THE GIT OF $4 \times 3 \times 3$ TENSORS (DETERMINANTAL CUBIC SURFACES)

CHARLOTTE SHERRATT

ABSTRACT. A $4 \times 3 \times 3$ tensor defines three determinantal varieties in $\mathbb{P}^3, \mathbb{P}^2$, and $\mathbb{P}^2$ where it drops rank: a cubic surface, and (in general) two length 6 subschemes. We study the Geometric Invariant Theory (GIT) quotient of $\mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ by $\mathrm{SL}(4) \times \mathrm{SL}(3) \times \mathrm{SL}(3)$. This moduli space is the $\mathrm{SL}(3)$-GIT-quotient of the final model of the Hilbert scheme of 6 points in $\mathbb{P}^2$ and admits a surjective rational map to the GIT moduli space of cubic surfaces, undefined at only one point.

Using the Hilbert-Mumford criterion and topological criterion for semistability, we explicitly describe the (semi)stable locus and the geometry of the GIT quotient. We also describe a blowup of the GIT quotient that resolves the rational map to the moduli space of cubic surfaces. This blowup is an $\mathrm{SL}(3)$-quotient of the Hilbert scheme of 6 points in $\mathbb{P}^2$.

The following document is an almost-complete draft. Some of the proofs in Subsection 2.3 are incomplete, and some of the general prose and citations throughout the document are missing. The majority of the new content of the paper is in Section 3, including a geometric classification of the stable, polystable, and semistable points of $\mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ under the $\mathrm{SL}(4) \times \mathrm{SL}(3) \times \mathrm{SL}(3)$-action, and a description of geometry of GIT quotient of the strictly semistable locus. Unless otherwise specified, all results and proofs in Section 3 are my own, completed with some advice from Dr Anand Deopurkar, with the exception of the degree computation in Proposition 3.4.

CONTENTS

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1. Introduction

Recall the well-known fact that any smooth cubic surface S is isomorphic to the blowup of $\mathbb{P}^2$ at 6 distinct points $X \subset \mathbb{P}^2$ in general position with respect to plane curves. These 6 points are cut out by an ideal sheaf $\mathcal{I}_X$ generated by four cubic forms, which by the Hilbert-Burch theorem has a free resolution of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-4)^{\oplus 3} \xrightarrow{\phi_1} \mathcal{O}_{\mathbb{P}^2}(-3)^{\oplus 4} \longrightarrow \mathcal{I}_X \longrightarrow 0.$$

Similarly, the ideal sheaf of a twisted cubic C inside a smooth cubic surface $S \subset \mathbb{P}^3$ has a free resolution of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 3} \xrightarrow{\phi_2} \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 3} \longrightarrow \mathcal{I}_{C/S} \longrightarrow 0.$$

The matrix ϕ_1 is a 4×3 matrix of linear forms in 3 variables, the matrix ϕ_2 is a 3×3 matrix of linear forms in 4 variables. The surface S is cut out by the cubic polynomial $\det(\phi_2)$. The sheaf $\mathcal{I}_{C/S}$ is a line bundle on its support, and its twist $\mathcal{I}_{C/S}(2)$ (the line bundle associated to a twisted cubic) induces a morphism $p : S \to \mathbb{P}^2$. Up to $\mathrm{PGL}(3)$, this morphism is the blowup of $\mathbb{P}^2$ at X, and the matrices ϕ_1 and ϕ_2 are adjoints of the same $4 \times 3 \times 3$ tensor (up to a change of basis).

These facts were observed in the language of classical geometry by Segre in 1906 [Seg06], and Thrall published a rough classification of $4 \times 3 \times 3$ tensors in 1941 [Thr41]. This was completed by Ng in 2002 [Ng02], with an explicit classification of (almost) all $4 \times 3 \times 3$ tensors up to projective equivalence.

In 1989, Gimigliano introduced a sheaf-theoretic approach to smooth determinantal surfaces [Gim89], viewing the various matrices as maps of vector bundles on $\mathbb{P}^2$ and $\mathbb{P}^3$. This was extended by Dolgachev and Kapranov in [DK93], detailing the specific geometry of smooth determinantal cubic surfaces. Dolgachev gives a detailed summary of this approach in [Dol12, §9.3]. Eisenbud and Popescu's series of papers on Gale transform in the late 90s [EP99; EP00] give a modern treatment of the connections between the Gale transform, free resolutions of ideals of points and determinantal varieties, of which $4 \times 3 \times 3$ tensors give the cleanest example.

More recently, the connection between determinantal representations of cubic surfaces were studied by Lehn, Lehn, Sorger and Straten in [LLSS17], who describe the geometry of the partial GIT quotient of $\mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ by $\mathrm{SL}(3) \times \mathrm{SL}(3)$ to describe the moduli space of twisted cubics on a fixed smooth cubic fivefold.

1.1. **Conventions and organisation.** Here k denotes a fixed algebraically closed field of characteristic zero. The projective space $\mathbb{P}V$ refers to the projective space of 1-dimensional subspaces of V, i.e. $\mathbb{P}V = \mathrm{Proj}(k[V^\vee]_\bullet)$ and $H^0(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(1)) = V^\vee$. For a vector space V, we will always let $\mathrm{SL}(V)$ act with the standard left action $v \mapsto g \cdot v$, and on $V^\vee$ by the left action $\nu \mapsto \nu \circ g^{-1}$ of precomposition.

1.2. **Notation.**

Definition 1.1. Let U, V, W be complex vector spaces with $\dim U = 4$ and $\dim V = \dim W = 3$. Note that V and W play symmetric roles in this paper. Whenever we fix bases of $U^\vee, V^\vee, W^\vee$, we will write the basis vectors as u_0, u_1, u_2, u_3, v_0, v_1, v_2, and w_0, w_1, w_2.

Definition 1.2. Let $G = \mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$.

Definition 1.3. We let $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ be a $4 \times 3 \times 3$ tensor. The tensor φ has several adjoints we are interested in, we use the following notation to describe them: for each decomposition of $U \otimes V \otimes W$ as $A \otimes B$, we write φ_A for the adjoint linear map

$$\varphi_A : A \longrightarrow B^\vee. \tag{1}$$

For example, $\varphi_U : U \to V^\vee \otimes W^\vee$, or $\varphi_{U\otimes W} : U \otimes W \to V^\vee$. In this notation, $\varphi = \varphi_{U\otimes V\otimes W}$. When $A, B, C \in \{U, V, W\}$ are three distinct vector spaces, we view $\varphi_A : A \to B^\vee \otimes C^\vee = \mathrm{Hom}(B, C^\vee)$ as an element of $A^\vee \otimes \mathrm{Hom}(B, C^\vee)$, which after choosing bases for B and C can be identified with a $\dim(C) \times \dim(B)$ matrix of linear forms in A.

We also interpret φ as a map of sheaves. We write

$$\widetilde{\varphi}_{\mathbb{P}A,B} : B \otimes \mathcal{O}_{\mathbb{P}A}(-1) \longrightarrow C^\vee \otimes \mathcal{O}_{\mathbb{P}A}, \tag{2}$$

where $\widetilde{\varphi}_{\mathbb{P}A,B}$ is presented by the matrix of linear forms $\varphi_A \in A^\vee \otimes \mathrm{Hom}(B, C^\vee)$.

Definition 1.4 (Associated subschemes)**.** We are interested in the subschemes of $\mathbb{P}A$ where φ_A drops rank. The most important ones will be given special names: let $S_\varphi \subset \mathbb{P}U$ be the subscheme where φ_U drops rank, it is a cubic surface. In a basis, S_φ is cut out by the cubic polynomial $\det(\varphi_U) \in k[U^\vee]_\bullet$ (the determinant of the matrix $\varphi_U \in U^\vee \otimes \mathrm{Hom}(V, W^\vee)$). The surface S_φ is called the *associated cubic surface* to φ. We let $R_\varphi \subset S_\varphi$ be the subscheme where φ_U has rank at most 1, it may be empty.

We let $X_\varphi \subset \mathbb{P}V$ be the subscheme where φ_V drops rank, it is cut out by the vanishing of the four maximal 3×3 minors of φ_V, which are cubic polynomials in $k[V^\vee]_\bullet$. We let $\mathcal{I}_{X_\varphi} \subset \mathcal{O}_{\mathbb{P}V}$ be the ideal sheaf cutting out X_φ, generated by these four minors. Similarly, we let $Y_\varphi \subset \mathbb{P}W$ be the subscheme on which φ_W drops rank, it is similarly cut out by an ideal sheaf $\mathcal{I}_{Y_\varphi} \subset \mathcal{O}_{\mathbb{P}W}$. The subschemes X_φ and Y_φ are called the *associated length* 6 *subschemes* to φ.

We also let $Z_\varphi \subset \mathbb{P}(V \otimes W)$ be the subscheme on which $\varphi_{V\otimes W}$ has rank at most 1. The subscheme Z_φ is contained in the Segre variety $\mathbb{P}V \times \mathbb{P}W \subset \mathbb{P}(V \otimes W)$.

Proposition 1.5. *The subscheme $R_\varphi \subset S_\varphi$ is contained in the singular locus of S_φ.*

Proof. The derivative of the determinant vanishes when all the 2×2 minors vanish. □

1.3. **Determinantal representations of cubic surfaces.**

Definition 1.6. Let $S \subset \mathbb{P}U$ be a cubic surface. A *determinantal representation* of S is a tensor $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ such that $S = S_\varphi$.

Let $X \subset \mathbb{P}V$ (resp. $Y \subset \mathbb{P}W$) be a length 6 subscheme. A *determinantal representation* of X (resp. Y) is a tensor $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ such that $X = X_\varphi$ (resp. $Y = Y_\varphi$).

Definition 1.7 (Determinantal representations)**.** Let $S \subset \mathbb{P}U$ be a cubic surface. A *determinantal representation* of S is a tensor $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ such that $S = S_\varphi$.

Let $X \subset \mathbb{P}V$ (resp. $Y \subset \mathbb{P}W$) be a length 6 subscheme. A *determinantal representation* of X (resp. Y) is a tensor $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ such that $X = X_\varphi$ (resp. $Y = Y_\varphi$).

The determinantal representations of cubic surfaces with at worst canonical (ADE) singularities is rather elegant, and can be easily understood from their associated length 6 subschemes. We introduce some language to describe these subschemes, closely following Section 2.5 of [DPT21].

Definition 1.8 (Canonical, admissible subschemes)**.** A cubic surface $S \subset \mathbb{P}^3$ is said to be *canonical* if S is smooth or has only canonical singularities.

A length 6 subscheme $X \subset \mathbb{P}^2$ is said to be *canonical* if:

(1) X is curvilinear,
(2) $h^0(\mathbb{P}^2, \mathcal{I}_X(3)) = 4$,
(3) and $\mathcal{I}_X(3)$ is generated by its global sections.

A canonical length 6 subscheme $X \subset \mathbb{P}^2$ is said to be *admissible* if it is not contained in a conic.

Proposition 1.9 (Proposition 2.5.2 of [DPT21])**.** *Let $X \subset \mathbb{P}^2$ be a canonical length 6 subscheme. Let $S = \mathrm{Bl}_X \mathbb{P}^2$, let $f : \mathrm{Bl}_X \mathbb{P}^2 \to \mathbb{P}^2$ be the blowup. The following hold:*

(1) S has at worst A_n singularities.
(2) $H^0(S, \omega_S^{-1})$ is 4-dimensional and basepoint-free.
(3) The image of the map $S \to \mathbb{P}^3$ induced by ω_S^{-1} is a cubic surface S_X.
(4) Let $R_\bullet = \bigoplus_{n \geq 0} R_n$ be the homogeneous coordinate ring of S_X. The pullback $R_n \to H^0(S, \omega_S^{-n})$ is an isomorphism.
(5) The map $S \to S_X$ is an isomorphism away from the following curves, which it contracts:
   - *The strict transform $f^*(C)$ of any conic $C \subset \mathbb{P}^2$ containing X, or*
   - *the strict transform $f^*(L)$ of any line $L \subset \mathbb{P}^2$ such that $\mathrm{len}(X \cap L) = 3$.*

(6) S_X is normal and has only canonical singularities. The Dynkin diagram associated to the singularities of S_X is a subdiagram of $\widetilde{E}_6$.

Definition 1.10 (Associated cubic surface)**.** Let $X \subset \mathbb{P}^2$ be a canonical length 6 subscheme. The *cubic surface associated to* X, denoted by S_X is the image of the map $\mathrm{Bl}_X \mathbb{P}^2 \to \mathbb{P}^3$ induced by the linear system $H^0(\mathrm{Bl}_X \mathbb{P}^2, \omega^{-1}_{\mathrm{Bl}_X \mathbb{P}^2})$.

