 \left[ \begin{array}{c|c}
 X^T X & X^T Z \\
 \hline
 P^T X & Z^T Z + I, \frac{\sigma_e^2}{G_{us}} \end{array} \right] \xrightarrow{\text{claim}}
 \left[ \begin{array}{c|c}
 \beta \\
 \hat{u}_s \end{array} \right] = \left[ \begin{array}{c|c}
 X \\
 Z^T \end{array} \right] \cdot \hat{a} = r
 \end{array}$$

left-hand side  
 right-hand side

$\text{var}(e) = R = I \cdot \sigma_e^2$   
 $\text{var}(u_s) = G = I \cdot \sigma_{us}^2$

Goal:  $\hat{a}$  as a solution to  $C \cdot \hat{a} = r$

- Multiply by  $C^{-1}$ :  $C^{-1} C \cdot \hat{a} = C^{-1} r$   
 $\hat{a} = C^{-1} r$   
 $\downarrow$   
 $\text{solve}(mat\_C, mat(R))$
- $(X^T Z)^T = Z^T (X^T)^T = Z^T X$
- $C^{-1} = \text{solve}(mat(C))$   
 $\text{mat\_a\_hat} \leftarrow \text{solve}(\overline{\text{mat\_C}}) \% * \% \text{mat\_R}$

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Example II:  $y = X\beta + Z u_s + e$

$E(e) = \emptyset, E(u_s) = \emptyset, E(y) = X\beta$

OHP Picture 5

Example II:

$$y = X\beta + ZU_5 + e$$

$$E(e) = \emptyset, E(U_5) = \emptyset, E(y) \neq X\beta$$

$$\text{Var}(e) = R = I \cdot \bar{V}e^2$$

$$\text{Var}(U_5) = \begin{bmatrix} \text{var}(u_1) & \text{cov}(u_1, u_4) & \text{cov}(u_1, u_5) \\ \text{cov}(u_4, u_1) & \text{var}(u_4) & \text{cov}(u_4, u_5) \\ \text{cov}(u_5, u_1) & \text{cov}(u_5, u_4) & \text{var}(u_5) \end{bmatrix} = G$$

$$U_5 = \begin{bmatrix} u_1 \\ u_4 \\ u_5 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{v}_{us}^2 & \frac{1}{2}\bar{v}_u^2 & \frac{1}{4}\bar{v}_u^2 \\ \frac{1}{2}\bar{v}_u^2 & \bar{v}_{us}^2 & \frac{1}{2}\bar{v}_u^2 \\ \frac{1}{4}\bar{v}_u^2 & \frac{1}{2}\bar{v}_u^2 & \bar{v}_{us}^2 \end{bmatrix}$$

$$u_i = \frac{1}{2}u_s + \frac{1}{2}u_d + m_i$$

$$\text{For sire model: } u_4 = \frac{1}{2}u_1 + m_4^*$$

$$\begin{aligned} \text{cov}(u_1, u_4) &= \text{cov}(u_1, \frac{1}{2}u_1 + m_4^*) \\ &= \text{cov}(u_1, \frac{1}{2}u_1) + \text{cov}(u_1, m_4^*) \\ &= \frac{1}{2}\text{cov}(u_1, u_1) = \frac{1}{2}\bar{v}_u^2 \end{aligned}$$

OHP Picture 6

Animal Model:

- LME
- random vector  $\underline{u}$  contains breeding values for all animals in the pedigree
- fixed effects, the same as for sire model

$$\begin{array}{c}
 \text{Animal} \\
 \hline
 3 \quad \begin{pmatrix} y \\ 4.5 \end{pmatrix} \\
 \rightarrow 4 \quad \begin{pmatrix} 2.9 \end{pmatrix} \\
 5 \quad \begin{pmatrix} 3.9 \end{pmatrix} \\
 6 \quad \begin{pmatrix} 3.5 \end{pmatrix}
 \end{array}
 = \begin{array}{c}
 X \\
 \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \\
 \beta \\
 \begin{pmatrix} \text{head} \\ \text{head} \\ \text{L} \end{pmatrix} \\
 \mathbb{I}
 \end{array}
 + \begin{array}{c}
 \underline{u} \\
 \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{pmatrix} \\
 \underline{u} \\
 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} \\
 \underline{e} \\
 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}
 \end{array}$$

