# THE GIT OF $4 \times 3 \times 3$ TENSORS (DETERMINANTAL CUBIC SURFACES)

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Abstract. A $4 \times 3 \times 3$ tensor defines three determinantal varieties in $\mathbb{P}^3, \mathbb{P}^2$, and $\mathbb{P}^2$ where it drops rank: a cubic surface, and (in general) two length 6 subschemes. We study the Geometric Invariant Theory (GIT) quotient of $\mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ by $\mathrm{SL}(4) \times \mathrm{SL}(3) \times \mathrm{SL}(3)$, the moduli space of linear determinantal representations of cubic surfaces. This moduli space is the SL(3)-GIT-quotient of the final model of the Hilbert scheme of 6 points in $\mathbb{P}^2$ and admits a surjective rational map to the GIT moduli space of cubic surfaces, undefined at only one point.

Using the Hilbert-Mumford criterion and topological criterion for semistability, we completely describe the (semi)stable locus and some of the geometry of the GIT quotient, including the ring of invariants up to finite extension. We also describe a blowup of the GIT quotient that resolves the rational map to the moduli space of cubic surfaces. This blowup is an SL(3)-quotient of the Hilbert scheme of 6 points in $\mathbb{P}^2$.

## 1. Introduction

The determinant of a $d \times d$ matrix of linear forms in $n$ variables is a degree $d$ polynomial. Going backwards, one can ask the question of when a given polynomial can be written as the determinant of a matrix of linear forms, and in how many ways. A quick dimension count, after quotienting out by the appropriate symmetry groups, shows that for most values of $d$ and $n$ the answer is either almost always no, or almost always yes and in infinitely many ways. But when $d = 3$ and $n = 4$, that is of cubic surfaces in $\mathbb{P}^3$, every smooth cubic surface can be written as a determinant in exactly 72 ways. This is in bijection with the 72 sets of six skew lines on a smooth cubic surface.

This fact has been known for over a century, and determinantal representations of cubic surfaces has been well-studied by both classical and modern algebraic geometers. Our contribution is a description of the Geometric Invariant Theory (GIT) moduli space of determinantal cubic surfaces.

A $3 \times 3$ matrix $\varphi_1$ of linear forms in 4 variables is a $4 \times 3 \times 3$ tensor $\varphi \in \mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, which has two adjoint $4 \times 3$ matrices $\varphi_2, \varphi_3$ of linear forms in 3 variables. We let $S_\varphi \subset \mathbb{P}^3, X_\varphi \subset \mathbb{P}^2, Y_\varphi \subset \mathbb{P}^2$ be the three varieties where $\varphi_1, \varphi_2, \varphi_3$ drop rank. For a sufficiently general $\varphi$, the following facts are true:

- $S_\varphi$ is a smooth cubic surface, and $X_\varphi$ and $Y_\varphi$ are reduced length 6 subschemes.
- The point configurations $X_\varphi$ and $Y_\varphi$ are Gale dual.
- The surface $S_\varphi$ is isomorphic to the blowup of $\mathbb{P}^2$ at both $X_\varphi$ and $Y_\varphi$.
- The ideals $I_{X_\varphi}$ and $I_{Y_\varphi}$ have free resolutions presented by the matrices $\varphi_2$ and $\varphi_3$.
- The two morphisms $S_\varphi \to \mathbb{P}^2$ associated to these blowups are induced by a pair of line bundles on $S_\varphi$ (associated to twisted cubics). These line bundles, as sheaves on $\mathbb{P}^3$, have free resolutions presented by $\varphi_1$ and its transpose.

Conversely, the ideal cutting out a general length 6 subscheme of $\mathbb{P}^2$ and the line bundle associated to a twisted cubic inside a smooth cubic surface have unique free resolutions of this form, up to

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basis change. All of the above correspondences have modular interpretations, generalise to some degenerate subschemes and surfaces, and induce rational maps between various moduli spaces.

The rational map

$$\mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3) \dashrightarrow |\mathcal{O}_{\mathbb{P}^3}(3)|$$

sending a $4 \times 3 \times 3$ tensor to its determinant cubic descends to a rational map, generically finite of degree 72, between GIT quotients

$$\det : \mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3) /\!/ (\mathrm{SL}(4) \times \mathrm{SL}(3) \times \mathrm{SL}(3)) \dashrightarrow |\mathcal{O}_{\mathbb{P}^3}(3)| /\!/ \mathrm{SL}(4) \tag{1}$$

The purpose of this paper is to study the GIT quotient on the left and describe this rational map. The main theorem is as follows:

**Theorem A.** *A point $\varphi \in \mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ is $\mathrm{SL}(4) \times \mathrm{SL}(3) \times \mathrm{SL}(3)$-stable if and only if its determinant cubic $\det(\varphi)$ is $\mathrm{SL}(4)$-stable, and $\mathrm{SL}(4) \times \mathrm{SL}(3) \times \mathrm{SL}(3)$-semistable if and only if either:*

- *the determinant cubic $\det(\varphi)$ is $\mathrm{SL}(4)$-semistable,*
- *the determinant cubic $\det(\varphi)$ has either $A_3$ or $A_1A_3$ singularities and the associated length 6 subschemes in $\mathbb{P}^2$ are supported at only two points, or*
- *the associated length 6 subschemes in $\mathbb{P}^2$ are smooth conics and the determinant cubic $\det(\varphi)$ is reducible.*

*The quotient of the strictly semistable locus is the disjoint union of a singleton and a subscheme whose normalisation is $\mathbb{P}(1, 2)$. The rational map (1) is undefined only at one point.*

1.1. 6 **points on a conic.** A common test family for moduli problems of cubic surfaces involves specialising 6 general points in $\mathbb{P}^2$ into some degenerate point configuration. When specialising so that 2 points collide, or 3 lie on a line, or all 6 on a conic, the associated family of cubic surfaces specialises to a singular cubic surface with an $A_1$ singularity. In the first two cases, this is also true for their determinantal representations, but all point configurations degenerating to lie on a conic $f = 0$ specialise to the same determinantal representation: a matrix $\mathrm{GL}(4) \times \mathrm{GL}(3)$-equivalent to

$$\varphi_2 = \begin{pmatrix} a & 0 & z & -y \\ b & -z & 0 & x \\ c & y & -x & 0 \end{pmatrix},$$

where $a, b, c \in \mathbb{C}[x, y, z]$ are linear forms such that $ax - by + cz = f$. The determinant cubic is a reducible surface, and the associated subschemes in $\mathbb{P}^2$ are the conic $f = 0$. When $f$ is a smooth conic, the $4 \times 3 \times 3$ tensor $\varphi_2$ is strictly $\mathrm{SL}(4) \times \mathrm{SL}(3) \times \mathrm{SL}(3)$-polystable and its image in the GIT quotient is the only point where the rational map (1) is undefined.

The birational map $\mathrm{Hilb}_6(\mathbb{P}^2) \dashrightarrow \mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)/\!/(\mathrm{SL}(4) \times \mathrm{SL}(3))$ contracts the divisor of length 6 subschemes contained in a conic to the orbits of tensors of the form (**??**). This is the final minimal model of $\mathrm{Hilb}_6(\mathbb{P}^2)$, a fact first observed in [ABCH13].

The rational map (1) is resolved by blowing up $\mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)/\!/(\mathrm{SL}(4) \times \mathrm{SL}(3) \times \mathrm{SL}(3))$, we call the resulting scheme $\mathcal{M}$. There is a surjective $\mathrm{SL}(3)$-invariant rational map

$$\mathrm{Hilb}_6(\mathbb{P}^2) \dashrightarrow \mathcal{M},$$

which is a quotient of $\mathrm{Hilb}_6(\mathbb{P}^2)$ and is an $\mathrm{SL}(3)$-orbit space in codimension 1.

1.2. **A brief history.** Much of the theory of determinantal representations of normal cubic surfaces was known by classical geometers such as Segre in 1906 [Seg06], who proved that every normal cubic surface with no $E_6$ singularity has a determinantal representation, and Thrall published a rough classification of $4 \times 3 \times 3$ tensors in 1941 [Thr41]. This was completed by Ng in 2002 [Ng02], who lists (almost) all $4 \times 3 \times 3$ tensors up to projective equivalence.

In 1989, Gimigliano introduced a sheaf-theoretic approach to smooth determinantal surfaces [Gim89], viewing the various matrices as maps of vector bundles on $\mathbb{P}^2$ and $\mathbb{P}^3$. This was extended by Dolgachev and Kapranov in [DK93], detailing the specific geometry of smooth determinantal cubic surfaces. Dolgachev gives a detailed summary of this approach for certain singular surfaces in [Dol12, §9.3]. Eisenbud and Popescu's series of papers on Gale transform in the late 90s [EP99; EP00] give a modern treatment of the connections between the Gale transform, free resolutions of ideals of points and determinantal varieties, of which a $4 \times 3 \times 3$ tensor gives excellent example.

More recently, the connection between determinantal representations of cubic surfaces and twisted cubics was studied by Lehn, Lehn, Sorger and Straten in [LLSS17], who describe the geometry of the GIT quotient of $\mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ by $\mathrm{SL}(3) \times \mathrm{SL}(3)$ to describe the moduli space of twisted cubics on a fixed smooth cubic fivefold.

1.3. **Structure of the paper.** Section 2 introduces important notation and recounts some known facts about determinantal representations. Most of these results, with more details and proofs, can be found in Section 9 of [Dol12] or in [Ng02].

Section 3 collates various known results to discuss the partial GIT quotient of $\mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ by $\mathrm{SL}(4) \times \mathrm{SL}(3)$. We use Gale duality to construct the birational contraction

$$\mathrm{Hilb}_6(\mathbb{P}^2) \dashrightarrow \mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)//(\mathrm{SL}(4) \times \mathrm{SL}(3).$$

Section 4 is devoted to understanding the full quotient by $\mathrm{SL}(4) \times \mathrm{SL}(3) \times \mathrm{SL}(3)$, and proving our main theorem. We describe the stable, semistable, and polystable loci, and identification of orbits in the GIT quotient. We also describe the ring of invariants up to finite extension.

1.4. **Conventions and organisation.** We are working over the complex numbers. The projective space $\mathbb{P}V$ refers to the projective space of 1-dimensional subspaces of $V$, i.e. $\mathbb{P}V = \mathrm{Proj}(\mathbb{C}[V^\vee]_\bullet)$ and $H^0(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(1)) = V^\vee$. We do not distinguish between a vector bundle and its associated sheaf, and use the same convention for the projectivisation of a vector bundle. For a vector space $V$, we will always let $\mathrm{SL}(V)$ act with the standard left action $v \mapsto g \cdot v$, and on $V^\vee$ by the left action $\nu \mapsto \nu \circ g^{-1}$ of precomposition.

All schemes are of finite type over $\mathbb{C}$. Unless otherwise specified, all tensor products are taken either over $\mathbb{C}$ or the structure sheaf of the ambient scheme. All *points* are $\mathbb{C}$-points unless otherwise specified, and by a *finite scheme* we mean a scheme with finite structure morphism to $\mathrm{Spec}\,\mathbb{C}$. When $X$ is a finite scheme, the *length* of a point $x \in X$ is the length of the non-reduced subscheme above $x$. When $X \subset Y$ is a closed subscheme, we let $\mathcal{I}_X \subset \mathcal{O}_Y$ be the ideal sheaf that cuts out $X$.

To a coherent sheaf $\mathcal{F}$ on a scheme $X$, we write $|\mathcal{F}| = \mathbb{P}(H^0(X, \mathcal{F}))$ for the complete linear system. For a vector subspace $A^\vee \subset H^0(X, \mathcal{F})$ we will write $(A^\vee, \mathcal{F})$ as shorthand for the linear series.

1.5. **Acknowledgements.** This paper is an extension of the author's undergraduate thesis. I thank my advisor, Anand Deopurkar, for introducing me to this topic and for countless meetings and helpful discussions.

Contents



## 2. Preliminaries

### 2.1. **Notation.**

**Definition 2.1.** Let $U, V, W$ be complex vector spaces with $\dim U = 4$ and $\dim V = \dim W = 3$. Whenever we fix bases of $U^\vee, V^\vee, W^\vee$, we will write the basis vectors as $u_0, u_1, u_2, u_3$ and $v_0, v_1, v_2$ and $w_0, w_1, w_2$. We often write elements of $U^\vee \otimes V^\vee \otimes W^\vee$ as a $4 \times 3 \times 3$ matrix of linear forms in $U$, where $V$ corresponds to the columns and $W$ corresponds to the rows.

Note that $V$ and $W$ play completely symmetric roles in this paper.

**Definition 2.2.** Let $G = \mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$.

**Definition 2.3.** We let $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ be a $4 \times 3 \times 3$ tensor. The tensor $\varphi$ has several adjoints we are interested in, we use the following notation to describe them: for each decomposition of $U \otimes V \otimes W$ as $A \otimes B$, we write $\varphi_A$ for the adjoint linear map

$$\varphi_A : A \longrightarrow B^\vee.$$

For example, $\varphi_U \in \mathrm{Hom}(U, V^\vee \otimes W^\vee)$ and $\varphi_{U \otimes W} \in \mathrm{Hom}(U \otimes W, V^\vee)$. In this notation, $\varphi = \varphi_{U \otimes V \otimes W}$. When $A, B, C \in \{U, V, W\}$ are three distinct vector spaces, we view $\varphi_A : A \to B^\vee \otimes C^\vee = \mathrm{Hom}(B, C^\vee)$ as an element of $A^\vee \otimes \mathrm{Hom}(B, C^\vee)$ which, after choosing bases for $B$ and $C$, can be identified with a $\dim(C) \times \dim(B)$ matrix of linear forms in $A$.

We also interpret $\varphi$ as a map of sheaves. We write

$$\widetilde{\varphi}_{\mathbb{P}A,B} : B \otimes \mathcal{O}_{\mathbb{P}A}(-1) \longrightarrow C^\vee \otimes \mathcal{O}_{\mathbb{P}A},$$

where $\widetilde{\varphi}_{\mathbb{P}A,B}$ is presented by the matrix of linear forms $\varphi_A \in A^\vee \otimes \mathrm{Hom}(B, C^\vee)$.

**Definition 2.4** (Associated subschemes)**.** We are interested in the subschemes of $\mathbb{P}A$ where $\varphi_A$ drops rank. The most important ones will be given special names: let $S_\varphi \subset \mathbb{P}U$ be the subscheme where $\varphi_U$ drops rank, it is a cubic surface. In a basis, $S_\varphi$ is cut out by the cubic polynomial $\det(\varphi_U) \in \mathbb{C}[U^\vee]_\bullet$ (the determinant of the matrix $\varphi_U \in U^\vee \otimes \mathrm{Hom}(V, W^\vee)$). The surface $S_\varphi$ is called the *associated cubic surface* to $\varphi$. We let $R_\varphi \subset S_\varphi$ be the subscheme where $\varphi_U$ has rank at most 1, it may be empty.

We let $X_\varphi \subset \mathbb{P}V$ be the subscheme where $\varphi_V$ drops rank, it is cut out by the vanishing of the four maximal $3 \times 3$ minors of $\varphi_V$, which are cubic polynomials in $\mathbb{C}[V^\vee]_\bullet$. We let $\mathcal{I}_{X_\varphi} \subset \mathcal{O}_{\mathbb{P}V}$ be the ideal sheaf cutting out $X_\varphi$, generated by these four minors. Similarly, we let $Y_\varphi \subset \mathbb{P}W$ be the subscheme on which $\varphi_W$ drops rank, it is similarly cut out by an ideal sheaf $\mathcal{I}_{Y_\varphi} \subset \mathcal{O}_{\mathbb{P}W}$. The subschemes $X_\varphi$ and $Y_\varphi$ are called the *associated length* 6 *subschemes* to $\varphi$.

For a vector space $A$, we will not distinguish between non-zero elements $a \in A$ and their projective equivalence class $\overline{a} \in \mathbb{P}A$ – we are only interested in maps between projective spaces, the associated subschemes to a $4 \times 3 \times 3$ tensor, and vector subspaces that satisfy certain rank conditions. We will also interchangeably classify points up to $\mathrm{GL}(-), \mathrm{PGL}(-)$, and $\mathrm{SL}(-)$-equivalence.

