MH4518

Simulation Techniques in Finance

Project Report AY23/24

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1. Introduction

This project aims to price the derivative through simulation. The Geometric Brownian Motion (GBM) model provided a foundation for understanding stochastic price movements, while the Heston model introduced volatility dynamics to better capture market complexities. The Short Rate Model, specifically the Vasicek model, allowed us to simulate interest rates, a critical factor in pricing financial instruments like the Drop-Back Certificate. Furthermore, our exploration of variance reduction techniques aimed to improve computational efficiency and reduce simulation noise. Lastly, we also estimate the Greeks to better help understand the fluctuations of the price.

1.1. Summary of Product

The CHF Drop-Back Certificate from Credit Suisse is a derivative that tracks the Swiss Market Index (SMI) as its underlying asset (which contains the top 20 stocks in Switzerland).

This product allows investors to invest gradually in the index fund over the product's life in case of a market correction, and has a term of 3 years which will last from 11 June 2021 to 11 June 2024. Thus, S_0 = 11,841.30 at the initial fixing date (11 June 2021), and S_T = Price of the reference index at the final fixing date (11 June 2024). As the derivative is tracking the SMI, it does not offer capital protection. If the value of the index falls to 0, the derivative will lose its value as well, and investors will lose all capital that is invested. This product is hence suitable for investors who are expecting a correction of the reference index, but will have a positive performance overall.

1.2. Payoff Function

The denomination of the product is D = 1000, and it has three trigger levels that will slowly expose the investors to the reference index in cases of downside correction. On the initial fixing date, CHF 550 per denomination will be invested in the reference index at the initial level. The three trigger levels are set at 95%, 90%, and 85% of the reference index from the initial fixing date. Upon falling below each of these trigger levels, an additional investment of CHF 150 will be made at each respective level. After the 3rd trigger event, the derivative will be fully invested in the reference index.

There will be an interest rate of 2.50% per annum, accrued daily on the cash component on the final fixing date if the reference does not fall below any of the trigger levels. The derivative has already passed the first and second trigger level on 7^{th} March 2022 and 16^{th} June 2022 respectively. CHF 850 per domination is thus already invested in the reference index. The payoff function relevant for our study for the period between 9 August 2023 and 9 November 2023 will then be:

$$\begin{aligned}\text{Payoff} &= 550 \cdot \frac{S_T}{S_0} + 150 \cdot \frac{S_T}{S_1} + 150 \cdot \frac{S_T}{S_2} \\ &+ 150 \cdot \left[1_{\{\text{3rd event triggered}\}} \cdot \frac{S_T}{S_3} + \left(1 - 1_{\{\text{3rd event triggered}\}}\right) \cdot (1 + r)^T \right]\end{aligned}$$

Equation 1: Payoff Function

In this equation, $1_{3rd\ event\ triggered} = 1_{S_{min} < 10065.1050}$. The remaining 150 CHF will either be invested in the index if the 3rd event is triggered, or will accrue the interest rate if not triggered.

2. Simulation

2.1. GBM model in risk-neutral world [Qingtian]

We first use exact GBM to simulate the underlying stock price from j = 1 to j = mΔt, where m is the number of days between current date and maturity date.

$$S_j = S_{j-1} exp(\nu \Delta t + \delta Z_j \sqrt{\Delta t})$$

Equation 2

Next, we put the minimum value and final stock price to the pay-off function and evaluate pay-off at time T.

$$P = \chi(S_T, S_{min})$$

Equation 3

Then we discount using the rate found in the government bond until maturity.

$$V = exp(-rm\Delta t)P$$

Equation 4

Repeating steps 1-3 for 100 times, we use the average value of V and use this value for the price valuation for that day. We perform the same steps everyday in the three month period from Aug 9 to Nov 9, using a three-month rolling window to estimate the sigma, while finding the risk-free interest rate r from government bonds for each day. The ν of equation 2 is calculated by the following equation:

