Phase-Space Trajectories of Physical Pendulum with Gravitational, Frictional, and External Force Torque

Jerry Kim

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Abstract

In order to describe the dynamics of a physical pendulum that is influenced by the presence of gravity, friction, and an external force, we use phase-space trajectories. Our approach does not use the small-angle approximation. We examine this under various initial conditions and see how the trajectory is influenced depending on which forces are included and excluded. Matlab is used to generate graphs of the phase space trajectories and to solve the equations of motion by the 4th-order Runge Kutta method.

Contents

1 Introduction

Models of physical pendulums are often used in physics and engineering. Pendulums have been used for clocks and timekeeping, in addition to scientific instruments such as accelerometers and seismometers [2].

The focus of this paper is the application of phase-space trajectories for a physical pendulum subjected to various torques. The dynamics of the pendulum are governed by the gravitational torque, frictional torque, and external force torque. The differential equation that describes the dynamics of a pendulum under these forces is second-order. However, the standard form we use to solve this is to transform it into two simultaneous first-order differential equations. One reason we convert this to two first-order equations is because it is easy to solve on the computer [1]. The conventional solution to an equation of motion is the position $x(t)$ and velocity $v(t)$ as functions of time, and they can be visualized using a phase space. We use phase-space plots for several reasons. First, the phase-space plots can form geometric objects whereas it can be difficult to describe the position and position without the phase-space plot. Second, phase-space plots are useful in the visualization of the motion of the object. For example, for the one-dimensional harmonic oscillator without friction or external torque, the solution follows closed elliptical orbits in phase-space, in which the size of the ellipse depends on the system's energy. The structure of this paper is as follows. In Section 2 the differential equations governing the dynamics of the pendulum are stated and then the second-order differential equation is formulated as a pair of first-order differential equations. Our scheme for solving these equations under various conditions, such as including or excluding the friction and/or external torque is also described. Finally, the details of the phase-space trajectories are also described. Sections 3, 4, and 5 contain the results, discussion, and conclusions, respectively.

2 Methods

We consider a pendulum that is driven through viscous air by an external force. The pendulum oscillates at a frequency ω_o if there is no external force or friction. The strength of the external torque is given by f while the viscosity of the air is given by α. We assume that the dynamics of the pendulum are governed by the gravitational torque, frictional torque, and external force torque. We assume that the frequency of the external force is ω, the length of the pendulum is l, the angle that the pendulum makes with the vertical is θ, the constant of gravity is g, the mass of the object on the end of the pendulum is m, and I is the moment of inertia. This gives us the following differential equation due to Newton's law [1]:

$$\frac{d^2\theta}{dt^2} = -\omega_o^2 sin\theta - \alpha\frac{d\theta}{dt} + fcos\omega t \tag{2.1}$$

where

$$\omega_o = \frac{mgl}{I}, \alpha = \frac{\beta}{I}, f = \frac{\tau_o}{I} \tag{2.2}$$

Equation (2.1) is a second-order differential equation, but it can be converted into two simultaneous first-order differential equations [1]

$$y_1 = \theta, y_2 = \frac{d\theta}{dt} \tag{2.3}$$

$$\frac{dy_1}{dt}(t) = y_2(t) \tag{2.4}$$

$$\frac{dy_2}{dt} = -\omega_o^2 siny_1(t) - \alpha y_2(t) + f cos\omega t \tag{2.5}$$

It is easy to solve (2.4) and (2.5) on the computer [1].

While it is conventional to solve for the equations of motion for the position $x(t)$ and velocity $v(t)$, in this paper we will focus on the phase-space trajectories.

We explore the pendulum in four different cases. First, when the pendulum is not subjected to friction or external forces. Second, when the pendulum is subjected to friction. Third, when we ignore friction but exclude the external force. Fourth, when we include both friction and the external driving force.