### 2.2. Determinantal representations of cubic surfaces.

**Definition 2.5.** Let $S \subset \mathbb{P}U$ be a cubic surface. A *determinantal representation* of $S$ is a tensor $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ such that $S = S_\varphi$.

Let $X \subset \mathbb{P}V$ (resp. $Y \subset \mathbb{P}W$) be a length 6 subscheme. A *determinantal representation* of $X$ (resp. $Y$) is a tensor $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ such that $X = X_\varphi$ (resp. $Y = Y_\varphi$).

The determinantal representations of cubic surfaces with at worst canonical (ADE, du Val) singularities is rather elegant, and can be easily understood from their associated length 6 subschemes. We introduce some language to describe these subschemes, closely following Subsection 2.5 of [DPT21].

**Definition 2.6** (Canonical, admissible subschemes)**.** A cubic surface $S \subset \mathbb{P}^3$ is said to be *canonical* if $S$ is smooth or has only canonical (ADE) singularities.

A length 6 subscheme $X \subset \mathbb{P}^2$ is said to be *canonical* if:

(1) $X$ is curvilinear,
(2) $h^0(\mathbb{P}^2, \mathcal{I}_X(3)) = 4$,
(3) and $\mathcal{I}_X(3)$ is generated by its global sections.

A canonical length 6 subscheme $X \subset \mathbb{P}^2$ is said to be *admissible* if it is not contained in a conic.

The configuration of singularities on a canonical cubic surface is associated to a subgraph of the Dynkin diagram $\widetilde{E}_6$. When this graph is the union of multiple Dynkin diagrams, we will use the notation $mA_1nA_k$ for $m$ copies of $A_1$ and $n$ copies of $A_k$ (omitting $m$ or $n$ if they are equal to 1).

**Proposition 2.7.** *Let $X \subset \mathbb{P}^2$ be a canonical length 6 subscheme. Let $S = \mathrm{Bl}_X \mathbb{P}^2$, let $f : \mathrm{Bl}_X \mathbb{P}^2 \to \mathbb{P}^2$ be the blowup. The following hold:*

*(1) $S$ has at worst $A_n$ singularities.*
*(2) $H^0(S, \omega_S^{-1})$ is 4-dimensional and basepoint-free.*
*(3) $H^i(X, \omega_S^{-n}) = 0$ for all $n \geq 0$ and $i > 0$.*
*(4) The image of the map $S \to \mathbb{P}^3$ induced by $\omega_S^{-1}$ is a cubic surface $S_X$, the map $S \to S_X$ is birational.*

*(5) The map $S \to S_X$ is an isomorphism away from the following curves, which it contracts:*
- *The strict transform $f^*(C)$ of any conic $C \subset \mathbb{P}^2$ containing $X$, or*
- *the strict transform $f^*(L)$ of any line $L \subset \mathbb{P}^2$ such that $\operatorname{len}(X \cap L) = 3$.*

*(6) Let $R_\bullet = \bigoplus_{n \geq 0} R_n$ be the homogeneous coordinate ring of $S_X$. The pullback $R_n \to H^0(S, \omega_S^{-n})$ is an isomorphism.*

*(7) $S_X$ is normal and has only canonical singularities.*

*Proof.* The details can be found in Proposition 2.5.2 of [DPT21]. We have added (5), which is easy to verify using basic facts about linear systems, noting that the map $\operatorname{Bl}_X \mathbb{P}^2 \to S \subset \mathbb{P}^3$ is the resolution of the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ given by the linear system $|\mathcal{I}_X(3)|$ of cubics through $X$. The case when $S_X$ is smooth can be found in Proposition IV.9 of [Bea78].

When $L$ is a line with $\operatorname{len}(X \cap L) = 3$, the restriction of $\mathcal{I}_X$ to $L$ is generated by one cubic. Similarly, if $X$ is contained in a conic $f_2 = 0$ then $X$ is the intersection of $f_2$ with a cubic $f_3 = 0$. The homogeneous ideal $\mathcal{I}_X$ is generated in degree 3 by $xf_2, yf_2, zf_2, f_3$, and the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ contracts the conic $f_2 = 0$ to the point $[0 : 0 : 0 : 1]$. $\square$

**Definition 2.8** (Associated cubic surface)**.** Let $X \subset \mathbb{P}^2$ be a canonical length 6 subscheme. The *cubic surface associated to* $X$, denoted by $S_X$ is the image of the map $\operatorname{Bl}_X \mathbb{P}^2 \to \mathbb{P}^3$ induced by the linear system $H^0(\operatorname{Bl}_X \mathbb{P}^2, \omega^{-1}_{\operatorname{Bl}_X \mathbb{P}^2})$.

There are two convenient equivalent definitions of $S_X$. One is that $S_X$ is the closure of the image of the map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ induced by the linear system $|\mathcal{I}_X(3)|$. The other follows from part (6) of Proposition 2.7, where

$$S_X = \operatorname{Proj}\left(\bigoplus_{n \geq 0} H^0(\operatorname{Bl}_X \mathbb{P}^2, \omega^{-n}_{\operatorname{Bl}_X \mathbb{P}^2})\right).$$

**Remark 2.9.** This construction can also be done in families. For ease of notation we only write it over affine schemes. Let $\operatorname{Spec} A$ be an affine scheme, and $X \subset \mathbb{P}^2_A = \mathbb{P}^2 \times \operatorname{Spec} A$ a closed subscheme, flat over $\operatorname{Spec} A$, whose fibres are length 6 canonical subschemes. The blowup $\operatorname{Bl}_X \mathbb{P}^2_A$ is flat over $\operatorname{Spec} A$, and the associated family of cubic surfaces

$$\mathcal{S}_X = \operatorname{Proj}_A \left(\bigoplus_{n \geq 0} H^0(\operatorname{Bl}_X \mathbb{P}^2_A, \omega^{-n}_{\operatorname{Bl}_X \mathbb{P}^2_A})\right)$$

is $A$-flat and is embedded in the projectivisation of the rank 4, locally free $A$-module $H^0(\operatorname{Bl}_X \mathbb{P}^2_A, \omega^{-1}_{\operatorname{Bl}_X \mathbb{P}^2_A})$. Note that under the blowup $f : \operatorname{Bl}_X \mathbb{P}^2 \to \mathbb{P}^2$, we have $f^*\mathcal{I}_X(3) = \omega^{-1}_{\operatorname{Bl}_X \mathbb{P}^2_A}$, inducing an isomorphism of locally free $A$-modules

$$H^0(\mathbb{P}^2_A, \mathcal{I}_X(3)) \cong H^0(\operatorname{Bl}_X \mathbb{P}^2_A, \omega^{-1}_{\operatorname{Bl}_X \mathbb{P}^2_A})$$

because the map $H^0(\mathbb{P}^2, \mathcal{I}_{X_a}(3)) \to H^0((\operatorname{Bl}_X \mathbb{P}^2_A)_a, \omega^{-1}_{(\operatorname{Bl}_X \mathbb{P}^2_A)_a})$ is an isomorphism on every fibre.

This induces a rational map

$$\operatorname{Hilb}_6(\mathbb{P}V) \dashrightarrow |\mathcal{O}_{\mathbb{P}U}(3)| /\!/ \operatorname{SL}(U),$$

defined on the locus of canonical subschemes whose associated cubic surface is $\operatorname{SL}(U)$-semistable.

Let $S$ be a canonical cubic surface. It is well-known that the set of birational maps $S \dashrightarrow \mathbb{P}^2$ are in bijection with canonical length 6 subschemes $X \subset \mathbb{P}^2$ whose associated cubic surface is $S$.

These birational maps are the inverses of the map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ induced by the linear system of cubics through $X$.

When $X$ is admissible, i.e. not contained in a conic, then by the Hilbert-Burch theorem the ideal sheaf $\mathcal{I}_X$ has a free resolution presented by a $4 \times 3$ matrix of linear forms in $\mathbb{C}[V^\vee]_\bullet$, which induces a determinantal representation. However, when $X$ is contained in a conic no $4 \times 3$ resolution exists and the subscheme $X$ is not determinantal. The following theorem is known, a proof can be found in Subsection 9.3.2 of [Dol12].

**Theorem 2.10.** *Fix a tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$. The following are equivalent:*

- *$S_\varphi$ is canonical.*
- *$X_\varphi$ is admissible.*
- *$Y_\varphi$ is admissible.*

*In this case, $S_\varphi$ is the associated cubic surface to $X_\varphi$ and $Y_\varphi$.*

*Every admissible length 6 subscheme of $\mathbb{P}^2$ has a unique determinantal representation, up to $G$-equivalence. The set of determinantal representations of a canonical cubic surface $S$ is in bijection (up to $G$-equivalence) with the set of admissible length 6 subschemes $X \subset \mathbb{P}V$ whose associated cubic surface is $S$.*

**Definition 2.11.** A tensor $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ is said to be *canonical* if $S_\varphi$ is canonical (or equivalently if $X_\varphi$ or $Y_\varphi$ is admissible).

**Remark 2.12** (Theorem 7.6 of [Thr41])**.** Every cubic surface has a determinantal representation, except for the cubic surface with an $E_6$ singularity (projectively equivalent to $u_0^2u_3+u_0u_2^2+u_1^3=0$). Every canonical length 6 subscheme associated to the $E_6$ cubic surface is contained in a conic. This was also known to Segre.

The proof is straightforward, and can be split into two cases: either a given cubic surface $S$ is canonical, in which case it has an associated admissible length 6 subscheme unless $S$ has an $E_6$ singularity; or it is not canonical, which reduces to a small case-by-case computation (there are 14 non-canonical cubic surfaces up to $\mathrm{SL}(U)$-equivalence).

We give a sketch of the various morphisms induced by a determinantal representation of a cubic surface. Fix a tensor $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ such that $S_\varphi$ is canonical. The following kernels all vanish: $\ker(\widetilde{\varphi}_{\mathbb{P}U,W}) = \ker(\widetilde{\varphi}_{\mathbb{P}U,V}) = 0$ and $\ker(\widetilde{\varphi}_{\mathbb{P}V,W}) = 0$ and $\ker(\widetilde{\varphi}_{\mathbb{P}W,V}) = 0$. We then have the following four exact sequences

$$0 \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}U}(-1) \xrightarrow{\widetilde{\varphi}_{\mathbb{P}U,W}} V^\vee \otimes \mathcal{O}_{\mathbb{P}U} \longrightarrow \mathcal{F}_\varphi \longrightarrow 0$$

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}U}(-1) \xrightarrow{\widetilde{\varphi}_{\mathbb{P}U,V}} W^\vee \otimes \mathcal{O}_{\mathbb{P}U} \longrightarrow \mathcal{G}_\varphi \longrightarrow 0$$

$$0 \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}V}(-1) \xrightarrow{\widetilde{\varphi}_{\mathbb{P}V,W}} U^\vee \otimes \mathcal{O}_{\mathbb{P}V} \longrightarrow \mathcal{I}_{X_\varphi}(3) \longrightarrow 0$$

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}W}(-1) \xrightarrow{\widetilde{\varphi}_{\mathbb{P}W,V}} U^\vee \otimes \mathcal{O}_{\mathbb{P}W} \longrightarrow \mathcal{I}_{Y_\varphi}(3) \longrightarrow 0.$$

The cokernels $\mathcal{F}_\varphi$ and $\mathcal{G}_\varphi$ are supported on $S_\varphi$, and are line bundles when restricted to $S_\varphi \setminus R_\varphi$. Taking cohomology, we have $V^\vee = H^0(S_\varphi, \mathcal{F}_\varphi)$ and $U^\vee = H^0(\mathbb{P}V, \mathcal{I}_{X_\varphi}(3))$. Both sheaves are rank 1, and the linear system $|\mathcal{F}_\varphi|$ induces a rational map $f : S_\varphi \dashrightarrow \mathbb{P}V$. Because $\mathcal{F}_\varphi = \operatorname{coker}(\widetilde{\varphi}_{\mathbb{P}U,W})$, then on points $f$ sends $u \mapsto v$ such that $\varphi(u, v, -) = 0$, and is defined on the open subscheme $S_\varphi \setminus R_\varphi$. The linear system $|\mathcal{I}_{X_\varphi}(3)|$ induces a rational map $g : \mathbb{P}V \dashrightarrow S_\varphi \subset \mathbb{P}U$, defined on $\mathbb{P}V \setminus X_\varphi$, which on points sends $v \mapsto u$ such that $\varphi(u, v, -) = 0$. So $(f \circ g)(v) = v$ on points and $f$ is birational with birational inverse $g$.

**Definition 2.13.** Fix a tensor $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ such that $S_\varphi$ is canonical. We define the birational maps $p_V : S_\varphi \dashrightarrow \mathbb{P}V$ and $p_W : S_\varphi \dashrightarrow \mathbb{P}W$ from the linear systems $|\mathcal{F}_\varphi|$ and $|\mathcal{G}_\varphi|$ on $S_\varphi$, as above.

**Remark 2.14.** Under the assumption that $S_\varphi$ is canonical, the tensor $\varphi_U$ is rank 1 at finitely many points on $S_\varphi$ and never rank 0. And a consequence of Proposition 6.2 in [EP00] is that at all points $v \in \mathbb{P}V$ and $w \in \mathbb{P}W$, the tensors $\varphi_V$ and $\varphi_W$ are rank at least 2 everywhere, dropping rank only on $X_\varphi$ and $Y_\varphi$. Consequently, the sheaves $\mathcal{F}_\varphi, \mathcal{G}_\varphi$ pull back to ample line bundles on the blowup $\mathrm{Bl}_{R_\varphi} S_\varphi$, and similarly $\mathcal{I}_{X_\varphi}(3)$ and $\mathcal{I}_{Y_\varphi}(3)$ pull back to the anti-canonical line bundles on $\mathrm{Bl}_{X_\varphi} \mathbb{P}V$ and $\mathrm{Bl}_{Y_\varphi} \mathbb{P}W$.

The morphism $\mathrm{Bl}_{X_\varphi} \mathbb{P}V \to S_\varphi$ factors through the blowup $\mathrm{Bl}_{R_\varphi} S_\varphi \to S_\varphi$, and the map $\mathrm{Bl}_{X_\varphi} \mathbb{P}V \to \mathrm{Bl}_{R_\varphi} S_\varphi$ is an isomorphism. Note that $\mathrm{Bl}_{R_\varphi} S_\varphi$ is smooth if and only if $X_\varphi$ (or equivalently $Y_\varphi$) is reduced, with $A_n$ singularities in bijection with length $n+1$ points of $X_\varphi$.

The determinantal structure on $S_\varphi$ is deeply related to the line configuration on $S_\varphi$. We quickly introduce some notation:

**Definition 2.15.** Let $A, B$ be vector spaces. A subspace $C \subset A^\vee \otimes B^\vee$ is said to be *of block type* $(a, b)$ if there exist subspaces $A' \subset A$ and $B' \subset B$ with $\dim(A') = a$ and $\dim(B') = b$, such that the pairing $c : A \otimes B \to \mathbb{C}$ annihilates $A' \otimes B'$ for every $c \in C$.

Because of the linear-algebraic definitions of $p_V, p_W$ and their birational inverses, the fibres of these four maps can only be singletons or open subvarieties of lines. The fibres of $p_V$ and $p_W$ above $X_\varphi$ and $Y_\varphi$ are projective lines of type $(1,3)$ and $(3,1)$, respectively. The fibres of $p_V^{-1}$ and $p_W^{-1}$ are singletons everywhere, except at points $u \in \mathbb{P}U$ where $\varphi_U(u)$ is rank 1, where the fibres are (open subvarieties of) the projectivisations of the left and right kernels of $\varphi_U(u) \in V^\vee \otimes W^\vee$.

When $S_\varphi$ is smooth, both $p_V$ and $p_W$ are morphisms and $X_\varphi$ and $Y_\varphi$ are reduced. We can write $X_\varphi = \{x_1, \dots, x_6\}$ and $Y_\varphi = \{y_1, \dots, y_6\}$, and let $L_{ij} \subset \mathbb{P}V$ (resp. $L'_{ij} \subset \mathbb{P}W$) be the line through $x_i$ and $x_j$ (resp. through $y_i$ and $y_j$), and $Q_i$ (resp. $Q'_i$) be the conic through $X_\varphi \setminus \{x_i\}$ (resp. $Y_\varphi \setminus \{y_i\}$).

