Digital Systems Design and Laboratory
[5. Karnaugh Maps]

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Outline

- ❑ **Minimum Forms of Switching Functions**
- ❑ Two- and Three-Variable Karnaugh Maps
- ❑ Four-Variable Karnaugh Maps
- ❑ Determination of Minimum Expressions Using Essential Prime Implicants
- ❑ Five-Variable Karnaugh Maps
- ❑ Other Forms of Karnaugh Maps

Recap: Logic Design

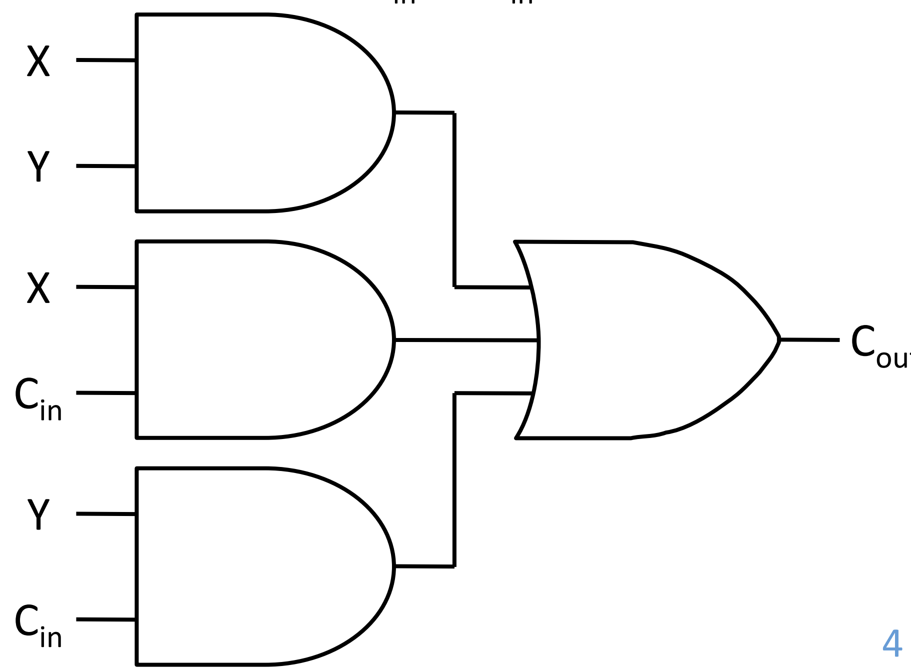
- Design a combinational logic circuit starting with a word description of the desired circuit behavior
- Steps
  - Translate the word description into a switching function (Unit 4)
    - Truth table
    - Boolean expression
      - SOP/POS derived from minterm or maxterm expansion (Unit 4)
  - Simplify the function
    - Boolean algebra (Units 2 and 3)
    - Karnaugh map (Unit 5)
    - Quine-McCluskey (Unit 6)
    - Other methods
  - Realize it using available logic gates

Difficulties in Algebraic Simplification

Problems

- Difficult to apply in a systematic way
- Difficult to tell when you have arrived at a **minimum** solution
  - Minimum SOP/POS
    - Minimum # of terms (i.e., # of gates)
    - Minimum # of literals (i.e., # of gate inputs)

$$C_{out} = X'YC_{in} + XY'C_{in} + XYC_{in}' + XYC_{in}$$
$$= XY + XC_{in} + YC_{in}$$



Solutions: systematic methods

- Karnaugh map (K-map) (Unit 5)
  - Especially useful for 3 or 4 variables
- Quine-McCluskey (Unit 6)
- Other methods

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Two-Variable Karnaugh Maps (1/2)

- Truth table = minterm expansion = Karnaugh map
  - Each square of the K-map corresponds to a combination of values of inputs
  - Each square = a minterm = a row in truth table

B \ A	0	1
0	a_0	a_2
1	a_1	a_3

A=0, B=0

A=1, B=0

A=0, B=1

A=1, B=1

B \ A	0	1
0	m_0 = A'B'	m_2 = AB'
1	m_1 = A'B	m_3 = AB

m_i	A B	F
0	0 0	a_0
1	0 1	a_1
2	1 0	a_2
3	1 1	a_3

Two-Variable Karnaugh Maps (2/2)

❑ Example

m_i	A B	F
0	0 0	1
1	0 1	1
2	1 0	0
3	1 1	0

B \ A	0	1
0	1	0
1	1	0

B \ A	0	1
0	1	0
1	1	0

F = A'

Three-Variable Karnaugh Maps (1/2)

- Minterms in adjacent squares of K-map differ in only ONE bit
  - Combine them: XY'+XY = X(Y'+Y) = X

BC \ A	0	1
00	A'B'C'	AB'C'
01	A'B'C	AB'C
11	A'BC	ABC
10	A'BC'	ABC'

→

BC \ A	0	1
00	m_0	m_4
01	m_1	m_5
11	m_3	m_7
10	m_2	m_6

Three-Variable Karnaugh Maps (2/2)

- Example

m_i	A B C	F
0	0 0 0	0
1	0 0 1	1
2	0 1 0	0
3	0 1 1	1
4	1 0 0	0
5	1 0 1	1
6	1 1 0	0
7	1 1 1	0