There are two convenient equivalent definitions of S_X. One is that S_X is the closure of the image of the map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ induced by the linear system $H^0(\mathbb{P}^2, \mathcal{I}_X(3))$. The other follows from part (4) of Proposition 1.9, where

$$S_X = \mathrm{Proj}\left(\bigoplus_{n \geq 0} H^0(\mathrm{Bl}_X \mathbb{P}^2, \omega^{-n}_{\mathrm{Bl}_X \mathbb{P}^2})\right).$$

Remark 1.11. This construction can also be done in families. For ease of notation we only write it over affine schemes. Let $\mathrm{Spec}\, A$ be an affine scheme, and $X \subset \mathbb{P}^2_A = \mathbb{P}^2 \times \mathrm{Spec}\, A$ a closed subscheme, flat over $\mathrm{Spec}\, A$, whose fibres are length 6 canonical subschemes. The blowup $\mathrm{Bl}_X \mathbb{P}^2_A$ is flat over $\mathrm{Spec}\, A$, and the associated family of cubic surfaces

$$\mathcal{S}_X = \mathrm{Proj}_A\left(\bigoplus_{n \geq 0} H^0(\mathrm{Bl}_X \mathbb{P}^2_A, \omega^{-n}_{\mathrm{Bl}_X \mathbb{P}^2_A})\right)$$

is A-flat and is embedded in the projectivisation of the rank 4, locally free A-module $H^0(\mathrm{Bl}_X \mathbb{P}^2_A, \omega^{-1}_{\mathrm{Bl}_X \mathbb{P}^2_A})$. Note that under the blowup $f : \mathrm{Bl}_X \mathbb{P}^2 \to \mathbb{P}^2$, we have $f^*\mathcal{I}_X(3) = \omega^{-1}_{\mathrm{Bl}_X \mathbb{P}^2_A}$, inducing an isomorphism of locally free A-modules

$$H^0(\mathbb{P}^2_A, \mathcal{I}_X(3)) \cong H^0(\mathrm{Bl}_X \mathbb{P}^2_A, \omega^{-1}_{\mathrm{Bl}_X \mathbb{P}^2_A})$$

because the map $H^0(\mathbb{P}^2, \mathcal{I}_{X_a}(3)) \to H^0((\mathrm{Bl}_X \mathbb{P}^2_A)_a, \omega^{-1}_{(\mathrm{Bl}_X \mathbb{P}^2_A)_a})$ is an isomorphism on every fibre.

This induces a rational map

$$\mathrm{Hilb}_6(\mathbb{P}V) \dashrightarrow |\mathcal{O}_{\mathbb{P}U}(3)| /\!/ \mathrm{SL}(U), \tag{3}$$

defined on the locus of canonical subschemes whose associated cubic surface is $\mathrm{SL}(U)$-semistable.

Let S be a canonical cubic surface. It is well-known that the set of birational maps $S \dashrightarrow \mathbb{P}^2$ are in bijection with canonical length 6 subschemes $X \subset \mathbb{P}^2$ whose associated cubic surface is S. these birational maps are the inverses of the map $\mathbb{P}^2 \dashrightarrow |\mathcal{I}_X(3)|$ induced by the linear system of cubics through X.

When X is admissible, i.e. not contained in a conic, then the ideal sheaf $\mathcal{I}_X$ has a free resolution presented by a 4×3 matrix of linear forms in $k[V^\vee]_\bullet$, which induces a determinantal representation. However, when X is contained in a conic, no such resolution exists, and the subscheme X is not determinantal. The following theorem is known, a proof can be found in Subsection 9.3.2 of [Dol12].

Theorem 1.12. *Fix a tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$. The following are equivalent:*

- *S_φ is canonical.*
- *X_φ is curvilinear and length 6.*
- *Y_φ is curvilinear and length 6.*

In this case, S_φ is the associated cubic surface to X_φ and Y_φ.

Every admissible length 6 subscheme of $\mathbb{P}^2$ has a unique determinantal representation, up to G-equivalence. The set of determinantal representations of a canonical cubic surface S is in bijection (up to G-equivalence) with the set of admissible length 6 subschemes $X \subset \mathbb{P}V$ whose associated cubic surface is S.

Proof. Dolgachev proves that S_φ is canonical if and only if X_φ is admissible. Any length 6 curvilinear subscheme $X \subset \mathbb{P}V$ whose ideal is generated by four cubic curves (a necessary condition to be of the form X_φ for some φ) must contain no length 4 collinear subscheme, which is equivalent to being canonical □

Remark 1.13 ([Thr41, Theorem 7.6])**.** Every cubic surface has a determinantal representation, except for the cubic surface with an E_6 singularity. Every canonical length 6 subscheme associated to the E_6 cubic surface is contained in a conic.

We give a sketch of the various morphisms induced by a determinantal representation of a cubic surface. Fix a tensor $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ such that S_φ is canonical. The following kernels all vanish: $\ker(\widetilde{\varphi}_{\mathbb{P}U,W}) = \ker(\widetilde{\varphi}_{\mathbb{P}U,V}) = 0$ and $\ker(\widetilde{\varphi}_{\mathbb{P}V,W}) = 0$ and $\ker(\widetilde{\varphi}_{\mathbb{P}W,V}) = 0$. We then have the following four exact sequences

$$0 \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}U}(-1) \xrightarrow{\widetilde{\varphi}_{\mathbb{P}U,W}} V^\vee \otimes \mathcal{O}_{\mathbb{P}U} \longrightarrow \mathcal{F}_\varphi \longrightarrow 0 \tag{4}$$

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}U}(-1) \xrightarrow{\widetilde{\varphi}_{\mathbb{P}U,V}} W^\vee \otimes \mathcal{O}_{\mathbb{P}U} \longrightarrow \mathcal{G}_\varphi \longrightarrow 0 \tag{5}$$

$$0 \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}V}(-1) \xrightarrow{\widetilde{\varphi}_{\mathbb{P}V,W}} U^\vee \otimes \mathcal{O}_{\mathbb{P}V} \longrightarrow \mathcal{I}_{X_\varphi}(3) \longrightarrow 0 \tag{6}$$

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}W}(-1) \xrightarrow{\widetilde{\varphi}_{\mathbb{P}W,V}} U^\vee \otimes \mathcal{O}_{\mathbb{P}W} \longrightarrow \mathcal{I}_{Y_\varphi}(3) \longrightarrow 0. \tag{7}$$

The cokernels $\mathcal{F}_\varphi$ and $\mathcal{G}_\varphi$ are supported on S_φ, and are line bundles when restricted to $S_\varphi \setminus R_\varphi$. Taking cohomology, we have $V^\vee = H^0(S_\varphi, \mathcal{F}_\varphi)$ and $U^\vee = H^0(\mathbb{P}V, \mathcal{I}_{X_\varphi}(3))$. Both sheaves are rank 1, and the linear system $|\mathcal{F}_\varphi|$ induces a rational map $f : S_\varphi \dashrightarrow \mathbb{P}V$. Because $\mathcal{F}_\varphi = \operatorname{coker}(\widetilde{\varphi}_{\mathbb{P}U,W})$, then on points f sends $u \mapsto v$ such that $\varphi(u, v, -) = 0$, and is defined on the open subscheme $S_\varphi \setminus R_\varphi$. The linear system $|\mathcal{I}_{X_\varphi}(3)|$ induces a rational map $g : \mathbb{P}V \dashrightarrow S_\varphi \subset \mathbb{P}U$, defined on $\mathbb{P}V \setminus X_\varphi$, which on points sends $v \mapsto u$ such that $\varphi(u, v, -) = 0$ – i.e. $f \circ g = \mathrm{id}$ where defined and f is birational.

Definition 1.14. Fix a tensor $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ such that S_φ is canonical. We define the birational maps $p_V : S_\varphi \dashrightarrow \mathbb{P}V$ and $p_W : S_\varphi \dashrightarrow \mathbb{P}W$ from the linear systems $|\mathcal{F}_\varphi|$ and $|\mathcal{G}_\varphi|$ on S_φ, as above.

Remark 1.15. The tensor φ_U is rank 1 at finitely many points on S_φ, and never rank 0. And a consequence of Proposition 6.2 in [EP00] is that at all points $v \in \mathbb{P}V$ and $w \in \mathbb{P}W$, the tensors φ_V and φ_W are rank at least 2 everywhere, dropping rank only on X_φ and Y_φ. Consequently, the sheaves $\mathcal{F}_\varphi, \mathcal{G}_\varphi$ pull back to ample line bundles on the blowup $\mathrm{Bl}_{R_\varphi} S_\varphi$, and similarly $\mathcal{I}_{X_\varphi}(3)$ and $\mathcal{I}_{Y_\varphi}(3)$ pull back to the anti-canonical line bundles on $\mathrm{Bl}_{X_\varphi} \mathbb{P}V$ and $\mathrm{Bl}_{Y_\varphi} \mathbb{P}W$.

The morphism $\mathrm{Bl}_{X_\varphi} \mathbb{P}V \to S_\varphi$ factors through the blowup $\mathrm{Bl}_{R_\varphi} S_\varphi \to S_\varphi$, and the map $\mathrm{Bl}_{X_\varphi} \mathbb{P}V \to \mathrm{Bl}_{R_\varphi} S_\varphi$ is an isomorphism. Note that $\mathrm{Bl}_{R_\varphi} S_\varphi$ is smooth if and only if X_φ (or equivalently Y_φ) is reduced.

The determinantal structure on S_φ is deeply related to the line configuration on S_φ. We quickly introduce some notation:

Definition 1.16. Let A, B be vector spaces. A subspace $C \subset A^\vee \otimes B^\vee$ is said to be *of block type* (a, b) if there exist subspaces $A' \subset A$ and $B' \subset B$ with $\dim(A') = a$ and $\dim(B') = b$, such that the pairing $c : A \otimes B \to \mathbb{C}$ annihilates $A' \otimes B'$ for every $c \in C$.

Because of the linear-algebraic definitions of p_V, p_W, and their birational inverses, the fibres of these four maps can only be singletons or open subvarieties of lines. The fibres of p_V and p_W above X_φ and Y_φ are projective lines of type $(1, 3)$ and $(3, 1)$, respectively. The fibres of p_V^{-1} and p_W^{-1} are singletons everywhere, except at points $u \in \mathbb{P}U$ where $\varphi_U(u)$ is rank 1, where the fibres are (open subvarieties of) the projectivisations of the left and right kernels of $\varphi_U(u) \in V^\vee \otimes W^\vee$.

When S_φ is smooth, both p_V and p_W are morphisms and X_φ and Y_φ are reduced. We can write $X_\varphi = \{x_1, \dots, x_6\}$ and $Y_\varphi = \{y_1, \dots, y_6\}$, and let $L_{ij} \subset \mathbb{P}V$ (resp. $L'_{ij} \subset \mathbb{P}W$) be the line through x_i and x_j (resp. through y_i and y_j), and Q_i (resp. Q'_i) be the conic through $X_\varphi \setminus \{x_i\}$ (resp. $Y_\varphi \setminus \{y_i\}$).

The map $p_V : S_\varphi \to \mathbb{P}V$ is the blowup of $\mathbb{P}V$ at the 6 points X_φ. Viewing $\varphi_U(U) \subset V^\vee \otimes W^\vee$, the 27 lines on S_φ split into the following three types:

- 6 lines of block type $(1, 3)$. These are the fibres $p_V^{-1}(x_i)$, and equivalently the strict transforms of the conics $p_W^{-1}(Q'_i)$.
- 6 lines of block type $(1, 3)$. These are the fibres $p_W^{-1}(y_i)$, and equivalently the strict transforms of the conics $p_V^{-1}(Q_i)$.
- 15 lines of block type $(2, 2)$. These are the strict transforms $p_V^{-1}(L_{ij})$ and $p_W^{-1}(L'_{ij})$.

Relabelling $\{y_1, \dots, y_6\}$ so that $p_V^{-1}(x_i) \cap p_W^{-1}(y_i) = \emptyset$, we have $p_V^{-1}(x_i) = p_W^{-1}(Q'_i)$ and $p_V^{-1}(Q_i) = p_W^{-1}(y_i)$ and $p_V^{-1}(L_{ij}) = p_W^{-1}(L'_{ij})$. The ordered tuples $\{x_1, \dots, x_6\}$ and $\{y_1, \dots, y_6\}$ are Gale dual (associated sets of points) in the sense of Coble [Cob22].

Gale duality gives an alternate way to construct a determinantal representation of a length 6 subscheme, working for most length 6 Gorenstein subschemes of $\mathbb{P}^2$, and works in families. We outline the construction below, all technical details and many related results can be found in sections 5 and 6 of [EP00].

Let $X \subset \mathbb{P}V$ be a length 6 locally Gorenstein subscheme. We consider the trace map $\tau : H^0(X, \omega_X) \to \mathbb{C}$ which, after composition with Serre duality, induces a perfect pairing

$$H^0(X, \mathcal{O}_X(1)) \otimes H^0(X, \omega_X(-1)) \longrightarrow H^0(X, \omega_X) \xrightarrow{\tau} \mathbb{C}.$$

We let $V'^{\vee} = (V^{\vee})^{\perp}$ be the annihilator with respect to this pairing, and say that the linear series $(V^{\vee}, \mathcal{O}_X(1))$ and $(V'^{\vee}, \omega_X(-1))$ are *Gale dual*.

Definition 1.17. Let $X \subset \mathbb{P}V$ be a locally Gorenstein subscheme. The image X' of X inside $\mathbb{P}V'$ under the linear series $(V'^{\vee}, \omega_X(-1))$, as above, is called the *Gale dual* of $X \subset \mathbb{P}V$.

If a projective isomorphism $\mathbb{P}W \to \mathbb{P}V'$ induces an isomorphism between subschemes $Y \subset \mathbb{P}W$ and $X' \subset \mathbb{P}V'$, then $Y \subset \mathbb{P}W$ is also said to be Gale dual to $X \subset \mathbb{P}V$.

To a length 6 locally Gorenstein subscheme $X \subset \mathbb{P}V$, we associate the linear map $\phi : V^{\vee} \otimes V'^{\vee} \to \ker(\tau) \subset H^0(X, \omega_X)$. The vector space $\ker(\tau)$ is 5-dimensional, and under the additional assumption that ϕ is surjective (this is satisfied, for example, if X contains a length 4 subscheme in linearly general position [EP00, Proposition 5.8]), we consider the inclusion of its 4-dimensional kernel

$$\varphi_U : \ker(\phi) \to V^{\vee} \otimes V'^{\vee}.$$

Choosing isomorphisms $\ker(\phi) \cong U$ and $V' \cong W$, we obtain a $4 \times 3 \times 3$ tensor $\varphi \in U^{\vee} \otimes V^{\vee} \otimes W^{\vee}$.

