

OHP Picture 1

Recap.

2023-11-10

④

□ Central topic :

Use linear mixed effects (LME) to predict breeding values

□ Why LME:

Because breeding values of related individuals are correlated, for animals i and j sharing some common ancestor, we have

$$\text{cov}(u_i, u_j) \neq \emptyset$$

□ LME: In matrix-vector notation

$$y = X\beta + Z_u + e$$

Annotations:

- y : known vector of observations
- $X\beta$: unknown vector of fixed effects (herd, sex, breed, ...)
- Z_u : unknown vector of random breeding values or covariants such as breast circumference
- e : unknown vector of random residuals

Matrices: X, Z known design matrices

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□ Additional Specification / Assumptions for LME ②

- For each random effect, expected value and variance-covariance matrix must be specified
- Random effects are: u, e, y

$$\left. \begin{array}{l} E(u) = 0 \\ E(e) = 0 \\ E(y) = E(X\beta + Zu + e) = X\beta \end{array} \right\} \text{together with model}$$

$$\left. \begin{array}{l} \text{var}(u) = G \\ \text{var}(e) = R \\ \text{cov}(u, e^T) = 0 \\ \text{cov}(y, u^T) = ZG \\ \text{cov}(y, e) = R \end{array} \right\} *$$

$$(*) \quad \text{var} \begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} V & ZG & R \\ ZG^T & G & 0 \\ R & 0 & R \end{bmatrix}, \quad V = ZGZ^T + R$$

$$E \begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} XB \\ 0 \\ 0 \end{bmatrix}$$

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Model : $y = X\beta + Bu + e$

$$E\begin{bmatrix} y \\ e \end{bmatrix} = \begin{bmatrix} X\beta \\ 0 \\ 0 \end{bmatrix}; \quad \text{var}\begin{bmatrix} y \\ e \end{bmatrix} = \begin{bmatrix} V & ZG & R \\ G^T & G & 0 \\ R & 0 & R \end{bmatrix}$$

Data : Introduce information from ~~used~~^{data} to the model for known parts :

$y = \begin{bmatrix} 4.5 \\ 2.9 \\ 3.9 \\ 3.5 \end{bmatrix}$; Known from data which is response variable (y)

Other information (except pedigree) are fixed
 \Rightarrow Herd with two levels

Based on y , X , Z we want estimates β for β and probabilities 0 for u

$\beta = \begin{bmatrix} \text{herd}_1 \\ \text{herd}_2 \end{bmatrix}; \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}; \quad e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$

$X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad Z = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

Mixed Model Equations (MME)

OHP Picture 4

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MME:

$$\begin{bmatrix} X^T R^{-1} X & X^T R^{-1} Z \\ Z^T R^{-1} X & Z^T R^{-1} Z + G^{-1} \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} X^T R^{-1} y \\ Z^T R^{-1} y \end{bmatrix}$$

- X, Z, y given by dataset
- R^{-1} is inverse of R , where $R = \text{var}(c) = I \cdot \sigma_e^2$
 $\Rightarrow R = \begin{bmatrix} \sigma_e^2 & 0 & \dots \\ 0 & \sigma_e^2 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$
- $\Rightarrow R^{-1} = \begin{bmatrix} 1/\sigma_e^2 & 0 & 0 & \dots \\ 0 & 1/\sigma_e^2 & \dots & \dots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}$ because $R \cdot R^{-1} = I$
- $= I \cdot 1/\sigma_e^2 = I \cdot \sigma_e^{-2}$; insert to MME

$$\begin{bmatrix} X^T I \sigma_e^{-2} X & X^T I \sigma_e^{-2} Z \\ Z^T I \sigma_e^{-2} X & Z^T I \sigma_e^{-2} Z + G^{-1} \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} X^T I \sigma_e^{-2} y \\ Z^T I \sigma_e^{-2} y \end{bmatrix}$$

$\downarrow \sigma_e^{-2}$
known via σ_e^2 and σ_e^{-2}

$$\boxed{\begin{bmatrix} X^T I \sigma_e^{-2} Z \\ Z^T I \sigma_e^{-2} X \\ D^T X \end{bmatrix} + G^{-1} \begin{bmatrix} \beta \\ \alpha \end{bmatrix}} = \begin{bmatrix} X^T y \\ Z^T y \\ D^T y \end{bmatrix}$$

$\boxed{?}$

OHP Picture 5

2. What is G and G^{-1}

(2)