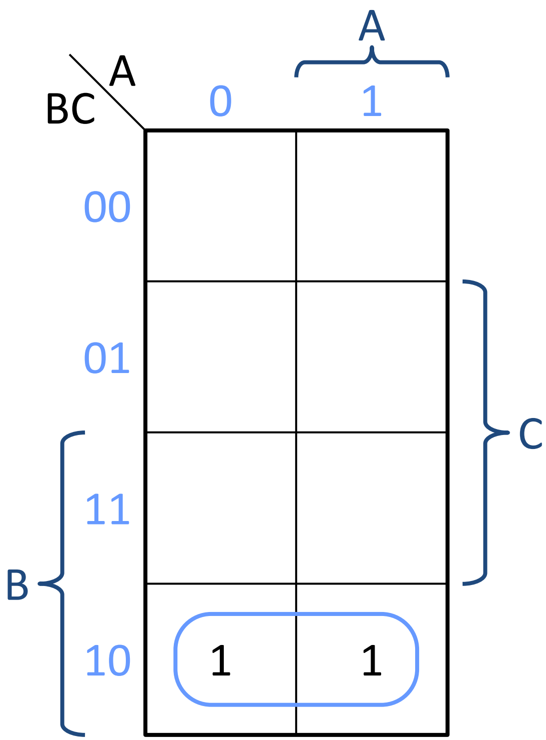
F = A'B'C + A'BC + AB'C

BC \ A	0	1
00	0	0
01	1	1
11	1	0
10	0	0

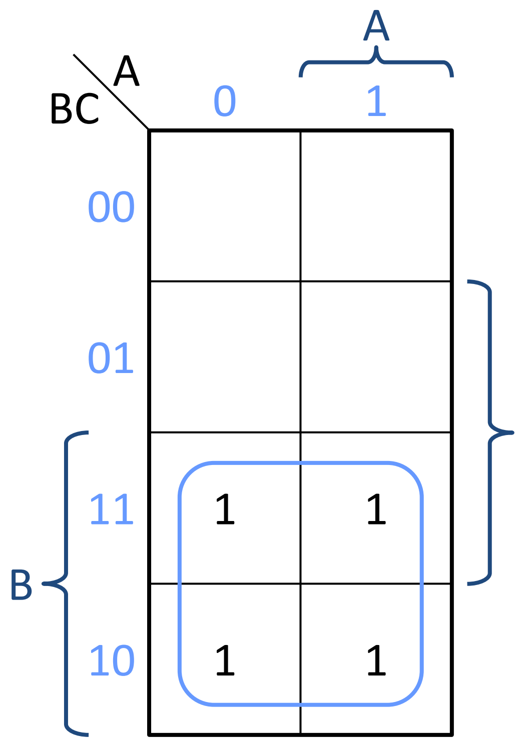
F = A'C+ B' C

Product Terms in Karnaugh Maps

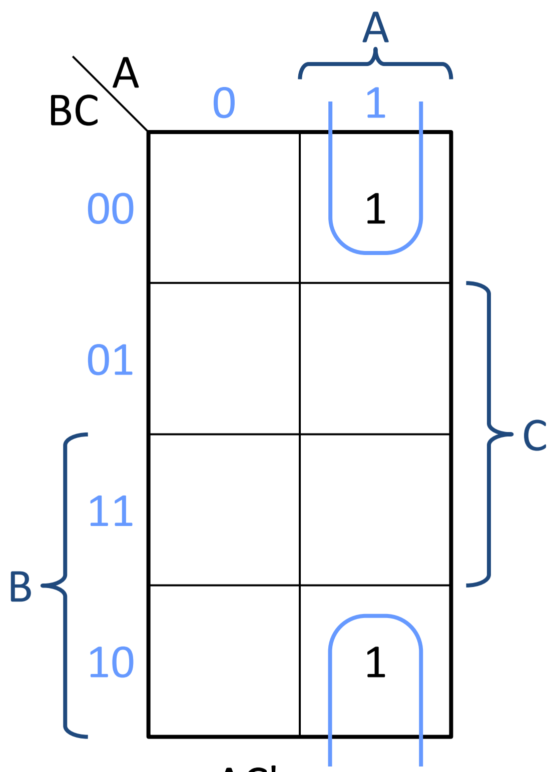
- Examples



BC'



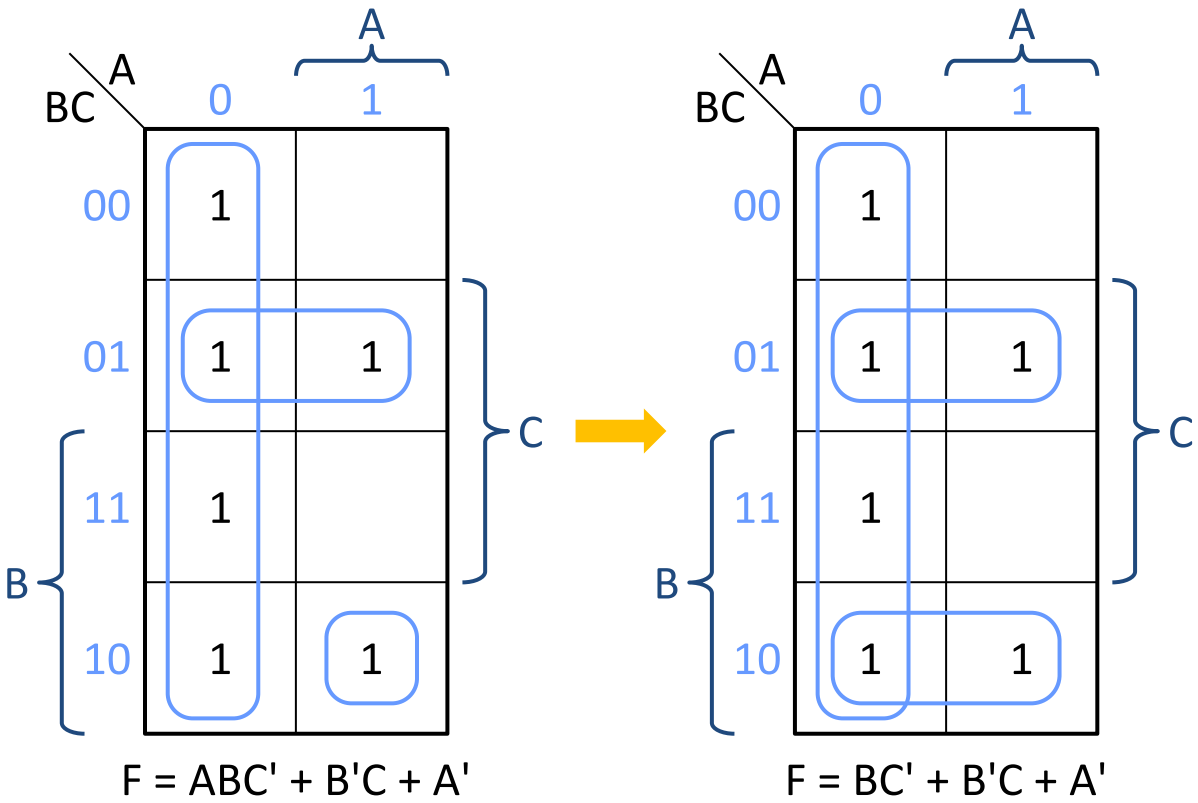
B



AC'

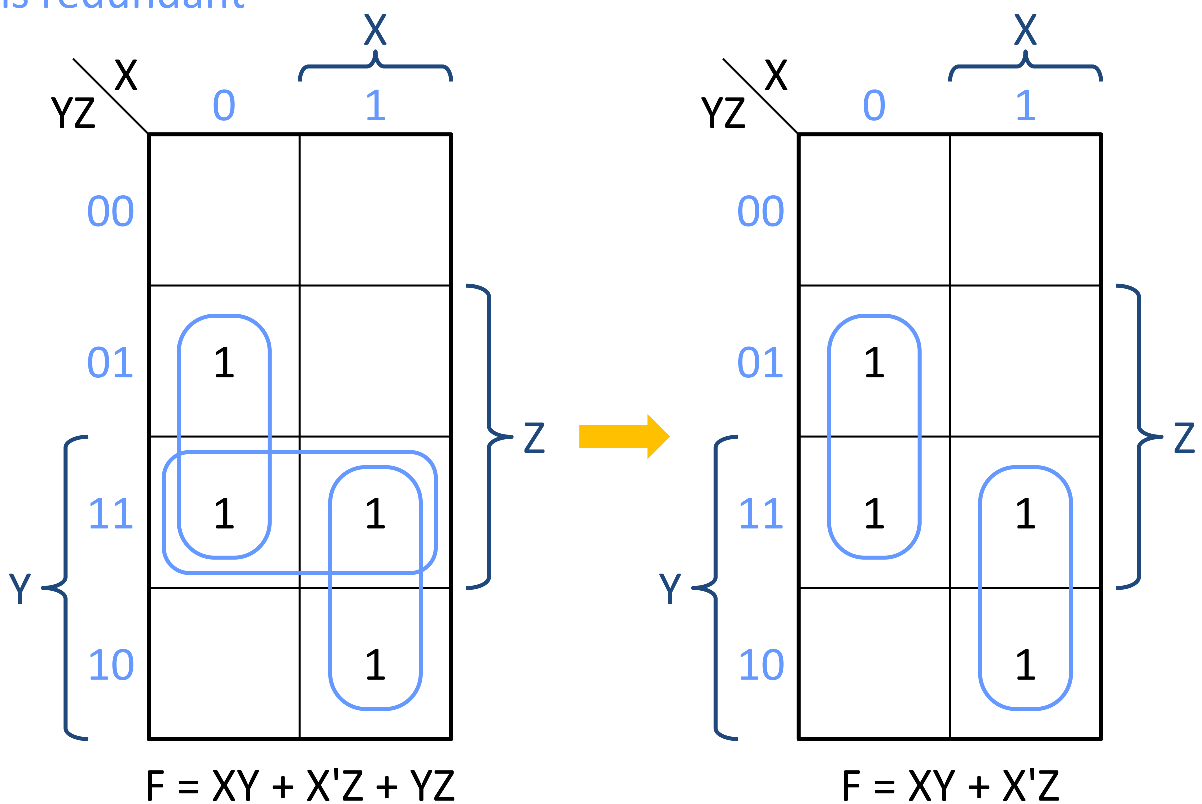
Another Example

- F = ABC' + B'C + A'
  - Mark 1's
  - Make circles (simplify)



