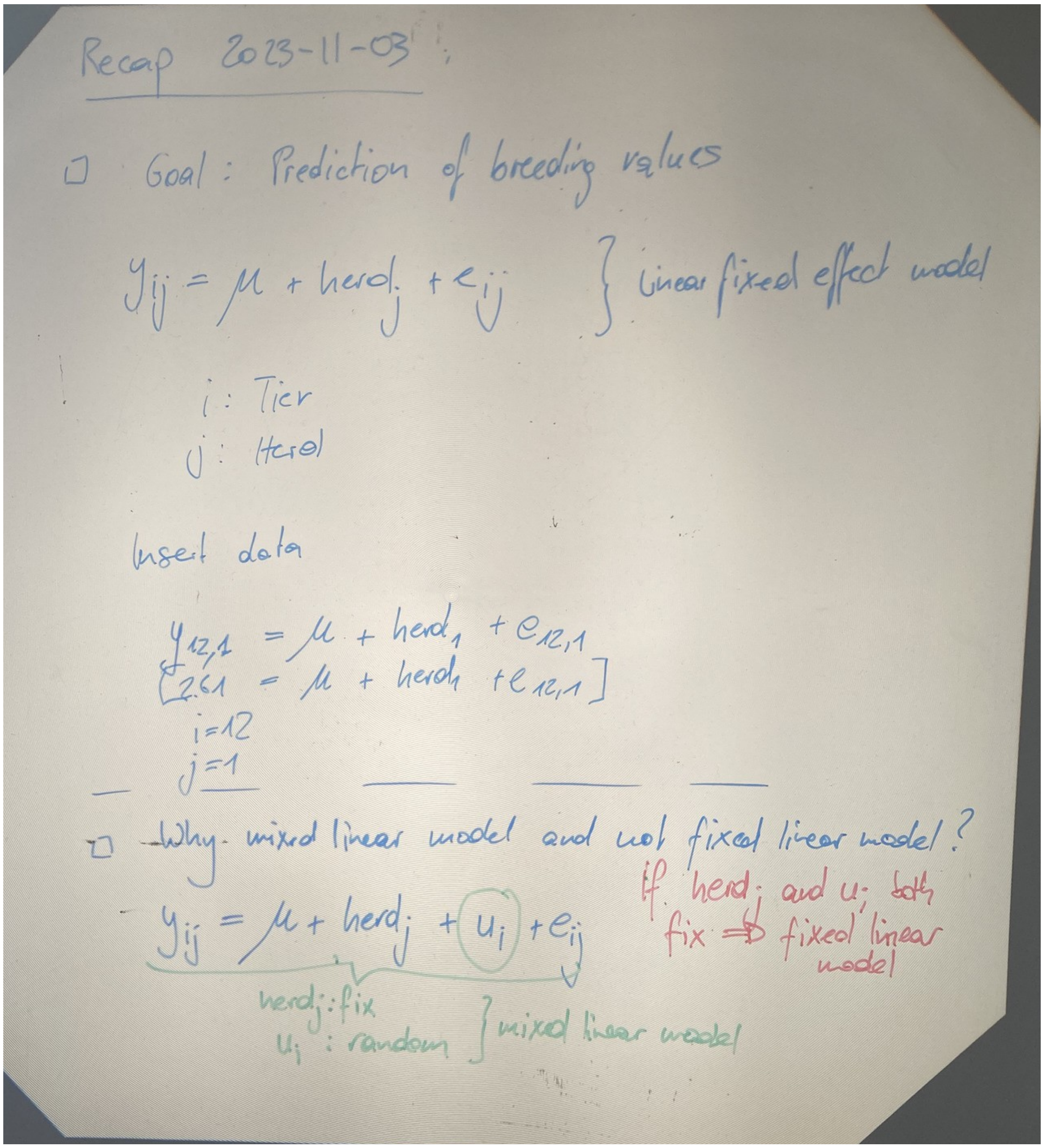
OHP Picture 1



Recap 2023-11-03:

- Goal: Prediction of breeding values

$$y_{ij} = \mu + herd_j + e_{ij}$$ } linear fixed effect model

i: Tier

j: Herd

Insert data

$$y_{12,1} = \mu + herd_1 + e_{12,1}$$

$$[2.61 = \mu + herd_1 + e_{12,1}]$$

$i=12$

$j=1$

- Why mixed linear model and not fixed linear model?