$$\nu = r - \frac{1}{2}\delta^2$$

Equation 5

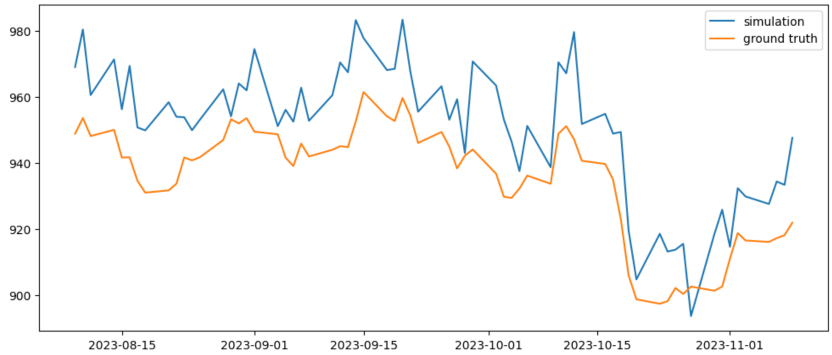


Figure 1: 3-month prices simulation using GBM

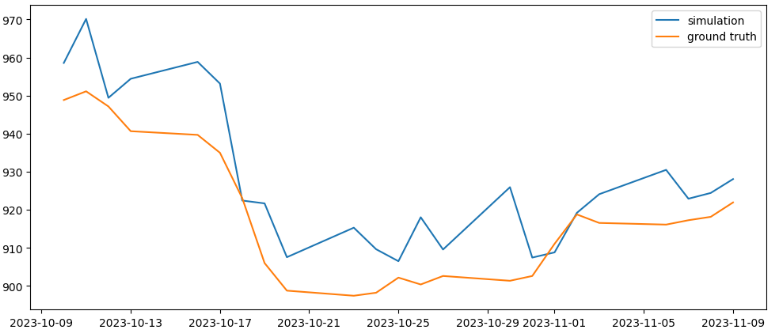


Figure 2: 1-month prices simulation using GBM

2.2. Heston Model [Qingtian]

The Heston Model is a mathematical model used in financial mathematics to describe the stochastic dynamics of asset prices. Heston Stochastic Volatility Model Equations are a system of stochastic differential equations. The key feature of the Heston Model is its incorporation of the stochastic volatility of the underlying asset to follow a random process.

The price still follows the stochastic process similar to GBM model, with the exception that volatility is now time-dependent:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^S$$

Equation 6

Then, the volatility process is modelled by:

$$dV(t) = \kappa(\theta - V(t))dt + \xi\sqrt{V(t)}\, dW_v(t)$$

Equation 7

Before simulating for a day's price, we need to calibrate the parameter set $\Theta = (\kappa, \theta, \xi, \rho, V_0)$. In this set, κ represents rate of reversion of price, θ means the long-term variance, ξ means the volatility of the variance process, V_0 is the initial volatility, and ρ is the correlation of volatility and price.

It may not be feasible to solve analytically for Heston parameters, so we attempted to build a machine learning model with these five parameters as trainable parameters. We used MSE (mean square error) as a loss function as below:

$$\frac{1}{n}\sum_{i=1}^{i=n}(\Theta_i - M_i)$$

Equation 8

Then we trained the model until its loss reaches global minima. But because the data we have is little (2 months of data was used for training and 1 month of data was used for testing), with a random sampling process, the training curve is not stable.

The performance of the Heston model also depends heavily on the initialisation of the parameters. For example, whether ρ taking a positive or negative number could significantly affect the training process. To avoid the model taking unreasonable Θ values, we attempted to clamp the parameters between 0 and 1 with ρ taking a value from -1 to 1. Additionally, due to time constraints, we only trained for about 5 epochs, so the training loss might not fully stabilise yet. We conclude that with the correct scheduler and more data, it should be able to train for longer and achieve better MSE loss.

For each day's unique Θ, we tested it against the correct data for from 10 October 2023 to 9 November 2023. The variance is again quite large, so the curve might look different depending on what the sampling process ends up sampling.

Figure 3 shows the variation of the parameters $\kappa, \theta, \xi, \rho, V_0$ respectively over the relevant period of simulation between 10 October 2023 to 9 November 2023.

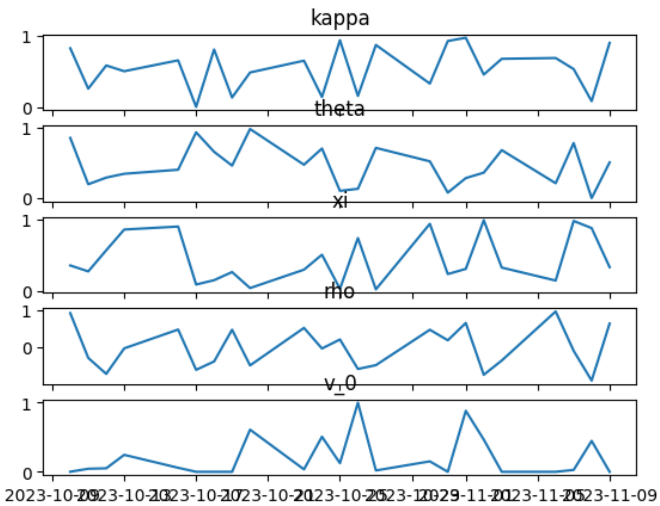


Figure 3: 1-month Variation of Heston model parameters

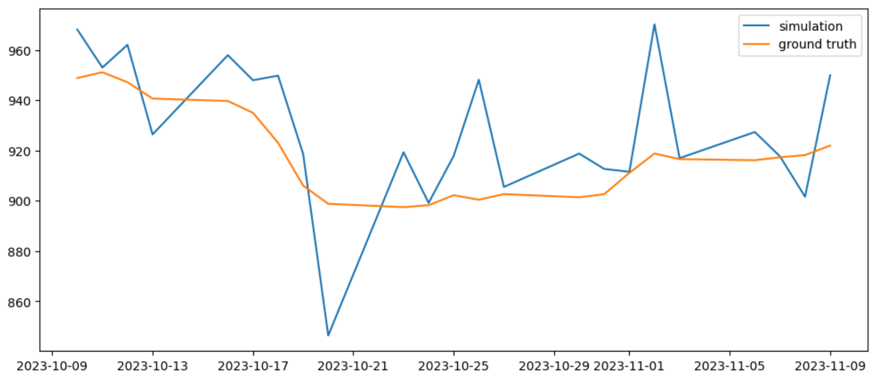


Figure 4: 1-month prices simulation using Heston

As the variance of the model is quite huge (around 159), the result might vary depending on the sampling.

