

BLUP

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Recap: Prediction of Breeding Values

- ▶ Own Performance Records (h^2 : Heritability)

$$\hat{u}_i = h^2 * (y_i - \mu)$$

- ▶ Repeated Measurements

$$\hat{u}_i = \frac{nh^2}{1 + (n-1)t} * (\bar{y}_i - \mu)$$

with n : number of records, t : repeatability

- ▶ Progeny Records

$$\hat{u}_i = \frac{2n}{n+k} * (\bar{y}_i - \mu) \quad \text{with } k = \frac{4-h^2}{h^2}$$

with n : number of progeny

General Principle

- ▶ All methods to predict breeding values follow the same principle
 1. Correct information sources for some population mean
 2. Multiply corrected information source by an appropriate factor

- ▶ Regression Method

$$\hat{u} = b(y - \mu)$$

- ▶ Selection Index

- ▶ uses all available information combined into an index

Selection Index

- ▶ will be presented later to estimate aggregate genotype
- ▶ Idea: all available information about an animals breeding value is combined into an index of merit (I)
- ▶ corresponds to multiple regression approach

$$\hat{u} = I = b_1 * y_1^* + b_2 * y_2^* + \dots + b_k * y_k^* = b^T y^*$$

where b the regression coefficients are computed such that the variance ($\text{var}(u - \hat{u})$) of the error is minimal.

Index Weights

- Minimization of the variance of the errors means

$$PEV = \text{var}(u - \hat{u}) = \text{var}(u - l) = \text{var}(u - b^T y^*)$$

$$= \text{var}(u) + \text{var}(b^T y^*) - 2\text{cov}(u, (y^*)^T b)$$

$$= \sigma_u^2 + b^T * \text{var}(y^*) * b - 2 * b^T * \text{cov}(u, (y^*)^T)$$

$$= \sigma_u^2 + b^T * P * b - 2 * b^T * G$$

Solution

► Compute $\frac{\partial PEV}{\partial b} = 0$

$$\frac{\partial PEV}{\partial b} = 2 * P * b - 2 * G = 0$$

$$\rightarrow b = P^{-1} * G$$

Problem with Correction

- ▶ Population mean is ideal as correction

$$y = \mu + u + e \quad \rightarrow \quad \bar{y} = \bar{\mu} + \bar{u} + \bar{e} = \mu$$

- ▶ Because performances are observed in different
 - ▶ environments and
 - ▶ time points
- ▶ Formation of comparison groups where animals are exposed to the same environments
- ▶ The more groups, the better the correction of environmental effects
- ▶ The more groups, the smaller the single groups

Bias

- ▶ With small comparison groups, it is more likely that mean breeding value of animals in a single group is not 0
- ▶ Average performance of all animals in a comparison group

$$\bar{y}_{CG} = \mu + \bar{u}_{CG} + \bar{e}_{CG}$$

* If \bar{u}_{CG} is not 0, the predicted breeding value \hat{u}_i of animal i is

$$\begin{aligned}\hat{u}_i &= I = b(y_i - (\mu + \bar{u}_{CG})) \\ &= b(y_i - \mu) - b\bar{u}_{CG} \\ &= \hat{u}_i - b\bar{u}_{CG}\end{aligned}$$

where $b\bar{u}_{CG}$ is called bias.

Solution - BLUP

- ▶ Solution to correction problem in selection index: BLUP
- ▶ Estimates environmental effects at the same time as breeding values are predicted
- ▶ Linear mixed effects model
- ▶ Meaning of BLUP
 - ▶ **B** stands for **best** → correlation between true (u) and its prediction (\hat{u}) is maximal or the prediction error variance ($\text{var}(u - \hat{u})$) is minimal.
 - ▶ **L** stands for **linear** → predicted breeding values are linear functions of the observations (y)
 - ▶ **U** stands for **unbiased** → expected values of the predicted breeding values are equal to the true breeding values
 - ▶ **P** stands for **prediction**

Example

Animal	Sire	Dam	Herd	Weaning Weight
12	1	4	1	2.61
13	1	4	1	2.31
14	1	5	1	2.44
15	1	5	1	2.41
16	1	6	2	2.51
17	1	6	2	2.55
18	1	7	2	2.14
19	1	7	2	2.61
20	2	8	1	2.34
21	2	8	1	1.99
22	2	9	1	3.10
23	2	9	1	2.81
24	2	10	2	2.14
25	2	10	2	2.41
26	3	11	2	2.54
27	3	11	2	3.16

Linear Models

- ▶ Simple linear model

$$y_{ij} = \mu + \textit{herd}_j + e_{ij}$$

- ▶ Result: Estimate of effect of herd j
- ▶ Try with given dataset

Linear Mixed Effects Model

- ▶ What about breeding value u_i for animal i ?
 - ▶ Problem: breeding values have a variance σ_u^2
 - ▶ Cannot be specified in simple linear model

→ **Linear Mixed Effects Model (LME)**

$$y_{ijk} = \mu + \beta_j + u_i + e_{ijk}$$

Matrix-Vector Notation

- LME for all animals of a population

→ use matrix-vector notation

$$y = X\beta + Zu + e$$

where

- y vector of length n of all observations
- β vector of length p of all fixed effects
- X $n \times p$ design matrix linking the fixed effects to the observations
- u vector of length n_u of random effects
- Z $n \times n_u$ design matrix linking random effect to the observations
- e vector of length n of random residual effects.

Expected Values and Variances

- ▶ Expected values

$$E(u) = 0 \text{ and } E(e) = 0 \rightarrow E(y) = X\beta$$

- ▶ Variances

$$\text{var}(u) = G \text{ and } \text{var}(e) = R$$

with $\text{cov}(u, e^T) = 0$,

$$\text{var}(y) = Z * \text{var}(u) * Z^T + \text{var}(e) = ZGZ^T + R = V$$

Estimates of unknown Parameters

$$\hat{u} = E(u|y) = GZ^T V^{-1}(y - X\hat{\beta})$$

$$\hat{\beta} = (X^T V^{-1} X)^{-1} X^T V^{-1} y$$

Mixed Model Equations

- ▶ Problem: V^{-1}
- ▶ Same solutions obtained with following set of equations

$$\begin{bmatrix} X^T R^{-1} X & X^T R^{-1} Z \\ Z^T R^{-1} X & Z^T R^{-1} Z + G^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X^T R^{-1} y \\ Z^T R^{-1} y \end{bmatrix}$$

Sire Model

- ▶ Breeding value of sire as random effect:

$$y = X\beta + Zs + e$$

Example

$$\begin{bmatrix} 2.61 \\ 2.31 \\ 2.44 \\ 2.41 \\ 2.51 \\ 2.55 \\ 2.14 \\ 2.61 \\ 2.34 \\ 1.99 \\ 3.1 \\ 2.81 \\ 2.14 \\ 2.41 \\ 2.54 \\ 3.16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \\ e_{10} \\ e_{11} \\ e_{12} \\ e_{13} \\ e_{14} \\ e_{15} \\ e_{16} \end{bmatrix}$$

Animal Model

- ▶ Breeding value for all animals as random effects

$$y = X\beta + Zu + e$$