→ Specification of LME:

$$\text{var}(u) = G \quad ; \text{ example } u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \end{bmatrix}, \text{ if}$$

we use animal model

$$\rightarrow \text{var}(u) = \text{var} \left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \end{bmatrix} \right) = \begin{bmatrix} \text{var}(u_1) & \text{cov}(u_1, u_2) & \text{cov}(u_1, u_3) \\ \text{cov}(u_2, u_1) & \text{var}(u_2) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

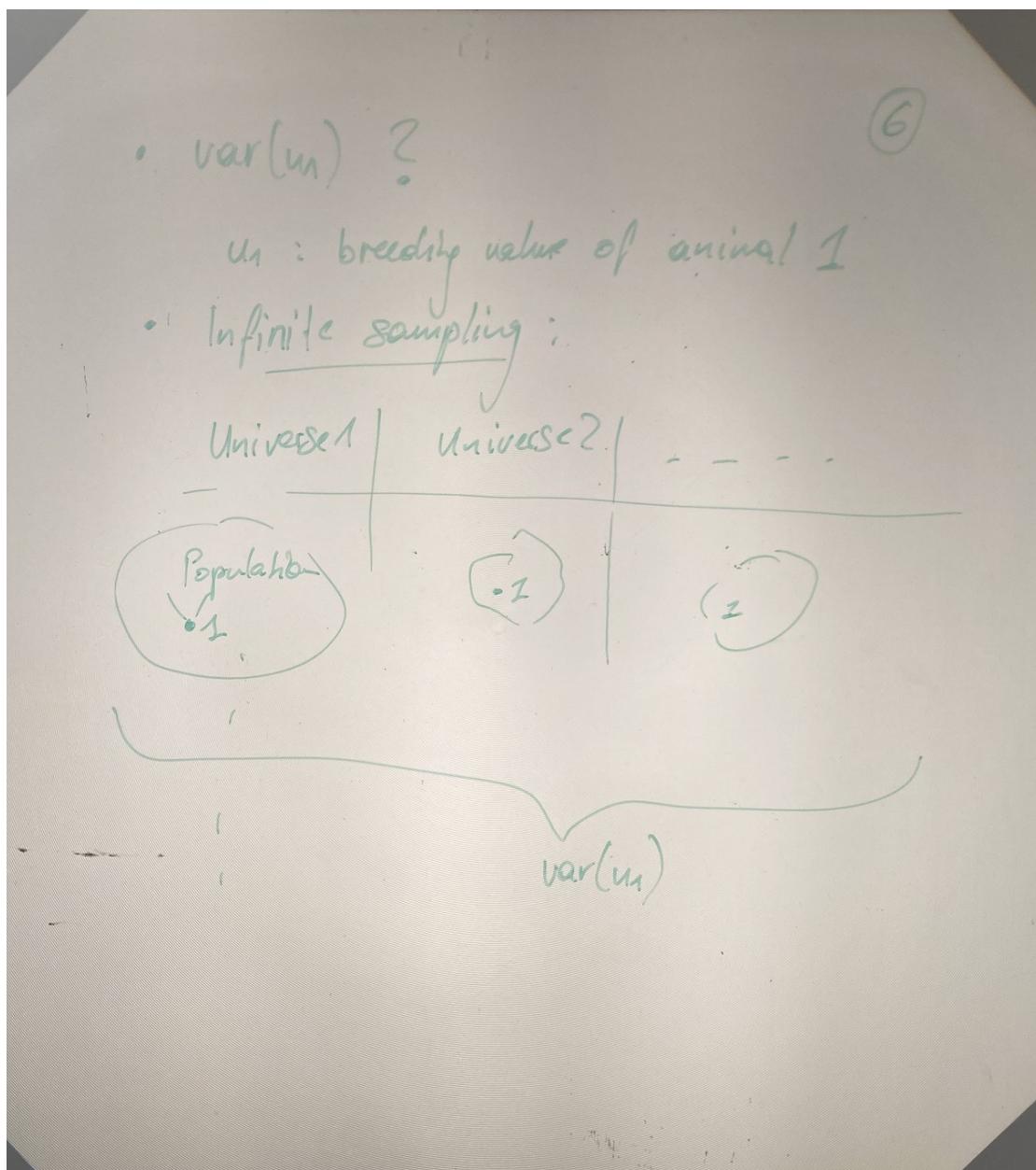
$$\text{var}(u_1) = \int (u_1 - E(u_1))^2 \cdot f(u_1) du_1$$

$$\text{cov}(u_1, u_2) = \iint (u_1 - E(u_1))(u_2 - E(u_2)) f(u_1, u_2) du_1 du_2$$

$$\text{cov}(u_2, u_1) = \iint (u_2 - E(u_2))(u_1 - E(u_1)) f(u_1, u_2) du_1 du_2$$

$$= \int (u_1 - E(u_1))^2 f(u_1) du_1 = \text{var}(u_1)$$

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(B)