2.2.1. Heston Model: Implementation Discussion and Evaluation

Upon further discussion, we realised that the heston model should have been trained on forward looking data, i.e. SMI call option data. However, we could not find them online, so currently the data that we used was historical price data of the past two months, which was not appropriate inputs for our Heston model. In future implementation, the option data for each day needs to be collected for each day of simulation as M_i in equation 8, where we will attempt to minimise the error. After we obtained Θ, we then simulate until maturity before discounting back to present, and then evaluate the true error to the price of the derivative that day.

2.3. Vasicek Short Rate Model (Interest Rate Model) [Wei May]

Since the drop back certificate pricing model involves interest rates and asset prices, we explored to model the interest rate with the Vasicek model and the asset prices with the GBM model. The Vasicek short rate model is used to simulate short term interest rate which is given by the SDE:

$$dr = \kappa(\theta - r)dt + \sigma dW.$$

Equation 9

The first part $\kappa(\theta - r)$ is the mean-reverting component responsible for pulling the short rate back towards the long term mean, θ. The mean-reversion rate is κ. The second part is the stochastic component comprising σ, the volatility and dW, a Wiener process.

The simulation approach is that we simulate the short rate paths with the Vasicek model, and simulate the underlying asset (SMI) price paths with some other model to model the asset dynamics. (For this project, we use GBM.) The last short rate value of the simulated short rate path is used in the calculation for the risk-neutral drift term μ, of the SDE for pricing financial instruments. (Note in the risk-neutral world, the μ is adjusted to become a risk-neutral drift (by the $-\frac{\sigma^2}{2}$ adjustment term).

$$\mu = \kappa(\theta - r) - \frac{\sigma^2}{2}$$

Equation 10

The main idea is that, instead of using the Swiss government bond risk-free interest rate to calculate the μ, we use the simulated short rate to calculate the μ. The standard κ=0.1, θ =0.025, σ=0.01 values commonly used in industry were also used for this simulation. The MSE was 528.75, and Variance was 86.7147.

The simulated short rates and corresponding drop back certificates prices for the period between 9 August 2023 to 9 November 2023 is plotted as follows:

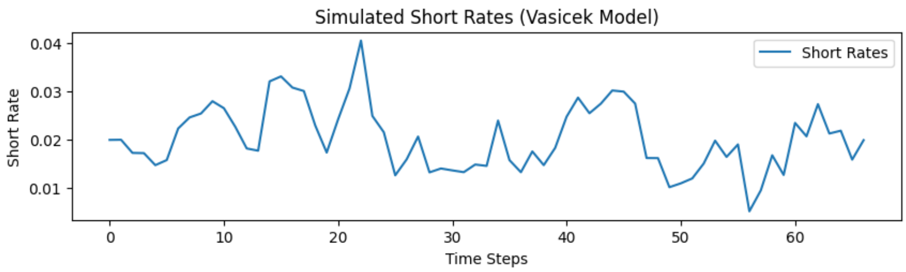
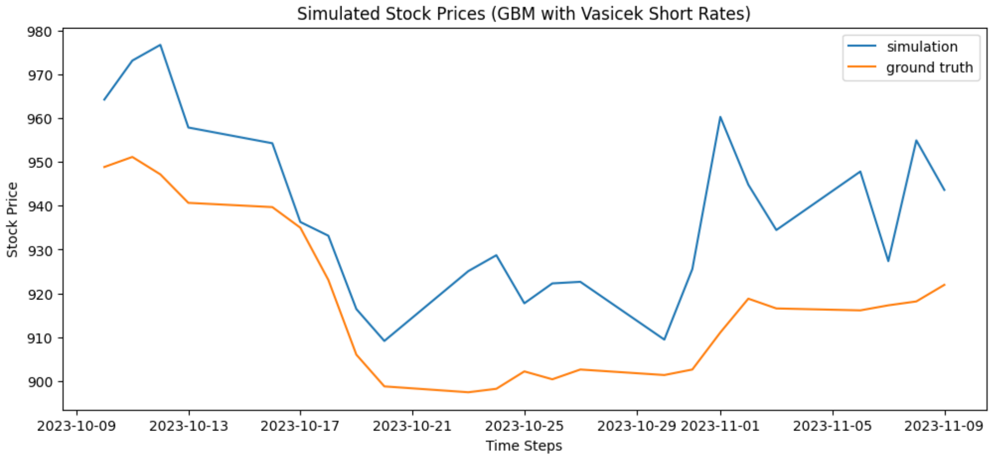


Figure 5: Simulated Vasicek model Short Rates Paths



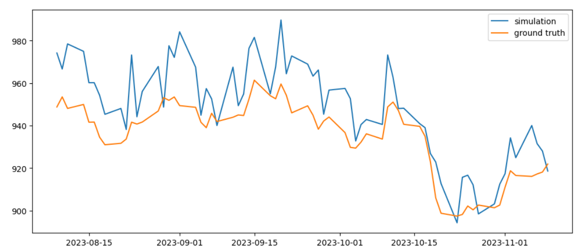
<u>Figure 6: GBM with Vasicek Short Rate model for simulating Drop back Certificate Price</u>

In future exploration, more sophisticated models that capture the interactions between interest rates and asset prices may be warranted to take into consideration the correlation between interest rates and the underlying asset.