Proposition 1.18. *Let $\varphi \in U^{\vee} \otimes V^{\vee} \otimes W^{\vee}$ be a tensor. If either X_φ or Y_φ is finite and φ_V or φ_W is injective (respectively) then it is length 6. The following statements are equivalent:*

- *X_φ and Y_φ are finite.*
- *X_φ is finite and locally Gorenstein.*
- *Y_φ is finite and locally Gorenstein.*

If one of the above statements hold, then the linear series $(W^{\vee}, \operatorname{coker}(\widetilde{\varphi}_{\mathbb{P}V,U})|_{X_\varphi})$ induces a canonical isomorphism $X_\varphi \to Y_\varphi$, under which the linear series $(V^{\vee}, \mathcal{O}_{X_\varphi}(1))$ and $(W^{\vee}, \mathcal{O}_{Y_\varphi}(1))$ are Gale dual in the sense of [EP00].

Proof. Everything follows from Theorem 6.1 and Proposition 6.2 of [EP00], except the claim that a finite subscheme is length 6. If φ_V is injective, then φ_V induces a linear embedding of $\mathbb{P}V \hookrightarrow \mathbb{P}(U^{\vee} \otimes W^{\vee})$. The subscheme X_φ is the scheme-theoretic intersection of $\mathbb{P}V$ with the degree 6, codimension 2 subvariety of $\mathbb{P}(U^{\vee} \otimes W^{\vee})$ of rank at most 2 pairings. This is length 6 if it is finite, the same argument applies for Y_φ by symmetry. □

Remark 1.19. The map sending a subscheme to its Gale dual can also be done in families. For ease of notation we only write the construction over affine schemes. Let $\operatorname{Spec} A$ be an affine scheme, and fix a locally Gorenstein subscheme $X_A \subset \mathbb{P}V \times \operatorname{Spec} A$ whose fibres are length 6. The annihilator of $V^{\vee} \otimes A$ with respect to the trace pairing $H^0(X_A, \mathcal{O}_{X_A}(1)) \otimes_A H^0(X_A, \omega_{X_A}(-1)) \to \mathbb{C}$ is a submodule $\mathcal{W}^{\vee} \subset H^0(X_A, \omega_{X_A}(-1))$. We let $Y_A \subset \mathbb{P}\mathcal{W}$ be the image of X_A under the linear series $(\mathcal{W}^{\vee}, \omega_{X_A}(-1))$. For each $a \in \operatorname{Spec} A$, the pairs $X_a \subset \mathbb{P}V \times \operatorname{Spec}(A/a)$ and $Y_a \subset \mathbb{P}\mathcal{W} \times \operatorname{Spec}(A/a)$ are Gale dual.

Under the additional assumption that each fibre X_a contains a length 4 subscheme in linearly general position, the map $(V^{\vee} \otimes \mathcal{W}^{\vee}) \otimes A/a \to H^0(X_a, \omega_{X_a})$ is surjective for each prime $a \in \operatorname{Spec} A$ and so the kernel $\mathcal{U} \to (V^{\vee} \otimes W^{\vee}) \otimes A \to H^0(X_A, \omega_{X_A})$ is a locally free, rank 4 A-module. This induces a map of A-modules

$$\mathcal{U} \longrightarrow (V^{\vee} \otimes \mathcal{W}^{\vee}) \otimes A,$$

which can be identified with a section $s_A : \operatorname{Spec} A \to \mathcal{U}^{\vee} \otimes V^{\vee} \otimes \mathcal{W}^{\vee}$.

1.4. **A classification of $4 \times 3 \times 3$ tensors.** The set of $4 \times 3 \times 3$ tensors can be split into the following types. For each $\varphi \in \mathbb{P}(U^{\vee} \otimes V^{\vee} \otimes W^{\vee})$ one of the following is true.

- S_φ is a smooth cubic surface. Equivalently, both X_φ and Y_φ are length 6 reduced subschemes, not contained in a conic, and contain no length 3 collinear subscheme.

- S_φ has only canonical singularities. Equivalently, both X_φ and Y_φ are curvilinear length 6 subschemes, not contained in a conic, containing no length 4 collinear subscheme.
- X_φ is a conic, and isomorphic to Y_φ as subschemes of $\mathbb{P}^2$. The surface S_φ is reducible.
- S_φ is not canonical. In this case, X_φ and Y_φ are not curvilinear, and may not be finite.
- $\det(\varphi_U)$ is the zero polynomial.

Kok Onn Ng published an almost complete classification of $4 \times 3 \times 3$ tensors [Ng02], with the goal to describe the GIT quotient $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G$. This paper never appeared, but we draw heavily on their classification of determinantal representations of cubic surfaces with at worst canonical singularities for our classification of G-(semi)stability. We derive our own results for tensors where S_φ is non-normal, reducible, or conical, as their classification of these tensors is incomplete.

Kok Onn Ng uses the correspondence between G-equivalence classes of tensors where S_φ has at most canonical singularities and curvilinear length 6 subschemes of $\mathbb{P}V$. Explicitly, they use Maple to find all admissible length 6 subschemes $X \subset \mathbb{P}V$ associated to each type of canonical cubic surface, and then compute the presentation matrix for the minimal resolution of the ideal I_X.

Definition 1.20. The configuration of singularities on a cubic surface with only canonical singularities is described by a Dynkin diagram $F \subset \widetilde{E}_6$. Two cubic surfaces with canonical singularities are said to be *of the same type* if they have the same configuration of singularities. Two length 6 subschemes are said to be *of the same type* if they are similarly degenerate (have the same number and types of non-reduced points, have the same number of length k collinear subschemes, impose the same numbers of conditions on plane curves).

We follow the notation in [Ng02], where a family of tensors is described by a Dynkin diagram $F \subset \widetilde{E}_6$ and a latin suffix to distinguish between different types of tensors, where $\varphi, \varphi' \in F(n)$ if and only if $S_\varphi, S_{\varphi'}$, $X_\varphi, X_{\varphi'}$, and $Y_\varphi, Y_{\varphi'}$ are all of the same type. Note that this is equivalent to only X_φ and $X_{\varphi'}$ being the same type. We let $F(n) \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ be the G-orbit of the corresponding family of tensors described in [Ng02].

Example 1.21. The subvariety $A_1A_2(a)$ is the G-orbit of tensors of the form

$$\varphi_U = \begin{pmatrix} \lambda(\lambda+1)u_1 + (\lambda+1)u_3 & u_3 & u_0 \\ \lambda u_1 + u_3 & -u_1 & u_3 \\ u_2 & 0 & u_3 \end{pmatrix},$$

parametrised by $\lambda \in \mathbb{A}^1 \setminus \{0, 1\}$.

Remark 1.22. If $|F| \geq 4$ and $F \neq 2A_2$ then all tensors of type $F(n)$ are G-equivalent. If $F = 2A_2$, then tensors of type $F(n)$ appear in a one-parameter family, otherwise if $|F| \leq 3$ then tensors of type $F(n)$ occur in a $(4-|F|)$-parameter family. Note that the parametrisations given in [Ng02] are not in bijection with unique G-equivalence classes of tensors, but they all have the correct dimension as a parameter space.

A necessary condition for $F'(n') \subset \overline{F(n)}$ is for $F \subset F'$ as graphs. The closure relations can easily be seen from associated length 6 subschemes, where $F'(n') \subset \overline{F(n)}$ if and only if a family of length 6 subschemes X_φ of type $F(n)$ can degenerate into a subscheme $X_{\varphi'}$ of type $F'(n')$.

Each subvariety $F(n)$ is connected. If $n \neq n'$, then $F(n') \cap \overline{F(n)} = \emptyset$, unless $F = D_4$ in which case $D_4(b) \subset \overline{D_4(a)}$.

Remark 1.23. The classification of determinantal representations of non-canonical cubic surfaces given in [Ng02] is incomplete. It omits tensors G-equivalent to

$$\varphi_U = \begin{pmatrix} u_0 & u_3 & -u_2 \\ -u_3 & u_0 & u_1 \\ u_2 & -u_1 & \lambda u_0 \end{pmatrix},$$

where either $\lambda = 0, 1$. We believe these are the only missing terms in their classification, their classification of determinantal representations of canonical cubic surfaces is correct and is the only one we use.

2. Partial GIT quotients

Before looking at $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G$, it is useful to look at the partial GIT quotients of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ by only one or two of the copies of $\mathrm{SL}(-)$. These are all known, with the latter being a special case of quiver GIT.

The GIT of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ under the $\mathrm{SL}(U), \mathrm{SL}(V), \mathrm{SL}(W)$-actions is mostly uninteresting: a tensor φ is semistable under the respective actions if and only if $\varphi_U, \varphi_V, \varphi_W$ is injective. All semistable points are stable.

The GIT quotient $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(V) \times \mathrm{SL}(W))$ is a moduli space of divisor-equivalent twisted cubics inside cubic surfaces, but is not a moduli space of sheaves. It has already been well studied in [LLSS17], we refer the reader to Section 3 of their paper for

The GIT quotient $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(U) \times \mathrm{SL}(W))$ is a moduli space of sheaves on $\mathbb{P}V$, being the final minimal model of the Hilbert scheme $\mathrm{Hilb}_6(\mathbb{P}V)$. The rational contraction $\mathrm{Hilb}_6(\mathbb{P}V) \dashrightarrow \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)/(\mathrm{SL}(U) \times \mathrm{SL}(W))$ contracts the divisor of subschemes contained in a conic. We give a concrete description of this map using Gale duality.

Definition 2.1. Let

$$\det : \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) \dashrightarrow |\mathcal{O}_{\mathbb{P}U}(3)| = \mathrm{Hilb}_{\text{cubic surfaces}}(\mathbb{P}U)$$

be the rational map sending a tensor φ to its determinant cubic $[S_\varphi]$.

Let $\det_V, \det_W, \det_{VW}$ be the induced rational maps to $|\mathcal{O}_{\mathbb{P}U}(3)|$ from the GIT quotients of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ by $\mathrm{SL}(V), \mathrm{SL}(W), \mathrm{SL}(V) \times \mathrm{SL}(W)$. Similarly, let $\det_U, \det_{UV}, \det_{UW}, \det_{UVW}$ be the induced rational maps to $|\mathcal{O}_{\mathbb{P}U}(3)|//\,\mathrm{SL}(U)$ from the GIT quotients of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ by $\mathrm{SL}(U), \mathrm{SL}(U) \times \mathrm{SL}(V), \mathrm{SL}(U) \times \mathrm{SL}(W)$, and $G = \mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$. We will see that none of these maps are morphisms.

Definition 2.2. Let $\pi_U, \pi_V, \pi_W, \pi_{UV}, \pi_{UW}, \pi_{VW}, \pi_{UVW}$ be the quotient maps from $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ to the GIT quotient by the respective subgroup of $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$, as in Definition 2.1. As an example,

$$\pi_{UV} : \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) \dashrightarrow \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//\left(\mathrm{SL}(U) \times \mathrm{SL}(V)\right).$$

To describe these GIT quotients, we quickly introduce some notation and facts about linear subspaces satisfying certain rank conditions.

2.1. **Linear subspaces of low rank.** Recall Definition 1.16: let A, B be vector spaces. A subspace $C \subset A^\vee \otimes B^\vee$ is said to be *of block type* (a, b) if there exist subspaces $A' \subset A$ and $B' \subset B$ with $\dim(A') = a$ and $\dim(B') = b$, such that the pairing $c : A \otimes B \to \mathbb{C}$ annihilates $A' \otimes B'$ for every $c \in C$.

Definition 2.3. Let A, B be vector spaces. A subspace $C \subset A^\vee \otimes B^\vee$ is said to be a *rank at most r subspace* if every $M \in C$ is of rank at most r. We say that C is a *rank r subspace* if C is a rank at most r subspace, and at least one element of C is rank r.

A subspace $C \subset A^\vee \otimes B^\vee$ is said to satisfy a *rank r block condition* if C is of block type (a, b), where $r = \dim A + \dim B - a - b$.

Proposition 2.4. *Any subspace satisfying a rank r block condition is a rank at most r subspace.*

Proof. Let A, B be vector spaces, and suppose $C \subset A^\vee \otimes B^\vee$ is of block type (a, b), and let $r = \dim A + \dim B - a - b$. Choose bases for A, B so that we can write any $M \in C$ as a block matrix

$$M = \left[\begin{array}{c|c} X & Y \\ \hline Z & 0 \end{array}\right] \tag{8}$$

where X is a $(\dim A - a) \times (\dim B - b)$ matrix, Y is a $(\dim A - a) \times b$ matrix, and Z is a $a \times (\dim B - b)$. Because $r + 1 > \dim A - a$ and $r + 1 > \dim B - b$, then all of the $(r + 1) \times (r + 1)$ minors of M vanish, except for maybe the top left-most minor. This $(r + 1) \times (r + 1)$ block matrix is

$$M' = \left[\begin{array}{c|c} X & Y' \\ \hline Z' & 0 \end{array}\right] \tag{9}$$

where Y' is a $(\dim A - a) \times (r + 1 - (\dim B - b + 1))$-matrix. But this is a $(\dim A - a) \times (\dim A - a + 1)$-matrix, so the $(r + 1) \times (\dim A - a + 1)$-submatrix $\left[\begin{array}{c} Y' \\ \hline 0 \end{array}\right]$ has rank at most $\dim A - a$. So M' has rank at most $(\dim A - a) + (\dim B - b) = r$, i.e. $\det(M')$ vanishes and $\operatorname{rank}(M) \leq r$. $\square$

However, not every rank at most r subspace satisfies a rank r block condition. The classification of rank r subspaces in general is very difficult, and is only known completely when $r \leq 7$, but we only need the classification for $r = 1, 2$ by Atkinson and Lloyd [AL81]:

Proposition 2.5 ([AL81, Lemma 2])**.** *Every rank 1 subspace satisfies a rank 1 block condition.*

Definition 2.6. Let $\varepsilon : \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \to \mathbb{C}$ be the Levi-Civita tensor, defined by its action on the basis $\{e_1, e_2, e_3\}$ by

$$\varepsilon(e_i, e_j, e_k) = \begin{cases} \operatorname{sgn}(ijk) & \text{if no indices repeat} \\ 0 & \text{otherwise,} \end{cases}$$

where $\operatorname{sgn}(ijk)$ is the sign of the permutation sending $123 \mapsto ijk$.