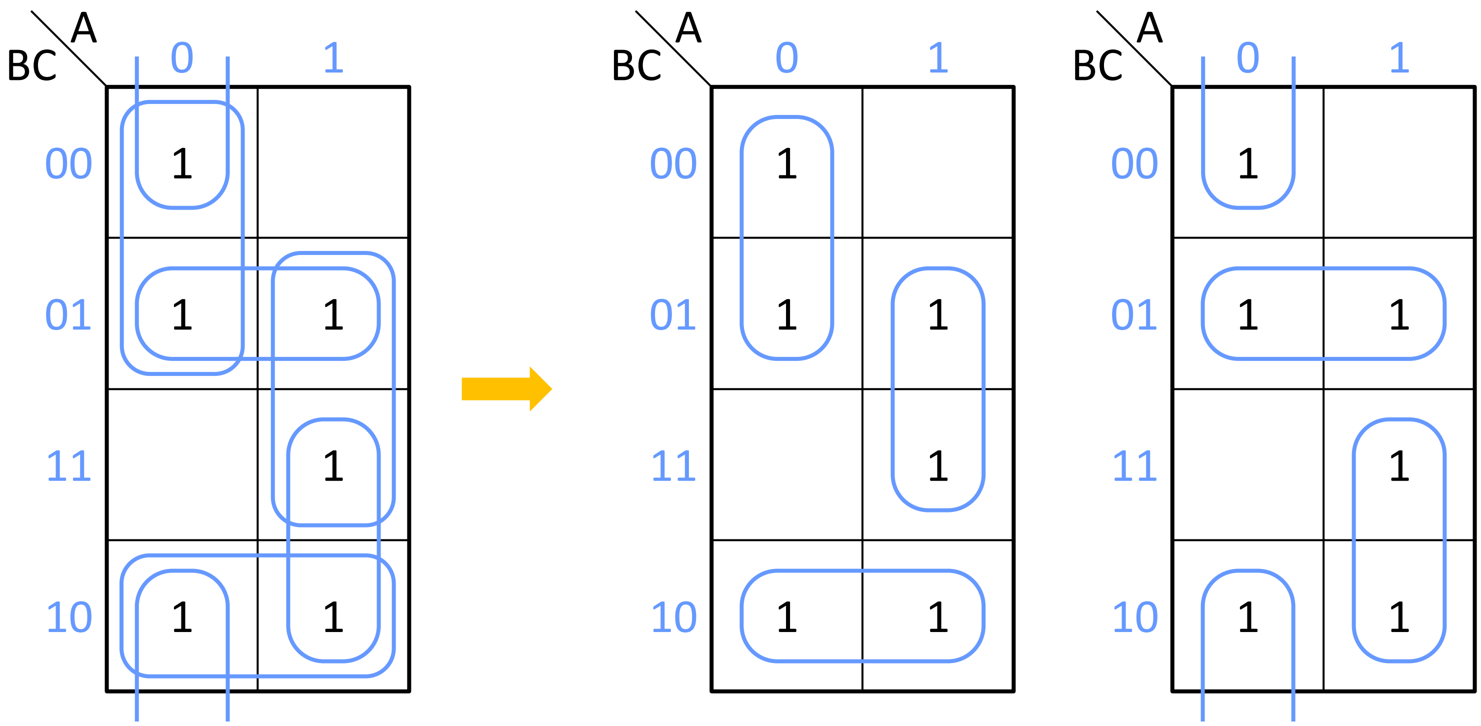
Consensus Theorem in Karnaugh Maps

- Overlapped circles imply redundant terms
- Consensus theorem
  - XY + X'Z + YZ = XY + X'Z
    - YZ is redundant



All Solutions in Karnaugh Maps

- All possible minimum SOPs can be determined from K-map
  - # of terms and # of literals
- Example: $F = \sum m(0, 1, 2, 5, 6, 7)$
  - Make each circle as large as possible
  - Select as few circles as possible to cover all minterms



Summary

- Truth table = minterm expansion = Karnaugh map
- Simplification in Karnaugh maps
  - Minimum SOP = (min # of terms, min # of literals)
  - Steps (make adjacent squares different in only one bit)
    - Mark 1's
    - Make circles
      - Make each circle as large as possible (# of literals)
      - Select as few circles as possible to cover all 1's (# of terms)
- Algebraic simplification also holds in Karnaugh maps
  - Combining terms: XY + XY' = X
  - Eliminating terms: X + XY = X; XY + X'Z + YZ = XY + X'Z
  - Eliminating literals: X + X'Y = X + Y
  - Adding redundant terms:
    Y = Y+ XX'; Y = Y(X + X'); XY + X'Z = XY + X'Z + YZ; X = X + XY