$$y_{ij} = \mu + herd_j + u_i + e_{ij}$$

If $herd_j$ and u_i both fix $\Rightarrow$ fixed linear model

$herd_j$: fix

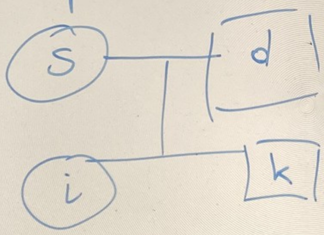
u_i: random } mixed linear model

OHP Picture 2

Breeding Values

$$y_{ij} = \mu + herd_j + u_i + e_{ij}$$

- Observation: weight of animal i in herd j
- u_i must be random, because $cov(u_i, u_k) \neq 0$ if animals i and k are related
- Example: animals i and k are full-sibs:



- For animal k: $y_{kl} = \mu + herd_l + u_j + e_{kl}$

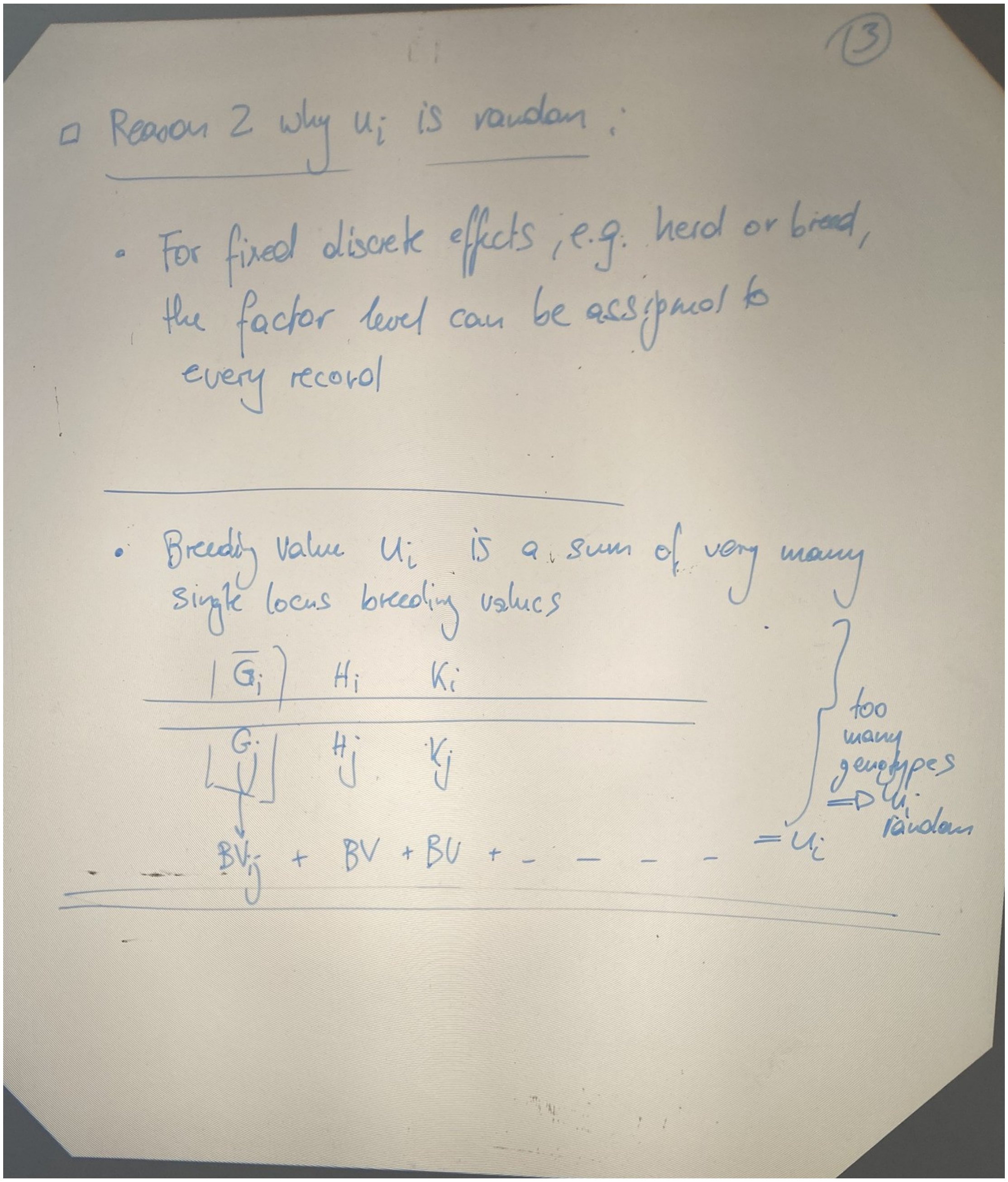
$cov(u_i, u_k) \neq 0$, because of decomposition of breeding values

