Digital Systems Design and Laboratory
[3. Boolean Algebra (Continued)]

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Outline

- **Multiplying Out and Factoring Expressions**
- Exclusive OR and Equivalence Operations
- Consensus Theorem
- Algebraic Simplification of Switching Expressions
- Proving Validity of an Equation

Multiplying Out and Factoring

- Distributive laws
  - $X(Y + Z) = XY + XZ$
  - $(X + Y)(X + Z) = X + YZ$
- Another useful theorem
  - $(X + Y)(X' + Z) = XZ + X'Y$
    - Proof? Check two cases: $X = 0$ or $X = 1$
- Multiplying out
  - Apply them from left terms to right terms
- Factoring
  - Apply them from right terms to left terms

Multiplying Out: from POS to SOP

- From product-of-sums (POS) to sum-of-products (SOP)
- Example

$$\underbrace{(\underline{A + B} + C')(\underline{A + B} + D)}_{2.}(A + B + E)\underbrace{(\underline{A} + D' + E)(\underline{A'} + C)}_{3.}$$

$$= \underbrace{(\underline{A + B} + C'D)(\underline{A + B} + E)}_{2.}[AC + A'(D' + E)]$$

$$= (A + B + C'DE)(AC + A'D' + A'E)$$

$$= AC + ABC + A'BD' + A'BE + A'C'DE$$

$$= AC + A'BD' + A'BE + A'C'DE$$

1. $X(Y + Z) = XY + XZ$
2. $(X + Y)(X + Z) = X + YZ$
3. $(X + Y)(X' + Z) = XZ + X'Y$

Factoring

- From sum-of-products (SOP) to product-of-sums (POS)
- Example

$$
\begin{aligned}
& AC + A'BD' + A'BE + A'C'DE \\
&= \underbrace{\underline{A}C + \underline{A'}(BD' + BE + C'DE)}_{3.} \\
&= (A + BD' + BE + C'DE)(A' + C) \\
&= \underbrace{[\underline{A + C'DE} + B(D' + E)]}_{2.}(A' + C) \\
&= (A + C'DE + B)(A + C'DE + D' + E)(A' + C) \\
&= \underbrace{(\underline{A + B} + C'DE)}_{2.}(A + D' + E)(A' + C) \\
&= (A + B + C')(A + B + D)(A + B + E)(A + D' + E)(A' + C)
\end{aligned}
$$

1. $X(Y + Z) = XY + XZ$
2. $(X + Y)(X + Z) = X + YZ$
3. $(X + Y)(X' + Z) = XZ + X'Y$

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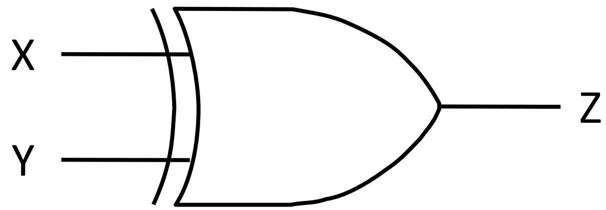
Exclusive-OR (1/2)

- Exclusive-OR (XOR) ($\oplus$)
  - $0 \oplus 0 = 0$, $0 \oplus 1 = 1$, $1 \oplus 0 = 1$, $1 \oplus 1 = 0$
  - Symbol (XOR gate)



  - Truth table

X	Y	Z = X $\oplus$ Y
0	0	0
0	1	1
1	0	1
1	1	0

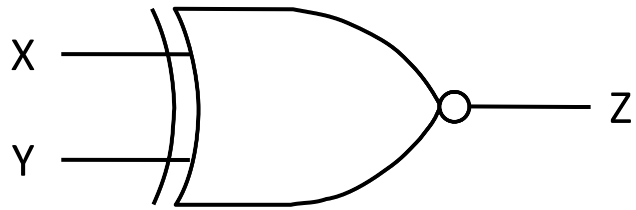
Exclusive-OR (2/2)

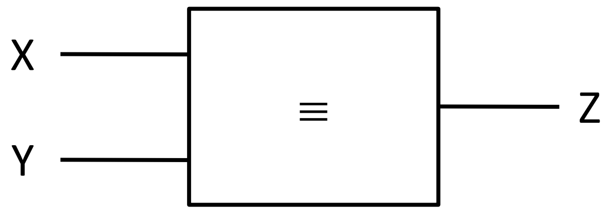
- $X \oplus Y = X'Y + XY'$
- Theorems
  - $X \oplus 0 = X$
  - $X \oplus 1 = X'$
  - $X \oplus X = 0$
  - $X \oplus X' = 1$
  - $X \oplus Y = Y \oplus X$ (commutative)
  - $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$ (associative)
  - $X(Y \oplus Z) = XY \oplus XZ$ (distributive)
  - $(X \oplus Y)' = X'Y' \oplus XY$
    - X and Y must be the same

Exclusive-NOR (1/2)

❑ Exclusive-NOR (XNOR) (≡)

- $0 \equiv 0 = 1$, $0 \equiv 1 = 0$, $1 \equiv 0 = 0$, $1 \equiv 1 = 1$
- Symbol (XNOR gate)





- Truth table

X	Y	$Z = X \equiv Y$
0	0	1
0	1	0
1	0	0
1	1	1

Simplification of XOR and XNOR

- $X \oplus Y = X'Y + XY'$
- $X \equiv Y = X'Y' + XY$
- Examples

$F = (A'B \equiv C) + (B \oplus AC')$
$= [A'BC + (A'B)'C'] + [B'AC' + B(AC')']$
$= A'BC + (A + B')C' + B'AC' + B(A' + C)$
$= B(A'C + A' + C) + C'(A + B' + AB')$
$= B(A' + C) + C'(A + B')$
$=$ … can be further simplified
$= B + C'$

$F = A' \oplus B \oplus C$
$= (A'B' + AB) \oplus C$
$= (A'B' + AB)C' + (A'B' + AB)'C$
$= A'B'C' + ABC' + A'BC + AB'C$
$=$ … can be further simplified?