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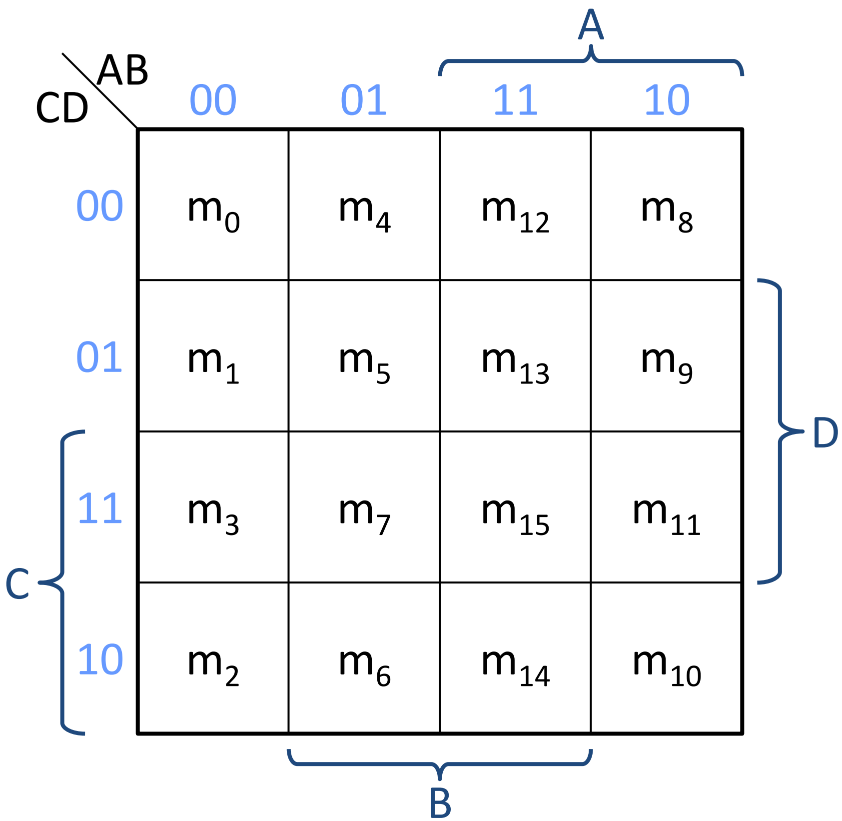
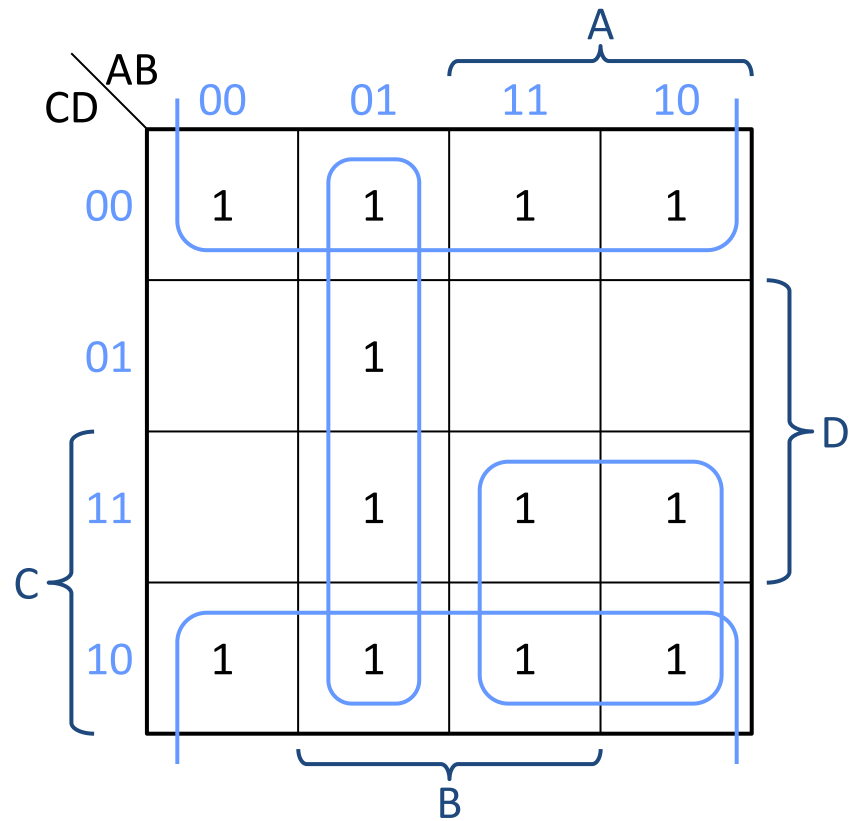
Four-Variable Karnaugh Maps

- Minterms in adjacent squares of K-map differ in only ONE bit

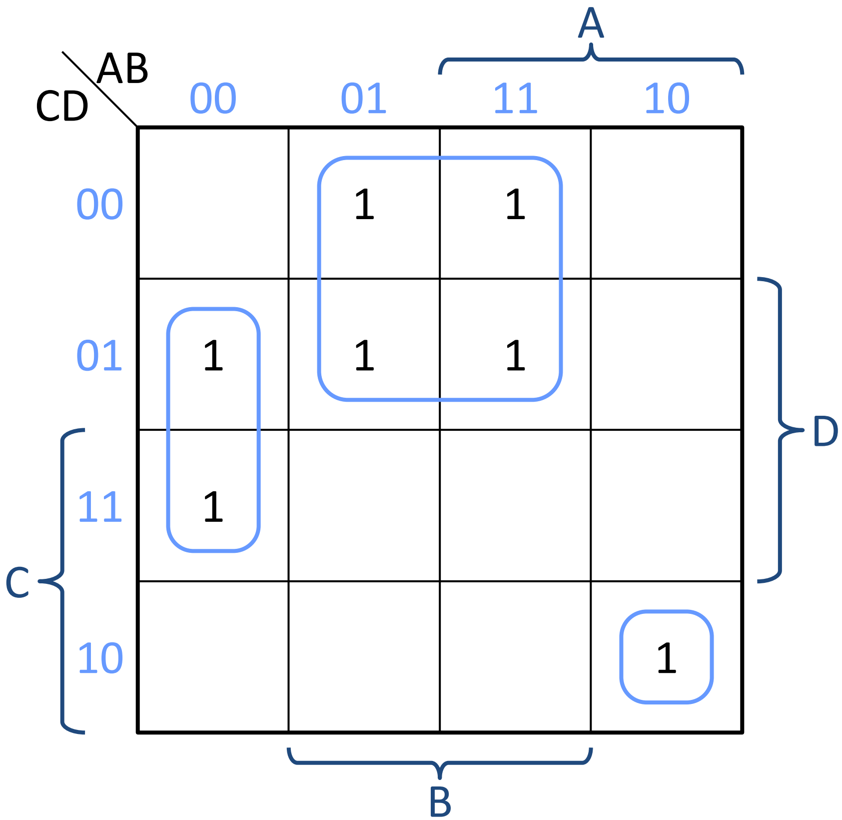




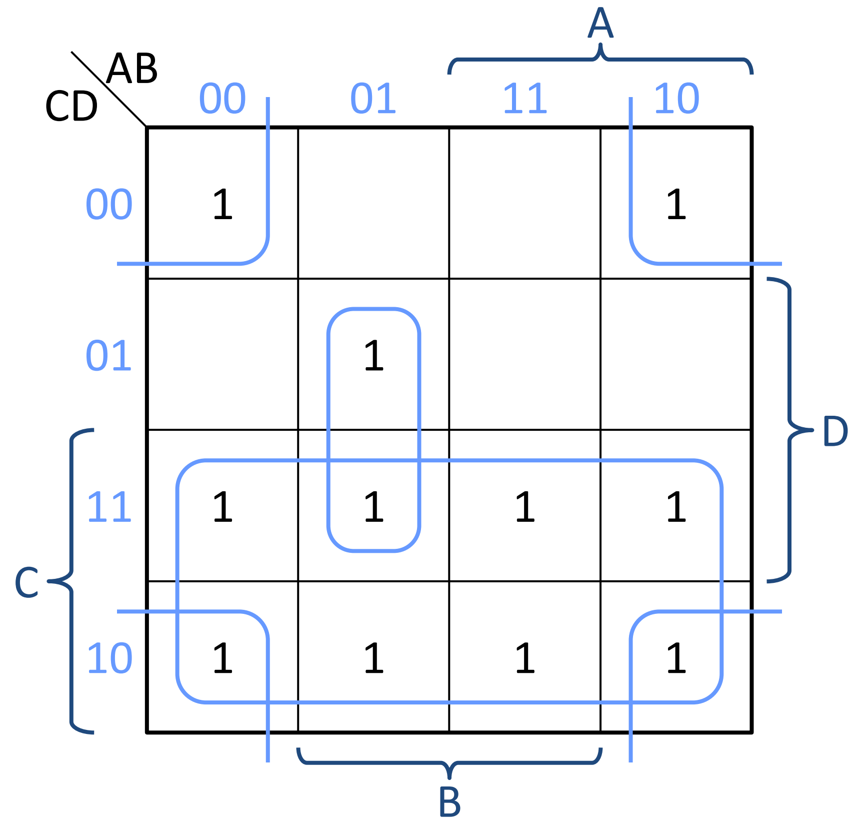
F = AC + A'B + D'