$$\left.\begin{aligned} u_i &= 1/2\, u_s + 1/2\, u_d + m_i \\ u_k &= 1/2\, u_s + 1/2\, u_d + m_k \end{aligned}\right\} cov(u_i, u_k) \neq 0$$

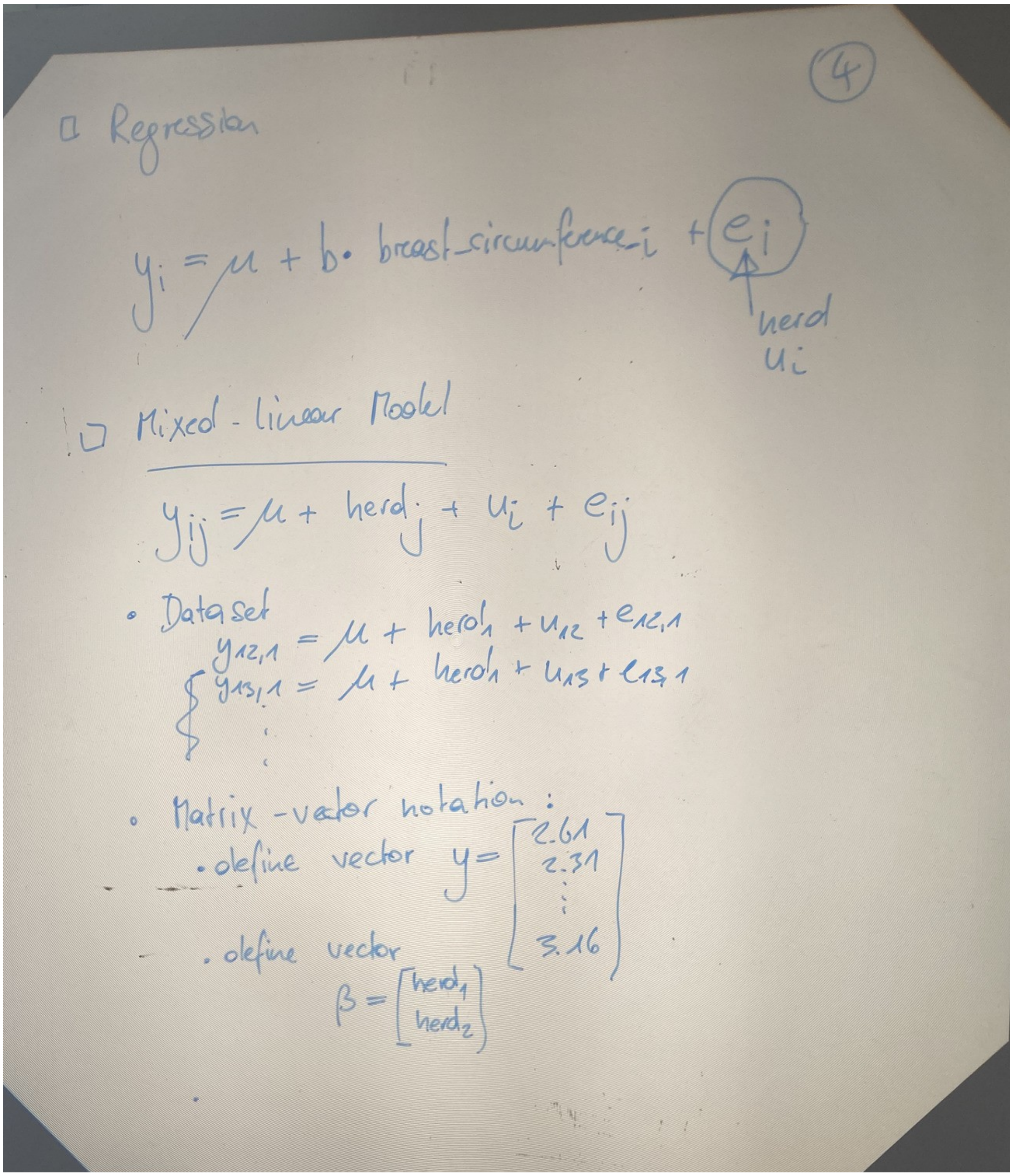
$$cov(u_i, u_k) = cov([1/2\, u_s + 1/2\, u_d + m_i], [1/2\, u_s + 1/2\, u_d + m_k])$$
$$= 1/4\, var(u_s) + 1/4\, var(u_d) = 1/2\, \sigma_u^2$$

provided s and d not related.