The Levi-Civita tensor is alternating in its three entries. Dualising the second two copies of $\mathbb{C}^3$, it has an adjoint $\varepsilon^* : \mathbb{C}^3 \to (\mathbb{C}^3)^\vee \otimes (\mathbb{C}^3)^\vee$. The image $\varepsilon^*(\mathbb{C}^3)$ is the 3-dimensional subspace of skew-symmetric pairings.

Let A, B be vector spaces, and $C \subset A^\vee \otimes B^\vee$ be a 3-dimensional subspace. If there exist injections $\iota_A : (\mathbb{C}^3)^\vee \to A^\vee$ and $\iota_B : (\mathbb{C}^3)^\vee \to B^\vee$ where $C = (\iota_A \otimes \iota_B)(\varepsilon^*(\mathbb{C}^3))$, then C is said to be a *skew-symmetric subspace* of $A^\vee \otimes B^\vee$.

A tensor $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ is said to be *skew-symmetric* if $\varphi_U(U)$ is a skew-symmetric subspace of $V^\vee \otimes W^\vee$.

Note that every skew-symmetric tensor is G-equivalent.

Proposition 2.7 ([AL81, Lemma 7])**.** *Every rank at most 2 subspace either satisfies a rank 2 block condition, or is a 3-dimensional skew-symmetric subspace.*

2.2. **The moduli space of divisor-equivalent twisted cubics inside cubic surfaces.** The ideal cutting out a generalised twisted cubic inside $\mathbb{P}U$ is generated by three quadrics, which arise as the minors of a 3×2 matrix of linear forms in U. This is identified with a $4 \times 3 \times 2$, which can be augmented to a $4 \times 3 \times 3$ tensor by adding a column. Conversely, any $4 \times 3 \times 3$ tensor can be reduced to a $4 \times 3 \times 2$ or $4 \times 2 \times 3$ tensor by deleting a column or row (restriction to a 2-dimensional subspace $W' \subset W$ or $V' \subset V$).

Fix a tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ such that S_φ is a cubic surface and a 2-dimensional subspace $W' \subset W$. We let $\varphi_{\mathbb{P}U,W'}$ denote the restriction of $\varphi_{\mathbb{P}U,W}$ to the subbundle $W' \otimes \mathcal{O}_{\mathbb{P}U}(-1)$, and $\operatorname{coker}(\widetilde{\varphi}_{\mathbb{P}U,W'}) := \mathcal{J}_{\varphi,W'}(2)$ denote the quotient sheaf. The ideal sheaf $\mathcal{J}_{\varphi,W'}$ cuts out a generalised twisted cubic $C_{W',\varphi} \subset S_\varphi$. The 2-dimensional linear system of twisted cubics divisor-equivalent to $C_{W',\varphi} \subset S_\varphi$ is in bijection with the Grassmannian $\mathrm{Gr}(2, W)$.

We have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & W' \otimes \mathcal{O}_{\mathbb{P}U}(-1) & \xrightarrow{\widetilde{\varphi}_{\mathbb{P}U,W'}} & V^\vee \otimes \mathcal{O}_{\mathbb{P}U} & \longrightarrow & \mathcal{J}_{\varphi,W'}(2) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & W \otimes \mathcal{O}_{\mathbb{P}U}(-1) & \xrightarrow{\widetilde{\varphi}_{\mathbb{P}U,W}} & V^\vee \otimes \mathcal{O}_{\mathbb{P}U} & \longrightarrow & \mathcal{F}_\varphi & \longrightarrow & 0,
\end{array}$$

where the map $W' \otimes \mathcal{O}_{\mathbb{P}U}(-1) \to W \otimes \mathcal{O}_{\mathbb{P}U}(-1)$ is the inclusion induced by $W' \to W$ and the map $\mathcal{J}_{\varphi,W'}(2) \to \mathcal{F}_\varphi$ is surjective. By the Snake lemma, there is an exact sequence

$$0 \longrightarrow W/W' \otimes \mathcal{O}_{\mathbb{P}U}(-1) \longrightarrow \mathcal{J}_{\varphi,W'}(2) \longrightarrow \mathcal{F}_\varphi \longrightarrow 0.$$

The inclusion $W/W' \otimes \mathcal{O}_{\mathbb{P}U}(-1) \to \mathcal{J}_{\varphi,W'}(2) \subset \mathcal{O}_{\mathbb{P}U}(2)$ identifies $W/W' \otimes \mathcal{O}_{\mathbb{P}U}(-1)$ with the twisted ideal sheaf $\mathcal{I}_{S_\varphi}(2) \subset \mathcal{J}_{\varphi,W'}(2)$, and we have $\mathcal{F}_\varphi = \mathcal{J}_{\varphi,W'}(2)/\mathcal{I}_{S_\varphi}(2)$.

Notice that the quotient $\mathcal{J}_{\varphi,W'}(2)/\mathcal{I}_{S_\varphi}(2)$ doesn't depend on the choice of W.

Lemma 2.8. *A tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is $\mathrm{SL}(V) \times \mathrm{SL}(W)$-semistable if and only if $\varphi_U(U)$ does not satisfy a rank 2 block condition.*

The tensor φ is stable if and only if $\varphi_U(U)$ does not satisfy a rank 3 block condition.

Proposition 2.9. *Let $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$. Then:*

- *φ is $\mathrm{SL}(V) \times \mathrm{SL}(W)$-stable if and only if $\varphi_U(U)$ contains no hyperplanes of type $(3,1)$ or $(1,3)$. Either*
  - *$\det(\varphi) = 0$ and φ is G-equivalent to a skew-symmetric tensor,*
  - *or S_φ is irreducible,*
  - *or its only planes are skew-symmetric subspaces and of type $(2,2)$.*
- *φ is strictly $\mathrm{SL}(V) \times \mathrm{SL}(W)$-semistable if and only if S_φ is a cubic surface and $\varphi_U(U)$ contains a plane of type $(1,3)$ or $(3,1)$. This is equivalent to either X_φ or Y_φ, respectively, not being locally Gorenstein.*

In particular, $\det(\varphi) \neq 0$ is sufficient for semistability, and S_φ being irreducible is sufficient for stability.

However, $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(V) \times \mathrm{SL}(W))$ is not a moduli space of sheaves on $\mathbb{P}^3$. The sequence

$$W \otimes \mathcal{O}_{\mathbb{P}U}(-1) \xrightarrow{\widetilde{\varphi}_{\mathbb{P}U,W}} V^\vee \otimes \mathcal{O}_{\mathbb{P}U} \longrightarrow \mathcal{F}_\varphi \longrightarrow 0$$

has a non-zero kernel if and only if $\det(\varphi) = 0$, which happens on stable points where $\varphi_U(U)$ is a skew-symmetric subspace. These tensors are uniquely determined up to $\mathrm{SL}(V) \times \mathrm{SL}(W)$-equivalent by their kernel $u \in \mathbb{P}U$, and are embedded as a copy of $\mathbb{P}U$ inside $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(V) \times \mathrm{SL}(W))$. In this case, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}U}(-2) \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}U}(-1) \xrightarrow{\widetilde{\varphi}_{\mathbb{P}U,W}} V^\vee \otimes \mathcal{O}_{\mathbb{P}U} \longrightarrow \mathcal{F}_\varphi \longrightarrow 0, \tag{10}$$

and $\mathcal{F}_\varphi$ is locally free of rank 1 on $\mathbb{P}U \setminus \{u\}$ but is rank 3 at $u \in \mathbb{P}U$. As a consequence,

Corollary 2.10. *The map* $\det_{VW} : \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(V) \times \mathrm{SL}(W)) \dashrightarrow |\mathcal{O}_{\mathbb{P}U}(3)|$ *is not a morphism, undefined on this copy of* $\mathbb{P}U$.

Proof. These are all of the $\mathrm{SL}(V) \times \mathrm{SL}(W)$-stable (and all of the $\mathrm{SL}(V) \times \mathrm{SL}(W)$-semistable) points where $\det(\varphi) = 0$. □

Remark 2.11. If $\det_{VW}$ was a morphism, then the $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$-semistable locus of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ would be the pullback of the $\mathrm{SL}(U)$-semistable locus of $|\mathcal{O}_{\mathbb{P}U}(3)|$. We will see that this is not the case.

2.3. **The final minimal model of** $\mathrm{Hilb}_6(\mathbb{P}^2)$**.** The $\mathrm{SL}(U) \times \mathrm{SL}(W)$-(semi)stability of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is a known result of quiver stability. The GIT quotient $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(V) \times \mathrm{SL}(W))$ is the final minimal model of $\mathrm{Hilb}_6(\mathbb{P}V)$, as outlined in [ABCH13].

Lemma 2.12. *A tensor* $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ *is* $\mathrm{SL}(U) \times \mathrm{SL}(W)$*-semistable if and only if for every* 3*-dimensional subspace* $U' \subset U$*, the image of* $\varphi_V(V)$ *under the projection to* $(U')^\vee \otimes W$ *does not satisfy a rank* 2 *block condition. Visually, this is the same as saying* φ *is not* $\mathrm{SL}(U) \times \mathrm{SL}(W)$*-equivalent to one of the following matrices:*

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} * & * & * \\ * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} * & * & * \\ * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix}.$$

All semistable points are stable.

We can classify $\mathrm{SL}(U) \times \mathrm{SL}(W)$-semistable tensors:

Corollary 2.13. *Suppose* $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ *is* $\mathrm{SL}(V) \times \mathrm{SL}(W)$*-unstable. Then there is a* $d \leq 3$ *where* X_φ *is the scheme theoretic union of a degree* d *plane curve, and a subscheme* X' *cut out by the minors of a* $(3-d) \times (2-d)$ *matrix of linear forms. In particular,* $\mathcal{I}_{X_\varphi}$ *is generated in degree* 3 *by at most* d *terms. If* X' *is finite, then it is length* $(3-d)(2-d)/2$.

Proof. This follows from computing the minors of the matrices of the form of Lemma 2.12. □

Proposition 2.14. *Let* $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$*, such that* $h^0(\mathbb{P}V, \mathcal{I}_{X_\varphi}(3)) = 4$*. Then either* X_φ *is finite, or* X_φ *is the scheme-theoretic union of a line and a length* 2 *subscheme.*

Proof. Suppose X_φ is not finite. If it contains a smooth conic $\{f = 0\}$, then $\mathcal{I}_{X_\varphi} \subset (f)\mathcal{O}_{\mathbb{P}V}$. By assumption, $h^0(\mathbb{P}V, \mathcal{I}_{X_\varphi}(3)) = 4$ so $\mathcal{I}_{X_\varphi} = (f, g)\mathcal{O}_{\mathbb{P}V}$ for some cubic g that is $\mathbb{C}$-linearly independent from lf for any line $l \in V^\vee$. But there are also no $V^\vee$-linear relations between g and lf, because f is irreducible, so φ has a column of zeros and one of its minors vanishes, a contradiction on $h^0(\mathbb{P}V, \mathcal{I}_{X_\varphi}(3)) = 4$.

Otherwise suppose that X_φ contains a line $\{l = 0\}$. □

Proposition 2.15. *A tensor* $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ *is* $\mathrm{SL}(U) \times \mathrm{SL}(W)$*-semistable if and only if either*

- $h^0(\mathbb{P}V, \mathcal{I}_{X_\varphi}(3)) = 4$, *or*
- X_φ *is a conic and* $h^0(\mathbb{P}V, \mathcal{I}_{X_\varphi}(3)) = 3$. *In this case, there is an identification of* $W \cong W$ *where* φ *is* $\mathrm{SL}(U) \times \mathrm{SL}(W)$*-equivalent to the following tensor*

$$\varphi_U = u_0 A + \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix}, \tag{11}$$

and $A \in \mathrm{Sym}^2(V^\vee)$ *is the non-zero quadratic form defining* X_φ.

Proof. A tensor φ is $\mathrm{SL}(U) \times \mathrm{SL}(W)$-semistable if and only if either all of the minors of φ_V are non-zero (i.e. $h^0(\mathbb{P}V, \mathcal{I}_{X_\varphi}(3)) = 4$), or the minors that vanish correspond to skew-symmetric blocks.