Two More Examples

- $F_1 = \sum m(1, 3, 4, 5, 10, 12, 13)$
- $F_2 = \sum m(0, 2, 3, 5, 6, 7, 8, 10, 11, 14 ,15)$



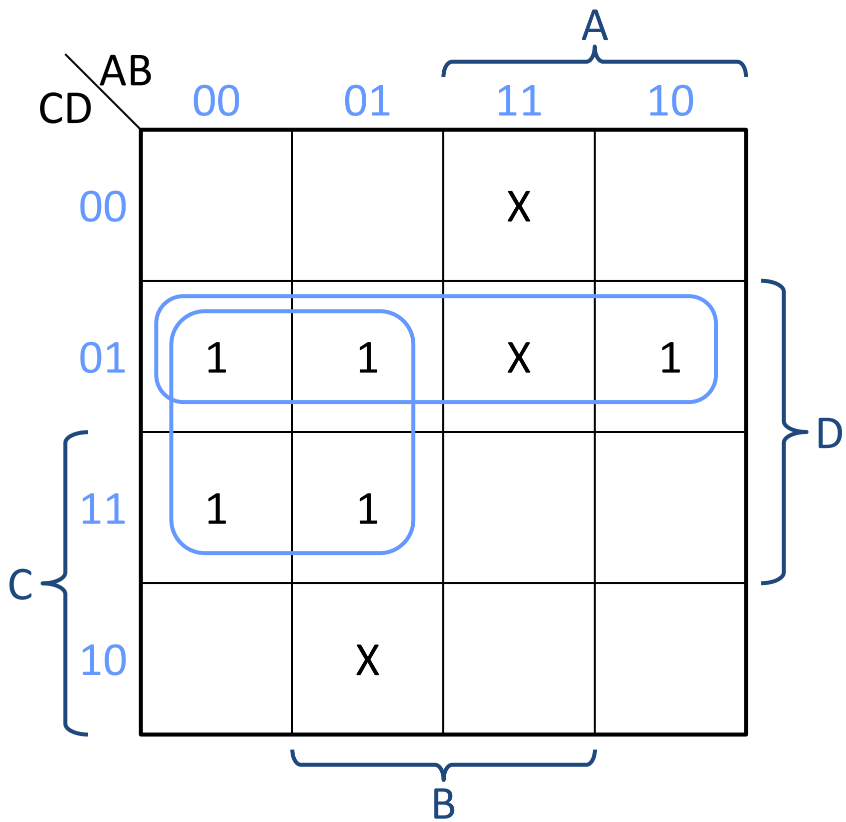
$F_1 = BC' + A'B'D + AB'CD'$



$F_2 = C + B'D' + A'BD$

Karnaugh Maps with Don't Cares

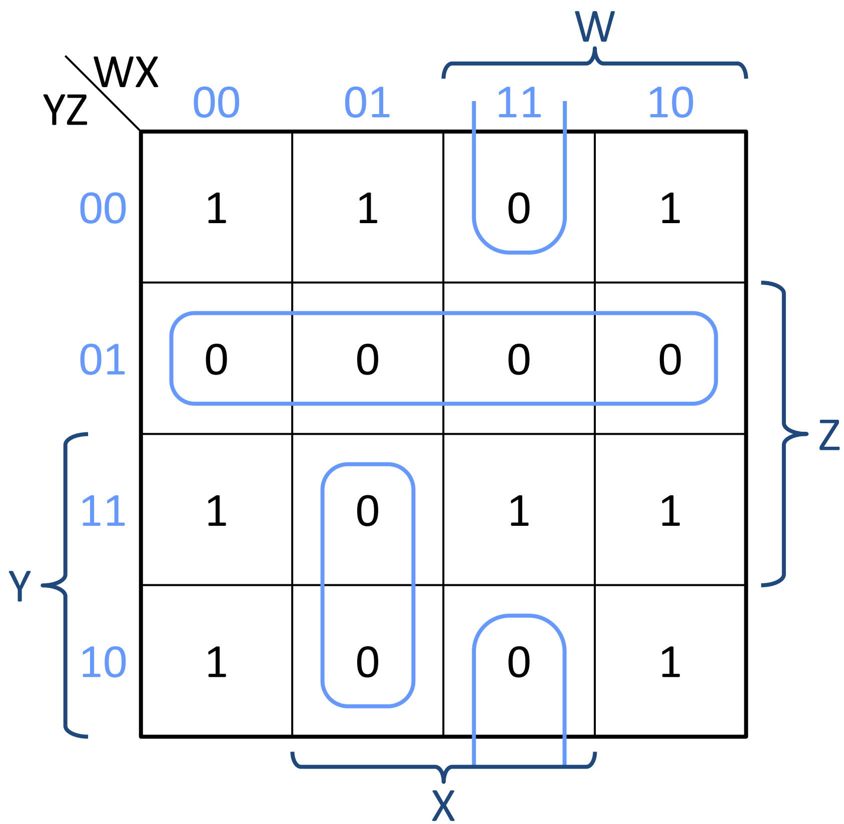
- Don't cares can be assigned with 0's or 1's
  - After assignment, the function becomes completely specified
- $F = \sum m(1, 3, 5, 7, 9) + \sum d(6, 12, 13)$



$F = A'D + C'D$

Minimum POS

- Minimum SOP = circle 1's of F
- Minimum POS = circle 0's of F
  - Find minimum SOP of F' and then complement it
  - Example: F = X'Z' + WYZ + W'Y'Z' + X'Y



F' = Y'Z + W'XY + WXZ'

By DeMorgan's law:

F = (Y + Z')(W + X' + Y')(W' + X' + Z)