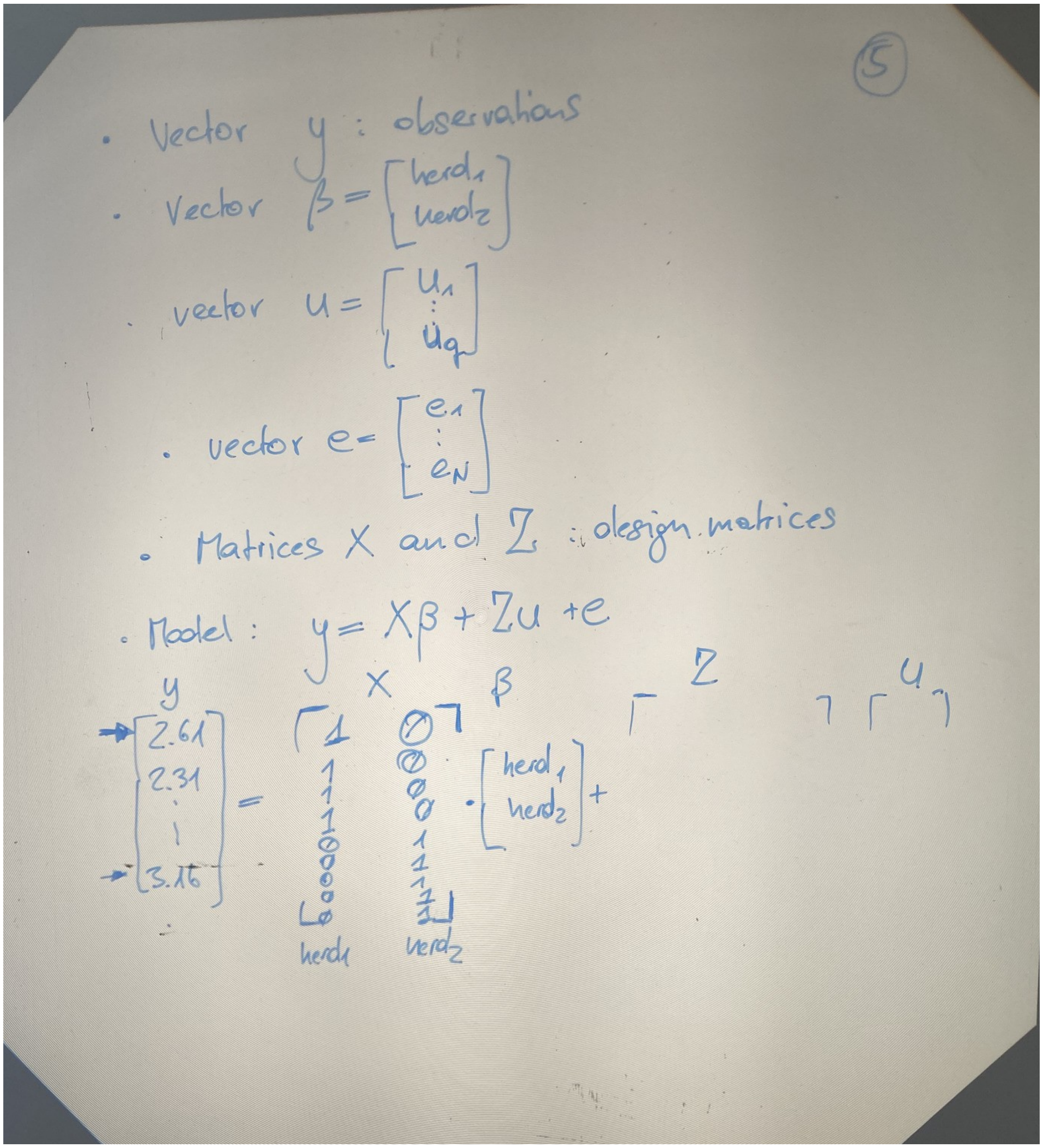
OHP Picture 3