The map $p_V : S_\varphi \to \mathbb{P}V$ is the blowup of $\mathbb{P}V$ at the 6 points $X_\varphi$. Viewing $\varphi_U(U) \subset V^\vee \otimes W^\vee$, the 27 lines on $S_\varphi$ split into the following three types:

- 6 lines of block type $(1,3)$. These are the fibres $p_V^{-1}(x_i)$, and equivalently the strict transforms of the conics $p_W^{-1}(Q'_i)$.
- 6 lines of block type $(1,3)$. These are the fibres $p_W^{-1}(y_i)$, and equivalently the strict transforms of the conics $p_V^{-1}(Q_i)$.
- 15 lines of block type $(2,2)$. These are the strict transforms $p_V^{-1}(L_{ij})$ and $p_W^{-1}(L'_{ij})$.

The twelve lines $p_V^{-1}(x_1), \dots, p_V^{-1}(x_6)$ and $p_W^{-1}(y_1), \dots, p_W^{-1}(y_6)$ form two halves of a double-six. Relabelling $\{y_1, \dots, y_6\}$ so that $p_V^{-1}(x_i) \cap p_W^{-1}(y_i) = \emptyset$, we have $p_V^{-1}(x_i) = p_W^{-1}(Q'_i)$ and $p_V^{-1}(Q_i) = p_W^{-1}(y_i)$ and $p_V^{-1}(L_{ij}) = p_W^{-1}(L'_{ij})$. The ordered tuples $(x_1, \dots, x_6)$ and $(y_1, \dots, y_6)$ are Gale dual (associated sets of points) in the sense of Coble [Cob22].

The tensor $\varphi$ can be recovered from the set of 6 points $\{x_1, \dots, x_6\}$ alone. Taking the lists $(x_1, \dots, x_6)$ and its Gale dual $(y_1, \dots, y_6)$, we consider the tensor products $x_i \otimes y_i \in V \otimes W$. The subspace $\mathrm{Span}\{x_i \otimes y_i\}_{i=1}^6 \subset V \otimes W$ is 5-dimensional, and can be written as the kernel of a linear map

$$\phi : V \otimes W \longrightarrow \mathbb{C}^4.$$

Up to choosing a basis $U^\vee \cong \mathbb{C}^4$, the map $\varphi_{V \otimes W}$ is given by $\phi$.

This generalises to most locally Gorenstein length 6 subschemes, following the work of Eisenbud and Popescu in [EP00]. We postpone this discussion to Subsection 3.1.

**Remark 2.16.** The $5 \times 3 \times 3$ tensor

$$\varphi^{\perp} : \ker(\varphi_{V \otimes W}) \longrightarrow V \otimes W$$

has some interesting properties. When $S_\varphi$ is smooth, the set of points $\{x_1 \otimes y_1, \ldots, x_6 \otimes y_6\} \in \mathbb{P}(V \otimes W)$ is the scheme-theoretic intersection of the linear subspace $\mathbb{P}\ker(\varphi_{V \otimes W}) \cong \mathbb{P}^4$ with the Segre variety $\mathbb{P}V \times \mathbb{P}W$. When viewed as a $3 \times 3$ matrix of linear forms in 5 variables, the determinant of the tensor $\varphi^{\perp}$ cuts out a cubic threefold with 6 rational double points at $x_i \otimes y_i$ (and no other singularities). Every non-degenerate cubic threefold with 6 singularities has a unique determinantal representation (up to interchanging $V$ and $W$) in this way.

This correspondence is at the heart of the study of determinantal threefolds, studied by Segre in [Seg87]. For a modern treatment we refer the reader to Section 3 of [HT10]. In general, to an injective map $\varphi_U \in \mathrm{Hom}(U, V^\vee \otimes W^\vee)$ one can associate its orthogonal complement $\varphi^\perp \in \mathrm{Hom}(\mathbb{C}^5, V \otimes W)$ with respect to the standard pairing between $V \otimes W$ and $V^\vee \otimes W^\vee$, unique up to $\mathrm{GL}(5)$. An element $\phi \in \mathrm{Hom}(A, B)$ is $\mathrm{SL}(A)$-semistable (and stable) if and only if $\phi$ is injective, so this correspondence induces an isomorphism between the GIT quotients

$$\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) /\!/ \mathrm{SL}(U) \cong \mathbb{P}(\mathbb{C}^5 \otimes V \otimes W) /\!/ \mathrm{SL}(5).$$

Of course, this is just the natural isomorphism between the Grassmannians $\mathrm{Gr}(4, V^\vee \otimes W^\vee)$ and $\mathrm{Gr}(5, V \otimes W)$. Because this isomorphism is $\mathrm{SL}(V) \times \mathrm{SL}(W)$-equivariant, it induces an isomorphism of moduli spaces

$$\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) /\!/ G \cong \mathbb{P}(\mathbb{C}^5 \otimes V \otimes W) /\!/ (\mathrm{SL}(5) \times \mathrm{SL}(V) \times \mathrm{SL}(W)).$$

After further quotienting by the involution swapping $V$ and $W$ we get a GIT moduli space of determinantal cubic threefolds. This construction can be similarly applied to reduce the GIT and classification of $m \times n \times l$ tensors to smaller dimensions. With it, we complete the GIT of $n \times 3 \times 3$ tensors, the only other non-trivial case of $3 \times 3 \times 3$ tensors was studied by Ng in [Ng95].

2.3. **A classification of $4 \times 3 \times 3$ tensors.** The set of $4 \times 3 \times 3$ tensors can be split into the following types. For each $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ one of the following is true.

- $S_\varphi$ is a smooth cubic surface. Equivalently, both $X_\varphi$ and $Y_\varphi$ are length 6 reduced subschemes not contained in a conic, and contain no length 3 collinear subscheme.
- $S_\varphi$ has only canonical singularities. Equivalently, both $X_\varphi$ and $Y_\varphi$ are curvilinear length 6 subschemes not contained in a conic, containing no length 4 collinear subscheme.
- $X_\varphi$ is a conic and is isomorphic to $Y_\varphi$ as subschemes of $\mathbb{P}^2$. The surface $S_\varphi$ is reducible.
- $S_\varphi$ is not canonical. In this case, $X_\varphi$ and $Y_\varphi$ are not curvilinear, and may not be finite.

Ng published an almost complete classification of $4 \times 3 \times 3$ tensors [Ng02], with the goal to describe the GIT quotient $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) /\!/ G$. This paper never appeared, but we draw heavily on their classification of determinantal representations of cubic surfaces with at worst canonical singularities for our classification of $G$-(semi)stability. We derive our own results for tensors where $S_\varphi$ is non-normal, reducible, or conical, as their classification of these tensors is incomplete.

Ng uses the correspondence between $G$-equivalence classes of tensors where $S_\varphi$ has at most canonical singularities and curvilinear length 6 subschemes of $\mathbb{P}V$. Explicitly, they use Maple to find all admissible length 6 subschemes $X \subset \mathbb{P}V$ associated to each type of canonical cubic surface, and then compute the $4 \times 3$ presentation matrix for the minimal resolution of the ideal $I_X$.

**Definition 2.17.** The configuration of singularities on a cubic surface with only canonical singularities is described by a Dynkin diagram $F \subset \widetilde{E}_6$. Two cubic surfaces with canonical singularities are said to be *of the same type* if they have the same configuration of singularities. Two length 6 subschemes are said to be *of the same type* if they are similarly degenerate (have the same number and types of non-reduced points, have the same number of length $k$ collinear subschemes, impose the same numbers of conditions on plane curves).

We follow the notation in [Ng02], where a family of canonical tensors is described by a Dynkin diagram $F \subset \widetilde{E}_6$ associated to the configuration of singularities on $S_\varphi$ and a latin suffix to distinguish between different types of tensors. These are chosen so that $\varphi, \varphi' \in F(n)$ if and only if $S_\varphi, S_{\varphi'}$, $X_\varphi, X_{\varphi'}$, and $Y_\varphi, Y_{\varphi'}$ are all of the same type. Note that this is equivalent to only $X_\varphi$ and $X_{\varphi'}$ being of the same type. We let $F(n) \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ be the $G$-orbit of the corresponding family of tensors described in [Ng02], equipped with the reduced subscheme structure.

**Example 2.18.** The subvariety $A_1A_2(a)$ is the $G$-orbit of tensors of the form

$$\varphi_U = \begin{pmatrix} \lambda(\lambda+1)u_1 + (\lambda+1)u_3 & u_3 & u_0 \\ \lambda u_1 + u_3 & -u_1 & u_3 \\ u_2 & 0 & u_3 \end{pmatrix},$$

parametrised by $\lambda \in \mathbb{A}^1 \setminus \{0,1\}$.

**Remark 2.19.** If $|F| \geq 4$ and $F \neq 2A_2$ then all tensors of type $F(n)$ are $G$-equivalent. If $F = 2A_2$, then tensors of type $F(n)$ appear in a one-parameter family, otherwise if $|F| \leq 3$ then tensors of type $F(n)$ occur in a $(4-|F|)$-parameter family. Note that the parametrisations given in [Ng02] are not in bijection with unique $G$-equivalence classes of tensors, but they all have the correct dimension as a parameter space.

A necessary condition for $F'(n') \subset \overline{F(n)}$ is that $F \subset F'$ as graphs. We will show in Proposition 4.18 that the closure relations can be read off from the associated length 6 subschemes, where $F'(n') \subset \overline{F(n)}$ if and only if a family of length 6 subschemes $X_\varphi$ of type $F(n)$ can degenerate into a subscheme $X_{\varphi'}$ of type $F'(n')$.

Each subvariety $F(n)$ is connected. If $n \neq n'$, then $F(n') \cap \overline{F(n)} = \emptyset$, unless $F = D_4$ in which case $D_4(b) \subset \overline{D_4(a)}$. Strictly speaking, the tensors $D_4(a)$ and $D_4(b)$ are the same type, this is the only case where our definition of "type" differs from the notation in [Ng02].

Another important type of tensors we will need frequently throughout the paper are the skew-symmetric tensors.

**Definition 2.20.** A tensor is said to be *skew-symmetric* if $\varphi$ is $G$-equivalent to a tensor of the form

$$\sigma_A := u_0 A + \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix},$$

where $A \in V^\vee \otimes W^\vee$ is any element. This choice of basis gives an identification of $V \cong W$, and $A$ can be chosen to be symmetric by changing the basis for $U$. In particular, we let

$$\sigma = \begin{pmatrix} u_0 & u_3 & -u_2 \\ -u_3 & u_0 & u_1 \\ u_2 & -u_1 & u_0 \end{pmatrix}.$$

We $\Sigma \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ be the subvariety of tensors $\mathrm{SL}(U) \times \mathrm{SL}(W)$-equivalent to $\sigma_A$ for some non-zero $A$, equipped with the reduced subscheme structure.

In this case, $X_\varphi$ and $Y_\varphi$ are the length 6 subschemes of $\mathbb{P}V \cong \mathbb{P}W$ cut out by the quadratic form $A \in \mathrm{Sym}^2(V^\vee)$. The cubic surface $S_\varphi$ is reducible, being the transverse union of a plane and a quadric surface cut out by a quadratic form of the same rank as $A$.

**Proposition 2.21.** *Every skew-symmetric tensor with the same rank quadratic form is $G$-equivalent.*

*Proof.* Fix a skew-symmetric tensor $\sigma_A$. Any symmetric complex matrix can be 'diagonalised' by some invertible (in fact unitary) matrix $B$, in the sense that $B^T AB = D$ is diagonal. Applying $B^T$ and $B$ to the skew-symmetric part preserves its skew-symmetry (this will just change the basis of the skew-symmetric subspace, which we can restore with an appropriate $\mathrm{SL}(U)$-action), and we have

$$B^T \sigma_A B = \begin{pmatrix} \lambda_1 u_0 & 0 & 0 \\ 0 & \lambda_2 u_0 & 0 \\ 0 & 0 & \lambda_3 u_0 \end{pmatrix} + B^T \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix} B$$

for some scalars $\lambda_i \in \mathbb{C}$. After changing basis if necessary, we have

$$\sigma_A \sim \begin{pmatrix} \lambda_1 u_0 & u_3 & -u_2 \\ -u_3 & \lambda_2 u_0 & u_1 \\ u_2 & -u_1 & \lambda_3 u_0 \end{pmatrix},$$

where each non-zero $\lambda_i$ can be rescaled to be 1, and $\sigma_A$ is uniquely determined up to $G$-equivalence by the number of non-zero $\lambda_i$s (the rank of $A$). $\square$

**Remark 2.22.** The classification of determinantal representations of non-canonical cubic surfaces given in [Ng02] is incomplete. It omits skew-symmetric tensors with rank 2 or 3 quadratic forms.

We believe these are the only missing terms in their classification, their classification of determinantal representations of canonical cubic surfaces is correct and is the only one we use.

2.4. **Low rank linear subspaces.** A useful invariant of a tensor is the number and type of linear subspaces on which it drops rank. We introduce some notation to describe these linear subspaces, all details can be found in the papers [AL80] and [AL81] by Atkinson and Lloyd.

Recall Definition 2.15: let $A, B$ be vector spaces. A subspace $C \subset A^\vee \otimes B^\vee$ is said to be *of block type* $(a, b)$ if there exist subspaces $A' \subset A$ and $B' \subset B$ with $\dim(A') = a$ and $\dim(B') = b$, such that the pairing $c : A \otimes B \to \mathbb{C}$ annihilates $A' \otimes B'$ for every $c \in C$. Up to a global choice of basis for $A$ and $B$, each subspace of block type $(a, b)$ is of the form

$$\left[\begin{array}{c|c} M & N \\ \hline P & 0 \end{array}\right] \subset A^\vee \otimes B^\vee$$

with a $a \times b$ block of zeros in the corner.

**Definition 2.23.** Let $A, B$ be vector spaces. A subspace $C \subset A^\vee \otimes B^\vee$ is said to be a *rank at most $r$ subspace* if every $M \in C$ is of rank at most $r$. We say that $C$ is a *rank $r$ subspace* if $C$ is a rank at most $r$ subspace, and at least one element of $C$ is rank $r$.

A subspace $C \subset A^\vee \otimes B^\vee$ is said to satisfy a *rank $r$ block condition* if $C$ is of block type $(a, b)$, where $r = \dim A + \dim B - a - b$.

It is easy to verify that any subspace satisfying a rank $r$ block condition is a rank at most $r$ subspace, but the converse is not true. The classification of rank $r$ subspaces in general is difficult, and is only known completely when $r \leq 7$, but we only need the classification for $r = 1, 2$:

**Proposition 2.24** ([AL81, Lemma 2]). *Every rank 1 subspace satisfies a rank 1 block condition.*

For $r = 2$, we introduce some notation to describe the only exception, which is the vector space of skew-symmetric $3 \times 3$ matrices.

**Definition 2.25.** Let $A, B$ be vector spaces, and $C \subset A \otimes B$ a subspace. If there exist injections $\iota_A : \mathbb{C}^3 \to A$ and $\iota_B : \mathbb{C}^3 \to B$ such that

$$C = (\iota_A \otimes \iota_B)(\mathbb{C}^3 \wedge \mathbb{C}^3)$$

then $C$ is said to be a *skew-symmetric subspace* of $A \otimes B$.

Note that this is consistent with Definition 2.20: a $4 \times 3 \times 3$ tensor $\varphi$ is skew-symmetric if and only if there is a three-dimensional subspace $U' \subset U$ where $\varphi_U(U')$ is a skew-symmetric subspace.

**Proposition 2.26** ([AL81, Lemma 7])**.** *Every rank at most* 2 *subspace either satisfies a rank* 2 *block condition, or is a* 3*-dimensional skew-symmetric subspace.*

## 3. Gale duality and the final minimal model of the Hilbert scheme of 6 points

For an introduction to geometric invariant theory, we refer the reader to Mumford and Fogarty's [MFK94], or Hoskin's notes [Hos23]. Before looking at $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G$, it is useful to understand the partial GIT quotients of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ by only one or two of the copies of $\mathrm{SL}(-)$. These are all known, with the latter being a special case of quiver GIT.