Suppose that one of the minors of φ_V vanishes. Then either φ is $\mathrm{SL}(U) \times \mathrm{SL}(W)$-unstable, or in a choice of basis for U, V, W we have

$$\varphi_V \equiv \left(\begin{array}{ccc} & A' & \\ \hline 0 & v_2 & -v_1 \\ -v_2 & 0 & v_0 \\ v_1 & -v_0 & 0 \end{array}\right)$$

for some element $A' \in V^\vee \otimes W^\vee$. Its adjoint is

$$\varphi_u = u_0 A' + \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix},$$

and after identifying $V^\vee = W^\vee$ by the bases $\{v_0, v_1, v_2\}$ and $\{w_0, w_1, w_2\}$, we can change the basis of U to eliminate the skew-symmetric part of A', and then φ_U is of the form (11) where $A = \frac{1}{2}A' + \frac{1}{2}A'^T$. Directly computing the minors of φ_V gives

$$\mathcal{I}_{X_\varphi} = \left(0, v_0 v^T A v, v_1 v^T A v, v_2 v^T A v\right).$$

Then either A is non-zero and X_φ is the subscheme cut out by $v^T A v = 0$, or $A = 0$ and φ is $\mathrm{SL}(U) \times \mathrm{SL}(W)$-unstable. □

Corollary 2.16. *Let* $X \subset \mathbb{P}V$ *be a length* 6 *subscheme. There is a unique* $\mathrm{SL}(U) \times \mathrm{SL}(W)$*-stable tensor* $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$*, up to* $\mathrm{SL}(U) \times \mathrm{SL}(W)$*-equivalence, such that* $X \subset X_\varphi$ *if and only if* X *contains no length* 4 *collinear subscheme.*

Proof. This is true by some case-by-case computations, to be written. □

Definition 2.17. Let $C_6 \subset \mathrm{Hilb}_6(\mathbb{P}V)$ be the divisor of length 6 subschemes contained in a conic, and $L_n \subset \mathrm{Hilb}_6(\mathbb{P}V)$ be the subscheme consisting of length 6 subschemes that contain a length n collinear subscheme. Note that $L_5 \subset L_4 \subset C_6$.

We let $\mathrm{Hilb}_6^{\mathrm{Gor}}(\mathbb{P}V)$ denote the locally Gorenstein locus of $\mathrm{Hilb}_6(\mathbb{P}V)$.

Proposition 2.18. *There is a rational map*

$$\gamma : \mathrm{Hilb}_6(\mathbb{P}V) \dashrightarrow \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) /\!/ (\mathrm{SL}(U) \times \mathrm{SL}(W)).$$

The map γ *is defined on the locally Gorenstein locus of* $\mathrm{Hilb}_6(\mathbb{P}V) \setminus L_5$*, sending a general subscheme* $X \mapsto \pi_{UW}(\varphi)$ *where* $X = X_\varphi$*, and contracts the divisor* $C_6 \setminus L_5$ *to a copy of* $\mathrm{Hilb}_{\mathrm{conics}}(\mathbb{P}V) \setminus \{\text{doubled lines}\}$. *The restriction of* γ *to* $\mathrm{Hilb}_6^{\mathrm{Gor}}(\mathbb{P}V) \setminus C_6$ *is an isomorphism onto its image.*

Proof. Recall the construction of Gale duality in families from Remark 1.19. For a scheme B, it takes a flat family of subschemes $X \subset \mathbb{P}V \times B$ where each fibre is length 6, locally Gorenstein, and contains a length 4 subscheme in linearly general position, to a non-zero section of a vector bundle $\mathcal{U}^\vee \otimes V^\vee \otimes \mathcal{W}^\vee$, where $\mathcal{U}$ and $\mathcal{W}$ are rank 4 and 3 vector bundles over B. Note that every element of $\mathrm{Hilb}_6^{\mathrm{Gor}}(\mathbb{P}V) \setminus L_5$ contains a length 4 subscheme in linearly general position.

If $X \subset \mathbb{P}V$ is contained in a conic $v^T A v = 0$, then the diagonal embedding $X \to \mathbb{P}V \times \mathbb{P}V \subset \mathbb{P}(\mathrm{Sym}^2 V)$ is contained in the hyperplane cut out by $A \in \mathrm{Sym}^2(V^\vee)$, and the tensor presenting the Gale dual of X is the skew-symmetric ...

By Corollary 2.16, for each locally Gorenstein length 6 subscheme $X \subset \mathbb{P}V$ where X has a length 4 subscheme in linearly general position, the assignment $X \mapsto \varphi$ such that $X \subset X_\varphi$ is unique up to $\mathrm{GL}(U) \times \mathrm{GL}(W)$. Each morphism $\mathrm{Spec}\, A \to \mathrm{Hilb}_6^{\mathrm{Gor}}(\mathbb{P}V) \setminus L_4$ induces a map $\mathrm{Spec}\, A \to \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$, whose image is contained in the $\mathrm{PGL}(U) \times \mathrm{PGL}(W)$-semistable locus. This descends to the GIT quotient by $\mathrm{PGL}(U) \times \mathrm{PGL}(W)$, and after gluing together an affine cover of $\mathrm{Hilb}_6^{\mathrm{Gor}}(\mathbb{P}V) \setminus L_4$, we have the morphism γ where the following diagram commutes:

$$\begin{tikzcd} \mathrm{Spec}\, A \arrow[d] \arrow[dr] & \\ \mathrm{Hilb}_6^{\mathrm{Gor}}(\mathbb{P}V) \setminus L_4 \arrow[r, "\gamma"] & \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{PGL}(U) \times \mathrm{PGL}(W)). \end{tikzcd}$$

The remainder of this proof is in the process of being written up. □

3. The full GIT quotient

3.1. **Classifying semistability.**

Theorem 3.1 ([Mum77, 1.14])**.** *A cubic surface $[S] \in |\mathcal{O}_{\mathbb{P}U}(3)|$ is $\mathrm{SL}(U)$-semistable if and only if S is normal with at worst A_2 singularities. A cubic surface is $\mathrm{SL}(U)$-stable if and only if S is normal with at worst A_1 singularities.*

Definition 3.2. Let $R \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ the subvariety of tensors φ such that there exists a non-zero $u \in U$ where $\varphi_U(u) \in V^\vee \otimes W^\vee$ has rank at most 1. If $\varphi \in R$, then S_φ is said to contain a rank 1 point.

Remark 3.3. The condition of S_φ containing a rank 1 point is equivalent to X_φ containing a length 3 collinear subscheme.

Proof. Suppose there is a $u \in \mathbb{P}U$ such that $\varphi_U(u)$ has rank at most 1. Take any subspace $V' \subset \ker(\varphi_U(u)) \subset V$. Then the line $\mathbb{P}V'$ is contracted to u under the rational map $\mathbb{P}V \dashrightarrow \mathbb{P}U$ given by the linear system of cubics through X_φ. So all four cubics agree on $\mathbb{P}V'$, i.e. they have the same three roots and $\mathrm{len}(X_\varphi \cap \mathbb{P}V') = 3$ or $\mathbb{P}V' \subset X_\varphi$.

Conversely, if there is a line $\mathbb{P}V'$ where $\mathrm{len}(X_\varphi \cap \mathbb{P}V') \geq 3$ then $\mathbb{P}V'$ is contracted to one point $u \in \mathbb{P}U$, so $V' \subset \ker(\varphi_U(u))$ and $\varphi_U(u)$ is rank at most 1. □

Proposition 3.4. *The subvariety $R \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is G-invariant divisor, cut out by a degree 24 polynomial $R_{24} \subset \mathbb{C}[U \otimes V \otimes W]^G_\bullet$.*

Proof. It is immediate that R is closed and G-invariant. To see that R is a divisor, we consider the intersection scheme

$$\widetilde{R} = \{(\varphi, u, V') \mid \varphi(u, V', -) = 0\} \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) \times \mathbb{P}U \times \mathrm{Gr}(2, V).$$

R is the image of $\widetilde{R}$ under the projection onto $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$. Now consider the projection $\widetilde{R} \to \mathrm{Gr}(2,V) \times \mathbb{P}U$, which is surjective. The fibre above a point $(u, V') \in \mathbb{P}U \times \mathrm{Gr}(2,V)$ is the subvariety of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ cut out by $\varphi(u, V', -) = 0$, which is a codimension 6 linear subspace of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$. Because $\dim(\mathbb{P}U \times \mathrm{Gr}(2,V)) = 5$ then R is codimension 1 as desired.

To compute the degree, consider a general line $l : \mathbb{P}^1 \to \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ parametrised by $[s : t] \mapsto s\varphi + t\varphi'$, for two general points $\varphi, \varphi' \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$. The degree of R is the intersection number of $l(\mathbb{P}^1)$ with R, which is the same as the intersection number of the Segre variety $\mathbb{P}V^\vee \times \mathbb{P}W^\vee \subset \mathbb{P}(V^\vee \otimes W^\vee)$ with the image of the linear map $f : \mathbb{P}^1 \times \mathbb{P}U \to \mathbb{P}(V^\vee \otimes W^\vee)$ that sends $([s:t], u) \mapsto s\varphi_U(u) + t\varphi'_U(u)$.

We compute this in the Chow ring. Let h_1, h_2, h_3 be the respective generators of $A_*(\mathbb{P}^1), A_*(\mathbb{P}U), A_*(\mathbb{P}(V^\vee \otimes W^\vee))$. We have $f^*(h_3) = h_1 + h_2$, and $f^*(\mathbb{P}V^\vee \times \mathbb{P}W^\vee) = f^*(6h_3^4) = 6f^*(h_3)^4$. And $(h_1+h_2)^4 = 4h_1h_2^3$, so $\deg(R) = 4 \cdot 6 = 24$. □

Proposition 3.5 (Sufficient criteria for G-semistability)**.** *A point $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$-semistable if:*

- *S_φ is an $\mathrm{SL}(U)$-semistable cubic surface.*
- *S_φ contains no rank 1 points ($\varphi \notin R$).*

Proof. The first condition is immediate: the pullback of an $\mathrm{SL}(U)$-semistable point in $|\mathcal{O}_{\mathbb{P}U}(3)|$ is $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$-semistable. The second condition is equivalent to the G-invariant polynomial R_{24} not vanishing at φ. □

We introduce some notation to easily describe sufficient conditions for instability with the Hilbert-Mumford criterion.

Definition 3.6 (Block form)**.** Let B be a $4 \times 3 \times 3$ array of 0s and $*$s, in the sense that $B = [B_0|B_1|B_2|B_3]$ and each B_i is a 3×3 matrix of 0s and $*$s.

A tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is said to be *of block form B* if there is a basis for U, V, W where $\varphi_U = \sum_{i=0}^{3} u_i \varphi_i$, and each $\varphi_i \in \mathrm{Hom}(V, W^\vee) \cong M_{3\times 3}(\mathbb{C})$ has non-zero entries only where B_i has an $*$. Note that we choose the ordering so that $\mathrm{SL}(V)$ acts on the columns of B_i and $\mathrm{SL}(W)$ acts on the rows of B_i.

Proposition 3.7 (Sufficient criteria for G-instability)**.** *A point $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$-unstable if*

(1) *φ_U is not injective.*
(2) *S_φ contains a plane of type $(1,3)$ or $(3,1)$.*
(3) *S_φ contains a plane of type $(2,2)$.*
(4) *φ is of the following block form:*

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(5) *φ is of the following block form:*

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & 0 & * & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 & * & 0 & 0 \\ * & * & 0 & * & * & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$(5)^T$ *φ is of the following block form:*

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & 0 \\ * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Any unstable determinantal representation of a normal cubic surface with finitely many lines is of block type (4), (5), *or its transpose* $(5)^T$.

Proof. We apply the Hilbert-Mumford criterion. A point φ is G-unstable if and only if there is a one-parameter subgroup λ of G that acts with only positive weights on φ. This one-parameter subgroup can always be chosen to act diagonally in some basis for U, V, W, with weight vector $\mathrm{wt}(\lambda) = ((a_0, a_1, a_2, a_3), (b_0, b_1, b_2), (c_0, c_1, c_2))$. In this basis we write $\varphi = \sum_{ijk} \varphi_{ijk}(u_i \otimes v_j \otimes w_k)$, acting with weight $a_i + b_j + c_k$ on $u_i \otimes v_j \otimes w_k$.

One can then check that the one-parameter subgroup with the following weight vector destabilises the respective φs, from above:

(1) Unstable by the weight $((1, 1, 1, -3), (0, 0, 0), (0, 0, 0))$ (note that this is also seen to be $\mathrm{SL}(U)$-unstable by quiver GIT).
(2) Unstable by the weight $((9, -3, -3, -3), (0, 0, 0), (4, 4, -8))$ (for type $(1, 3)$) and by the weight $((9, -3, -3, -3), (4, 4, -8), (0, 0, 0))$ (for type $(3, 1)$).
(3) Unstable by the weight $((9, -3, -3, -3), (8, -4, -4), (8, -4, -4))$.
(4) Unstable by the weight $((9, 9, -3, -15), (8, -4, -4), (8, -4, -4))$.
(5) Unstable by the weight $((9, 9, -3, -15), (12, 0, -12), (4, 4, -8))$.
$(5)^T$ Unstable by the weight $((9, 9, -3, -15), (4, 4, -8), (12, 0, -12))$.

□

These conditions for instability were found using an algorithmic implementation of the Hilbert-Mumford criterion using convex geometry, see Subsection 3.1 of [Hos23] for details. This algorithm outputs 18 maximal unstable block forms, reducing to 12 after taking transposes, which further reduces to the 5 forms above after suitable changes of basis. We provide these 12 forms in Appendix A for completeness, but we only do not need them for any proofs: we only use the Hilbert-Mumford criterion to prove tensors are unstable, and prove semistability geometrically using Proposition 3.5.