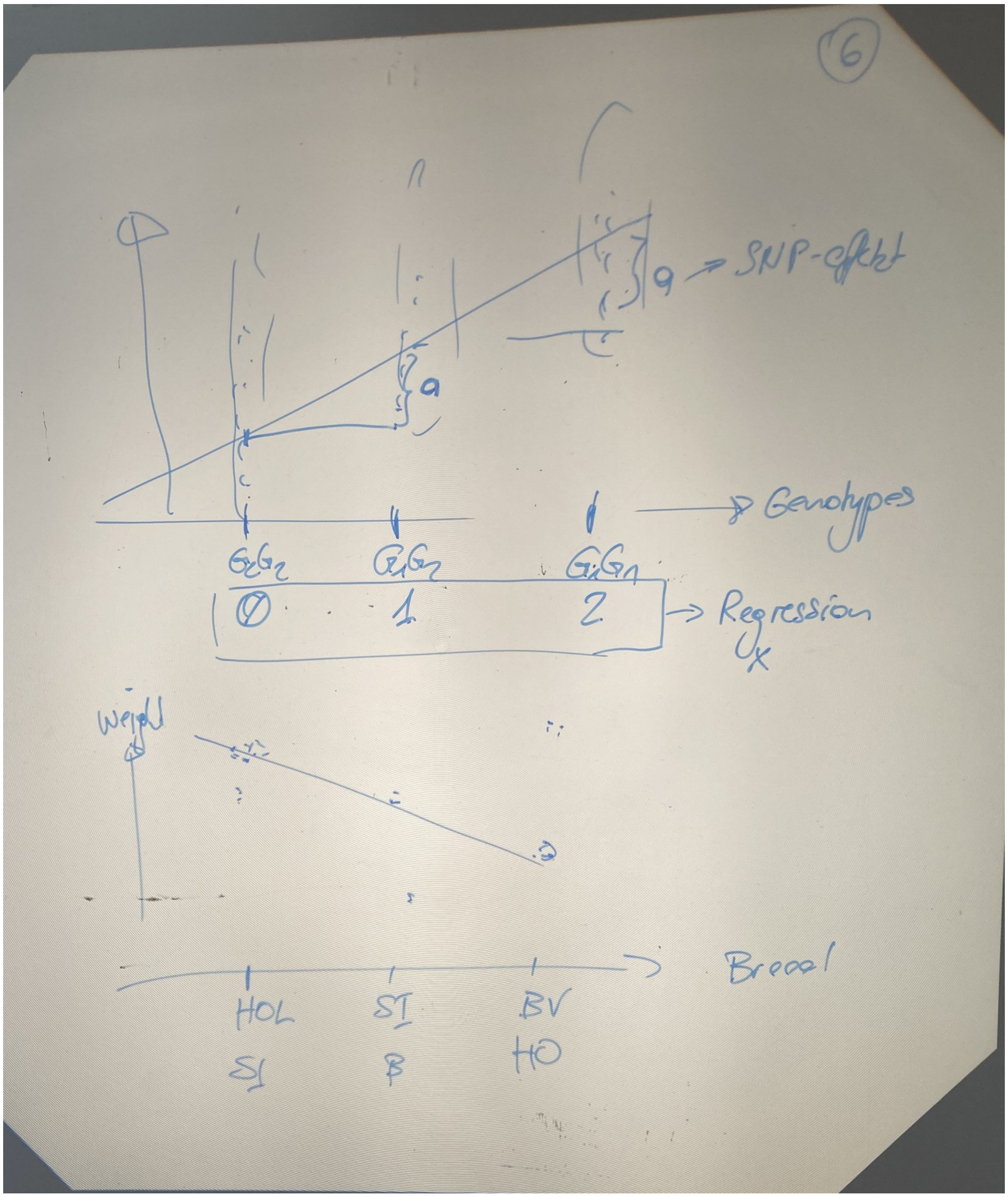
OHP Picture 4



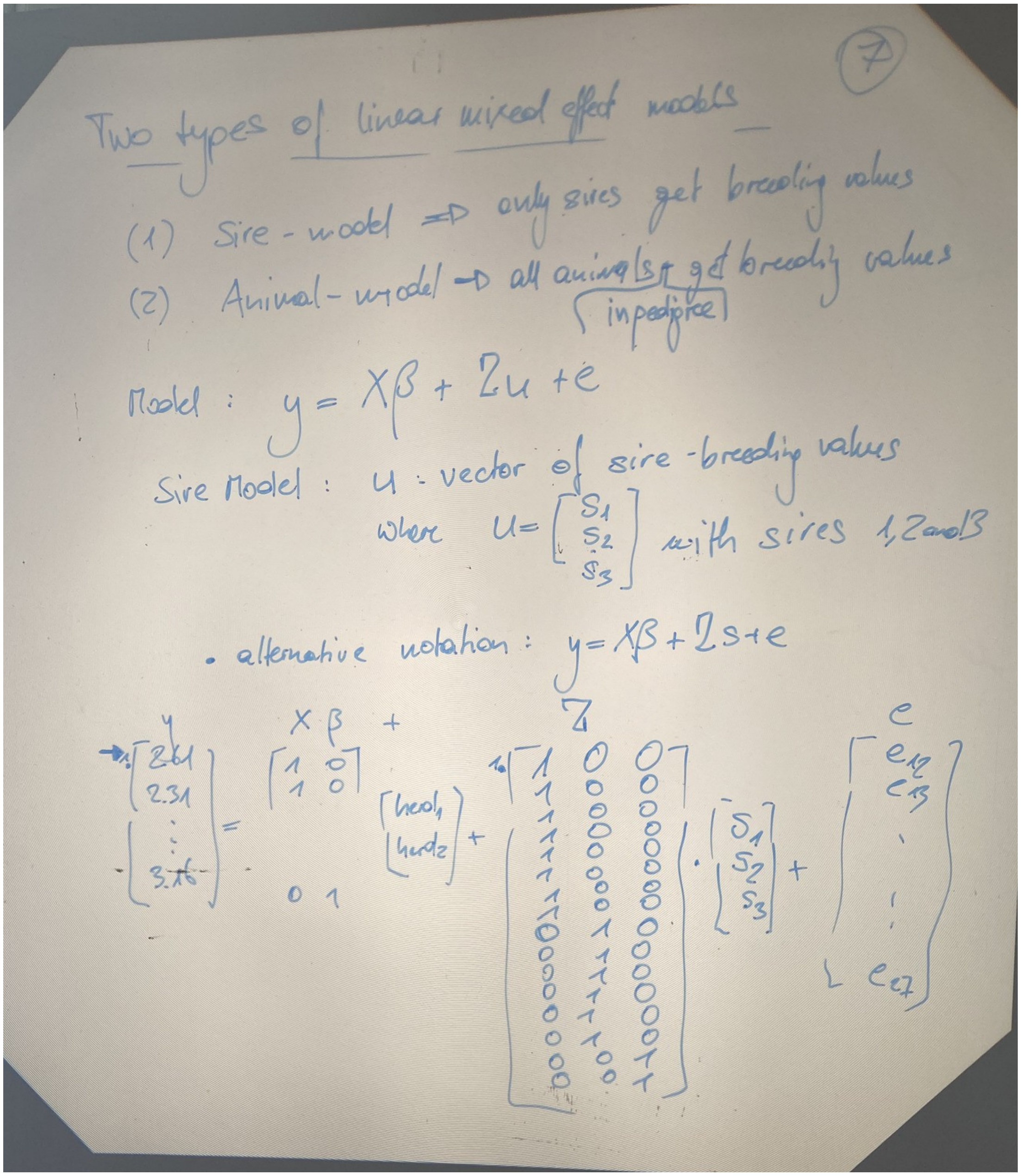
OHP Picture 5



OHP Picture 6



OHP Picture 7



(7)