**Definition 3.1.** As shorthand, for $A, B \in \{U, V, W\}$ we let $\pi_{AB}, \pi_{UVW}$ be the quotient maps

$$\pi_{AB} : \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) \dashrightarrow \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//\,(\mathrm{SL}(A) \times \mathrm{SL}(B))$$
$$\pi_{UVW} : \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) \dashrightarrow \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G.$$

**Definition 3.2.** We let

$$\det : \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) \dashrightarrow |\mathcal{O}_{\mathbb{P}U}(3)| = \mathrm{Hilb}_{\text{cubic surfaces}}(\mathbb{P}U)$$

be the rational map sending a tensor $\varphi$ to its determinant cubic $[S_\varphi]$. As above, we use subscripts after det to denote the induced rational maps between GIT quotients. As an example,

$$\det_{UVW} : \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G \dashrightarrow |\mathcal{O}_{\mathbb{P}U}(3)|//\,\mathrm{SL}(U).$$

The GIT of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ under the $\mathrm{SL}(U), \mathrm{SL}(V), \mathrm{SL}(W)$-actions is mostly uninteresting: a tensor $\varphi$ is semistable under the respective action if and only if $\varphi_U, \varphi_V, \varphi_W$ is injective. All semistable points are stable, and the respective GIT quotients are the Grassmannians $\mathrm{Gr}(4, V^\vee \otimes W^\vee), \mathrm{Gr}(3, U^\vee \otimes W^\vee)$, and $\mathrm{Gr}(3, U^\vee \otimes V^\vee)$.

The quotient $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(V) \times \mathrm{SL}(W))$ is a moduli space of divisor-equivalent twisted cubics inside cubic surfaces, but is not a moduli space of sheaves. It has already been well studied in [LLSS17], we refer the reader to Section 3 of their paper for a detailed discussion. While we don't need any of their results for our GIT quotient, it is useful to note the following fact

**Remark 3.3.** The rational map

$$\det_{VW} : \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(V) \times \mathrm{SL}(W)) \dashrightarrow |\mathcal{O}_{\mathbb{P}U}(3)|$$

is not a morphism – there are $\mathrm{SL}(V) \times \mathrm{SL}(W)$-stable tensors whose determinant cubic is zero. These are $G$-equivalent to

$$\sigma_0 = \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix}.$$

If $\det_{VW}$ was a morphism then a tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ would be $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$-semistable if and only if $S_\varphi$ was $\mathrm{SL}(U)$-semistable. This is not the case.

The only unstudied quotient is $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(U) \times \mathrm{SL}(W))$, which is an orbit space and a moduli space of sheaves on $\mathbb{P}V$. The $\mathrm{SL}(U) \times \mathrm{SL}(W)$-(semi)stability of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is also known from quiver GIT, a result of Drézet. The quotient $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(U) \times \mathrm{SL}(W))$ is the final minimal model of the Hilbert scheme of 6 points in $\mathbb{P}V$, as outlined by Aracara, Bertram, Coskun and Huizenga in [ABCH13]. As they explain, the rational map from the Hilbert scheme to this minimal model contracts the divisor of length 6 subschemes contained in a conic. We give a concrete description of this map using Gale duality.

3.1. **Gale duality.** Gale duality gives an alternate way to construct a determinantal representation of a length 6 subscheme, working for most length 6 locally Gorenstein subschemes of $\mathbb{P}^2$, and works in families. We outline the construction below, all technical details and many related results can be found in sections 5 and 6 of [EP00].

Let $X \subset \mathbb{P}V$ be a length 6 locally Gorenstein subscheme. We consider the trace map $\tau : H^0(X, \omega_X) \to \mathbb{C}$ which, after composition with Serre duality, induces a perfect pairing

$$H^0(X, \mathcal{O}_X(1)) \otimes H^0(X, \omega_X(-1)) \longrightarrow H^0(X, \omega_X) \xrightarrow{\tau} \mathbb{C}.$$

We let $V'^\vee = (V^\vee)^\perp$ be the annihilator with respect to this pairing, and say that the linear series $(V^\vee, \mathcal{O}_X(1))$ and $(V'^\vee, \omega_X(-1))$ are *Gale dual*.

**Definition 3.4.** Let $X \subset \mathbb{P}V$ be a locally Gorenstein subscheme. The image $X'$ of $X$ inside $\mathbb{P}V'$ under the linear series $(V'^\vee, \omega_X(-1))$, as above, is called the *Gale dual* of $X \subset \mathbb{P}V$.

If a projective isomorphism $\mathbb{P}W \to \mathbb{P}V'$ induces an isomorphism between subschemes $Y \subset \mathbb{P}W$ and $X' \subset \mathbb{P}V'$, then $Y \subset \mathbb{P}W$ is also said to be Gale dual to $X \subset \mathbb{P}V$.

To a length 6 locally Gorenstein subscheme $X \subset \mathbb{P}V$, we associate the linear map $\phi : V^\vee \otimes V'^\vee \to \ker(\tau) \subset H^0(X, \omega_X)$. The vector space $\ker(\tau)$ is 5-dimensional and, under the additional assumption that $\phi$ is surjective (this is satisfied, for example, if $X$ contains a length 4 subscheme in linearly general position [EP00, Proposition 5.8]), we consider the inclusion of its 4-dimensional kernel

$$\psi : \ker(\phi) \to V^\vee \otimes V'^\vee.$$

Choosing isomorphisms $\ker(\phi) \cong U$ and $V' \cong W$, the map $\psi$ is a $4 \times 3 \times 3$ tensor $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$, unique up to $\mathrm{GL}(U) \times \mathrm{GL}(W)$.

**Definition 3.5.** Let $X$ be a length 6 locally Gorenstein subscheme. Following the above construction, when $\ker(\tau)$ is 5-dimensional we call the tensor $\psi \in \mathrm{Hom}(\ker(\phi), V^\vee \otimes V'^\vee)$ the *associated tensor* to $X$.

This is consistent with Definition 2.4, where for a tensor $\varphi$ such that $X_\varphi$ is finite and locally Gorenstein, the associated tensor to $X_\varphi$ is $\mathrm{GL}(U) \times \mathrm{GL}(W)$-equivalent to $\varphi$. More precisely, by [EP00] we have

**Proposition 3.6.** *Let $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ be a tensor. If either $X_\varphi$ or $Y_\varphi$ is finite and $\varphi_V$ or $\varphi_W$ is injective (respectively) then it is length 6. The following statements are equivalent:*

- *$X_\varphi$ and $Y_\varphi$ are finite.*
- *$X_\varphi$ is finite and locally Gorenstein.*
- *$Y_\varphi$ is finite and locally Gorenstein.*

*If one of the above statements holds, then the linear series $(W^\vee, \operatorname{coker}(\widetilde{\varphi}_{\mathbb{P}V,U})|_{X_\varphi})$ induces a canonical isomorphism $X_\varphi \to Y_\varphi \subset \mathbb{P}W$, under which the linear series $(V^\vee, \mathcal{O}_{X_\varphi}(1))$ and $(W^\vee, \mathcal{O}_{Y_\varphi}(1))$ are Gale dual.*

*Proof.* Everything follows from Theorem 6.1 and Proposition 6.2 of [EP00], except the claim that a finite subscheme is length 6. If $\varphi_V$ is injective, then $\varphi_V$ induces a linear embedding of $\mathbb{P}V \hookrightarrow \mathbb{P}(U^\vee \otimes W^\vee)$. The subscheme $X_\varphi$ is the scheme-theoretic intersection of $\mathbb{P}V$ with the degree 6, codimension 2 subvariety of $\mathbb{P}(U^\vee \otimes W^\vee)$ of rank at most 2 pairings. This intersection is length 6 if it is finite, the same argument applies for $Y_\varphi$ by interchanging $V$ and $W$. □

**Remark 3.7.** The case of length 6 subschemes contained in a conic is different. Fix a length 6, locally Gorenstein subscheme $X \subset \mathbb{P}V$ that is contained in a unique conic whose equation is given by a quadratic form $A \in \operatorname{Sym}^2(V^\vee)$.

The Veronese embedding $\nu : X \to \mathbb{P}V \times \mathbb{P}V \subset \mathbb{P}(\operatorname{Sym}^2 V)$ is then contained in and spans the hyperplane $\mathbb{P}\Lambda \subset \mathbb{P}(\operatorname{Sym}^2 V)$ cut out by the quadratic form $A \in \operatorname{Sym}^2(V^\vee) = (\operatorname{Sym}^2 V)^\vee$. After choosing an isomorphism $\omega_X \cong \mathcal{O}_X(2)$, the linear series $\Lambda^\vee \subset H^0(X, \mathcal{O}_X(2))$ can be identified with a linear series $\Lambda^\vee \subset H^0(X, \omega_X)$. This isomorphism also induces an isomorphism $\omega_X(-1) \cong \mathcal{O}_X(1)$, allowing us to identify $V^\vee \subset H^0(X, \mathcal{O}_X(1))$ with $V^\vee \subset H^0(X, \omega_X(-1))$.

Under this identification, the linear map $\phi : V^\vee \otimes V^\vee \to \Lambda^\vee$ is surjective, induced from the Serre duality pairing

$$H^0(X, \mathcal{O}_X(1)) \otimes H^0(X, \omega_X(-1)) \longrightarrow H^0(X, \omega_X).$$

The linear map $\psi : \ker(\phi) \to V^\vee \otimes V^\vee$ gives a unique $4 \times 3 \times 3$ tensor $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$ up to $\mathrm{GL}(U) \times \mathrm{GL}(W)$ (independent from the identification $\omega_X \cong \mathcal{O}_X(2)$), after choosing isomorphisms $\ker(\phi) \cong U$ and $V \cong W$. This is $\mathrm{GL}(U) \times \mathrm{GL}(W)$-equivalent to the skew-symmetric tensor $\sigma_A \in \Sigma$, uniquely determined by the element $A \in \operatorname{Sym}^2(V^\vee)$.

**Remark 3.8.** The map sending a subscheme to its associated tensor can also be done in families. For ease of notation we only write the construction over affine schemes. Let $\operatorname{Spec} A$ be an affine scheme, and fix a locally Gorenstein subscheme $X_A \subset \mathbb{P}V \times \operatorname{Spec} A$ whose fibres are length 6. We let $\tau : H^0(X_A, \omega_{X_A}) \to A$ be the trace pairing.

The annihilator of $V^\vee \otimes A$ with respect to the trace pairing $H^0(X_A, \mathcal{O}_{X_A}(1)) \otimes_A H^0(X_A, \omega_{X_A}(-1)) \to A$ is a submodule $\mathcal{W}^\vee \subset H^0(X_A, \omega_{X_A}(-1))$. We let $Y_A \subset \mathbb{P}\mathcal{W}$ be the image of $X_A$ under the linear series $(\mathcal{W}^\vee, \omega_{X_A}(-1))$. For each $a \in \operatorname{Spec} A$, the pairs $X_a \subset \mathbb{P}V \times \operatorname{Spec}(A/a)$ and $Y_a \subset \mathbb{P}\mathcal{W} \times \operatorname{Spec}(A/a)$ are Gale dual.

Under the additional assumption that each fibre $X_a$ contains a length 4 subscheme in linearly general position, the map $\phi : (V^\vee \otimes \mathcal{W}^\vee) \otimes_A A/a \to \ker(\tau) \otimes_A A/a$ of locally free $A$-modules is surjective for each prime $a \in \operatorname{Spec} A$. We let $\mathcal{U} = \ker(\phi)$, a locally free rank 4 $A$-module, and consider the injection

$$\psi : \mathcal{U} \longrightarrow V^\vee \otimes \mathcal{W}^\vee.$$

The map $\psi$ can be identified with a section $s_A : \operatorname{Spec} A \to \mathcal{U}^\vee \otimes V^\vee \otimes_A \mathcal{W}^\vee$.

## 3.2. **The final minimal model of the Hilbert scheme of 6 points.**

**Definition 3.9.** The construction of Remark 3.8 induces a rational map

$$\gamma : \mathrm{Hilb}_6(\mathbb{P}V) \dashrightarrow \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(U) \times \mathrm{SL}(W)),$$

defined on the locally Gorenstein locus where the map $V^\vee \otimes W^\vee \to \ker(\tau) \subset H^0(X, \omega_X)$ is surjective and the associated tensor to $X$ is $\mathrm{SL}(U) \times \mathrm{SL}(W)$-semistable.

$\mathrm{SL}(U) \times \mathrm{SL}(W)$-(semi)stability is known, a result of Drézet [Dré87]:

**Lemma 3.10.** *A tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is $\mathrm{SL}(U) \times \mathrm{SL}(W)$-semistable if and only if $\varphi_V$ is not $\mathrm{SL}(U) \times \mathrm{SL}(W)$-equivalent to a matrix of the following form:*

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} * & * & * \\ * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} * & * & * \\ * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix}.$$

*All semistable points are stable.*

Note that $\varphi$ is $\mathrm{SL}(U) \times \mathrm{SL}(W)$-unstable if and only if there is a 3-dimensional subspace $U' \subset U$ such that the image of $\varphi_V(V)$ under the projection $U^\vee \otimes W^\vee \to U'^\vee \otimes W^\vee$ satisfies a rank 2 block condition.

**Remark 3.11.** A quick computation shows that $X_\varphi$ contains a positive-dimensional component whenever $\varphi$ is $\mathrm{SL}(U) \times \mathrm{SL}(W)$-unstable. Another sufficient condition for $\mathrm{SL}(U) \times \mathrm{SL}(W)$-semistability is that $\ker(\widetilde{\varphi}_{\mathbb{P}V,W}) = 0$, i.e. that the sequence

$$0 \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}V}(-1) \xrightarrow{\widetilde{\varphi}_{\mathbb{P}V,W}} U^\vee \otimes \mathcal{O}_{\mathbb{P}V} \longrightarrow \mathcal{I}_{X_\varphi}(3) \longrightarrow 0$$

is exact. As a result, $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(U) \times \mathrm{SL}(W))$ is a moduli space of coherent sheaves on $\mathbb{P}V$ and every ideal sheaf of a length 6 subscheme that has a $4 \times 3$ free resolution is $\mathrm{SL}(U) \times \mathrm{SL}(W)$-stable.

Because of this, the $\mathrm{SL}(U) \times \mathrm{SL}(W)$-invariant rational map

$$\chi : \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) \dashrightarrow \mathrm{Hilb}_6(\mathbb{P}V) \tag{2}$$

sending a tensor $\varphi$ to its associated subscheme $X_\varphi$ (defined whenever $X_\varphi$ is length 6) factors through the GIT quotient $\pi_{UW}$ where defined.

**Proposition 3.12.** *The induced map $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(U) \times \mathrm{SL}(W)) \dashrightarrow \mathrm{Hilb}_6(\mathbb{P}V)$ is the birational inverse of $\gamma$, i.e. the following diagram commutes:*

$$\begin{tikzcd} \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) \arrow[rr, dashed, "\chi"] \arrow[dr, dashed, "\pi_{UW}"] & & \mathrm{Hilb}_6(\mathbb{P}V) \arrow[dl, dashed, "\gamma"] \\ & \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(U) \times \mathrm{SL}(W)) & \end{tikzcd}$$

*Proof.* This is a consequence of Proposition 3.6. Concretely, fix a tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ such that $X_\varphi$ is length 6 and locally Gorenstein. The canonical sheaf of $X_\varphi$ is $\mathrm{coker}(\widetilde{\varphi}_{\mathbb{P}V,U})|_{X_\varphi}(1)$, and the pair of linear series $(V^\vee, \mathcal{O}_{X_\varphi}(1))$ and $(W^\vee, \mathrm{coker}(\widetilde{\varphi}_{\mathbb{P}V,U})|_{X_\varphi}) = (W^\vee, \omega_{X_\varphi}(-1))$ induces a surjective linear map

$$\phi : V^\vee \otimes W^\vee \longrightarrow \ker(\tau) \subset H^0(X_\varphi, \omega_X)$$

which embeds $X_\varphi \hookrightarrow \mathbb{P}V \times \mathbb{P}W$ as the scheme-theoretic intersection of the Segre variety $\mathbb{P}V \times \mathbb{P}W$ and the image of $\mathbb{P}\ker(\tau)^\vee$ under $\phi^\vee$ inside $\mathbb{P}(V \otimes W)$.