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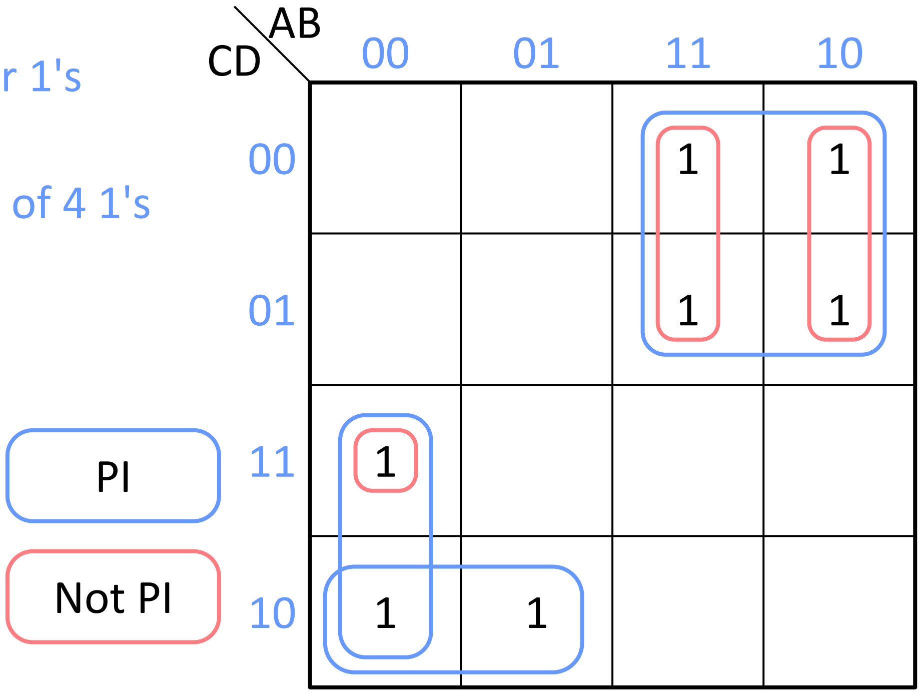
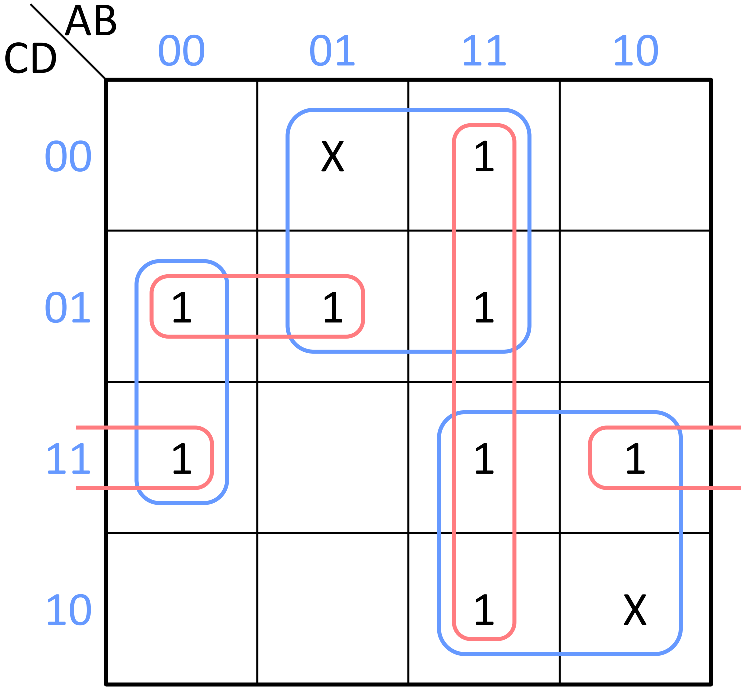
Prime Implicants (1/2)

- **Implicant**: a product term
  - Any single 1 or any group of 1's in the K-map
- **Prime implicant** (PI): an implicant that cannot be covered by other implicants
  - A circle that cannot be enlarged any more
    - A single 1 is a PI
      if not adjacent to any other 1's
    - Two adjacent 1's is a PI
      if not contained in a group of 4 1's



Prime Implicants (2/2)

- **Cover**: a set of prime implicants which covers all 1's
- A minimum SOP contains only prime implicants (why?)
  - Minimum cover = (min # of PIs, min # of literals)
- Don't cares are treated just like 1's here