Solutions from Mixed Model Equations

## OHP Picture 7

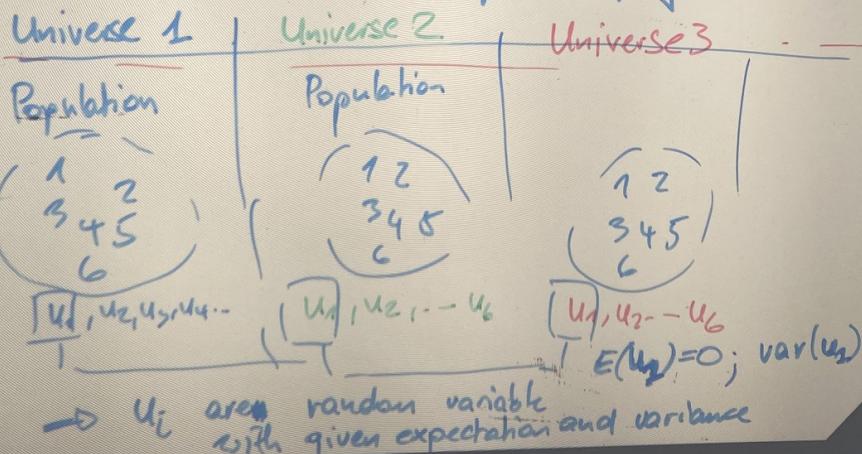
Solutions from Mixed model equations

$$R = I \cdot \sigma_e^2 ; G$$

$$\begin{bmatrix} X^T X & X^T Z \\ Z^T X & Z^T Z + G^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X^T y \\ Z^T y \end{bmatrix}$$

$$G_{6x6} = \text{var}(\underline{u}) = \begin{bmatrix} \text{var}(u_1) & \text{cov}(u_1, u_2) & \dots & \text{cov}(u_1, u_6) \\ \text{cov}(u_2, u_1) & \text{var}(u_2) & \dots & \text{cov}(u_2, u_6) \\ \vdots & \vdots & \ddots & \text{var}(u_6) \end{bmatrix}$$

\* Source of  $\text{var}(u_i)$  and  $\text{cov}(u_i, u_j)$  is based on the so-called "large-sampling" concept



OHP Picture 8

For animal  $i$  :  $E(u_i) = \emptyset$

$$\text{var}(u_i) = (1 + F_i) \bar{F_u}^2$$

- Inbreeding coefficient of animal  $i$
- $F_i = \frac{1}{2} A_{sd}$   
where  $A_{sd}$  is the relationship between parents s and d

$$G_i = \text{var}(u)$$

$$= \begin{bmatrix} \text{var}(u_1) & \text{cov}(u_1, u_2) & \text{cov}(u_1, u_3) & \text{cov}(u_1, u_6) \\ \text{cov}(u_2, u_1) & \text{var}(u_2) & & \\ & & & \end{bmatrix}$$

$$= \begin{bmatrix} (1+F_1)\bar{F_u}^2 & \emptyset & \frac{1}{2}\bar{F_u}^2 & \frac{1}{2}\bar{F_u}^2 \\ \emptyset & (1+F_2)\bar{F_u}^2 & & \end{bmatrix}$$

$\text{cov}(u_1, u_2) = \emptyset$ , because animals 1 and 2 do not share common ancestors

$$\begin{aligned} \text{cov}(u_1, u_3) &= \text{cov}(u_1, \frac{1}{2}u_1 + \frac{1}{2}u_2 + u_5) \\ &= \text{cov}(u_1, \frac{1}{2}u_1) + \text{cov}(u_1, \frac{1}{2}u_2) + \text{cov}(u_1, u_5) \\ &= \frac{1}{2}\text{cov}(u_1, u_1) + \frac{1}{2}\text{cov}(u_1, u_2) \\ &= \frac{1}{2}\text{var}(u_1) = \frac{1}{2}\bar{F_u}^2 \end{aligned}$$