There is an isomorphism $\ker(\phi) \cong U$ where the map $\ker(\phi) \to V^\vee \otimes W^\vee$ is given by $\varphi_U$, and so $\gamma \circ \chi(\varphi) = \pi_{UW}(\varphi)$ is the $\mathrm{SL}(U) \times \mathrm{SL}(W)$-orbit of $\varphi$. This construction can also be done over any base using Remark 3.8. Because $\gamma \circ \chi = \pi_{UW}$ on points and the source and target are reduced, then $\gamma \circ \chi = \pi_{UW}$. Of course, this was already known in [ABCH13]. $\square$

Recall that $\Sigma$ is the $\mathrm{SL}(U) \times \mathrm{SL}(W)$-orbit of tensors of the form

$$\sigma_A = u_0 A + \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix},$$

where $A \in \mathrm{Sym}^2(V^\vee)$ is non-zero.

**Proposition 3.13.** *The subvariety $\Sigma \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is a $\mathrm{PGL}(U) \times \mathrm{PGL}(W)$-bundle over its GIT quotient $\Sigma//(\mathrm{SL}(U) \times \mathrm{SL}(W)) \cong |\mathcal{O}_{\mathbb{P}V}(2)|$.*

*Proof.* The morphism $\Sigma \to \mathrm{Hilb}_{\mathrm{conics}}(\mathbb{P}V) = |\mathcal{O}_{\mathbb{P}V}(2)|$ that sends $\varphi \mapsto X_\varphi$ is surjective and $\mathrm{SL}(U) \times \mathrm{SL}(W)$-invariant. Since $\Sigma$ is contained in and closed inside the $\mathrm{SL}(U) \times \mathrm{SL}(W)$-stable locus, this induces a morphism $f : \Sigma//(\mathrm{SL}(U) \times \mathrm{SL}(W)) \to |\mathcal{O}_{\mathbb{P}V}(2)|$.

The $\mathrm{PGL}(U) \times \mathrm{PGL}(W)$-stabiliser of each $\varphi \in \Sigma$ is trivial, so $\Sigma$ is a $\mathrm{PGL}(U) \times \mathrm{PGL}(W)$-bundle over its quotient $\Sigma//(\mathrm{PGL}(U) \times \mathrm{PGL}(W))$, which is smooth. And $\Sigma//(\mathrm{PGL}(U) \times \mathrm{PGL}(W)) = \Sigma//(\mathrm{SL}(U) \times \mathrm{SL}(W))$ as schemes. The $\mathrm{SL}(U) \times \mathrm{SL}(W)$-orbits in $\Sigma$ are in bijection with non-zero quadratic forms $A \in \mathrm{Sym}^2(V^\vee)$, up to scaling. Since $\mathbb{P}\,\mathrm{Sym}^2(V^\vee) = |\mathcal{O}_{\mathbb{P}V}(2)|$ then $f$ is a bijective morphism between smooth varieties, hence an isomorphism. □

Along with the fact that $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(U) \times \mathrm{SL}(W))$ is an orbit space, we will use the following proposition to understand orbit closures inside $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ by passing to $\mathrm{Hilb}_6(\mathbb{P}V)$.

**Definition 3.14.** Let $C_6 \subset \mathrm{Hilb}_6(\mathbb{P}V)$ be the divisor of length 6 subschemes contained in a conic, and $C_6' \subset C_6$ be the subscheme of locally Gorenstein length 6 subschemes contained in a unique conic.

**Proposition 3.15.** *The birational map*

$$\gamma : \mathrm{Hilb}_6(\mathbb{P}V) \dashrightarrow \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//\left(\mathrm{SL}(U) \times \mathrm{SL}(W)\right)$$

*contracts $C_6'$ to $\Sigma//(\mathrm{SL}(U) \times \mathrm{SL}(W)) \cong |\mathcal{O}_{\mathbb{P}V}(2)|$.*

*The map $\gamma$ is defined on the locus of subschemes that are either determinantal or contained in a unique conic. The restriction of $\gamma$ to the locus of determinantal subschemes is an isomorphism onto its image.*

*Proof.* As constructed, the map $\gamma$ sends a general length 6 subscheme $X$ to the associated tensor $\varphi_U$ that presents the kernel of the map $V^\vee \otimes W^\vee \to \ker(\tau) \subset H^0(X, \omega_X)$. As outlined in Remark 3.7, when $X$ is contained in a unique conic cut out by a non-zero quadratic form $A \in \mathrm{Sym}^2(V^\vee)$ then $\gamma([X])$ is the $\mathrm{SL}(U) \times \mathrm{SL}(W)$-orbit of $\sigma_A$, so $\gamma(C_6') = \Sigma//(\mathrm{SL}(U) \times \mathrm{SL}(W))$.

The map $\gamma$ can also be defined for non-Gorenstein determinantal subschemes $X = X_\varphi \subset \mathbb{P}V$. In this case, the tensor $\varphi_W$ is the matrix of linear relations between the 4 cubic equations that cut out $X$, which is uniquely determined by $X$ up to $\mathrm{SL}(U) \times \mathrm{SL}(W)$. Concretely, fix an affine scheme $\mathrm{Spec}\, A$ and morphism $f : \mathrm{Spec}\, A \to \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ such that for each $a \in \mathrm{Spec}\, A$ the fibre $X_{f(a)}$ is a length 6 subscheme of $\mathbb{P}V \times \mathrm{Spec}(A/a)$. This induces a flat family of determinantal subschemes of $\mathbb{P}V$, and conversely every such flat family arises in this way (up to taking affine charts). The map of locally free $A$-modules

$$H^0(\mathbb{P}V_A, \mathcal{O}_{\mathbb{P}V_A}(1)) \otimes_A H^0(\mathbb{P}V_A, \mathcal{I}_X(3)) \longrightarrow H^0(\mathbb{P}V_A, \mathcal{I}_X(4))$$

is surjective on all fibres and has a 3-dimensional kernel $\mathcal{W}$. Taking the identification $H^0(\mathbb{P}V_A, \mathcal{O}_{\mathbb{P}V_A}(1)) = V^\vee \otimes A$ and writing $\mathcal{U}^\vee = H^0(\mathbb{P}V_A, \mathcal{I}_X(3))$, the kernel map $\mathcal{W} \to \mathcal{U}^\vee \otimes V^\vee$ gives a section

$$s : \mathrm{Spec}\, A \longrightarrow \mathcal{U}^\vee \otimes V^\vee \otimes_A \mathcal{W}^\vee,$$

and the fibres $s(a)$ are $\mathrm{GL}(4) \times \mathrm{GL}(3)$-equivalent to $f(a)$, i.e. $s(a) = (\pi_{UW} \circ f)(a)$. Since $\mathrm{Hilb}_6(\mathbb{P}V)$ is smooth this uniquely extends the map $\gamma$ to the locus of determinantal subschemes.

It is easy to check that $\gamma \circ \chi = \pi_{UW}$ on points on this form, so $\gamma$ is an isomorphism when restricted to the locus of determinantal subschemes. □

## 4. The $4 \times 3 \times 3$ GIT quotient

This section contains the majority of our results. Subsections 4.1, 4.2 and 4.3 provide necessary and sufficient conditions for $G$-semistability, stability and polystability, respectively, which combine to prove the main theorem:

**Theorem A.** *A tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is:*

- *$G$-stable if and only if $S_\varphi$ is $\mathrm{SL}(U)$-stable ($S_\varphi$ has at worst $4A_1$ singularities).*
- *$G$-semistable if and only if either $S_\varphi$ is $\mathrm{SL}(U)$-semistable or $\varphi \in A_3(d) \cup A_1A_3(f) \cup G \cdot \sigma$.*
- *Strictly $G$-polystable if and only if either $\varphi \in 3A_2(a)$ or $\varphi \in 2A_2(b) \cup A_1 2A_2(c) \cup G \cdot \sigma$.*

*In the GIT quotient $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G$, the strictly $G$-semistable locus is the disjoint union of a singleton and a subscheme whose normalisation is $\mathbb{P}(1,2)$. The map*

$$\det_{UVW} : \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) \dashrightarrow |\mathcal{O}_{\mathbb{P}U}(3)|// \mathrm{SL}(U)$$

*is undefined only at the singleton $\pi_{UVW}(\sigma)$.*

Finally, Subsection 4.4 describes a blowup of the GIT quotient $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G$ that resolves the rational map $\det_{UVW}$.

### 4.1. Classifying semistability.

It turns out that the criteria for $G$-semistability is relatively simple: a tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is $G$-semistable if and only if either $S_\varphi$ is $\mathrm{SL}(U)$-semistable or there are no non-zero points $u \in U$ where the matrix $\varphi_U(u)$ has rank less than 2. The $\mathrm{SL}(U)$-(semi)stability of cubic surfaces is well-known:

**Proposition 4.1** ([Mum77, Subsection 1.14])**.** *A cubic surface $[S] \in |\mathcal{O}_{\mathbb{P}U}(3)|$ is $\mathrm{SL}(U)$-semistable if and only if $S$ is normal with at worst $A_2$ singularities. A cubic surface is $\mathrm{SL}(U)$-stable if and only if $S$ is normal with at worst $A_1$ singularities.*

The ring of invariants of cubic surfaces has been known for much longer, dating back to Salmon and Clebsch's work [Sal60; Cle61b; Cle61a] in 1860.

**Proposition 4.2.** *The ring of $\mathrm{SL}(U)$-invariants of $\mathbb{C}[\mathrm{Sym}^3(U)]_\bullet$ is generated by the invariants $I_8, I_{16}, I_{24}, I_{32}, I_{40}, I_{100}$, of degree indicated by their subscripts. The square $I_{100}^2$ is a polynomial in the other five, which are algebraically independent.*

*The discriminant divisor, which vanishes wherever the cubic surface is singular, has degree* 32.

This implies the GIT quotient $|\mathcal{O}_{\mathbb{P}U}(3)|// \mathrm{SL}(U)$ is isomorphic to $\mathbb{P}(1,2,3,4,5)$. All strictly semistable points in the GIT quotient are identified to the $\mathrm{SL}(U)$-orbit of the $3A_2$ cubic surface. In $|\mathcal{O}_{\mathbb{P}U}(3)|// \mathrm{SL}(U) \cong \mathbb{P}(1,2,3,4,5)$ this is the point $[8:1:0:0:0]$.

There is a unique linear map $\det : U^\vee \otimes V^\vee \otimes W^\vee \to \mathrm{Sym}^3(U^\vee)$, up to rescaling, that sends a tensor to its determinant cubic. The invariants $I_d$ pullback to invariants $J_{3d} = I_d \circ \det \in \mathbb{C}[U \otimes V \otimes W]^G_\bullet$. The other invariant we use is the following:

**Definition 4.3.** Fix a tensor $\varphi \in U^\vee \otimes V^\vee \otimes W^\vee$. If there is a $u \in \mathbb{P}U$ where $\varphi_U(u)$ has rank (at most) 1, then $S_\varphi$ is said to *contain the rank (at most)* 1 *point* $u$. Let $R_{24} \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ the subvariety of tensors $\varphi$ that contain a rank at most 1 point.

The condition of $S_\varphi$ containing a rank at most 1 point is equivalent to $X_\varphi$ containing a length 3 collinear subscheme.

**Proposition 4.4.** *The subvariety $R_{24} \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is $G$-invariant divisor, cut out by a degree 24 polynomial $R_{24} \in \mathbb{C}[U \otimes V \otimes W]^G_\bullet$.*

*Proof.* It is immediate that $R_{24}$ is closed and $G$-invariant. To see that $R_{24}$ is a divisor, we consider the intersection scheme

$$\widetilde{R}_{24} = \{(\varphi, u, V') \mid \varphi(u, V', -) = 0\} \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) \times \mathbb{P}U \times \mathrm{Gr}(2, V).$$

$R_{24}$ is the image of $\widetilde{R}_{24}$ under the projection onto $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$. Now consider the projection $\widetilde{R}_{24} \to \mathrm{Gr}(2, V) \times \mathbb{P}U$, which is surjective. The fibre above a point $(u, V') \in \mathbb{P}U \times \mathrm{Gr}(2, V)$ is the subvariety of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ cut out by $\varphi(u, V', -) = 0$, which is a codimension 6 linear subspace of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$. Because $\dim(\mathbb{P}U \times \mathrm{Gr}(2, V)) = 5$ then $R_{24}$ is codimension 1 as desired.

To compute the degree, consider a general line $l : \mathbb{P}^1 \to \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ parametrised by $[s : t] \mapsto s\varphi + t\varphi'$, for two general points $\varphi, \varphi' \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$. The degree of $R_{24}$ is the intersection number of $l(\mathbb{P}^1)$ with $R_{24}$, which is the same as the intersection number of the Segre variety $\mathbb{P}V^\vee \times \mathbb{P}W^\vee \subset \mathbb{P}(V^\vee \otimes W^\vee)$ with the image of the linear map $f : \mathbb{P}^1 \times \mathbb{P}U \to \mathbb{P}(V^\vee \otimes W^\vee)$ that sends $([s : t], u) \mapsto s\varphi_U(u) + t\varphi'_U(u)$.

We compute this in the Chow ring. Let $h_1, h_2, h_3$ be the respective generators of $A_*(\mathbb{P}^1), A_*(\mathbb{P}U), A_*(\mathbb{P}(V^\vee \otimes W^\vee))$. We have $f^*(h_3) = h_1 + h_2$, and $f^*(\mathbb{P}V^\vee \times \mathbb{P}W^\vee) = f^*(6h_3^4) = 6f^*(h_3)^4$. And $(h_1 + h_2)^4 = 4h_1h_2^3$, so $\deg(R_{24}) = 4 \cdot 6 = 24$. □

**Definition 4.5.** We let $S_\bullet = \mathbb{C}[R_{24}, J_{24}, \ldots, J_{120}, J_{300}]_\bullet \subset \mathbb{C}[U \otimes V \otimes W]^G_\bullet$, where $J_{3d} = I_d \circ \det$ as above. The projective space $\mathrm{Proj}(S_\bullet)$ is a codimension 1 subscheme of $\mathbb{P}(1, 1, 2, 3, 4, 5)$, with a rational map $\det_{UVW}$ to $|\mathcal{O}_{\mathbb{P}U}(3)| /\!/ \mathrm{SL}(U)$, forgetting the first coordinate.

We let $\eta : \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) /\!/ G \dashrightarrow \mathrm{Proj}(S_\bullet)$ be the rational map induced by the inclusion of graded rings.

We will later see that $\eta$ is a finite morphism of even degree dividing 72. The $G$-invariant functions in $S_\bullet$ give geometric criteria for semistability:

**Proposition 4.6** (Sufficient criteria for $G$-semistability)**.** *A point $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$-semistable if:*

- *$S_\varphi$ is an $\mathrm{SL}(U)$-semistable cubic surface.*
- *$S_\varphi$ contains no rank 1 points ($\varphi \notin R_{24}$).*

*Proof.* The first condition is immediate: the preimage of an $\mathrm{SL}(U)$-semistable point in $|\mathcal{O}_{\mathbb{P}U}(3)|$ is $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$-semistable. The second condition is equivalent to the $G$-invariant polynomial $R_{24}$ not vanishing at $\varphi$. □

To prove this geometric criteria is sufficient, we use the Hilbert-Mumford criterion.

**Definition 4.7** (Block form)**.** Let $B$ be a $4 \times 3 \times 3$ array of 0s and $*$s, in the sense that $B = [B_0|B_1|B_2|B_3]$ and each $B_i$ is a $3 \times 3$ matrix of 0s and $*$s.

A tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is said to be *of block form $B$* if there is a basis for $U, V, W$ where $\varphi_U = \sum_{i=0}^{3} u_i\varphi_i$ and each $\varphi_i \in \mathrm{Hom}(V, W^\vee) \cong \mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^3)$ has non-zero entries only where $B_i$ has an $*$. Note that we choose the ordering so that $\mathrm{SL}(V)$ acts on the columns of $B_i$ and $\mathrm{SL}(W)$ acts on the rows of $B_i$.

**Proposition 4.8** (Sufficient criteria for $G$-instability)**.** *A point $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$-unstable if*

(1) *$\varphi_U$ is not injective.*
(2) *$S_\varphi$ contains a plane of type $(1,3)$ or $(3,1)$.*
(3) *$S_\varphi$ contains a plane of type $(2,2)$.*
(4) *$\varphi$ is of the following block form:*

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(5) *$\varphi$ is of the following block form:*

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & 0 & * & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 & * & 0 & 0 \\ * & * & 0 & * & * & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$(5)^T$ *$\varphi$ is of the following block form:*

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & 0 \\ * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

*Any $G$-unstable determinantal representation of a normal cubic surface with finitely many lines is of block type $(4)$, $(5)$, or its transpose $(5)^T$.*

*Proof.* We apply the Hilbert-Mumford criterion. A point $\varphi$ is $G$-unstable if and only if there is a one-parameter subgroup $\lambda$ of $G$ that acts with only positive weights on $\varphi$. This one-parameter subgroup can always be chosen to act diagonally in some basis for $U, V, W$, with weight vector $\mathrm{wt}(\lambda) = ((a_0, a_1, a_2, a_3), (b_0, b_1, b_2), (c_0, c_1, c_2))$. In this basis we write $\varphi = \sum_{ijk} \varphi_{ijk}(u_i \otimes v_j \otimes w_k)$, acting with weight $a_i + b_j + c_k$ on $u_i \otimes v_j \otimes w_k$.

One can then check that the one-parameter subgroup with the following weight vector destabilises the respective $\varphi$s, from above:

(1) Unstable by the weight $((1,1,1,-3),(0,0,0),(0,0,0))$ (note that this is also seen to be $\mathrm{SL}(U)$-unstable by quiver GIT).
(2) Unstable by the weight $((9,-3,-3,-3),(0,0,0),(4,4,-8))$ (for type $(1,3)$) and by the weight $((9,-3,-3,-3),(4,4,-8),(0,0,0))$ (for type $(3,1)$).
(3) Unstable by the weight $((9,-3,-3,-3),(8,-4,-4),(8,-4,-4))$.
(4) Unstable by the weight $((9,9,-3,-15),(8,-4,-4),(8,-4,-4))$.
(5) Unstable by the weight $((9,9,-3,-15),(12,0,-12),(4,4,-8))$.
$(5)^T$ Unstable by the weight $((9,9,-3,-15),(4,4,-8),(12,0,-12))$.

□

These conditions for instability were found using an algorithmic implementation of the Hilbert-Mumford criterion using convex geometry, see Subsection 3.1 of [Hos23] for details on the convex criterion. This algorithm outputs 18 maximal $G$-unstable block forms, reducing to 12 after taking transposes, which further reduces to the 5 forms above after suitable changes of basis. We provide these 12 forms in Appendix A for completeness. For the purpose of our proofs, we also list block form (10), a subtype of (4), and block form (12), a subtype of (5) and $(5)^T$.

(10) Unstable by the weight $((3,3,-3,-3),(0,0,0),(4,-2,-2))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(12) Unstable by the weight $(3,3,-3,-3),(2,2,-4),(2,2,-4))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & 0 & * & * & 0 \\ * & * & * & * & * & * & * & * & 0 & * & * & 0 \\ * & * & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Now, Proposition 4.6 actually gives necessary conditions for $G$-semistability, and Proposition 4.8 gives necessary conditions for $G$-instability. We prove this by a case-by-case computation, splitting the closed points of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ into the following four types of cubic surfaces, a result of Schläfi [Sch63].

**Proposition 4.9.** *Every cubic surface $[S] \in |\mathcal{O}_{\mathbb{P}U}(3)|$ is of one (or more) of the following types:*

- *$S$ is normal, with finitely many lines ($S$ is smooth or has only canonical singularities).*
- *$S$ is reducible.*
- *$S$ is a cone over a cubic curve.*
- *$S$ is non-normal, non-conical, and irreducible. In this case, the polynomial defining $S$ is $\mathrm{SL}(U)$-equivalent to either $u_2u_0^2 + u_1u_3^2$ or $u_2u_0^2 + u_3u_0u_1 + u_1^3$.*

*Proof.* For a concise proof see Section 2 of [BW79]. ☐

We then compute the $G$-semistable points for each type of cubic surface, and combine the classifications at the end. We use Ng's classification of $4 \times 3 \times 3$ tensors [Ng02] for cubic surfaces with at most canonical singularities:

**Lemma 4.10.** *Let $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ such that $S_\varphi$ is normal with finitely many lines. Then $\varphi$ is $G$-semistable if and only if one of the following holds:*

- *$S_\varphi$ has at worst $A_2$ singularities ($S_\varphi$ is $\mathrm{SL}(U)$-semistable).*
- *$S_\varphi$ is an $A_3$ cubic surface with no rank 1 points. In this case, $\varphi \in A_3(d)$ and $\varphi_U$ is $G$-equivalent to the following matrix, where $\alpha \in \mathbb{A}^1 \setminus \{0\}$ is a parameter that determines the isomorphism class of $S_\varphi$:*

$$\begin{pmatrix} (\alpha-1)u_0 + u_1 & -u_1 & u_0 \\ u_2 & 0 & u_3 \\ u_0 + u_3 & \alpha u_0 + u_1 + u_2 & 0 \end{pmatrix}$$

- *$S_\varphi$ is the $A_1A_3$ cubic surface with no rank 1 points. In this case, $\varphi \in A_1A_3(f)$ and $\varphi_U$ is $G$-equivalent to the following matrix:*

$$\begin{pmatrix} u_1 & u_0 & 0 \\ u_2 & 0 & u_3 - u_0 - u_1 \\ u_3 & u_2 & u_1 \end{pmatrix}$$

*Proof.* If $S_\varphi$ is $\mathrm{SL}(U)$-semistable, then there is a homogeneous, non-constant $\mathrm{SL}(U)$-invariant polynomial $f \in \mathbb{C}[\mathrm{Sym}^3(U)]_\bullet$ where $f([S_\varphi]) \neq 0$. We let $\det : U^\vee \otimes V^\vee \otimes W^\vee \to \mathrm{Sym}^3(U^\vee)$ be the determinant map (unique up to a constant multiple), and consider $f \circ \det \in \mathbb{C}[U \otimes V \otimes W]_\bullet$. Then $f \circ \det$ is homogeneous, non-constant and $G$-invariant, and $f \circ \det(\varphi) \neq 0$ so $\varphi$ is $G$-semistable.

Otherwise, suppose $S_\varphi$ is $\mathrm{SL}(U)$-unstable. A general $\mathrm{SL}(U)$-unstable cubic surface has one $A_3$ singularity, and it is easy to verify from the classification in [Ng02] that every determinantal representation of an $\mathrm{SL}(U)$-unstable cubic surface is contained in either $\overline{A_3(a)} \cup \overline{A_3(b)} \cup \overline{A_3(c)}$ or $\overline{A_3(d) \cup A_1A_3(f)}$.

Every point in $\overline{A_3(a)}, \overline{A_3(b)}, \overline{A_3(c)}$ is $G$-unstable of block types (4), (5), or $(5)^T$. If $\varphi \in A_3(d)$ or $\varphi \in A_1A_3(f)$ then $S_\varphi$ contains no rank 1 points ($X_\varphi$ has no length 3 collinear subscheme), and is $G$-semistable. $\square$

**Lemma 4.11.** *Let $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ such that $S_\varphi$ is reducible. Then $\varphi$ is $G$-semistable if and only if $\varphi$ is $G$-equivalent to*

$$\sigma = \begin{pmatrix} u_0 & u_3 & -u_2 \\ -u_3 & u_0 & u_1 \\ u_2 & -u_1 & u_0 \end{pmatrix}.$$

*Proof.* If $S_\varphi$ is reducible then it contains a hyperplane $\mathbb{P}U' \subset \mathbb{P}U$, and $\varphi_U(U') \subset V^\vee \otimes W^\vee$ is a rank at most 2 subspace. If $\varphi_U(U')$ is of types $(1,3), (3,1)$, or $(2,2)$ then $\varphi$ is $G$-unstable, so suppose $\varphi_U(U')$ is a skew-symmetric subspace.

Then $\varphi$ is a skew-symmetric tensor, and by Proposition 2.21 is $G$-equivalent to

$$\sigma_A = \begin{pmatrix} \lambda_1 u_0 & u_3 & -u_2 \\ -u_3 & \lambda_2 u_0 & u_1 \\ u_2 & -u_1 & \lambda_3 u_0 \end{pmatrix}$$

where each of the $\lambda_i$s is either 0 or 1. If one of the $\lambda_i$s is 0 then $\varphi$ is of block form (12) and is $G$-unstable. Otherwise, $\varphi$ is $G$-equivalent to $\sigma$. It is easy to check from the $2 \times 2$ minors of $\sigma$ that $S_\sigma$ has no rank 1 points, so $R_{24}(\sigma) \neq 0$ and $\sigma$ is $G$-semistable. $\square$

**Lemma 4.12.** *Let $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ such that $S_\varphi$ is a cone. Then $\varphi$ is $G$-unstable.*

*Proof.* We choose coordinates so that $\det(\varphi_U)$ has no $u_3$ term, i.e. that $S_\varphi$ is a cone over $[0:0:0:1]$. Then, in some basis, either $\varphi_U$ is not injective or $\varphi_U(0,0,0,1)$ is rank 1 or 2. If $\varphi_U$ is not injective then $\varphi$ is $\mathrm{SL}(U)$-unstable. So first suppose $\varphi_U(0,0,0,1)$ is rank 1.

Because $\det(\varphi_U)$ has no $u_3$ term, then $\varphi_U$ is $G$-equivalent to

$$\varphi_U = \begin{pmatrix} u_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where the $a_{ij}$s are linear forms in $u_0, u_1, u_2$ and $a_{22}a_{33} - a_{23}a_{32} = 0$. So $\mathrm{Span}\{a_{22}, a_{23}, a_{32}, a_{33}\}$ is an at most 2-dimensional subspace of $U^\vee$. Changing basis so that $a_{22}, a_{23}, a_{32}, a_{33}$ are linear forms in $u_0$ and $u_1$, we see that $\varphi_U$ is of block form (4) and is $G$-unstable.

Otherwise suppose that $\varphi_U(0,0,0,1)$ is rank 2. Then $\varphi_U$ is $G$-equivalent to

$$\varphi_U = \begin{pmatrix} u_3 & 0 & 0 \\ 0 & u_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$$

where the $a_{ij}$s are linear forms in $u_0, u_1, u_2$ and $a_{13}a_{31} + a_{23}a_{32} = 0$. Similarly, $\mathrm{Span}\{a_{13}, a_{31}, a_{23}, a_{32}\}$ is at most 2-dimensional and $\varphi_U$ is of block form (12) and is $G$-unstable. $\square$

**Lemma 4.13.** *Let $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ such that $S_\varphi$ is non-normal, non-conical, and irreducible. Then $\varphi$ is $G$-unstable.*

*Proof.* We choose coordinates so that $\det(\varphi_U)$ is at most linear in $u_2, u_3$. If the line $L \subset S_\varphi$ parametrised by $[0:0:u_2:u_3]$ is rank 1 then $\varphi$ is of block type (10) or its transpose and is $G$-unstable.

If $L$ is not rank 1 then it has at least two rank 2 points, suppose that $\varphi_U(0,0,1,0)$ and $\varphi_U(0,0,0,1)$ are both rank 2. We let their left and right kernels be $V_2, V_3 \subset V$ and $W_2, W_3 \subset W$. We have $\varphi_{V\otimes W}(V_i \otimes W_i) = 0$ for $i = 2,3$ because the determinant of $\varphi_U$ has no terms in $u_2^2$ or $u_3^2$ (this condition can be checked by cofactor expansion).

Now, there are three possibilities. If $V_2 \otimes W_2 = V_3 \otimes W_3$, then $\varphi$ is of block form (12) and is $G$-unstable. Otherwise, suppose that $L$ is a line of type $(1,3)$ or $(3,1)$ (i.e. that $V_2 = V_3$ or $W_2 = W_3$). Then $\varphi$ is of the following block form

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

which is contained in block form (2) or its transpose and is $G$-unstable. The only thing left to check is if $L$ is a line of type $(2,2)$, which occurs when $V_2 \neq V_3$ and $W_2 \neq W_3$. We choose coordinates so that

$$\varphi_U = \begin{pmatrix} 0 & u_3 & u_2 \\ u_3 & 0 & 0 \\ u_2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$$

where the $a_{ij}$s are linear forms in $u_0, u_1$. The coefficient of $u_2u_3$ in $\det(\varphi_U)$ is $a_{23} + a_{32}$, which must be zero, so after choosing coordinates where $a_{23} = -a_{32} = u_0$ we see that $\varphi$ has a plane of type $(2,2)$ and is $G$-unstable. □

Combining these four lemmas, we find that the conditions described in Proposition 4.6 are sufficient (and Proposition 4.8 also provides necessary conditions for instability):

**Proposition 4.14** (Classification of $G$-semistability)**.** *A point $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$-semistable if and only if one of the following holds:*

- *$S_\varphi$ is an $\mathrm{SL}(U)$-semistable cubic surface, or*
- *$S_\varphi$ contains no rank 1 points.*

*Proof.* If $S_\varphi = \mathbb{P}U$ then either $\varphi_U(U) \subset V^\vee \otimes W^\vee$ is a 4-dimensional rank 2 subspace or $\dim(\varphi_U(U)) \leq 3$. In the former case $\varphi$ satisfies a rank 2 block condition and is $\mathrm{SL}(V) \times \mathrm{SL}(W)$-unstable, in the latter case $\varphi_U$ is not injective and $\varphi$ is $\mathrm{SL}(U)$-unstable. So if $\varphi$ is $G$-semistable, then $S_\varphi$ is a cubic surface. We then split into the cases in Proposition 4.9 and use Lemmas 4.10, 4.11, 4.12 and 4.13. □

**Remark 4.15.** A necessary, but insufficient, condition for $G$-semistability is that $\varphi$ is stable under all partial group actions of $\mathrm{SL}(A)$ and $\mathrm{SL}(A) \times \mathrm{SL}(B)$.

Recall the ring $S_\bullet$ defined in Definition 4.5. By Proposition 4.14, the $G$-unstable locus is (set-theoretically) cut out by the invariants $R_{24}$ and $J_{3d}$. As a result,

**Proposition 4.16.** *The ring $\mathbb{C}[U \otimes V \otimes W]^G_\bullet$ is a finitely generated, integral, graded $S_\bullet$-module. The map $\eta$ is a finite morphism of even degree dividing 72, and the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G \arrow[r, "\eta"] \arrow[dr, dashed, "\det_{UVW}"'] & \operatorname{Proj}(S_\bullet) \arrow[d, dashed] \\
& |\mathcal{O}_{\mathbb{P}U}(3)|//\operatorname{SL}(U)
\end{tikzcd}$$

*For every subscheme $Z \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G$, we have an equality $\dim(Z) = \dim(\eta(Z))$, and an inequality $\dim(Z) \leq \dim(\det_{UVW}(Z)) + 1$.*

*Proof.* It is immediate that $\mathbb{C}[U \otimes V \otimes W]^G_\bullet$ is a finitely generated, integral, graded $S_\bullet$-module. On an open set, the generically finite, degree 72 rational map $\det_{UVW}$ factors through $\eta$, so $\eta$ has degree dividing 72.

To see that $\eta$ has even degree, we note that a larger automorphism group of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is the semidirect product $G \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ is the involution swapping $V$ and $W$ that acts as the Gale transform on $X_\varphi$. The ring $S_\bullet$ is invariant under this larger action, while $\mathbb{C}[U^\vee \otimes V^\vee \otimes W^\vee]^G_\bullet$ is not invariant under Gale duality, so the morphism $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G \to \operatorname{Proj}(S_\bullet)$ factors through the degree two morphism $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G \to \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(G \rtimes \mathbb{Z}/2\mathbb{Z})$.