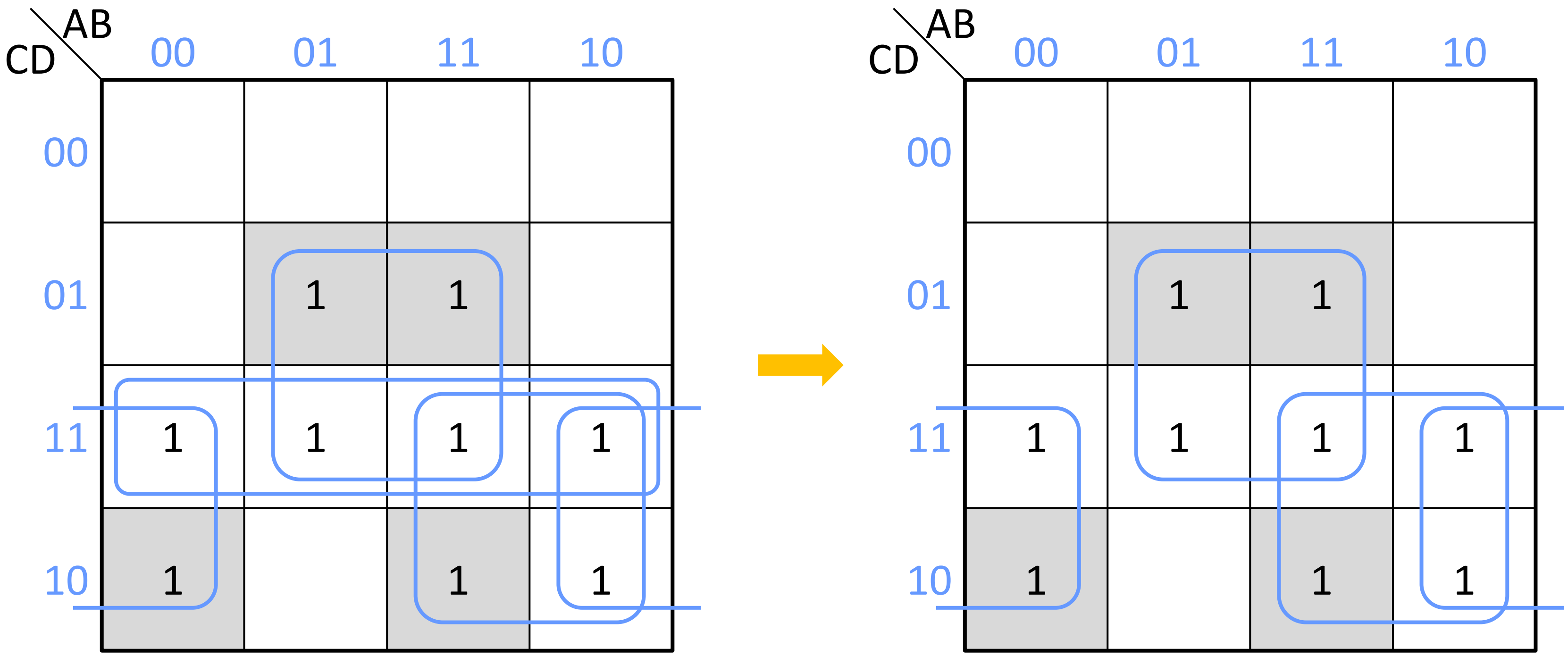


F = A'B'D + BC' + AC

F = A'C'D + AB + B'CD

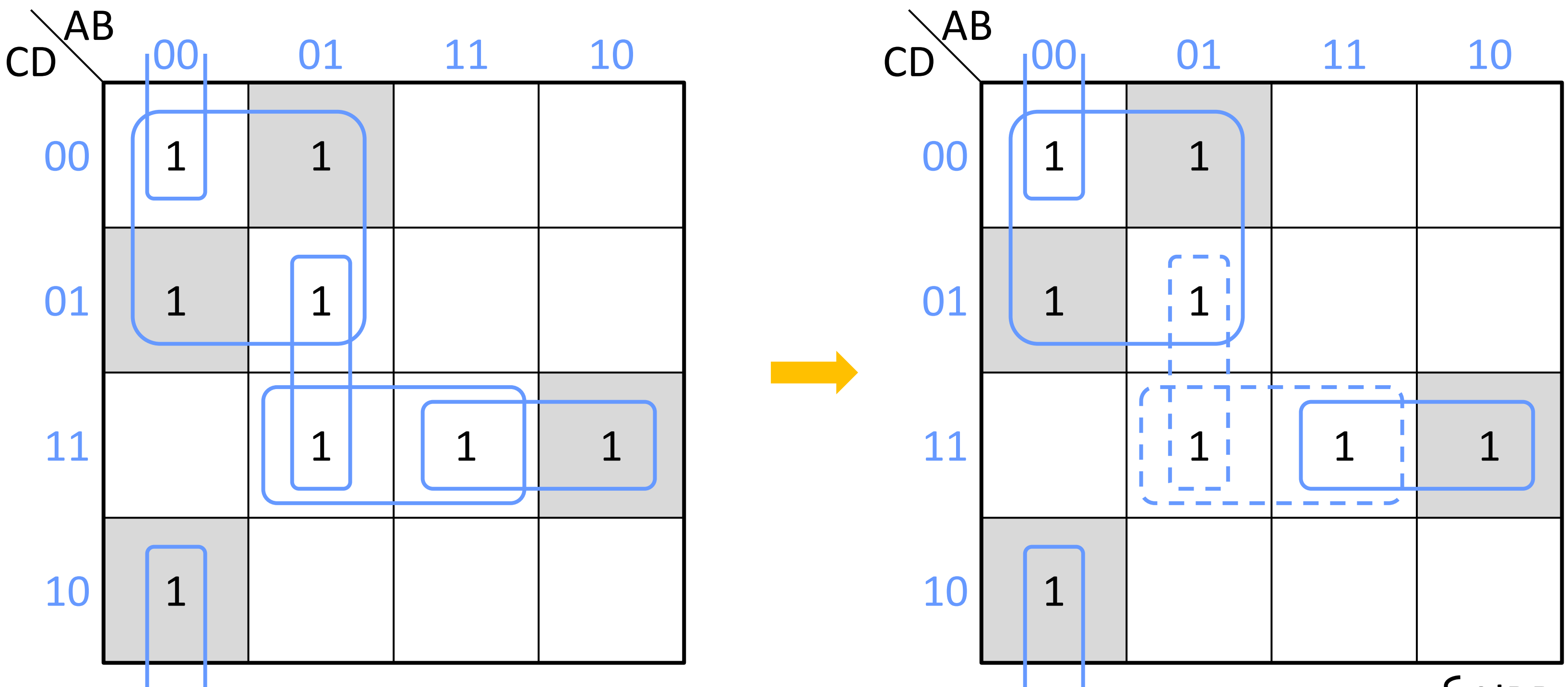
Essential Prime Implicants

- **Essential prime implicant**: if a minterm is covered by only one PI, the PI is essential
  - Essential PI must be included in minimum SOP
  - Find essential PI's = find the 1's circled only once
- Example: F = CD + BD + B'C + AC = BD + B'C + AC



Another Example

- Find minimum cover
  - Find all PI's
  - Find essential PI's
  - Find a minimum set of PI's to cover the remaining 1's



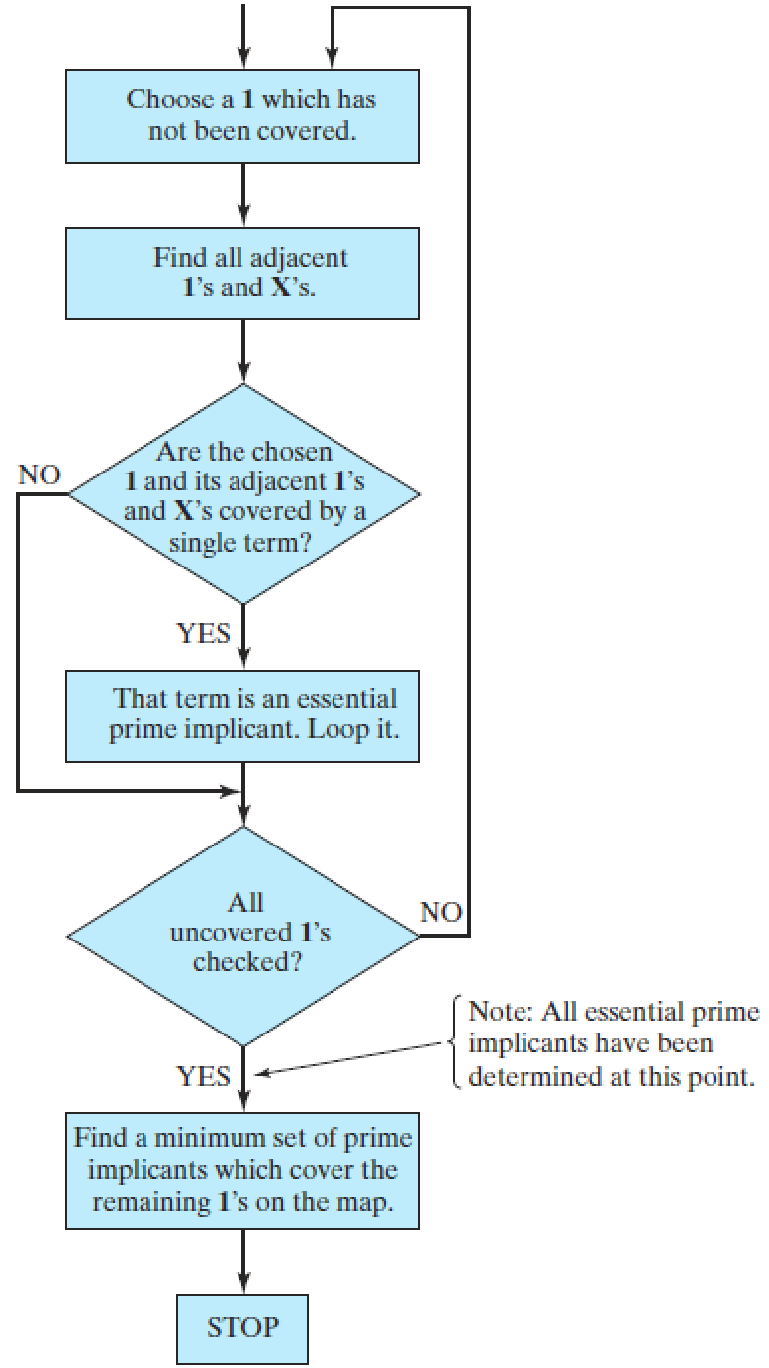
$$F = A'C' + A'B'D' + ACD + \begin{cases} A'BD \\ BCD \end{cases}$$

Summary

- Minimum SOP = minimum cover = a minimum set of PI's which cover all 1's
  - Minimum cover = (min # of PIs, min # of literals)
- Steps
  - Find all PI's
  - Find essential PI's
  - Find a minimum set of PI's to cover the remaining 1's
- Recap: steps of simplification in Karnaugh maps
  - Mark 1's
  - Make circles
    - Make each circle as large as possible **= find PI**
    - Select as few circles as possible to cover all 1's **= find minimum cover**

Flowchart

FIGURE 5-19 Flowchart for Determining a Minimum Sum of Products Using a Karnaugh Map