- $\text{var}(u) = G$
- $= \begin{bmatrix} \text{var}(u_1) & \text{cov}(u_1, u_2) & \dots \\ \text{cov}(u_2, u_1) & \text{var}(u_2) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$

$\text{var}(u_1) = (1+F_1) \cdot \bar{G}_u^2 = \bar{G}_u^2$; F_1 = Inzuchtgrad von Tier 1

$\text{var}(u_2) = (1+F_2) \cdot \bar{G}_u^2 = \bar{G}_u^2$

- If animals not related
 $\text{cov}(u_1, u_2) = \emptyset$
- $\text{cov}(u_1, u_3) = \text{cov}(u_1, [\frac{1}{2}u_1 + \frac{1}{2}u_2 + w_3])$
- $u_3 = \frac{1}{2}u_1 + \frac{1}{2}u_2 + w_3$
- $= \text{cov}(u_1, \frac{1}{2}u_1) + \text{cov}(u_1, \frac{1}{2}u_2) + \text{cov}(u_1, w_3)$
- $= \frac{1}{2} \text{cov}(u_1, u_1) + \emptyset + \emptyset$
- $= \frac{1}{2} \text{var}(u_1) = \frac{1}{2} \bar{G}_u^2$

$F_1 \neq \emptyset$; if parents of 1 are related

Animals 1 and 2 are not related
 $= \emptyset$

OHP Picture 8

$$\begin{aligned}
 \text{⑧} \\
 \text{cov}(u_3, u_4) &= \text{cov}\left(\left[\frac{1}{2}u_1 + \frac{1}{2}u_2 + m_3\right], \left[\frac{1}{2}u_1 + \frac{1}{2}u_2 + m_4\right]\right) \\
 u_3 &= \frac{1}{2}u_1 + \frac{1}{2}u_2 + m_3 \\
 u_4 &= \frac{1}{2}u_1 + \frac{1}{2}u_2 + m_4 \\
 &= \underbrace{\text{cov}\left(\frac{1}{2}u_1, \frac{1}{2}u_1\right)}_{=0} + \underbrace{\text{cov}\left(\frac{1}{2}u_1, \frac{1}{2}u_2\right)}_{=0} \\
 &\quad + \underbrace{\text{cov}\left(\frac{1}{2}u_1, m_4\right)}_{=0} \\
 &\quad + \underbrace{\text{cov}\left(\frac{1}{2}u_2, \frac{1}{2}u_1\right)}_{=0} + \underbrace{2\text{cov}\left(\frac{1}{2}u_2, \frac{1}{2}u_2\right)}_{=0} \\
 &\quad + \underbrace{\text{cov}\left(\frac{1}{2}u_2, m_4\right)}_{=0} + \underbrace{\text{cov}\left(m_3, \frac{1}{2}u_1\right)}_{=0} \\
 &\quad + \underbrace{\text{cov}\left(m_3, \frac{1}{2}u_2\right)}_{=0} + \underbrace{\text{cov}\left(m_3, m_4\right)}_{=0} \\
 &= \text{cov}\left(\frac{1}{2}u_1, \frac{1}{2}u_1\right) + \underbrace{\text{cov}\left(\frac{1}{2}u_2, \frac{1}{2}u_2\right)}_{=0} \\
 &= \frac{1}{4}\text{cov}(u_1, u_1) + \frac{1}{4}\text{cov}(u_2, u_2) \\
 &= \frac{1}{4}\text{var}(u_1) + \frac{1}{4}\text{var}(u_2) \\
 &= \frac{1}{4}\sigma_{u_1}^2, \frac{1}{4}\sigma_{u_2}^2 - \frac{1}{2}\sigma_{u_1}^2
 \end{aligned}$$

OHP Picture 9

③

Summary: All elements of G depend
on $\bar{u}^2 \Rightarrow G = A \cdot \bar{u}^2$

\nwarrow Numerator Relationship Matrix

\Rightarrow Definition of A :

- Diagonal elements: $(A)_{ii} = 1 + F_i$
- off diagonal: $(A)_{ij} = \frac{\text{cov}(u_i, u_j)}{\bar{u}^2}$
element in row i and column j