The inequalities on dimensions follow from $\eta$ being a finite morphism and the rational projection $\operatorname{Proj}(S_\bullet) \dashrightarrow |\mathcal{O}_{\mathbb{P}U}(3)|//\operatorname{SL}(U)$ having at most 1-dimensional fibres. ☐

4.2. **Classifying stability.** We use the geometric description of the $G$-semistable locus to describe the ring of $G$-invariants up to finite extension, give a geometric description of $G$-stability, and describe some of the geometry of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$. Similar to Subsection 4.1, we prove $G$-stability geometrically and prove strict $G$-semistability using the Hilbert-Mumford criterion.

**Definition 4.17.** For a subvariety $F \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)^{G-ss}$, we let $\overline{F}^{G-ss}$ be its closure inside the $G$-semistable locus.

**Proposition 4.18.** *The preimage of the $\operatorname{SL}(U)$-stable locus of $|\mathcal{O}_{\mathbb{P}U}(3)|$ admits the following stratification, where a box indicates a maximally singular type of tensor inside $\det^{-1}_{UVW}\left(|\mathcal{O}_{\mathbb{P}U}(3)|^{\operatorname{SL}(U)-s}\right)$. An arrow $F(n) \to F'(n')$ indicates an inclusion $F'(n') \subset \overline{F(n)}^{G-ss}$.*

$$\begin{tikzcd}
A_1(a) \arrow[r] \arrow[dr] \arrow[ddr] & 2A_1(a) \arrow[r] \arrow[dr] & 3A_1(a) \arrow[r] & \boxed{4A_1(a)} \\
& 2A_1(b) \arrow[r] \arrow[ur] \arrow[dr] & 3A_1(b) \arrow[dr] & \\
& 2A_1(c) \arrow[r] \arrow[ur] \arrow[dr] & 3A_1(c) \arrow[r] & \boxed{4A_1(b)} \\
A_1(b) \arrow[r] \arrow[uur] \arrow[ur] & 2A_1(d) \arrow[r] \arrow[ur] & \boxed{3A_1(d)} &
\end{tikzcd}$$

*The set of $G$-semistable tensors $\varphi$ where $S_\varphi$ is not $\operatorname{SL}(U)$-stable can be decomposed into the disjoint union $\overline{A_2(a)}^{G-ss} \cup \overline{A_2(b)}^{G-ss}$. It has the following stratification, where a box indicates a maximally singular type of tensor inside $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)^{G-ss}$. An arrow $F(n) \to F'(n')$ indicates an* *inclusion* $F'(n') \subset \overline{F(n)}^{G-ss}$.

$$\begin{tikzcd}
 & A_1A_2(a) \arrow[r] \arrow[dr] & 2A_1A_2(a) \arrow[dr] & & \\
 & A_1A_2(b) \arrow[ur] \arrow[dr] \arrow[dddr] & 2A_1A_2(b) \arrow[r] \arrow[ddr] & A_12A_2(a) \arrow[dr] & \\
A_2(a) \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] & A_1A_2(c) \arrow[ur] \arrow[dr] & 2A_1A_2(c) \arrow[ur] \arrow[dr] & & \boxed{3A_2(a)} \\
 & A_1A_2(d) \arrow[ur] \arrow[r] \arrow[dr] & 2A_1A_2(d) \arrow[r] & A_12A_2(b) \arrow[ur] & \\
 & & 2A_2(a) \arrow[uuur] \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & A_1A_2(e) \arrow[r] & 2A_1A_2(f) \arrow[dr] & \\
A_2(b) \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & A_1A_2(f) \arrow[r] & 2A_1A_2(e) \arrow[r] & \boxed{A_12A_2(c)} \\
 & A_1A_2(g) \arrow[r] \arrow[dr] & 2A_2(b) \arrow[ur] \arrow[r] & \boxed{G \cdot \sigma} \\
 & A_3(d) \arrow[r] & A_1A_3(f) \arrow[ur] &
\end{tikzcd}$$

*Proof.* We analyse the closure relations using the associated length 6 subschemes. Specifically, the $G$-semistable locus of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is contained in the $\mathrm{SL}(U) \times \mathrm{SL}(W)$-stable locus, so for two types of $G$-semistable tensors $F(n)$ and $F'(n')$ we have $F'(n') \subset \overline{F(n)}$ if and only if $\pi_{UW}(F'(n')) \subset \overline{\pi_{UW}(F(n))}$. And because $\gamma : \mathrm{Hilb}_6(\mathbb{P}V) \dashrightarrow \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(U) \times \mathrm{SL}(W))$ is an isomorphism when restricted to curvilinear subschemes not contained in a conic, then we can compute these closures inside $\mathrm{Hilb}_6(\mathbb{P}V)$.

As an example, to see that $\overline{A_2(a)}^{G-ss} \cap \overline{A_2(b)}^{G-ss} = \emptyset$, we note that if $\varphi \in \overline{A_2(a)}^{G-ss}$ then $X_\varphi$ has a collinear length 4 cycle but no length 3 point, and if $\varphi \in \overline{A_2(b)}^{G-ss}$ then $X_\varphi$ has a length 3 point but no collinear length 4 cycle.

The only things left to check are the closure relations involving $G \cdot \sigma$. The $G$-orbit of a $G$-semistable tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is codimension at most 6 if $S_\varphi$ has canonical singularities, and $G \cdot \sigma$ is codimension 7, so $G \cdot \varphi \not\subset \overline{G \cdot \sigma}$ if $S_\varphi$ has canonical singularities. Fix a $G$-semistable type of tensors $F(n)$, where each $\varphi \in F(n)$ has no rank 1 points, and consider a family of $B \subset \mathrm{Hilb}_6(\mathbb{P}V)$ length 6 subschemes of type $F(n)$ that specialise to lie on a smooth conic (which always exists). Then the family $\gamma(B)$ specialises from $\pi_{UW}(F(n))$ to $\pi_{UW}(G \cdot \sigma)$, and $G \cdot \sigma \subset \overline{F(n)}$. Conversely, if one tensor $\varphi \in F(n)$ contains (and hence all tensors contain) a rank 1 point, then $F(n) \subset R_{24}$. But $\sigma \notin R_{24}$ and $R_{24}$ is closed, so $G \cdot \sigma \not\subset \overline{F(n)}$. $\square$

**Remark 4.19.** The rational map $\det_{UVW}$ is undefined only at $\pi_{UVW}(\overline{A_3(d)}^{G-ss})$. We will see in the proof of Proposition 4.26 that this is a singleton.

Now, to understand $G$-stability and polystability, we need to be able to compute the dimension of the $G$-orbit and stabiliser of a given tensor.

**Proposition 4.20.** *Let $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ be a canonical tensor. There is an equality*

$$\dim \mathrm{Stab}_G(\varphi) = \dim \mathrm{Stab}_{\mathrm{SL}(U)}(S_\varphi).$$

*Proof.* By quiver GIT (see [Dré87], or Subsection 3.3 of [LLSS17] for a detailed discussion), if $\varphi$ is a canonical tensor then $\varphi$ is $\mathrm{SL}(V) \times \mathrm{SL}(W)$-stable. So the dimensions of the $G$-stabiliser of $\varphi$ and the $\mathrm{SL}(U)$-stabiliser of $\pi_{VW}(\varphi)$ are equal. The $\mathrm{SL}(U)$-equivariant rational map

$$\det_{VW} : \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(V) \times \mathrm{SL}(W)) \dashrightarrow |\mathcal{O}_{\mathbb{P}U}(3)|$$

is generically finite when restricted to (the GIT quotient of) canonical tensors, so the $\mathrm{SL}(U)$-stabilisers of the source and target have the same dimension. ☐

Because every canonical cubic surface $S$ is anticanonically embedded in $\mathbb{P}^3$, the automorphism group of the surface $S$ is equal to the subgroup of $\mathrm{PGL}(4)$ that fixes the subscheme $S \subset \mathbb{P}^3$. This has the same dimension as the $\mathrm{SL}(4)$-stabiliser. Sakamaki computed the automorphism groups of canonical cubic surfaces with no parameters in Theorem 3 of [Sak10], and the dimension of every automorphism group of a canonical cubic surface can be found in Section 8 of [CP21]. Combining their results and using Proposition 4.20, we find that

**Proposition 4.21.** *Fix a subgraph $F \subset \widetilde{E}_6$. The subvariety $F(n) \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is codimension $|F|$ and for each $\varphi \in F(n)$ we have $\dim \mathrm{Stab}_G(\varphi) = \max\{0, 4 - |F|\}$ unless*

- *$F = 2A_2$, in which case $\operatorname{codim} 2A_2(n) = 4$ and $\dim \mathrm{Stab}_G(\varphi) = 1$, or*
- *$F(n) = D_4(b)$, in which case $\operatorname{codim} D_4(b) = 5$ and $\dim \mathrm{Stab}_G(\varphi) = 1$.*

*The subvariety $G \cdot \sigma$ is codimension 7, and $\dim \mathrm{Stab}_G(\sigma) = 3$.*

*Proof.* The stabiliser dimensions follow from [Sak10; CP21]. When $|F| \geq 4$ and $F \neq 2A_2$ then the subvariety $F(n)$ is a $G$-orbit and its dimension follows from orbit-stabiliser (noting that $\dim \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee) - \dim G = 4$), otherwise we add the number of parameters for a tensor type $F(n)$ to the dimension of an individual $G$-orbit.

For $G \cdot \sigma$, we note that the quotient $\pi_{UW}$ contracts $G \cdot \sigma$ to the open subscheme of $\Sigma//(\mathrm{SL}(U) \times \mathrm{SL}(W)) \cong |\mathcal{O}_{\mathbb{P}V}(2)|$ parametrising smooth conics. This is codimension 7 inside $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//(\mathrm{SL}(U) \times \mathrm{SL}(W))$, an orbit space, so $G \cdot \sigma$ is codimension 7 inside $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$. Alternatively, one can directly compute that $\mathrm{Stab}_G(\sigma)$ is a degree 27 extension of $\mathrm{SO}(3, \mathbb{C})$ and use orbit-stabiliser. ☐

**Proposition 4.22** (Sufficient criteria for $G$-stability)**.** *Every determinantal representation of an $\mathrm{SL}(U)$-stable cubic surface is $G$-stable.*

*Proof.* We first prove the stabiliser is finite. By Proposition **??**, we have $\dim \mathrm{Stab}_G(\varphi) = \dim \mathrm{Stab}_{\mathrm{SL}(V)}(X_\varphi)$. The schemes $X_\varphi$ associated to a tensor of type $4A_1(a)$ and $4A_1(b)$ are supported at (at least) four points in linearly general position, so have trivial $\mathrm{SL}(V)$-stabilisers. Similarly, for $3A_1(d)$ these schemes are three length 2 points in linearly general position, which has a finite $\mathrm{SL}(V)$-stabiliser.

Now to prove the $G$-orbits are closed. Let $\varphi$ be a determinantal representation of an $\mathrm{SL}(U)$-stable cubic surface, and $\varphi'$ be a $G$-semistable tensor. If $S_\varphi$ and $S_{\varphi'}$ are not $\mathrm{SL}(U)$-equivalent, then $f(\varphi) \neq f(\varphi')$, so $\varphi' \notin \overline{G \cdot \varphi}$.

So suppose $S_\varphi$ and $S_{\varphi'}$ are $\mathrm{SL}(U)$-equivalent. The $G$-orbits $G\cdot\varphi$ and $G\cdot\varphi'$ are irreducible codimension 4 subvarieties of $\mathbb{P}(U^\vee\otimes V^\vee\otimes W^\vee)$, so if $G\cdot\varphi'\subset\overline{G\cdot\varphi}$ then $G\cdot\varphi'\cap G\cdot\varphi$ is non-empty and $\varphi$ and $\varphi'$ are $G$-equivalent. So if $\varphi'\in\overline{G\cdot\varphi}$ then $\varphi'\in G\cdot\varphi$, so $G\cdot\varphi$ is closed. □

**Proposition 4.23** (Sufficient criteria for strict $G$-semistability)**.** *A point $\varphi\in\mathbb{P}(U^\vee\otimes V^\vee\otimes W^\vee)$ is strictly $G$-semistable if $\varphi$ is $G$-semistable and $\varphi$ is of one of the following two block forms:*

*(1)*

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0\\ * & * & * & * & * & * & * & * & * & * & 0 & 0\\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{array}\right]$$

*(2)*

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & 0\\ * & * & * & * & * & 0 & * & * & 0 & * & 0 & 0\\ * & * & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0\end{array}\right]$$

*Proof.* We use the Hilbert-Mumford criterion. The first type is seen to not be $G$-stable by the weight vector $((3,0,0,-3),(2,-1,-1),(1,1,-2))$, the second type is not $G$-stable by the weight vector $((1,0,0,-1),(1,0,-1),(1,0,-1))$, which act with only non-negative weights where there is an $*$. A complete list of the block types output by the computer is found in Appendix A □

**Corollary 4.24** (Classification of $G$-stability)**.** *A point $\varphi\in\mathbb{P}(U^\vee\otimes V^\vee\otimes W^\vee)$ is $G$-stable if and only if $S_\varphi$ is an $\mathrm{SL}(U)$-stable cubic surface.*

*Proof.* We look at the set of tensors $\varphi$ where $S_\varphi$ is not $\mathrm{SL}(U)$-stable. The subvariety $A_2(a)$ consists of tensors $G$-equivalent to

$$\varphi_U=\begin{pmatrix} u_3 & -\alpha_1u_0+u_1 & u_1\\ 0 & u_2 & -u_0+u_1\\ u_1-\alpha_2u_2 & 0 & u_0\end{pmatrix},$$

where $\alpha_1,\alpha_2\in\mathbb{A}^1\setminus\{0,1\}$, which is of strictly $G$-semistable block type (1). The subvariety $A_2(b)$ consists of tensors $G$-equivalent to

$$\varphi_U=\begin{pmatrix} -\beta_1u_3 & u_3 & u_2\\ (1-\beta_1)u_3 & (1-\beta_2)u_0+\beta_2u_1+u_2 & u_0\\ u_1 & u_0 & 0\end{pmatrix},$$

where $\beta_1\in\mathbb{A}^1\setminus\{0,1\}$ and $\beta_2\in\mathbb{A}^1\setminus\{0,1/(1-\beta_1)\}$, which is of strictly $G$-semistable block type (2). All other $G$-semistable tensors where $\varphi$ is not $\mathrm{SL}(U)$-stable are in either $\overline{A_2(a)}^{G-ss}$ or $\overline{A_2(b)}^{G-ss}$, according to the closure relations given in Proposition 4.18, and are strictly $G$-semistable. □

## 4.3. **Polystability and the geometry of the GIT quotient.**

**Proposition 4.25.** *The subvariety $3A_2(a)\subset\overline{A_2(a)}^{G-ss}$ is the unique $G$-polystable orbit inside $\overline{A_2(a)}^{G-ss}$, the set $\pi_{UVW}(\overline{A_2(a)}^{G-ss})\in\mathbb{P}(U^\vee\otimes V^\vee\otimes W^\vee)//G$ is a singleton.*

*Proof.* All points in $3A_2(a)$ are $G$-equivalent. They are $G$-polystable because a tensor of type $3A_2(a)$ is maximally singular inside the $G$-semistable locus. So it suffices to prove that $\pi_{UVW}(\overline{A_2(a)}^{G-ss})$ is a finite scheme, since it is connected then it is a singleton.