Two types of linear mixed effect models

(1) Sire - model $\Rightarrow$ only sires get breeding values

(2) Animal - model $\to$ all animals (in pedigree) get breeding values

Model : $y = X\beta + Zu + e$

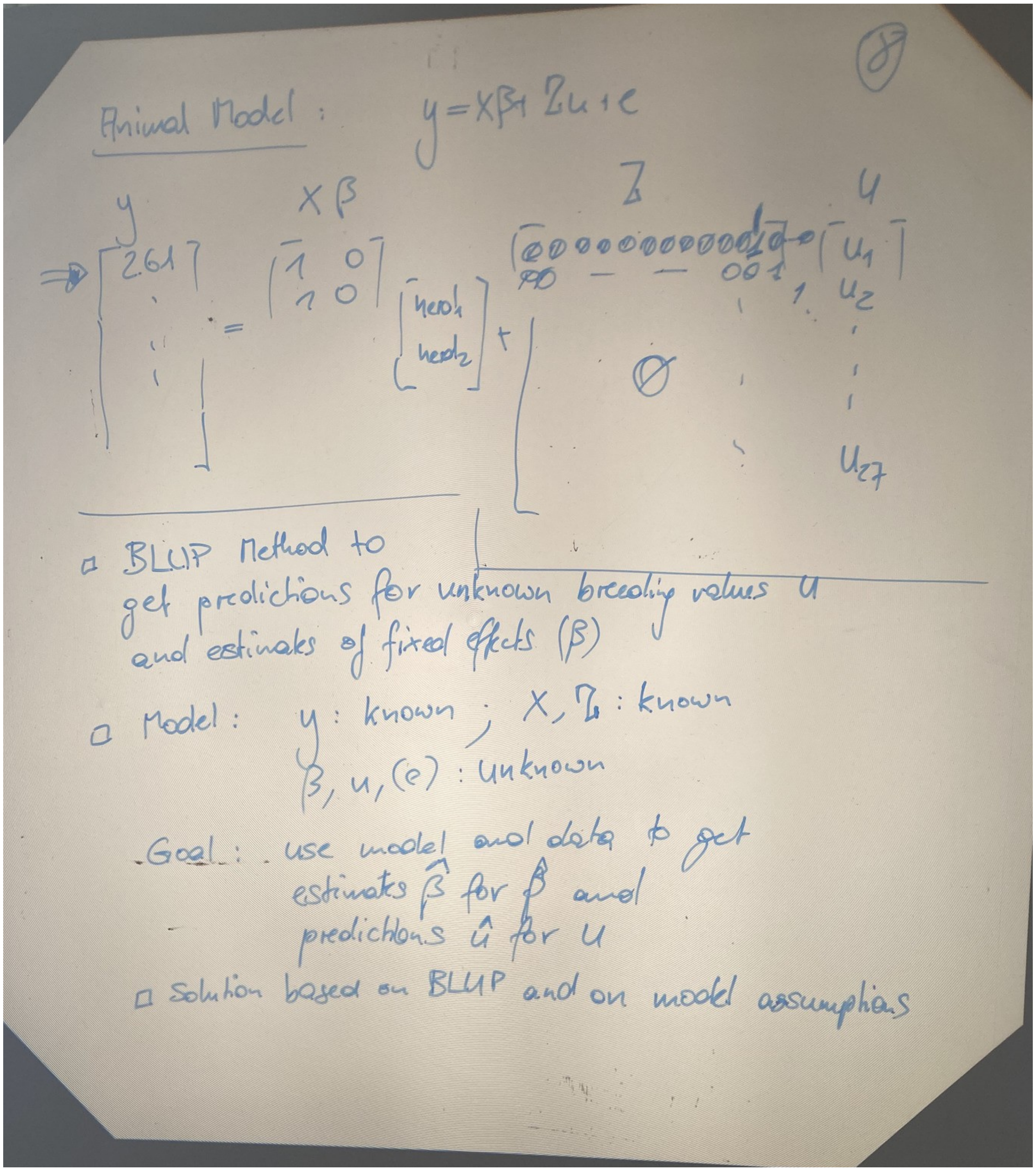
Sire Model : u : vector of sire - breeding values

where $u = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$ with sires 1, 2 and 3

- alternative notation : $y = X\beta + Zs + e$

$$\begin{array}{c} y \\ \begin{bmatrix} 2.61 \\ 2.31 \\ \vdots \\ 3.16 \end{bmatrix} \end{array} = \begin{array}{c} X \\ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ & \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \beta \\ \begin{bmatrix} herd_1 \\ herd_2 \end{bmatrix} \end{array} + \begin{array}{c} Z \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \cdot \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} + \begin{array}{c} e \\ \begin{bmatrix} e_{12} \\ e_{13} \\ \vdots \\ e_{27} \end{bmatrix} \end{array}$$

OHP Picture 8



8

Animal Model: $y = X\beta + Zu + e$

$$\underbrace{\begin{bmatrix} 2.61 \\ \vdots \\ \vdots \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} herd_1 \\ herd_2 \end{bmatrix}}_{X\beta} + \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \\ & & \emptyset & & & & 1 \\ & & & & & & \end{bmatrix}}_{Z} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{27} \end{bmatrix}}_{u}$$