3. Variance Reduction

Variance reduction techniques are used to improve the accuracy and efficiency of simulation models. Reducing variance will help the model achieve reliable and accurate results.

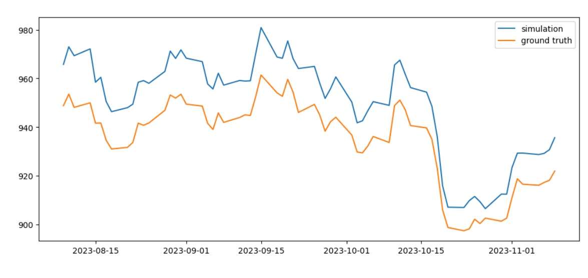
Monte Carlo simulation was used as the baseline for comparison, to assess the effectiveness of the variance reduction techniques that will be used.

Average variance over three months using monte carlo simulation = 85.4660. The Average variance refers to finding the variance of each day's simulated price, sum then up, followed by dividing the number of working days in the three months.	

3.1. Antithetic Variates (AV) [Qingtian]

Antithetic Variates (AV) approach involves generating n number of Z_j in equation 2, followed by setting the rest n numbers of random variable $-Z_j$. The two variables will have the same distribution. The negative correlation between the negatively correlated pairs is exploited to reduce the overall variance of the simulation. Assume $H_{AV} = \frac{1}{2n}\sum_{i=1}^{i=n}(H(Z_j) + H(-Z_j))$, if $cov(H(Z_j) - H(-Z_j))$ is negative, then $var(H_{AV})$will be smaller than monte carlo simulation.

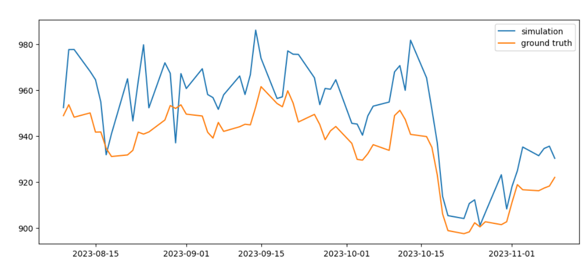
Average variance over three months is 84.6623, indicating some variance reduction, but it is not large. It could be due to the fact that $cov(H(Z_j), H(-Z_j))$ is only slightly negative, resulting in very small variance reduction.



3.2. Control Variates (CV) [Qingtian]

In the control variates (CV) variance reduction technique, the goal is to reduce the variance of simulation estimates by introducing auxiliary information often in the form of a known or easily computable variable, which is called a control variate. We first find the control variate mean and variance (in our case is final stock price S_T) by simulating 100 pilot prices $S_T^{\,i}$ and its corresponding derivative prices P^i. We then estimate the c value through formula $c = \frac{Cov(P^i, S_T^{\,i})}{var(S^i_T)}$. Finally, we conduct the proper simulation to obtain $CV^i = P^i + c(S_T^{\,i} - \overline{S}_T)$ for each path i, and then we take the average of CV^i. As long as $corr(S_T^{\,i}, P^i)$ is not equal to zero, there will be some degree of variance reduction.

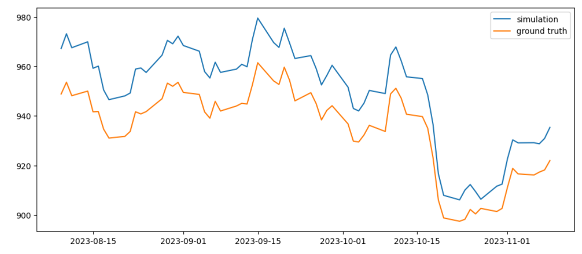
The average variance over three months = 86.1925. It did not result in reduction of variance, which might be because $corr(S_T^{\,i}, P^i)$ is very close to 0.



3.3. Empirical Martingale Simulation (EMS) [Jacky]

EMS exploits the martingale property of $\mathbb{E}^Q[e^{-rt}S(t)|S(0)] = S(0)$ for any t, meaning that the expected value of the next observation is equal to the current observation, given all past information. This concept is fundamental where under the risk-neutral measure, the discounted price of a derivative is often assumed to be a martingale. EMS involves adjusting a simulated price path of an underlying asset so that it satisfies the martingale property based on empirical data. This should lead to a more stable and accurate estimation.

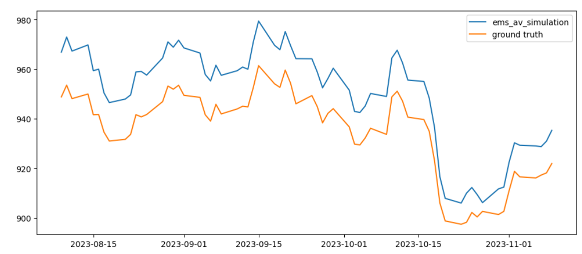
The average variance over three months was 84.8933, indicating some degree of variance reduction.