OHP Picture 9

$$G = \begin{bmatrix} 1+F_1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \dots \\ 0 & 1 & & & & \\ \frac{1}{2} & & 1 & 0 & & \\ \frac{1}{2} & & 0 & 1 & 0 & \\ \frac{1}{4} & & & 0 & 1 & \\ \vdots & & & & & \ddots \end{bmatrix} / \cdot \bar{\sigma}_u^2$$

$$G = A \cdot \bar{\sigma}_u^2 \quad \text{used in animal model}$$

(A)  $\rightarrow$  numerator relationship matrix

For Mixed Model Equations, we need  $G^{-1}$

$$G^{-1} = (A \cdot \bar{\sigma}_u^2)^{-1} = A^{-1} \cdot \bar{\sigma}_u^{-2}$$

$$\begin{bmatrix} X^T X & X^T Z \\ Z^T X & Z^T Z + A^{-1} \cdot \lambda \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X^T y \\ Z^T y \end{bmatrix}$$

$$\lambda = \bar{\sigma}_e^2 / \bar{\sigma}_u^2$$

Compute Numerator Relationship Matrix A

$$\text{Background: } \text{var}(\underline{u}) = G = A \cdot \bar{\sigma}_u^2$$

OHP Picture 10

Compute Numerator Relationship Matrix A

Background:  $\text{var}(\mathbf{u}) = \mathbf{G} = \mathbf{A} \cdot \mathbf{G_u}^2$

- Diagonal elements of  $\mathbf{A}$ :

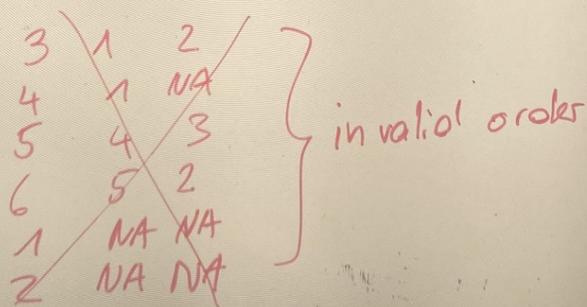
$$(\mathbf{A})_{ii} = (1 + F_i) \text{ with } F_i = \frac{1}{2} (\mathbf{A})_{\text{sd}}$$

- Off diagonal:  $(\mathbf{A})_{ij} = \frac{\text{cov}(u_i, u_j)}{(\mathbf{G}_u)^2}$

Recipe to compute  $\mathbf{A}$  based on given pedigree:

• Step 1: Complete all animals in pedigree

Result: Pedigree must be sorted such that parents are before offspring (topological sort)



OHP Picture 11

- Step 2: Empty Matrix A: square  
Dimension is the number of animals in pedigree

$$A_{6 \times 6} = \begin{array}{|c|c|c|c|c|c|} \hline & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & | & | & | & | & | & | \\ \hline 2 & | & | & | & | & | & | \\ \hline 3 & | & | & | & | & | & | \\ \hline 4 & | & | & | & | & | & | \\ \hline 5 & | & | & | & | & | & | \\ \hline 6 & | & | & | & | & | & | \\ \hline \end{array} \quad \left. \begin{array}{l} (A)_{12} \\ (A)_M \end{array} \right\}$$

- Step 3: Diagonal element  $(A)_{11} = 1 + F_1 = 1$   
Because animal 1 has unknown parents  
 $F_1 = \emptyset$

- Step 4: Off-diagonal:

$$(A)_{12}^{12} = \frac{1}{2} \left[ \underbrace{(A)_{1,NA}}_{=0} + \underbrace{(A)_{1,NA}}_{=0} \right] = \emptyset$$

$$\begin{aligned} (A)_{13}^{13} &= \frac{1}{2} \left[ (A)_{1,1} + (A)_{1,2} \right] \\ &= \frac{1}{2} [1 + \emptyset] = \frac{1}{2} \end{aligned}$$

$$(A)_{14}^{14} = \frac{1}{2} \left[ (A)_{1,1} + (A)_{1,NA} \right] = \frac{1}{2}$$

- Step 5: Copy first row into first column

OHP Picture 12

Step 5: Copy first row into first column

$$A = \left[ \begin{array}{cccccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} & \cancel{\frac{1}{4}} \\ \frac{1}{2} & & & & & \\ \frac{1}{2} & & & & & \\ \frac{1}{2} & & & & & \\ \frac{1}{4} & & & & & \end{array} \right]$$