- BLUP Method to get predictions for unknown breeding values u and estimates of fixed effects (β)
- Model: y: known; X, Z: known; $\beta, u, (e)$: unknown

Goal: use model and data to get estimates $\hat{\beta}$ for β and predictions $\hat{u}$ for u

- Solution based on BLUP and on model assumptions

OHP Picture 9

Model Assumptions: (must be specified for mixed model) (9)

- Expected values
  - vector u of breeding values, defined as deviations $\Rightarrow E(u) = 0 \Leftrightarrow E\begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}$
  - vector e of residuals defined as deviations $\Rightarrow E(e) = 0$

$$\begin{aligned} \Rightarrow E(y) &= E[X\beta + Zu + e] \\ &= E(X\beta) + E(Zu) + \underbrace{E(e)}_{0} \\ &= X\beta + Z \cdot \underbrace{E(u)}_{0} \\ &= X\beta \end{aligned}$$

- Abbreviation:

$$E\begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} X\beta \\ 0 \\ 0 \end{bmatrix}$$

- Variance: $var(u) = G$; $var(e) = R$; $var(y) = V$
  where G and R are known variance-covariance matrices.
  $cov(u, e') = 0$

OHP Picture 10

10

- $var(u)$: variance-covariance matrix

$$G = var(u) = \begin{bmatrix} var(u_1) & cov(u_1,u_2) & cov(u_1,u_3) & \cdots \\ cov(u_2,u_1) & var(u_2) & cov(u_2,u_3) & \cdots \\ \vdots & & & \end{bmatrix}$$

where $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{27} \end{bmatrix}$

- $var(e) = R = \begin{bmatrix} var(e_1) & cov(e_1,e_2) & \cdots \\ cov(e_2,e_1) & var(e_2) & \cdots \\ & & \end{bmatrix}$

- $var(e_1) = var(e_2) = \ldots = \sigma_e^2$

$cov(e_1,e_2) = cov(e_1,e_3) \ldots = 0$

$$\Rightarrow R = \begin{bmatrix} \sigma_e^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_e^2 & & & \\ \vdots & & & & \end{bmatrix} = I \cdot \sigma_e^2$$

OHP Picture 11

11

$$\mathrm{cov}(y,u) = \mathrm{cov}(X\beta + Zu + e, u^T)$$

$$= \underbrace{\mathrm{cov}(\underbrace{X\beta}_{fix}, u^T)}_{=\emptyset} + \mathrm{cov}(Zu, u^T) + \underbrace{\mathrm{cov}(e, u^T)}_{=\emptyset}$$

$$=$$

$$= \mathrm{cov}(Zu, u^T) = Z\underbrace{\mathrm{cov}(u,u^T)}_{var(u)} = ZG$$

$$\mathrm{cov}(y,e^T) = \mathrm{cov}(X\beta + Zu + e, e^T)$$

$$= \mathrm{cov}(X\beta, e^T) + \mathrm{cov}(Zu, e^T) + \mathrm{cov}(e, e^T)$$

$$= \emptyset \quad + Z\underbrace{\mathrm{cov}(u,e^T)}_{=\emptyset} + \mathrm{var}(e)$$

$$= R$$

$$\mathrm{var}\begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} V & ZG & R \\ GZ & G & \emptyset \\ R & \emptyset & R \end{bmatrix}; \quad V = ZGZ^T + R$$

OHP Picture 12

12

- Using properties of BLUP:
  solutions for $\hat{u}$ and $\hat{\beta}$:

$$\hat{u} = G Z^T V^{-1} (y - X\hat{\beta})$$

$$\hat{\beta} = (X^T V^{-1} X)^{-} X^T V^{-1} y$$

- Practical Problem: Large data sets (10^7 Mio records)
  ($6 \cdot 10^7$ Mio
  V^{-1}: cannot be computed

- Same solutions from system of equations:

$$\begin{bmatrix} X^T R^{-1} X & X^T R^{-1} Z \\ Z^T R^{-1} X & Z^T R^{-1} Z + G^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X^T R^{-1} y \\ Z^T R^{-1} y \end{bmatrix}$$

- $R = I\sigma_e^2 \Rightarrow R^{-1} = I \cdot 1/\sigma_e^2$