3.4. EMS + AV [Jacky]

EMS can be combined with other variance reduction techniques like AV and CV due to the complementary nature of these methods. Each technique addresses different aspects of variance in simulations, and when used together, they can enhance the efficiency of the simulation process. AV generated the pairs of the negatively correlated variables, and the average will be taken. EMS will then be applied, and the variance will then be calculated.

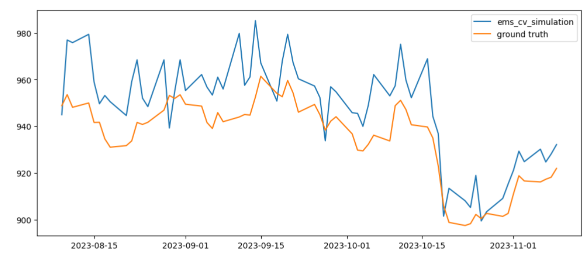
A combination of two variance reduction techniques was explored. EMS + AV was carried out, which resulted in a variance of 83.1904, indicating some variance reduction.



3.5. EMS + CV [Jacky]

Similar to the previous technique, the control variate technique will first be applied before applying EMS. It should leverage the strengths of each method to produce more reliable and efficient results.

The EMS + CV simulation approach was also applied, and resulted in a variance of 84.3852, indicating some variance reduction. However, it is not as effective as the previous EMS + AV technique.



3.6. Summary

In summary, EMS + AV had the lowest variance of 83.1904.

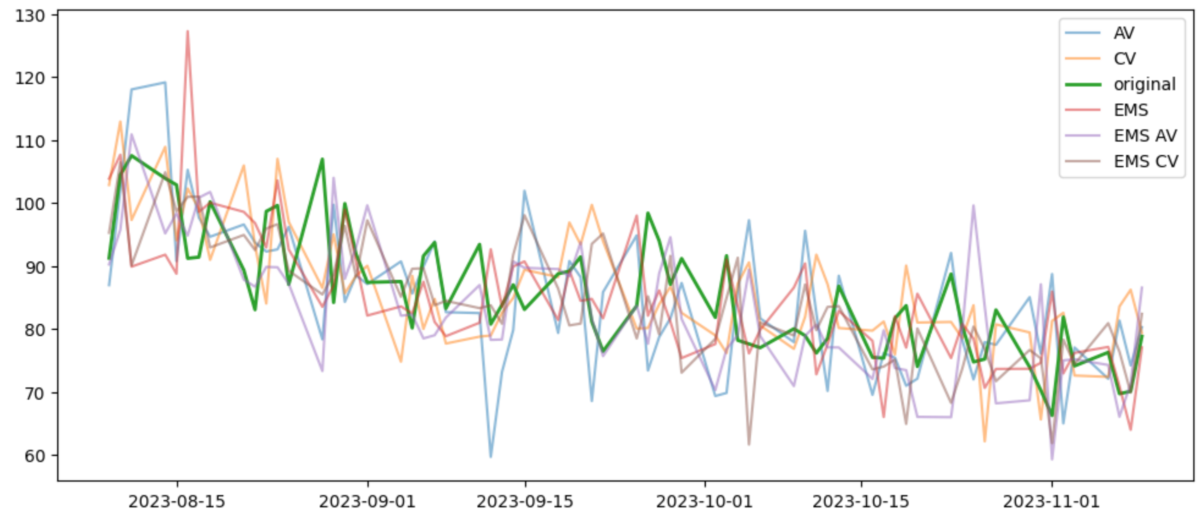


Figure 7: Graph with all Simulations using Various Reduction Techniques, with y-axis referring to variance of each day

The average scores for three months from each of the variance reduction techniques are recorded in the table.

Technique	MC	AV	CV	EMS	EMS + AV	EMS + CV
Variance	85.4660	84.6623	86.1925	84.8933	83.1904	84.3852

Table 1: Summary of Variance from each technique

4. Estimating Sensitivities [Wei May]

Greeks values provide insights into how the derivative's price is expected to change in response to movements in the underlying stock price, which offers useful information for risk management and investment decisions.

In the calculation of the Greeks values which are measures of sensitivities, the stock price is perturbed slightly in both the upward and downward directions by an arbitrarily small number h (=0.01 was used). Then the derivative price is recalculated for each perturbed stock price. Delta ($\delta = \frac{\partial f}{\partial S}$) represents the ratio of the derivative's price movement relative to the underlying asset, while Gamma ($\Gamma = \frac{\partial^2 f}{\partial S^2}$) represents the rate of change of delta concerning changes in the underlying asset's price. (Gamma is a measure of delta's sensitivity.)

4.1. Finite Difference Method (FDM) for calculating Delta and Gamma

$\delta = \frac{f(S+h) - f(S)}{h}$ or $\delta = \frac{f(S) - f(S-h)}{h}$ or $\delta = \frac{(Sf+h) - f(S-h)}{2h}$ $(central)$

$\Gamma = \frac{f(S+h) - 2f(S) + f(S-h)}{h^2}$

The FDM was used to approximate Delta and Gamma values over the rolling window estimation period of 3 months from 9 August 2023 to 9 November 2023 via the above formulae.