Now, Proposition 3.5 actually gives necessary conditions for G-semistability, and Proposition 3.7 gives necessary conditions for G-instability. We prove this by a case-by-case computation, splitting the closed points of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ into the following four types of cubic surfaces:

Proposition 3.8. *Every cubic surface $[S] \in |\mathcal{O}_{\mathbb{P}U}(3)|$ is of one (or more) of the following types:*

- *S is normal, with finitely many lines (S is smooth or has only canonical singularities).*
- *S is reducible.*
- *S is a cone over a cubic curve.*
- *S is non-normal, non-conical, and irreducible. In this case, the polynomial defining S is $\mathrm{SL}(U)$-equivalent to either $u_2u_0^2 + u_1u_3^2$ or $u_2u_0^2 + u_3u_0u_1 + u_1^3$.*

We then compute the G-semistable points for each type of cubic surface, and combine the classifications at the end. We use Kok Onn Ng's classification of $4 \times 3 \times 3$ tensors [Ng02] for cubic surfaces with at most canonical singularities:

Lemma 3.9. *Let $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ such that S_φ is normal with finitely many lines. Then φ is G-semistable if and only if one of the following holds:*

- *S_φ has at worst A_2 singularities (S_φ is $\mathrm{SL}(U)$-semistable).*
- *$\varphi \in A_3(d)$. There is a one-parameter family of such tensors.*
- *$\varphi \in A_1A_3(f)$, which is contained in the closure of the family $A_3(d)$.*

Proof. If S_φ is $\mathrm{SL}(U)$-semistable, then there is a homogeneous, non-constant $\mathrm{SL}(U)$-invariant polynomial $f \in \mathbb{C}[\mathrm{Sym}^3(U)]_\bullet$ where $f([S_\varphi]) \neq 0$. We let $\det : U^\vee \otimes V^\vee \otimes W^\vee \to \mathrm{Sym}^3(U^\vee)$ be the determinant map (unique up to a constant multiple), and consider $f \circ \det \in \mathbb{C}[U \otimes V \otimes W]_\bullet$. Then $f \circ \det$ is homogeneous, non-constant and G-invariant, and $f \circ \det(\varphi) \neq 0$ so φ is G-semistable.

Otherwise, suppose S_φ is $\mathrm{SL}(U)$-unstable. A general $\mathrm{SL}(U)$-unstable cubic surface has one A_3 singularity, and it is easy to verify from [Ng02] that every determinantal representation of an $\mathrm{SL}(U)$-unstable cubic surface is contained in $\overline{A_3(a)} \cup \overline{A_3(b)} \cup \overline{A_3(c)}$ or $A_3(d) \cup A_1A_3(f)$. Every point in $\overline{A_3(a)}, \overline{A_3(b)}, \overline{A_3(c)}$ is G-unstable of block types (4), (5), or $(5)^T$.

If $\varphi \in A_3(d)$ or $\varphi \in A_1A_3(f)$ then S_φ contains no rank 1 points (X_φ has no length 3 collinear subscheme), and is G-semistable. Otherwise, $\varphi \in \overline{A_3(a)} \cup \overline{A_3(b)} \cup \overline{A_3(c)}$ and is G-unstable. $\square$

Definition 3.10. Fixing a basis for U, V, W, we let σ be defined by

$$\sigma_U = \begin{pmatrix} u_0 & u_3 & -u_2 \\ -u_3 & u_0 & u_1 \\ u_2 & -u_1 & u_0 \end{pmatrix}.$$

The cubic surface S_σ is cut out by $\det(\sigma_U) = u_0(u_0^2 + u_1^2 + u_2^2 + u_3^3) = 0$, it is the union of a plane and a smooth quadric surface whose intersection is a smooth conic. The subschemes X_σ and Y_σ are the smooth conics cut out by $v_0^2 + v_1^2 + v_2^2 = 0$ and $w_0^2 + w_1^2 + w_2^2 = 0$.

Lemma 3.11. *Let $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ such that S_φ is reducible. Then φ is G-semistable if and only if φ is G-equivalent to σ.*

Proof. If S_φ is reducible it contains a hyperplane, and φ is G-unstable unless this hyperplane is a skew-symmetric subspace of $\mathbb{P}(V^\vee \otimes W^\vee)$. As such, without loss of generality we choose a basis such that

$$\varphi_U = u_0 A + \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix} \tag{12}$$

for some 3×3 matrix A. By changing basis of U, we can remove the skew-symmetric part of A, and so A can be assumed to be symmetric. Any symmetric complex matrix can be "diagonalised" by some invertible (in fact unitary) matrix B, in the sense that $B^TAB = D$ is diagonal. Applying B^T and B to the skew-symmetric part preserves its skew-symmetry (this will just change the basis of the skew-symmetric subspace, which we can restore with an appropriate $\mathrm{SL}(U)$-action), and we have

$$B^T\varphi_U B = \begin{pmatrix} \lambda_1 u_0 & 0 & 0 \\ 0 & \lambda_2 u_0 & 0 \\ 0 & 0 & \lambda_3 u_0 \end{pmatrix} + B^T \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix} B \tag{13}$$

for some scalars $\lambda_i \in \mathbb{C}$. And then equivalently, in a new basis we have

$$\varphi_U \sim \begin{pmatrix} \lambda_1 u_0 & u_3 & -u_2 \\ -u_3 & \lambda_2 u_0 & u_1 \\ u_2 & -u_1 & \lambda_3 u_0 \end{pmatrix}. \tag{14}$$

If one of the λ_is are non-zero, then φ_U is of block type (12) and is unstable. Otherwise, we can choose coordinates to rescale each λ_i to 1, and φ_U is of the form

$$\varphi_U \sim \sigma = \begin{pmatrix} u_0 & u_3 & -u_2 \\ -u_3 & u_0 & u_1 \\ u_2 & -u_1 & u_0 \end{pmatrix}. \tag{15}$$

But S_σ has no rank 1 points, so σ is G-semistable by Proposition 3.5. □

Lemma 3.12. *Let $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ such that S_φ is a cone. Then φ is G-unstable.*

Proof. We choose coordinates so that $\det(\varphi_U)$ has no u_3 term, i.e. that S_φ is a cone over $[0:0:0:1]$. Then in some basis, φ_U is either not injective, or $\varphi_U(0,0,0,1)$ is rank 1 or 2. If φ_U is not injective then φ is unstable. So first suppose $\varphi_U(0,0,0,1)$ is rank 1.

Because $\det(\varphi_U)$ has no u_3 term, then φ_U is G-equivalent to

$$\varphi_U = \begin{pmatrix} u_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{16}$$

where the a_{ij}s are linear forms in u_0, u_1, u_2 and $a_{22}a_{33} - a_{23}a_{32} = 0$. So $\mathrm{Span}\{a_{22}, a_{23}, a_{32}, a_{33}\}$ is an at most 2-dimensional subspace of $U^\vee$. So φ_U satisfies block condition (4) and is unstable.

Otherwise suppose that $\varphi_U(0,0,0,1)$ is rank 2. Then φ_U is G-equivalent to

$$\varphi_U = \begin{pmatrix} u_3 & 0 & 0 \\ 0 & u_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} \tag{17}$$

where the a_{ij}s are linear forms in u_0, u_1, u_2 and $a_{13}a_{31} + a_{23}a_{32} = 0$. Similarly, $\mathrm{Span}\{a_{13}, a_{31}, a_{23}, a_{32}\}$ is at most 2-dimensional and φ_U is of block form (12) and is unstable. □

Lemma 3.13. *Let $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ such that S_φ is non-normal, non-conical, and irreducible. Then φ is G-unstable.*

Proof. We choose coordinates so that $\det(\varphi_U)$ is at most linear in u_2, u_3. If the line $L \subset S_\varphi$ parametrised by $[0:0:u_2:u_3]$ is rank 1 then φ is of block type (10) and is unstable.

If L is not rank 1 then it has at least two rank 2 points, suppose that $\varphi_U(0,0,1,0)$ and $\varphi_U(0,0,0,1)$ are both rank 2. We choose coordinates so that $\varphi_U(0,0,0,1)$ is a 2×2 block in the top left corner, so the bottom right entry of φ_U is zero everywhere.

First suppose the line L is of type $(3,1)$. There are two cases: either L is also a line of type $(1,3)$, or it is not. If it is not, then in a basis two of the bottom entries are zero, so S_φ has a plane of type $(3,1)$ and is unstable. Otherwise, $\varphi_U(0,0,u_2,u_3)$ only has non-zero entries in the top left 2×2 block and the bottom right entry is zero. But then φ is of block type (12) and is unstable. The argument for L being a line of type $(1,3)$ but not $(3,1)$ is identical, after taking the transpose.

The only thing left to check is if L is a line of type $(2,2)$. We choose coordinates so that

$$\varphi_U = \begin{pmatrix} 0 & u_3 & u_2 \\ u_3 & 0 & 0 \\ u_2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}, \tag{18}$$

where the a_{ij}s are linear forms in u_0, u_1. The coefficient of u_2u_3 in $\det(\varphi_U)$ is $a_{23} + a_{32}$, which must be zero, so φ has a plane of type $(2,2)$ and is unstable. □

Combining these four lemmas, we find that the conditions described in Proposition 3.5 are sufficient (and Proposition 3.7 also provides necessary conditions for instability):

Proposition 3.14 (Classification of G-semistability)**.** *A point $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$-semistable if and only if one of the following holds:*

- *S_φ is an $\mathrm{SL}(U)$-semistable cubic surface, or*
- *S_φ contains no rank 1 points.*

Proof. If $S_\varphi = \mathbb{P}U$ then either $\varphi_U(U) \subset V^\vee \otimes W^\vee$ is a 4-dimensional rank 2 subspace or $\dim(\varphi_U(U)) \leq 3$. In the former case φ satisfies a rank 2 block condition and is $\mathrm{SL}(V) \times \mathrm{SL}(W)$-unstable, in the latter case φ_U is not injective and φ is $\mathrm{SL}(U)$-unstable. So if φ is G-semistable, then S_φ is a cubic surface. We then split into the cases in Proposition 3.8 and use Lemmas 3.9, 3.11, 3.12 and 3.13. $\square$

Remark 3.15. A necessary, but insufficient, condition for G-semistability is that φ is stable under all partial group actions.

Remark 3.16. The instability of some tensors can be nicely described geometrically. For example, by [EP00], X_φ (resp. Y_φ) is finite but not locally Gorenstein if and only if S_φ contains a plane of type $(1,3)$ (resp. $(3,1)$), in which case φ is G-unstable.

3.2. **Classifying stability.** We use the geometric description of the G-semistable locus to describe the ring of G-invariants up to finite extension, give a geometric description of G-stability, and describe some of the geometry of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$.

Proposition 3.17. *The preimage of the $\mathrm{SL}(U)$-stable locus of $|\mathcal{O}_{\mathbb{P}U}(3)|$ admits the following stratification, where a box indicates a maximally singular type of tensor inside $\det^{-1}_{UVW}\left(|\mathcal{O}_{\mathbb{P}U}(3)|^{\mathrm{SL}(U)-s}\right)$.*

$$\begin{tikzcd}
A_1(a) \arrow[r] \arrow[dr] \arrow[ddr] & 2A_1(a) \arrow[r] \arrow[dr] & 3A_1(a) \arrow[r] & \boxed{4A_1(a)} \\
 & 2A_1(b) \arrow[r] \arrow[ur] \arrow[dr] & 3A_1(b) \arrow[dr] & \\
 & 2A_1(c) \arrow[r] \arrow[ur] \arrow[dr] & 3A_1(c) \arrow[r] & \boxed{4A_1(b)} \\
A_1(b) \arrow[r] \arrow[ur] \arrow[uur] & 2A_1(d) \arrow[r] \arrow[ur] & \boxed{3A_1(d)} &
\end{tikzcd}$$

The G-semistable tensors φ where S_φ is not $\mathrm{SL}(U)$-stable can be decomposed into the disjoint union $\overline{A_2(a)}^{G-ss} \cup \overline{A_2(b)}^{G-ss}$. It has the following stratification, where a box indicates a maximally singular *type of tensor inside* $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)^{G-ss}$.

$$\begin{tikzcd}
 & A_1A_2(a) \arrow[r] \arrow[dr] & 2A_1A_2(a) \arrow[dr] & & \\
 & A_1A_2(b) \arrow[ur] \arrow[dr] \arrow[dddr] & 2A_1A_2(b) \arrow[r] \arrow[ddr] & A_12A_2(a) \arrow[dr] & \\
A_2(a) \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] & A_1A_2(c) \arrow[ur] \arrow[dr] & 2A_1A_2(c) \arrow[ur] \arrow[dr] & & \boxed{3A_2(a)} \\
 & A_1A_2(d) \arrow[ur] \arrow[r] \arrow[dr] & 2A_1A_2(d) \arrow[r] & A_12A_2(b) \arrow[ur] & \\
 & & 2A_2(a) \arrow[uuur] \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & A_1A_2(e) \arrow[r] & 2A_1A_2(f) \arrow[dr] & \\
A_2(b) \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & A_1A_2(f) \arrow[r] & 2A_1A_2(e) \arrow[r] & \boxed{A_12A_2(c)} \\
 & A_1A_2(g) \arrow[r] \arrow[dr] & 2A_2(b) \arrow[r] \arrow[ur] & \boxed{G \cdot \sigma} \\
 & A_3(d) \arrow[r] & A_1A_3(f) \arrow[ur] &
\end{tikzcd}$$

Proof. We analyse the closure relations using the associated length 6 subschemes. Specifically, the G-semistable locus of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is contained in the $\mathrm{SL}(U) \times \mathrm{SL}(W)$-stable locus, so for two types of G-semistable tensors $F(n)$ and $F'(n')$, we have $F'(n') \subset \overline{F(n)}$ if and only if $\pi_{UW}(F'(n')) \subset \overline{\pi_{UW}(F(n))}$. And because $\gamma : \mathrm{Hilb}_6(\mathbb{P}V) \dashrightarrow \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(U) \times \mathrm{SL}(W))$ is an isomorphism when restricted to curvilinear subschemes not contained in a conic, then we can compute these closures inside $\mathrm{Hilb}_6(\mathbb{P}V)$.

The associated length 6 subschemes to each type of tensor can be found in [Ng02]. As an example, to see that $\overline{A_2(a)}^{G-ss} \cap \overline{A_2(b)}^{G-ss} = \emptyset$, we note that if $\varphi \in \overline{A_2(a)}^{G-ss}$ then X_φ has a collinear length 4 cycle but no length 3 point, and if $\varphi \in \overline{A_2(b)}^{G-ss}$ then X_φ has a length 3 point but no collinear length 4 cycle.