This is immediate: $\eta\circ\pi_{UVW}(\overline{A_2(a)}^{G-ss})=\{[0:8:1:0:0:0]\}$ is a singleton, so $\dim(\pi_{UVW}(\overline{A_2(a)}^{G-ss}))=0$. □

**Proposition 4.26.** *A point $\varphi \in \overline{A_2(b)}^{G-ss}$ is G-polystable if and only if $\varphi \in \overline{2A_2(b)}^{G-ss}$. There is a bijective morphism $\mathbb{P}(1,2) \to \overline{2A_2(b)}^{G-ss}//G$.*

*Proof.* If $\varphi \in \overline{2A_2(b)}^{G-ss}$, then $\varphi$ is $G$-equivalent to a tensor of the form

$$\varphi_{[s:t]} = \begin{pmatrix} 0 & u_3 & u_2 \\ u_3 & su_1 + tu_2 & u_0 \\ u_1 & u_0 & 0 \end{pmatrix},$$

parametrised by $[s:t] \in \mathbb{P}^1$. The involution $\tau \in G$ that swaps $u_1$ and $u_2$, $v_0$ and $v_2$, and $w_0$ and $w_2$ acts on $\mathbb{P}^1$ by sending $[s:t] \mapsto [t:s]$. Geometrically, this swaps the two $A_2$ singularities of $S_\varphi$ and swaps the two points of $X_\varphi$ and $Y_\varphi$. If $[s:t] \neq [s':t'], [t':s']$, then $\varphi_{[s:t]}$ and $\varphi_{[s':t']}$ are not $G$-equivalent. The quotient of $\mathbb{P}^1$ by the subgroup $\{1, \tau\} \cong \mathbb{Z}/2\mathbb{Z}$ is isomorphic to $\mathbb{P}(1,2)$. We now need to prove that a tensor $\varphi \in \overline{A_2(b)}$ is $G$-polystable if and only if $\varphi \in \overline{2A_2(b)}^{G-ss}$.

We first prove that every tensor of type $\overline{2A_2(b)}^{G-ss}$ is $G$-polystable. If $\varphi \in G \cdot \sigma$ then $\varphi$ is $G$-polystable because this is a maximally singular $G$-orbit inside the $G$-semistable locus. The $G$-stabiliser of any tensor $\varphi \in \overline{2A_2(b)}^{G-ss} \setminus G \cdot \sigma$ is 1-dimensional, so $G \cdot \varphi$ is an irreducible codimension 5 subvariety by orbit-stabiliser. By the same argument as the second half of the proof of Proposition 4.22, then if $\varphi, \varphi' \in \overline{2A_2(b)}^{G-ss} \setminus G \cdot \sigma$ we have $\varphi' \in \overline{G \cdot \varphi}$ if and only if $\varphi' \in G \cdot \varphi$. And the points $\pi_{UVW}(\sigma)$ and $\pi_{UVW}(\varphi)$ are distinct because all the invariants $J_d \in S_\bullet$ vanish at $\sigma$ but not at $\varphi$. So $\sigma \notin \overline{G \cdot \varphi}$, hence $G \cdot \varphi$ is closed inside the $G$-semistable locus if $\varphi \in \overline{2A_2(b)}^{G-ss}$.

For the other direction, we only need to prove that every tensor in $\overline{A_2(b)}^{G-ss} \setminus \overline{2A_2(b)}^{G-ss}$ has a non-closed $G$-orbit. By an argument similar to the proof of Proposition 4.22, every determinantal representation of a cubic surface with at worst $A_1A_2$ singularities has a finite $G$-stabiliser. Because they are strictly $G$-semistable with finite $G$-stabiliser, then tensors of types $A_2(b), A_1A_2(e), A_1A_2(f), A_1A_2(g)$ must have a non-closed $G$-orbit. For type $A_3(d)$, we note that $G \cdot \sigma \subset \overline{A_3(d)}^{G-ss}$ and that $\eta \circ \pi_{UVW}(\overline{A_3(d)}^{G-ss}) = \eta \circ \pi_{UVW}(G \cdot \sigma) = \{[1:0:0:0:0:0]\}$ is a singleton. Because $\overline{A_3(d)}^{G-ss}$ is connected, then $\pi_{UVW}(\overline{A_3(d)}^{G-ss}) = \pi_{UVW}(G \cdot \sigma)$ and for each $\varphi \in \overline{A_3(d)}^{G-ss}$ we have $G \cdot \sigma \subset \overline{G \cdot \varphi}$. In particular, $G \cdot \varphi$ is not closed unless $\sigma \in G \cdot \varphi$.

It is then immediate that $\pi_{UVW}(\overline{A_2(b)}^{G-ss}) = \pi_{UVW}(\overline{2A_2(b)}^{G-ss})$ as these are the only $G$-polystable points inside $\overline{A_2(b)}^{G-ss}$. The morphism $\mathbb{P}^1 \to \overline{2A_2(b)}^{G-ss}$ sending $[s:t] \mapsto \pi_{UVW}(\varphi_{[s:t]})$ descends to a map of quotients $f : \mathbb{P}(1,2) \to \overline{2A_2(b)}^{G-ss}$. The morphism $f$ is a bijection on $\mathbb{C}$-points. ∎

4.4. **Resolving the determinant map between GIT quotients.** To resolve the rational map $\det_{UVW}$, we consider the closure of its graph. A natural question to ask is whether this space has a modular interpretation, which we partially resolve.

**Definition 4.27.** Let $\mathcal{M} \subset \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G \times |\mathcal{O}_{\mathbb{P}U}(3)|// \operatorname{SL}(U)$ be the closure of the graph of $\det_{UVW}$, equipped with a projection

$$\rho : \mathcal{M} \longrightarrow \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)//G.$$

The projection $\rho$ is an isomorphism away from the fibre above $\pi_{UVW}(\sigma)$. To probe this fibre, we use a specific test family of length 6 subschemes:

**Definition 4.28.** Let $\mathcal{H} \subset \mathrm{Hilb}_6(\mathbb{P}V)$ be the open subscheme parametrising length 6 subschemes $X \subset \mathbb{P}V$ such that:

- $X$ is canonical,
- $X$ is not contained in a singular conic, and
- the associated cubic surface $S_X$ is SL(4)-semistable.

The image of $\mathcal{H}$ under the contraction $\gamma$ lies in the SL($V$)-semistable locus of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)/\!/\,(\mathrm{SL}(U) \times \mathrm{SL}(W))$, and so the composition of $\gamma$ with the quotient map $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)/\!/\,(\mathrm{SL}(U) \times \mathrm{SL}(W)) \dashrightarrow \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)/\!/G$ induces a morphism $f : \mathcal{H} \to \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)/\!/G$. On points, if $X$ is contained in a conic then $f([X]) = \pi_{UVW}(\sigma)$ and otherwise $f([X]) = \pi_{UVW}(\varphi)$ for some $\varphi$ where $X_\varphi = X$.

And, following the construction in Remark 2.9, the family of associated cubic surfaces over $\mathcal{H}$ induces a morphism $g : \mathcal{H} \to |\mathcal{O}_{\mathbb{P}U}(3)|/\!/\,\mathrm{SL}(U)$. For each $[X] \in \mathcal{H}$, the point $g([X])$ is the image of the associated cubic surface $[S_X]$ under the quotient map $|\mathcal{O}_{\mathbb{P}U}(3)| \dashrightarrow |\mathcal{O}_{\mathbb{P}U}(3)|/\!/\,\mathrm{SL}(U)$. If $X$ is contained in a smooth conic then $S_X$ has an $A_1$ singularity, and otherwise we have $\det_{UVW} \circ f([X]) = g([X])$.

Both the morphisms $f$ and $g$ are surjective, soand the graph of $\det_{UVW}$ is contained in the closure of the image of $(f, g)$., so the product morphism

$$(f, g) : \mathcal{H} \longrightarrow \mathcal{M}$$

is surjective.

**Proposition 4.29.** *The product morphism $(f, g) : \mathcal{H} \longrightarrow \mathcal{M}$ is surjective.*

*The fibre of $\rho : \mathcal{M} \to \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)/\!/G$ above $\pi_{UVW}(\sigma)$ is the product $\{\pi_{UVW}(\sigma)\} \times \Delta$, where $\Delta \subset |\mathcal{O}_{\mathbb{P}U}(3)|/\!/\,\mathrm{SL}(U)$ is the GIT quotient of the discriminant divisor.*

*Proof.* We have $f^{-1}(\{\pi_{UVW}(\sigma)\}) = \mathcal{H} \cap C_6$, so the fibre $\rho^{-1}(\{\pi_{UVW}(\sigma)\})$ is the closure of the image of $\mathcal{H} \cap C_6$ under $(f, g)$. It is known (see Section 1 of [Sak10]) that every cubic surface with an $A_1$ singularity is associated to a length 6 canonical subscheme that is the intersection of a smooth conic with a (possibly degenerate) plane cubic, and conversely that every canonical subscheme contained in a smooth conic is associated to a cubic surface with an $A_1$ singularity. In particular, every cubic surface with only $A_1$ singularities or both $A_1$ and $A_2$ singularities is associated to an element of $\mathcal{H} \cap C_6$.

So the image of $\mathcal{H} \cap C_6$ under the map $g$ is contained in the GIT moduli space of singular cubic surfaces $\Delta \subset |\mathcal{O}_{\mathbb{P}U}(3)|/\!/\,\mathrm{SL}(U)$. On points, we have an equality $g(\mathcal{H} \cap C_6) = \Delta$, and $\Delta \cong \mathbb{P}(1, 2, 3, 5)$ is reduced giving an equality $g(\mathcal{H} \cap C_6) = \Delta$. Because $\Delta$ is closed, then the fibre of $\rho$ above $\pi_{UVW}(\sigma)$ is $\{\pi_{UVW}(\sigma)\} \times \Delta$. $\square$

**Remark 4.30.** Let $\mathcal{H}^s \subset \mathcal{H}$ be the open subscheme of subschemes whose associated cubic surface is SL(4)-stable. The morphism $(f, g) : \mathcal{H} \to \mathcal{M}$ is SL($V$)-equivariant, and $(f, g)(\mathcal{H}^s) \subset \mathcal{M}$ is an SL($V$)-orbit space for $\mathcal{H}^s$. Conversely, the fibre $(f, g)^{-1}(\{m\})$ for each $m \in \mathcal{M}$ is an SL($V$)-orbit if and only if $m \in (f, g)(\mathcal{H}^s)$.

This suggests that $\mathcal{M}$ is a GIT quotient of $\mathrm{Hilb}_6(\mathbb{P}V)$, or at least birational in codimension 2 to one, prompting the following question:

**Question 4.31.** *Is the map $(f, g) : \mathrm{Hilb}_6(\mathbb{P}V) \dashrightarrow \mathcal{M}$ a GIT quotient? If so, with respect to which line bundle? And is the semistable locus $\mathcal{H}$, is the stable locus $\mathcal{H}^s$?*

GIT quotients of Hilbert schemes of points have been studied by Durgin in her PhD thesis [Dur15] (the asymptotic semistability of $n = 6$) and Gallardo and Schmidt in [GS24] ($n = 5, 7$). The Picard group of the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{P}V)$ is free of rank 2, generated by the line bundles associated to the divisors $H$ and $B/2$, where $H$ parametrises subschemes that intersect a fixed line $L \subset \mathbb{P}V$ and $B$ parametrises non-reduced subschemes (the exceptional divisor of the Hilbert-Chow morphism). The ample cone is spanned by line bundles $\mathcal{L}_m$ associated to the divisors $mH - B/2$ for $m > n - 1$.

Durgin's thesis gives an explicit description for $\mathrm{SL}(V)$-semistability with respect to $\mathcal{L}_m$, for sufficiently large $m \gg 0$, but $\mathcal{M}$ is not this GIT quotient. Any length 6 subscheme containing a length 3 point is $\mathrm{SL}(V)$-unstable for $m \gg 0$. But the morphism $\mathcal{H} \to \mathcal{M}$ is defined for a general subscheme with a length 3 point, contracting this family to the strict transform of $\pi_{UVW}(\overline{2A_2}^{G-ss})$ under $\rho$.

## Appendix A. Block forms

All computations were carried out in SageMath version 10.3 [Sage].

***G*-instability:** Our computational implementation of the convex Hilbert-Mumford criterion for $G$-semistability of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ outputs the following 12 maximal block types (and their transposes) for $G$-instability. A tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is $G$-unstable if and only if it is of one of the following block forms:

(1) Unstable by the weight $((1,1,1,-3),(0,0,0),(0,0,0))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \end{array}\right]$$

(2) Unstable by the weight $((9,-3,-3,-3),(0,0,0),(4,4,-8))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(3) Unstable by the weight $((9,-3,-3,-3),(8,-4,-4),(8,-4,-4))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & 0 & 0 & * & 0 & 0 & * & 0 & 0 \\ * & * & * & * & 0 & 0 & * & 0 & 0 & * & 0 & 0 \end{array}\right]$$

(4) Unstable by the weight $((9,9,-3,-15),(8,-4,-4),(8,-4,-4))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(5) Unstable by the weight $((9,9,-3,-15),(12,0,-12),(4,4,-8))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & 0 & * & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 & * & 0 & 0 \\ * & * & 0 & * & * & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

*Contained in type* (2):

(6) Unstable by the weight $((1,1,1,-3),(4,0,-4),(4,0,-4))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & 0 \\ * & * & 0 & * & * & 0 & * & * & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(7) Unstable by the weight $((12,0,-12),(16,4,-20),(9,9,-3,-15))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & 0 \\ * & * & * & * & * & * & * & * & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(8) Unstable by the weight $((3,3,-3,-3),(4,-2,-2),(6,0,-6))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(9) Unstable by the weight $((3,3,3,-9),(8,-4,-4),(4,4,-8))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

*Contained in type* (4):

(10) Unstable by the weight $((3,3,-3,-3),(0,0,0),(4,-2,-2))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(11) Unstable by the weight $((5,1,1,-7),(4,0,-4),(4,0,-4))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 & * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

*Contained in type* (5):

(12) Unstable by the weight $(3,3,-3,-3),(2,2,-4),(2,2,-4))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & 0 & * & * & 0 \\ * & * & * & * & * & * & * & * & 0 & * & * & 0 \\ * & * & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Types (6) to (9) are contained in type (2) because there is a change of basis that converts the bottom row to be of the form

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(10) $\subset$ (4): by changing basis on the columns so that the last matrix is of the form

$$\begin{bmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(11) $\subset$ (4): by taking a linear combination of the middle two matrices so that $\varphi$ is of the form

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 & * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(12) $\subset$ (5): any line in $\mathbb{P}((\mathbb{C}^2)^\vee \otimes (\mathbb{C}^2)^\vee)$ intersects the Segre variety of rank 1 pairings at least once, so has a rank 1 point. For each $\varphi$ of block type (12), we choose a change of basis so that

$$\varphi \in \left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & 0 & * & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and this is contained in block type (5).

The proofs for the transposes of types (7) to (10) are identical. Block type (2) is the condition of a plane of type $(3,1)$, block type $(2)^T$ is the condition for a plane of type $(1,3)$.

**Strict $G$-semistability:** Our computational implementation of the convex Hilbert-Mumford criterion for $G$-(semi)stability of $\mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ outputs the following 7 maximal block types (and their transposes) for strict $G$-semistability, in addition to the block forms for $G$-instability. A $G$-semistable tensor $\varphi \in \mathbb{P}(U^\vee \otimes V^\vee \otimes W^\vee)$ is strictly $G$-semistable if and only if it is of one of the following block forms:

(1) Not stable by the weight $((3,0,0,-3),(2,-1,-1),(1,1,-2))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(2) Not stable by the weight $((1,0,0,-1),(1,0,-1),(1,0,-1))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & 0 \\ * & * & * & * & * & 0 & * & * & 0 & * & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

*Contained in type* (1)*:*

(3) Not stable by the weight $((1,1,0,-2),(1,0,-1),(1,0,-1))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(4) Not stable by the weight $((2,0,-1,-1),(1,0,-1),(1,0,-1))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & 0 & * & * & 0 \\ * & * & * & * & * & 0 & * & 0 & 0 & * & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

*Contained in unstable type* (2)*:*

(5) Not stable by the weight $((0,0,0,0),(2,-1,-1),(1,1,-2))$:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * \\ * & 0 & 0 & * & 0 & 0 & * & 0 & 0 & * & 0 & 0 \end{array}\right]$$

(3) $\subset$ (1): by taking a linear combination of the first two matrices so that the bottom row is of the form

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & 0 & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(4) $\subset$ (1): by taking a linear combination of the last two matrices so that the last matrix is of the form

$$\begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Not stable type (5) is contained in $G$-unstable type (2) by taking a change of basis that converts the bottom row to be of the form

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

## References


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