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Consensus Theorem

- XY + X'Z + YZ = XY + X'Z
  - YZ is redundant, referred to as the **consensus term**
- Example

  A'C'D + A'BD + BCD + ABC + ACD'

  = A'C'D + **A'**BD + BCD + **A**BC + ACD'

  = A'C'D + A'BD + ABC + ACD'
- Example

  A'B' + AC + BC' + B'C + AB

  = **A'**B' + **A**C + BC' + B'C + AB

  = A'B' + AC + BC' + AB

  = A'B' + A**C** + B**C'** + AB

  = A'B' + AC + BC'

Ordering Does Matter

- Example (one ordering)

  A'C'D + A'BD + BCD + ABC + ACD'

  = A'C'D + **A'**BD + BCD + **A**BC + ACD'

  = A'C'D + A'BD + ABC + ACD'

- Example (another ordering)

  A'**C'**D + A'BD + B**C**D + ABC + ACD'

  = A'C'D + BCD + ABC + ACD'

  = A'C'D + BC**D** + ABC + AC**D'**

  = A'C'D + BCD + ACD'

Dual Form of Consensus Theorem

- $XY + X'Z + YZ = XY + X'Z$
  - YZ is redundant
- $(X + Y)(X' + Z)(Y + Z) = (X + Y)(X' + Z)$
  - $(Y + Z)$ is redundant
- Example

$(A + B + C')(A + B + D')(B + C + D')$

$= (A + B + \underline{C'})(A + B + D')(B + \underline{C} + D')$

$= (A + B + C')(B + C + D')$

= ... can be further simplified

$= B + AC + C'D'$

Redundancy Insertion

- Example
  - ABCD + B'CDE + A'B' + BCE'
  - Consensus terms
    - A**B**CD + **B'**CDE → ACDE
    - **A**BCD + **A'**B' → 0; A**B**CD + A'**B'** → 0
    - **B'**CDE + **B**CE' → 0; B'CD**E** + BC**E'** → 0
    - A'**B'** + **B**CE → A'CE
  - No redundancy
  - Redundancy insertion

  ABCD + B'CDE + A'B' + BCE' + ACDE

  = ABCD + B'CDE + A'B' + BC**E'** + ACD**E**

  = B'CDE + A'B' + BCE' + ACDE

  = B'CDE + **A'**B' + BCE' + **A**CDE

  = A'B' + BCE' + ACDE

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Three Basic Ways

- **Goal**
  - Simplifying an expression reduces the cost of realizing the expression using gates
- **Combining terms: $XY + XY' = X(Y + Y') = X$**
  - $ABC'D' + ABCD' = ABD'$
  - $AB'C + ABC + A'BC = AB'C + ABC + ABC + A'BC = AC + BC$
  - $(A + BC)(D + E') + A'(B' + C')(D + E') = D + E'$
    - DeMorgan's Law
- **Eliminating terms: $X + XY = X$ and $XY + X'Z + YZ = XY + X'Z$**
  - $A'B + A'BC = A'B$
  - $A'BC' + BCD + A'BD = A'BC' + BCD$
- **Eliminating literals: $X + X'Y = X + Y$**
  - $A'B + A'B'C'D' + ABCD' = A'B + A'C'D' + ABCD' = A'B + A'C'D' + BCD'$

Adding Redundant Terms

- Y = Y + XX'
- Y = Y(X + X')
- XY + X'Z = XY + X'Z + YZ
- X = X + XY
- Example

  W<u>**X**</u> + XY + <u>**X'**</u>Z' + WY'Z'

  = WX + XY + X'Z' + WY'Z' + WZ'

  = W<u>**X**</u> + XY + <u>**X'**</u>Z' + WZ'

  = WX + XY + X'Z' …can you derive this directly?

Quick Note

- No easy way to determine if a Boolean expression has a minimum number of terms or literals
  - Systematic (graphical) method will be discussed

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- ❑ **Proving Validity of an Equation**

Equation Validity (1/2)

- Several methods
  - Construct a truth table
    - Proof by cases (equivalent for all combinations of values of variables)
  - Manipulate one side until it is identical to the other side
  - Reduce both sides independently to the same expression
  - Perform the same operation on both sides if the operation is reversible
    - Complement is reversible
    - AND is not reversible: $XY = XZ$ does not imply $Y = Z$
    - OR is not reversible: $X + Y = X + Z$ does not imply $Y = Z$
    - $X \oplus Y = X \oplus Z$ implies $Y = Z$?
- Prove invalidity
  - Try to find **one** combination of values of variables such that two sides have different values

Equation Validity (2/2)

- Example: prove A'BD' + BCD + ABC' + AB'D = AD + A'BC + BC'D'

  A'BD' + BCD + ABC' + AB'D

  = A'BD' + BCD + ABC' + AB'D + A'BC + BC'D' + ABD

  = AD + A'BD' + BCD + ABC' + A'BC + BC'D'

  = AD + A'BC + BC'D'

- Example: prove the equivalence between
  - A'BC'D + (A' + BC)(A + C'D') + BC'D + A'BC'
  - ABCD + A'C'D' + ABD + ABCD' + BC'D

Q&A