The only things left to check are the closure relations involving $G \cdot \sigma$. The G-orbit of a G-semistable tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is codimension at most 6 if S_φ has canonical singularities, and $G \cdot \sigma$ is codimension 7, so $G \cdot \varphi \not\subset \overline{G \cdot \sigma}$ if S_φ has canonical singularities. Fix a G-semistable type of tensors $F(n)$, where each $\varphi \in F(n)$ has no rank 1 points, and consider a family of $B \subset \mathrm{Hilb}_6(\mathbb{P}V)$ length 6 subschemes of type $F(n)$ that specialise to lie on a smooth conic (which always exists). Then the family $\gamma(B)$ specialises from $\pi_{UW}(F(n))$ to $\pi_{UW}(G \cdot \sigma)$, and $G \cdot \sigma \subset \overline{F(n)}$. Conversely, if one tensor $\varphi \in F(n)$ contains (and hence all tensors contain) a rank 1 point, then $F(n) \subset R$. But $\sigma \notin R$ and R is closed, so $G \cdot \sigma \not\subset \overline{F(n)}$. $\square$

Theorem 3.18. *The ring of* $\mathrm{SL}(U)$*-invariants of* $\mathbb{C}[\mathrm{Sym}^3(U)]_\bullet$ *is generated by the invariants* $I_8, I_{16}, I_{24}, I_{32}, I_{40}, I_{100}$*, of degree given by their subscripts. The square* I_{100}^2 *is a polynomial in the other five, which are algebraically independent.*

The GIT quotient $|\mathcal{O}_{\mathbb{P}U}(3)|//\,\mathrm{SL}(U)$ *is isomorphic to* $\mathbb{P}(1,2,3,4,5)$.

Definition 3.19. Consider the map of vector spaces $\det : U^\vee \otimes V^\vee \otimes W^\vee \to \mathrm{Sym}^3(U^\vee)$. Let $J_{3d} = I_d \circ \det \in \mathbb{C}[U \otimes V \otimes W]_\bullet$ for each invariant $I_d \in \mathbb{C}[\mathrm{Sym}^3(U)]_\bullet$.

We let $S_\bullet = \mathbb{C}[R_{24}, J_{24}, \ldots, J_{120}, J_{300}]_\bullet \subset \mathbb{C}[U \otimes V \otimes W]_\bullet^G$. The projective space $\mathrm{Proj}(S_\bullet)$ is a codimension 1 subscheme of $\mathbb{P}(1,1,2,3,4,5)$, with a rational map to $\mathbb{P}(1,2,3,4,5)$, denoted by $\det_{UVW}$ in our notation, induced by the injection $\det^* : \mathbb{C}[\mathrm{Sym}^3(U)]_\bullet \to S_\bullet$ that sends $I_d \mapsto J_{3d}$.

We let $\eta : \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G \dashrightarrow \mathrm{Proj}(S_\bullet)$ be the rational map induced by the inclusion of graded rings.

Proposition 3.20. $S_\bullet$ *is a finitely generated, integral, graded* $\mathbb{C}[U \otimes V \otimes W]_\bullet^G$*-module. The map* η *is a finite morphism of even degree dividing 72, and the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G \arrow[r, "\eta"] \arrow[dr, dashed, "\det_{UVW}"'] & \mathrm{Proj}\,(S_\bullet) \arrow[d, dashed] \\
& |\mathcal{O}_{\mathbb{P}U}(3)|//\,\mathrm{SL}(U)
\end{tikzcd}$$

For every subscheme $Z \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G$*, we have an equality* $\dim(Z) = \dim(\eta(Z))$*, and an inequality* $\dim(Z) \leq \dim(\det_{UVW}(Z)) + 1$.

Proof. By Theorem 3.14, a point $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is G-unstable if and only if φ is contained in the vanishing locus of the homogeneous ideal generated by $S_\bullet$. So $S_\bullet$ generates $\mathbb{C}[U \otimes V \otimes W]_\bullet^G$ up to radical, and $S_\bullet$ is a finitely generated, integral, graded $\mathbb{C}[U \otimes V \otimes W]_\bullet^G$-module.

A larger automorphism group of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is the semidirect product of $G \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ is the involution swapping V and W, acting as the Gale transform on X_φ. The ring $S_\bullet$ is invariant under this larger action, while $\mathbb{C}[U^\vee \otimes V^\vee \otimes W^\vee]_\bullet^G$ is not invariant under Gale duality, so the morphism $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G \to \mathrm{Proj}(S_\bullet)$ factors through the degree two morphism $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G \to \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(G \rtimes \mathbb{Z}/2\mathbb{Z})$. And because $\det_{UVW}$ is a generically finite rational map of degree 72 that factors through η, then the degree of η divides 72. $\square$

Remark 3.21. The rational map $\det_{UVW}$ is undefined only at $\pi_{UVW}(\overline{A_3(d)}^{G-ss})$. We will see in the proof of Proposition 3.27 that this is a singleton.

Remark 3.22. We can compute the dimension of subschemes inside $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G$ by analysing their images under η and $\det_{UVW}$. In particular, $\dim(Z) = \dim(\eta(Z))$ because η is finite.

The map $\det_{UVW}$ is only defined at the finitely many points φ where $\eta(\varphi) = [1:0:0:0:0:0]$.

Proposition 3.23 (Sufficient criterion for G-stability)**.** *Every determinantal representation of an* $\mathrm{SL}(U)$*-stable cubic surface is* G*-stable.*

Proof. We first prove the stabiliser is finite. Let $G_\varphi \subset G$ be the G-stabiliser of a tensor φ, and $\mathrm{SL}(U)_\varphi, \mathrm{SL}(V)_\varphi, \mathrm{SL}(W)_\varphi$ be its projection onto the three factors of G. These are subgroups of the respective stabilisers of $S_\varphi, X_\varphi, Y_\varphi$, and so we have an inequality $\dim(G_\varphi) \leq \dim(\mathrm{SL}(U)_{S_\varphi}) + \dim(\mathrm{SL}(V)_{X_\varphi}) + \dim(\mathrm{SL}(W)_{Y_\varphi})$. It suffices to show that all of these stabilisers are finite for G_φ to be finite. Since the dimension of the stabiliser is upper semicontinuous, we only need to check this for tensors of type $4A_1(a), 4A_1(b)$, and $3A_1(d)$.

The PGL(U)-stabiliser of every normal cubic surface with no parameters was found by Yoshiyuki Sakamaki in [Sak10]. The respective PGL(U)-stabiliser of a $4A_1$ cubic surface is $\mathfrak{S}_4$, so the SL(U)-stabilisers are finite. The schemes X_φ and Y_φ associated to a tensor of type $4A_1(a)$ and $4A_1(b)$ are supported at (at least) four points in linearly general position, so have trivial PGL(V) and PGL(U)-stabilisers. Similarly, for $3A_1(d)$ these schemes are three length 2 points in linearly general position, which has a finite stabiliser.

Now to prove the G-orbits are closed. Let φ be a determinantal representation of an SL(U)-stable cubic surface, and φ' be a G-semistable tensor. If S_φ and $S_{\varphi'}$ are not SL(U)-equivalent, then $f(\varphi) \neq f(\varphi')$, so $\varphi' \notin \overline{G \cdot \varphi}$.

So suppose S_φ and $S_{\varphi'}$ are SL(U)-equivalent. The G-orbits $G \cdot \varphi$ and $G \cdot \varphi'$ are irreducible codimension 4 subvarieties of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$, so if $G \cdot \varphi' \subset \overline{G \cdot \varphi}$ then $G \cdot \varphi' \cap G \cdot \varphi$ is non-empty and φ and φ' are G-equivalent. So if $\varphi' \in \overline{G \cdot \varphi}$ then $\varphi' \in G \cdot \varphi$, so $G \cdot \varphi$ is closed. $\square$

Proposition 3.24 (Sufficient criteria for strict G-semistability)**.** *A point $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is strictly G-semistable if φ is G-semistable and φ is of one of the following two block forms:*

(1)

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(2)

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & 0 \\ * & * & * & * & * & 0 & * & * & 0 & * & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Proof. We use the Hilbert-Mumford criterion. The first type is seen to not be G-stable by the weight vector $((3,0,0,-3),(2,-1,-1),(1,1,-2))$, the second type is not G-stable by the weight vector $((1,0,0,-1),(1,0,-1),(1,0,-1))$, which act with only non-negative weights where there is an $*$. A complete list of the block types output by the computer is found in Appendix A $\square$

Corollary 3.25 (Classification of G-stability)**.** *A point $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is G-stable if and only if S_φ is an* SL(U)*-stable cubic surface.*

Proof. We look at the set of tensors φ where S_φ is not SL(U)-stable. The subvariety $A_2(a)$ consists of tensors G-equivalent to

$$\varphi_U = \begin{pmatrix} u_3 & -\alpha_1 u_0 + u_1 & u_1 \\ 0 & u_2 & -u_0 + u_1 \\ u_1 - \alpha_2 u_2 & 0 & u_0 \end{pmatrix},$$

where $\alpha_1, \alpha_2 \in \mathbb{A}^1 \setminus \{0,1\}$, which is of strictly G-semistable block type (1). The subvariety $A_2(b)$ consists of tensors G-equivalent to

$$\varphi_U = \begin{pmatrix} -\beta_1 u_3 & u_3 & u_2 \\ (1-\beta_1)u_3 & (1-\beta_2)u_0 + \beta_2 u_1 + u_2 & u_0 \\ u_1 & u_0 & 0 \end{pmatrix},$$

where $\beta_1 \in \mathbb{A}^1 \setminus \{0,1\}$ and $\beta_2 \in \mathbb{A}^1 \setminus \{0, 1/(1-\beta_1)\}$, which is of strictly G-semistable block type (2). All other G-semistable tensors where φ is not SL(U)-stable are in either $\overline{A_2(a)}^{G-ss}$ or $\overline{A_2(b)}^{G-ss}$, according to the closure relations given in Proposition 3.17, and are strictly G-semistable. $\square$

3.3. **The geometry of the GIT quotient.**

Proposition 3.26. *The subvariety* $3A_2(a) \subset \overline{A_2(a)}^{G-ss}$ *is the unique* G*-polystable orbit inside* $\overline{A_2(a)}^{G-ss}$, *the set* $\pi_{UVW}(\overline{A_2(a)}^{G-ss}) \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G$ *is a singleton.*

Proof. All points in $3A_2(a)$ are G-equivalent. They are G-polystable because a tensor of type $3A_2(a)$ is maximally singular inside the G-semistable locus. So it suffices to prove that $\pi_{UVW}(\overline{A_2(a)}^{G-ss})$ is a finite scheme, since it is connected then it is a singleton.

This is immediate: $\eta \circ \pi_{UVW}(\overline{A_2(a)}^{G-ss}) = \{[0:8:1:0:0:0]\}$ is a singleton, so $\dim(\pi_{UVW}(\overline{A_2(a)}^{G-ss})) = 0$ because η is a finite morphism. □

Proposition 3.27. *A point* $\varphi \in \overline{A_2(b)}^{G-ss}$ *is* G*-polystable if and only if* $\varphi \in \overline{2A_2(b)}^{G-ss}$. *There is an isomorphism* $\pi_{UVW}(\overline{A_2(b)}^{G-ss}) = \overline{A_2(b)}//G \cong \mathbb{P}(1,2)$.

Proof. If $\varphi \in \overline{2A_2(b)}^{G-ss}$, then φ is G-equivalent to a tensor of the form

$$\varphi_{[s:t]} = \begin{pmatrix} 0 & u_3 & u_2 \\ u_3 & su_1 + tu_2 & u_0 \\ u_1 & u_0 & 0 \end{pmatrix}, \tag{19}$$

parametrised by $[s:t] \in \mathbb{P}^1$. The involution $\tau \in G$ that swaps u_1 and u_2, v_0 and v_2, and w_0 and w_2 acts on $\mathbb{P}^1$ by sending $[s:t] \mapsto [t:s]$. Geometrically, this swaps the two A_2 singularities of S_φ, and sends X_φ to its Gale dual. If $[s:t] \neq [s':t'], [t':s']$, then $\varphi_{[s:t]}$ and $\varphi_{[s':t']}$ are not G-equivalent. The quotient of $\mathbb{P}^1$ by the subgroup $\{1,\tau\} \cong \mathbb{Z}/2\mathbb{Z}$ is isomorphic to $\mathbb{P}(1,2)$. We now need to prove that a tensor $\varphi \in \overline{A_2(b)}$ is G-polystable if and only if $\varphi \in \overline{2A_2(b)}^{G-ss}$.

We first prove that every tensor of type $\overline{2A_2(b)}^{G-ss}$ is G-polystable. If $\varphi \in G \cdot \sigma$ then φ is G-polystable because this is a maximally singular G-orbits inside the G-semistable locus. The G-stabiliser of any tensor $\varphi \in \overline{2A_2(b)}^{G-ss} \setminus G\cdot\varphi$ is 1-dimensional, so $G\cdot\varphi$ is an irreducible codimension 5 subvariety by orbit-stabiliser. By the same argument as the second half of the proof of Proposition 3.23, then if $\varphi, \varphi' \in \overline{2A_2(b)}^{G-ss} \setminus G\cdot\varphi$ we have $\varphi' \in \overline{G\cdot\varphi}$ if and only if $\varphi' \in G\cdot\varphi$. And the points $\pi_{UVW}(\sigma)$ and $\pi_{UVW}(\varphi)$ are distinct because all the invariants J_d vanish at σ but not at φ. So $\sigma \notin \overline{G\cdot\varphi}$, hence $G\cdot\varphi$ is closed inside the G-semistable locus if $\varphi \in \overline{2A_2(b)}^{G-ss}$.