$A = \begin{bmatrix} 1 & \dots & j & \dots & 1 \\ \vdots & & \vdots & & \vdots \\ i & \rightarrow (A)_{ij} & & & \end{bmatrix}$

\nwarrow relationship coefficient between animals i and j

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• Parents is add of animal i

$A = \begin{bmatrix} 1 & 1/2(A)_{sd} \\ 1/2(A)_{sd} & 1 \end{bmatrix}$

$F_i = 1/2(A)_{sd}$

$(A)_{ji} = 1/2(A)_{js} + 1/2(A)_{jd}$

- Meaning of $(A)_{ji} = \frac{\text{cov}(u_j, u_i)}{\sigma_u^2}$; with $u_i = 1/2 u_s + 1/2 u_d + m_i$.

$= \frac{\text{cov}(u_j, [1/2 u_s + 1/2 u_d + m_i])}{\sigma_u^2} = \frac{\text{cov}(u_j, 1/2 u_s) + \text{cov}(u_j, 1/2 u_d)}{\sigma_u^2}$

OHP Picture 11

$$\begin{aligned}
 & G = A \cdot \sigma_u^2 \\
 \textcircled{*} \quad & \text{cov}(u_j, u_i) = \underbrace{(A)_{ji}}_{\text{j}} \cdot \sigma_u^2 \\
 & \left[\begin{array}{c} G \\ \vdots \\ \text{--- } \text{cov}(u_j, u_i) \end{array} \right] = \left[\begin{array}{ccc} s & A_d & \vdots \\ \vdots & \ddots & \vdots \\ (A)_{js} & (A)_{jd} & (A)_{ji} \end{array} \right] \cdot \sigma_u^2 \\
 & u_i = \frac{1}{2} u_s + \frac{1}{2} u_d + m_i \quad \text{insert to } \textcircled{*} \\
 \text{cov}(u_j, u_i) &= \text{cov}(u_j, \left[\frac{1}{2} u_s + \frac{1}{2} u_d + m_i \right]) \quad \underbrace{=}_{=0} \\
 &= \text{cov}(u_j, \frac{1}{2} u_s) + \text{cov}(u_j, \frac{1}{2} u_d) + \text{cov}(u_j, m_i) \\
 &= \frac{1}{2} \text{cov}(u_j, u_s) + \frac{1}{2} \text{cov}(u_j, u_d) \\
 \text{cov}(u_j, u_s) &= (A)_{js} \cdot \sigma_u^2 ; \quad \text{cov}(u_j, u_d) = (A)_{jd} \cdot \sigma_u^2 \\
 \text{cov}(u_j, u_i) &= \frac{1}{2} (A)_{js} \cdot \sigma_u^2 + \frac{1}{2} (A)_{jd} \cdot \sigma_u^2 = \frac{1}{2} [(A)_{js} + (A)_{jd}] \sigma_u^2 \\
 &= \boxed{(A)_{ji} \sigma_u^2}
 \end{aligned}$$

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(12)

| | | | | | | |
|---|---|----|----|---------|--|--|
| | | | | | | |
| 3 | 1 | 2 | -3 | 12 | | |
| 4 | 1 | NA | 4 | 1 NA | | |
| 5 | 4 | 3 | 5 | 4 3 | | |
| 6 | 5 | 2 | 6 | 5 2 | | |
| | | | | 1 NA NA | | |
| | | | | 2 NA NA | | |

~~NA NA NA NA 1 2 1 2 1 2 1 2 1 2 1 2~~
~~1 2 3 4 5 6~~

$$A = \begin{bmatrix} 1 & \boxed{1} & 0 & 1/2 & 1/2 & 1/4 \\ 2 & 0 & \boxed{1} & 1/2 & 1/2 & 1/4 \\ 3 & 1/2 & & & & \\ 4 & 1/2 & & & & \\ 5 & 1/2 & & & & \\ 6 & 1/4 & & & & \end{bmatrix}$$

$(A)_{11} = 1 + F_1$
 $= 1 + 0 = 1$

$(A)_{12} = \frac{1}{2}(A)_{11N} + \frac{1}{2}(A)_{12N}$
 $= \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

$(A)_{13} = \frac{1}{2}(A)_{11} + \frac{1}{2}(A)_{12}$
 $= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}$

$(A)_{14} = \frac{1}{2}(A)_{11N} + 0 = \frac{1}{2}$

$(A)_{15} = \frac{1}{2}(A)_{11N} + \frac{1}{2}(A)_{12N} = \frac{1}{2} \cdot \frac{1}{2} + 0 = \frac{1}{4}$