4.2. Interpretation of Calculated Delta and Gamma

Delta of 0.08740 (GBM value) indicates that for a 1% increase in the underlying SMI stock price, the drop-back certificate derivative price is expected to increase by 0.08740, assuming all other factors remain constant.

A gamma of = 1.5817e-09 indicates that delta is expected to increase by approximately 1.5817e-09 for a 1% increase in the underlying stock price. The positive gamma indicates the delta is increasing as the stock price increases.

4.3. Greeks: Results and Discussion

The results of the Greeks show that all 3 models produce positive Delta of around 0.088 which indicates that generally, the certificate's value increases as the SMI's value increases. The Gamma values obtained were very small ($\approx 10^{-9}$), and positive for GBM and Heston models but negative for Vasicek models. Hence, this provides room for discussion.

Negative gamma would mean the dropback certificate's exposure to the SMI is increasing at decelerating rate, while Positive gamma indicates the dropback certificate's exposure to the SMI is increasing at accelerating rate.

Given the certificate description, we can make an informed assessment on which is more probable based on its investment structure and how it adjusts exposure to the SMI. The certificate enables gradual investment in the SMI over the product's lifetime, hence increasing the product's sensitivity to the SMI reference index's performance. And with each trigger event, the direct exposure (participation rate) of the product to the SMI reference index increases in a nonlinear way (accelerates with each trigger event). Hence, this gradual and incremental nature of the investments with each trigger event directly increases exposure to the SMI, and are potential reasons for the structured product exhibiting a positive gamma characteristic.

Hence, it appears that the GBM and Heston models are better (resulting in more appropriate/probable positive sign of gamma) for this simulation of the drop back certificate performance, which is in line with the conclusion drawn about the models performance as well from the MSE comparisons above.

4.4. Greeks Means and Plots

The mean Deltas and Gammas were computed for the GBM, Heston model and Vasicek interest rate model under risk-neutral measure.

	GBM	**Heston**	**Vasicek Short Rate**
Delta	0.08740	0.08761	0.08810

Gamma	1.5817e-09	1.08743e-09	-1.1199e-09

The plots of the Deltas and Gammas over the period from 9 August 2023 to 9 November 2023 can be found in the Appendix.

5. Comparative Study

When we compare the 3 models by their MSE, from our simulation, it looks like the GBM model performs the best as it has the lowest MSE (273.38). The Vasicek short rate model performs the worst, with highest MSE (542.47), which may be because the presence of trigger events leading to changes in investment levels in the certificate may introduce market discontinuities. But as the Vasicek model is continuous-time in nature, it might struggle to accurately model scenarios with sudden jumps or step changes in the underlying process.

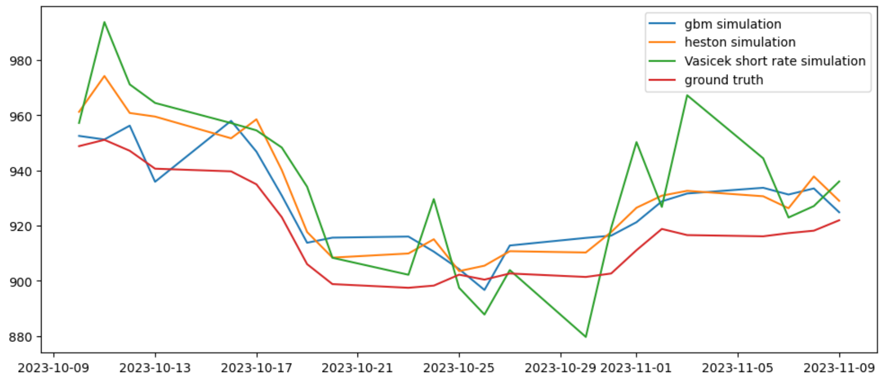


Figure 8: Graph of simulations using the 3 models

Model Name	**GBM Model**	**Heston Model**	**Short Rate Model**
MSE	273.38	350.73	542.47

Table 2: Summary of MSE from each model simulation

Additional remarks on comparison between GBM and Heston Model (theory) is available in the Appendix.

6. Conclusion

In conclusion, our project delved into the simulation of Drop-Back Certificate prices, employing three distinct models—Geometric Brownian Motion (GBM), Heston, and a Short Rate Model. Through extensive simulations, we explored the dynamic pricing behaviours under different market conditions. Additionally, we investigated various variance reduction techniques to enhance the efficiency of our simulations.

This project not only deepened our understanding of derivative pricing but also showcased the importance of selecting appropriate models and employing advanced techniques to optimise simulation outcomes. The insights gained contribute to the broader field of financial modelling and pave the way for more sophisticated analyses in risk management and investment strategy development.