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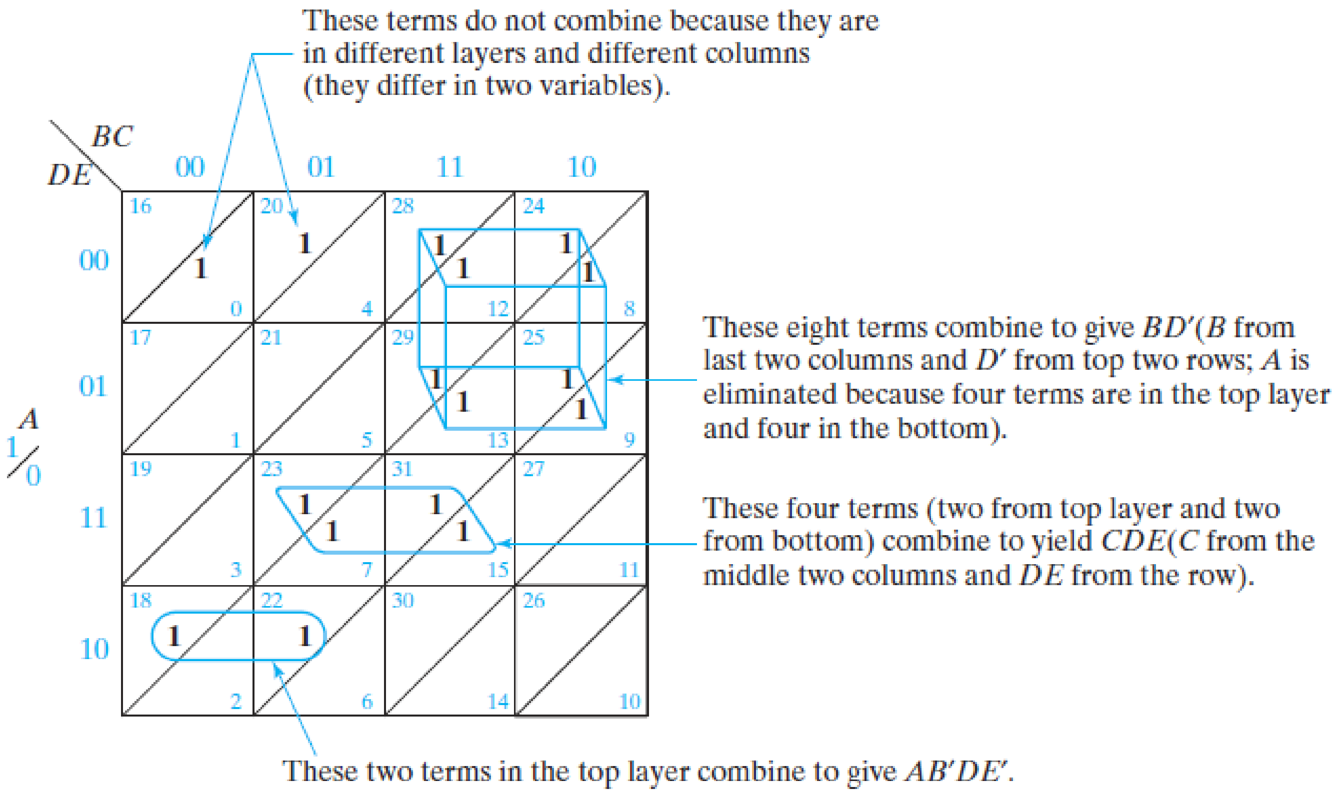
Five-Variable Karnaugh Maps (1/2)

- Example
  - F = BD' + CDE + AB'DE' + AB'CD'E' + A'B'C'D'E'

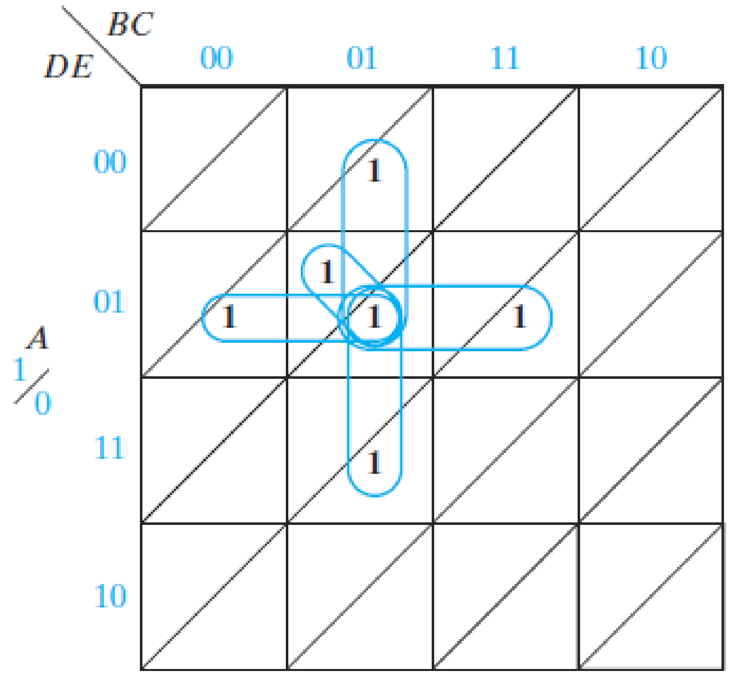
FIGURE 5-21
A Five-Variable Karnaugh Map



Five-Variable Karnaugh Maps (2/2)

- Example
  - F = A'B' CD' + A'B'CE + A'B'D'E + A'CD'E + B'CD'E

FIGURE 5-22



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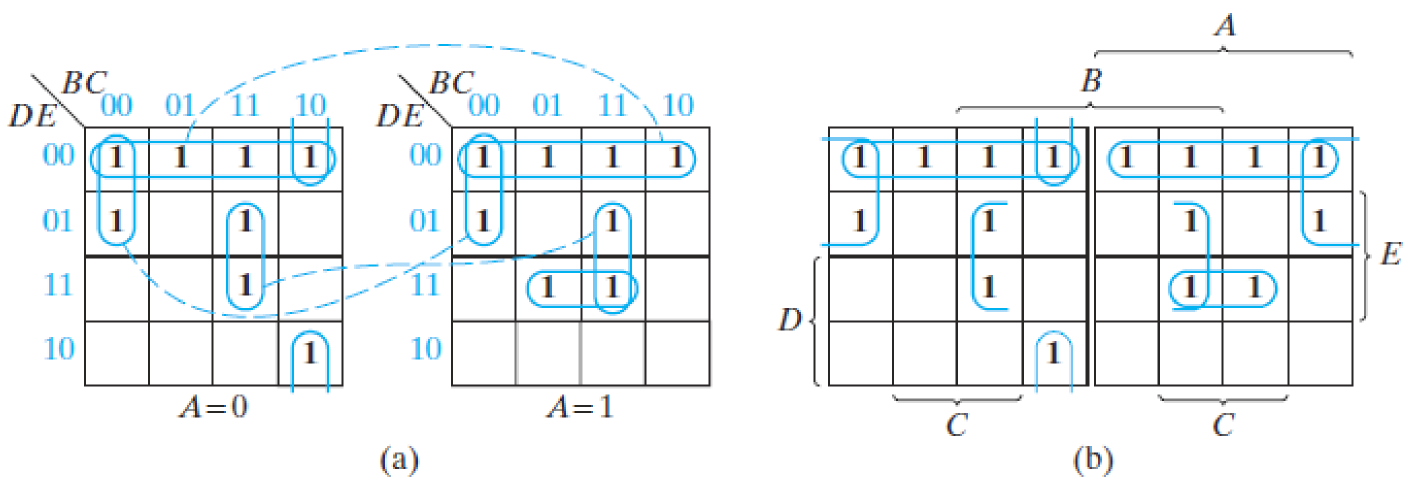
Other Forms of Karnaugh Maps

- ❑ Side-by-side maps
- ❑ Mirror image maps

FIGURE 5-28
Other Forms of Five-Variable Karnaugh Maps



Q&A