For the other direction, we only need to prove that every tensor in $\overline{A_2(b)}^{G-ss} \setminus \overline{2A_2(b)}^{G-ss}$ has a non-closed G-orbit. Similar to the first half of the proof of Proposition 3.23, every determinantal representation of a cubic surface with at worst A_1A_2 singularities has a finite G-stabiliser. Because they are strictly G-semistable with finite G-stabiliser, then tensors of types $A_2(b), A_1A_2(e), A_1A_2(f), A_1A_2(g)$ must have a non-closed G-orbit. For type $A_3(d)$, we note that $G\cdot\sigma \subset \overline{A_3(d)}^{G-ss}$ and that $\eta \circ \pi_{UVW}(\overline{A_3(d)}^{G-ss}) = \eta\circ\pi_{UVW}(G\cdot\sigma) = \{[1:0:0:0:0:0]\}$ is a singleton. Because $\overline{A_3(d)}^{G-ss}$ is connected, then $\pi_{UVW}(\overline{A_3(d)}^{G-ss}) = \pi_{UVW}(G\cdot\sigma)$ and for each $\varphi \in \overline{A_3(d)}^{G-ss}$ we have $G\cdot\sigma \subset \overline{G\cdot\varphi}$. In particular, $G\cdot\varphi$ is not closed.

It is then immediate that $\pi_{UVW}(\overline{A_2(b)}^{G-ss}) = \pi_{UVW}(\overline{2A_2(b)}^{G-ss})$ as these are the only G-polystable points inside $\overline{A_2(b)}^{G-ss}$. And we have $\overline{2A_2(b)}^{G-ss}/G \cong \mathbb{P}^1/(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{P}(1,2)$. □

These facts can all be summarised in the following theorem:

Theorem 3.28. *A tensor* $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ *is:*

- *G-stable if and only if S_φ is $\mathrm{SL}(U)$-stable (S_φ has at worst $4A_1$ singularities).*
- *G-semistable if and only if either S_φ is $\mathrm{SL}(U)$-semistable or $\varphi \in A_3(d) \cup A_1A_3(f) \cup G\cdot\sigma$.*
- *Strictly G-polystable if and only if either $\varphi \in 3A_2(a)$ or $\varphi \in 2A_2(b) \cup A_1 2A_2(c) \cup G\cdot\sigma$.*

In the GIT quotient $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G$, the strictly G-semistable locus is the disjoint union of a singleton and a copy of $\mathbb{P}(1,2)$.

The rational map $\det_{UVW} : \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) \dashrightarrow |\mathcal{O}_{\mathbb{P}U}(3)|//\mathrm{SL}(U)$ is undefined only at $\pi_{UVW}(\sigma)$, a non-singular point of this copy of $\mathbb{P}(1,2)$.

3.4. **Resolving the determinant map between GIT quotients.**

Definition 3.29. Let $\mathcal{M} \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G \times |\mathcal{O}_{\mathbb{P}U}(3)|//\mathrm{SL}(U)$ be the closure of the graph of $\det_{UVW}$.

To probe $\mathcal{M}$, we use a specific test family of length 6 subschemes:

Definition 3.30. Let $\mathcal{H} \subset \mathrm{Hilb}_6(\mathbb{P}V)$ be the open subscheme parametrising length 6 subschemes $X \subset \mathbb{P}V$ such that:

- X is canonical,
- X is not contained in a singular conic, and
- the associated cubic surface S_X is SL(4)-semistable.

The image of $\mathcal{H}$ under the contraction γ lies in the $\mathrm{SL}(V)$-semistable locus of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(U) \times \mathrm{SL}(W))$, and so we have a morphism $f : \mathcal{H} \to \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G$. This sends a point $[X] \mapsto f([X])$, where if X is contained in a conic then $f([X]) = \pi_{UVW}(\sigma)$, and otherwise $f([X]) = \pi_{UVW}(\varphi)$ for some φ where $X_\varphi = X$.

And, following the construction in Remark 1.11, the family of associated cubic surfaces over $\mathcal{H}$ induces a morphism $g : \mathcal{H} \to |\mathcal{O}_{\mathbb{P}U}(3)|//\mathrm{SL}(U)$. For each $[X] \in \mathcal{H}$, the point $g([X])$ is the image of the associated cubic surface $[S_X]$ under the quotient map $|\mathcal{O}_{\mathbb{P}U}(3)| \dashrightarrow |\mathcal{O}_{\mathbb{P}U}(3)|//\mathrm{SL}(U)$. If X is contained in a smooth conic, then S_X has an A_1 singularity, and otherwise we have $\det_{UVW} \circ f([X]) = g([X])$.

Both the morphisms f and g are surjective and the graph of $\det_{UVW}$ is contained in the image of (f,g), so the product morphism

$$(f,g) : \mathcal{H} \longrightarrow \mathcal{M}$$

is surjective.

Proposition 3.31. *The fibre of $\rho : \mathcal{M} \to \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G$ above $\pi_{UVW}(\sigma)$ is the product $\{\pi_{UVW}(\sigma)\} \times \Delta$, where $\Delta \subset |\mathcal{O}_{\mathbb{P}U}(3)|//\mathrm{SL}(U)$ is the GIT quotient of the discriminant divisor.*

Proof. We have $f^{-1}(\{\pi_{UVW}(\sigma)\}) = \mathcal{H} \cap C_6$, so the fibre $\rho^{-1}(\{\pi_{UVW}(\sigma)\})$ is the image of $\mathcal{H} \cap C_6$ under (f,g). It is known (see either [BW79] or [Sak10] for proofs) that every cubic surface with an A_1 singularity is associated to a length 6 canonical subscheme $X \subset \mathbb{P}V$ that is the intersection of a smooth conic with a (possibly degenerate) plane cubic, in particular every cubic surface with only A_1 singularities or both A_1 and A_2 singularities is associated to an element of $\mathcal{H} \cap C_6$.

The family of associated cubic surfaces to $\mathcal{H} \cap C_6$ does not include cubic surfaces with only A_2 singularities, but $g(\mathcal{H} \cap C_6) = \Delta$ because all SL(4)-semistable cubic surfaces with A_2 singularities are identified with those with A_1A_2 singularities in the GIT quotient. □

Remark 3.32. Let $\mathcal{H}^s \subset \mathcal{H}$ be the open subscheme of subschemes whose associated cubic surface is SL(4)-stable. The morphism $(f,g) : \mathcal{H} \to \mathcal{M}$ is $\mathrm{SL}(V)$-equivariant, and $(f,g)(\mathcal{H}^s) \subset \mathcal{M}$ is an $\mathrm{SL}(V)$-orbit space for $\mathcal{H}^s$. Conversely, the fibre $(f,g)^{-1}(m)$ for each $m \in \mathcal{M}$ is an $\mathrm{SL}(V)$-orbit if and only if $m \in (f,g)(\mathcal{H}^s)$.

This suggests that $\mathcal{M}$ is a GIT quotient of $\mathrm{Hilb}_6(\mathbb{P}V)$, or at least birational in codimension 2 to one, prompting the following question:

Question 3.33. *Is the map $(f,g) : \mathrm{Hilb}_6(\mathbb{P}V) \dashrightarrow \mathcal{M}$ a GIT quotient? If so, with respect to which line bundle? And is the semistable locus $\mathcal{H}$, is the stable locus $\mathcal{H}^s$?*

GIT quotients of Hilbert schemes of points have been studied by Durgin in her PhD thesis [Dur15] (the asymptotic stability of $n = 6$) and Gallardo and Schmidt in [GS24] ($n = 5, 7$). The Picard group of the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{P}V)$ is free of rank 2, generated by the line bundles associated to the divisors H and $B/2$, where H parametrises subschemes that intersect a fixed line $L \subset \mathbb{P}V$ and B parametrises non-reduced subschemes (the exceptional divisor of the Hilbert-Chow morphism). The ample cone parametrises line bundles $\mathcal{L}_m$ associated to the divisors $mH - B/2$ for $m > n-1$, and admits a wall-and-chamber stratification as m varies. Durgin's thesis gives an explicit description for $\mathrm{SL}(V)$-semistability with respect to $\mathcal{L}_m$, for sufficiently large $m \gg 0$, but this differs from our setup in two key ways:

- Any length 6 subscheme containing a length 3 point is $\mathrm{SL}(V)$-unstable for $m \gg 0$. But the morphism $\mathcal{H} \to \mathcal{M}$ is defined for certain subschemes that have length 3 points, contracting them to the strict transform of $\pi_{UVW}(\overline{2A_2}^{G-ss})$ under ρ, which is isomorphic to $\mathbb{P}(1,2)$.
- Letting L, L' be two different lines, and choosing 3 distinct points on L and L', their union is a canonical length 6 subscheme associated to an A_2 cubic surface (and the two-parameter family of A_2 cubic surfaces arises in this way). Subschemes of this type are $\mathrm{SL}(V)$-stable for $m \gg 0$. The morphism $\mathcal{H} \to \mathcal{M}$ isn't immediately defined for points of this type, but if it could be extended then it would identify them all to the singleton $(\pi_{UVW}(\sigma), S_{3A_2})$.

Appendix A. Block forms

All computations were carried out in SageMath version 10.3 [Sage].

***G*-instability:** Our computational implementation of the convex Hilbert-Mumford criterion for G-semistability of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ outputs the following 12 maximal block types (and their transposes) for G-instability, where a tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is G-unstable if and only if it is of one of the following block forms:

(1) Unstable by the weight $((1,1,1,-3),(0,0,0),(0,0,0))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \end{array}\right]$$

(2) Unstable by the weight $((9,-3,-3,-3),(0,0,0),(4,4,-8))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(3) Unstable by the weight $((9,-3,-3,-3),(8,-4,-4),(8,-4,-4))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & 0 & 0 & * & 0 & 0 & * & 0 & 0 \\ * & * & * & * & 0 & 0 & * & 0 & 0 & * & 0 & 0 \end{array}\right]$$

(4) Unstable by the weight $((9,9,-3,-15),(8,-4,-4),(8,-4,-4))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(5) Unstable by the weight $((9,9,-3,-15),(12,0,-12),(4,4,-8))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & 0 & * & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 & * & 0 & 0 \\ * & * & 0 & * & * & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Contained in type (2):

(6) Unstable by the weight $((1,1,1,-3),(4,0,-4),(4,0,-4))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & 0 \\ * & * & 0 & * & * & 0 & * & * & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(7) Unstable by the weight $((12,0,-12),(16,4,-20),(9,9,-3,-15))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & 0 \\ * & * & * & * & * & * & * & * & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(8) Unstable by the weight $((3,3,-3,-3),(4,-2,-2),(6,0,-6))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(9) Unstable by the weight $((3,3,3,-9),(8,-4,-4),(4,4,-8))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Contained in type (4):

(10) Unstable by the weight $((3,3,-3,-3),(0,0,0),(4,-2,-2))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(11) Unstable by the weight $((5,1,1,-7),(4,0,-4),(4,0,-4))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 & * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Contained in type (5):

(12) Unstable by the weight $(3,3,-3,-3),(2,2,-4),(2,2,-4))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & 0 & * & * & 0 \\ * & * & * & * & * & * & * & * & 0 & * & * & 0 \\ * & * & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Types (6) to (9) are contained in type (2) because there is a change of basis that converts the bottom row to be of the form

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(10) $\subset$ (4): by changing basis on the columns so that the last matrix is of the form

$$\begin{bmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(11) $\subset$ (4): by taking a linear combination of the middle two matrices so that φ is of the form

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 & * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(12) $\subset$ (5): any line in $\mathbb{P}((\mathbb{C}^2)^\vee \otimes (\mathbb{C}^2)^\vee)$ intersects the Segre variety of rank 1 pairings at least once, so has a rank 1 point. For each φ of block type (12), we choose a change of basis so that

$$\varphi \in \left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & 0 & * & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and this is contained in block type (5).

The proofs for the transposes of types (7) to (10) are identical. Block type (2) is the condition of a plane of type $(3,1)$, block type $(2)^T$ is the condition for a plane of type $(1,3)$.

Strict G-semistability: Our computational implementation of the convex Hilbert-Mumford criterion for G-(semi)stability of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ outputs the following 7 maximal block types (and their transposes) for strict G-semistability, where a G-semistable tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is strictly G-semistable if and only if it is of one of the following block forms:

(1) Not stable by the weight $((3,0,0,-3),(2,-1,-1),(1,1,-2))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(2) Not stable by the weight $((1,0,0,-1),(1,0,-1),(1,0,-1))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & 0 \\ * & * & * & * & * & 0 & * & * & 0 & * & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Contained in type (1)*:*

(3) Not stable by the weight $((1,1,0,-2),(1,0,-1),(1,0,-1))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(4) Not stable by the weight $((2,0,-1,-1),(1,0,-1),(1,0,-1))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & 0 & * & * & 0 \\ * & * & * & * & * & 0 & * & 0 & 0 & * & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Contained in unstable type (2)*:*

(5) Not stable by the weight $((0,0,0,0),(2,-1,-1),(1,1,-2))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & 0 & 0 & * & 0 & 0 & * & 0 & 0 & * & 0 & 0 \end{array}\right]$$

$(3) \subset (1)$: by taking a linear combination of the first two matrices so that the bottom row is of the form

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$(4) \subset (1)$: by taking a linear combination of the last two matrices so that the last matrix is of the form

$$\begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Not stable type (5) is contained in unstable type (2) by taking a change of basis that converts the bottom row to be of the form

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

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Mathematical Sciences Institute, Australian National University, Canberra, Australia