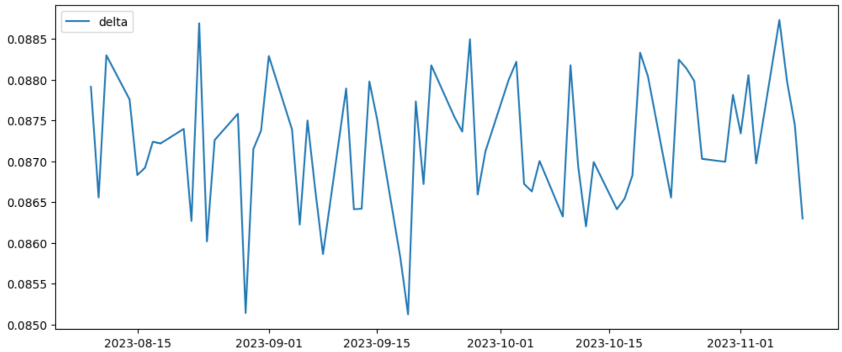
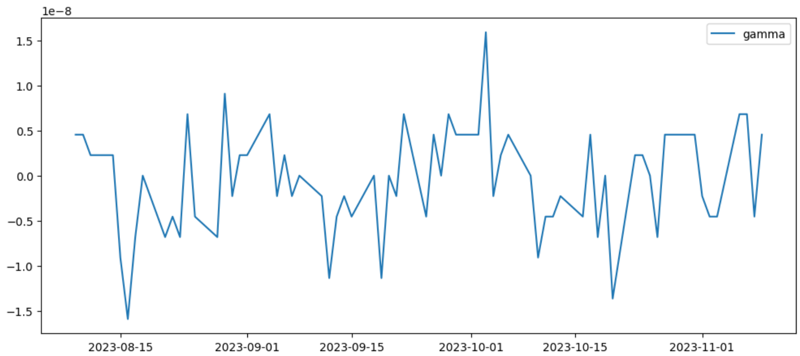
7. Appendix

7.1. Google Colab Notebooks:

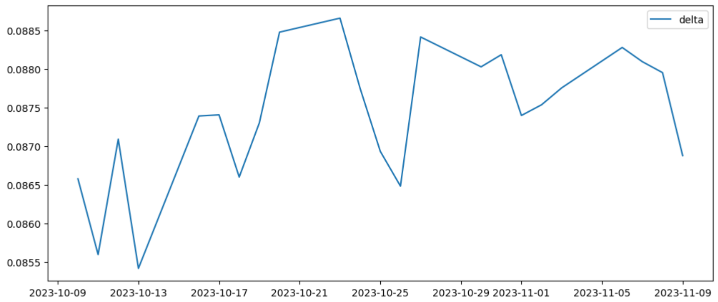
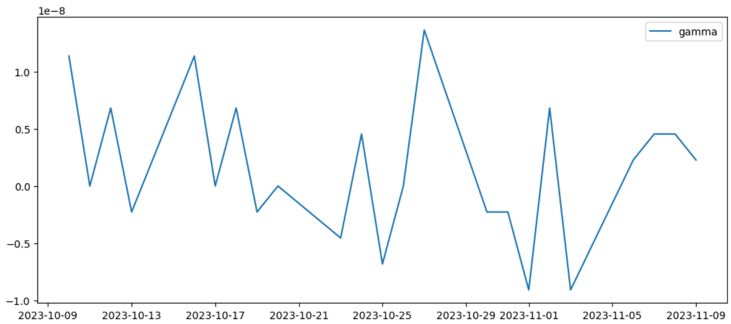
Dropback_Certificate_Pricing_Models_final_revised.ipynb
https://drive.google.com/file/d/1koqtXWqXvWS4ne7vkwikMLTH0WMdKOxC/view?usp=sharing
1MonthModels.ipynb
https://drive.google.com/file/d/19XvXz4OKUI4ghkqA7UC7VsuxRvpTcQgq/view?usp=drive_link

7.2. Additional Greeks Plots

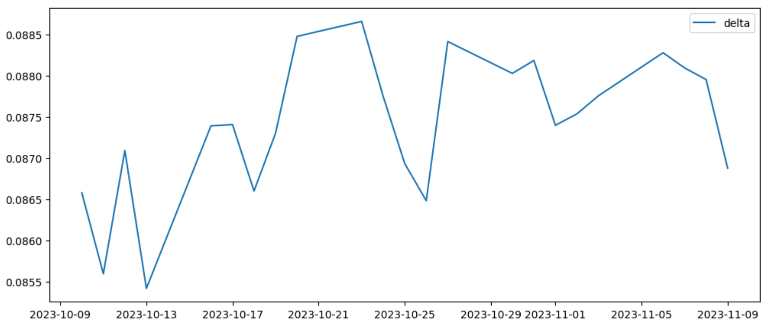
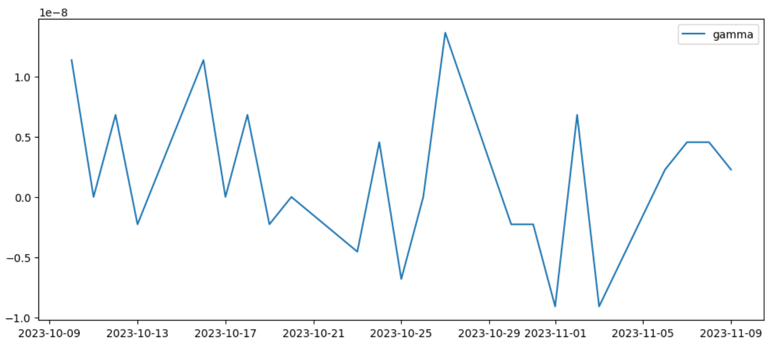
7.2.1. For GBM (3 months)

	Mean	**Plot**
Delta	0.08721	
Gamma	-7.1266e-10	

7.2.2. For GBM (1 month)

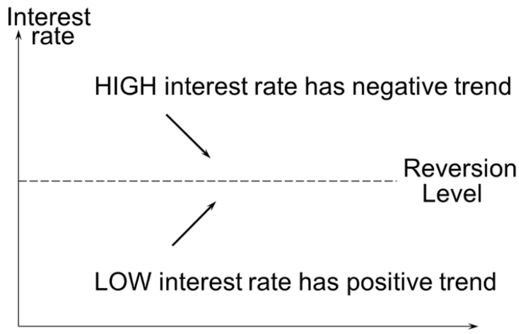
	Mean	Plot
Delta	0.08740	
Gamma	1.5817299161268318e-09	

7.2.3. For Heston Model (1 month)

	Mean	Plot
Delta	0.08740	
Gamma	1.58173e-09	

7.2.4. Vasicek Short Rate Model Theory

The Vasicek (1977) interest rate model is a single-factor short-rate model used to predict where the interest rates will end up at the end of a given period, outlining an interest rate's evolution as a factor composed of market risk, time and equilibrium value.



Single-factor equilibrium model indicates that the process for r involves only one source of uncertainty. This is an assumption of the Vasicek interest rate model. This simplification is made to facilitate analytical tractability.

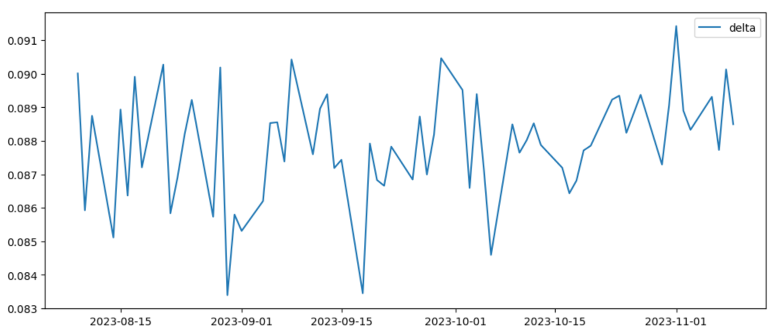
Other assumptions of the Vasicek interest rate model include mean-reversion, which is the phenomenon where interest rates appear to be pulled back to a long-run average level over time. It is an important difference between interest rates and stock prices. When *r* is high, mean reversion tends to cause negative drift, while mean reversion causes positive drift when *r* is low.

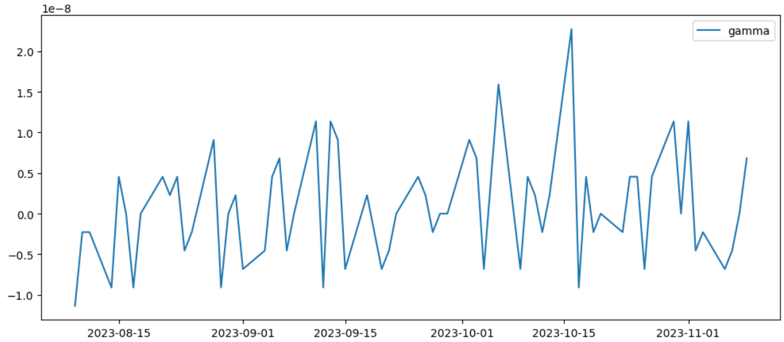
The Risk-neutral process for *r* in the Vasicek's model is:

$$dr = a(b - r)dt + \sigma dz$$

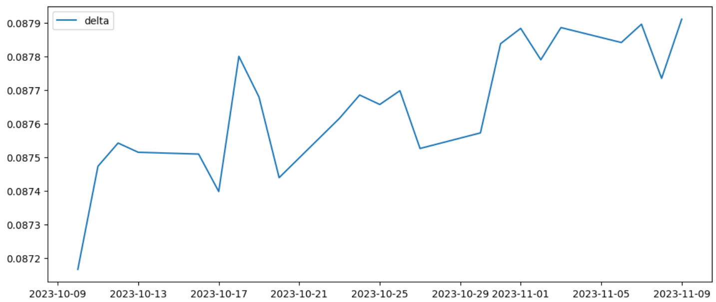
Where a, b and σ are typical industry constants, dz is a Weiner process.
$a(b - r)$ is the mean-reversion term in a risk-neutral world. σ is the interest rate volatility.
In the Vasicek's model, mean-reversion happens as the short rate is pulled to a level b at a rate a. The 'pull' has a normally distributed stochastic term σdz superimposed on it.

7.2.5. For Vasicek Short Rate Model (3 months)

	Mean	Plot
Delta	0.08810	

Gamma	-1.1199e-09	

7.2.6. For Vasicek Short Rate Model (1 month)

	Mean	**Plot**
Delta	0.087655	
Gamma	-9.88581e-11