In the first case, when there is no friction or external force, equation (2.5) is reduced to

$$\frac{dy_2}{dt}(t) = -\omega_o^2 siny_1(t) \tag{2.6}$$

Before we examine the phase-space trajectory, if we use the small angle approximation, that is, $sin(\theta) \approx \theta$ for small values of θ, we observe that these equations can allow us to determine the total energy of the system [1]:

$$E = K + V = \frac{1}{2}mv^2 + \frac{1}{2}\omega^2 m^2 x^2 \tag{2.7}$$

$$= \frac{\omega^2 m^2 A^2}{2m} cos^2(\omega t) + \frac{1}{2}\omega^2 m^2 A^2 sin^2(\omega t) \tag{2.8}$$

$$= \frac{1}{2} m\omega^2 A^2 \tag{2.9}$$

The system of equations (2.4) and (2.6) can also be solved for $x(t)$ and $v(t)$ by using numerical methods instead of the small-angle approximation. In this paper, we will use the 4th-order Runge-Kutta method to solve for $x(t)$ and $v(t)$.

From equation 2.7, we observe that the pendulum follows closed elliptical orbits in phase space, and the size of the ellipse depends on the total energy of the system.

When we now include friction but still exclude the external force, equation (2.5) becomes

$$\frac{dy_2}{dt}(t) = -\omega_o^2 siny_1(t) - \alpha y_2(t) \tag{2.10}$$

When we now include the small driving torque but exclude friction, equation (2.5) becomes

$$\frac{dy_2}{dt}(t) = -\omega_o^2 siny_1(t) + fcos(\omega t) \tag{2.11}$$

Finally, when we include both friction and the external torque, we get equation 2.5.

3 Results

We consider four types of experiments of phase-space trajectories: the pendulum system without friction or external torque, with just friction, with just external force, and with both friction and external torque. By examining the phase space diagrams, we are able to examine stunning displays of motion, including chaotic motion.

3.1 Without Friction Or External Torque

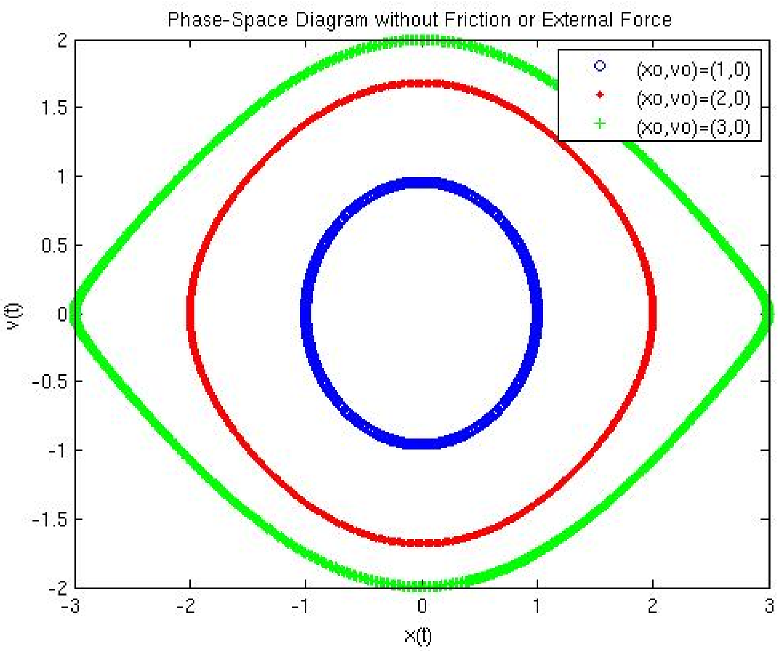

Figure 3.1: Phase-space plot of velocity vs position created using Matlab. The data points in blue are for the case (x_0, v_0) = (1,0), the data points in red are for the case (x_0, v_0) = (2,0), and the data points in green are for the case (x_0, v_0) = (3,0). All of these data points are generated by solving the pendulum equation by using the 4th-order Runge Kutta method

We examined three different initial condition cases, using (x_0, v_0) = (1,0), (2,0), and (3,0). In all three cases, we also used the parameters $\omega_o = 1, \alpha =$

.2, $f = .52, w = .666$; with timesteps of $t = .01$ using the 4th-order Runge Kutta method. The resulting phase-space trajectories can be seen in Figure 3.1

3.2 With Friction

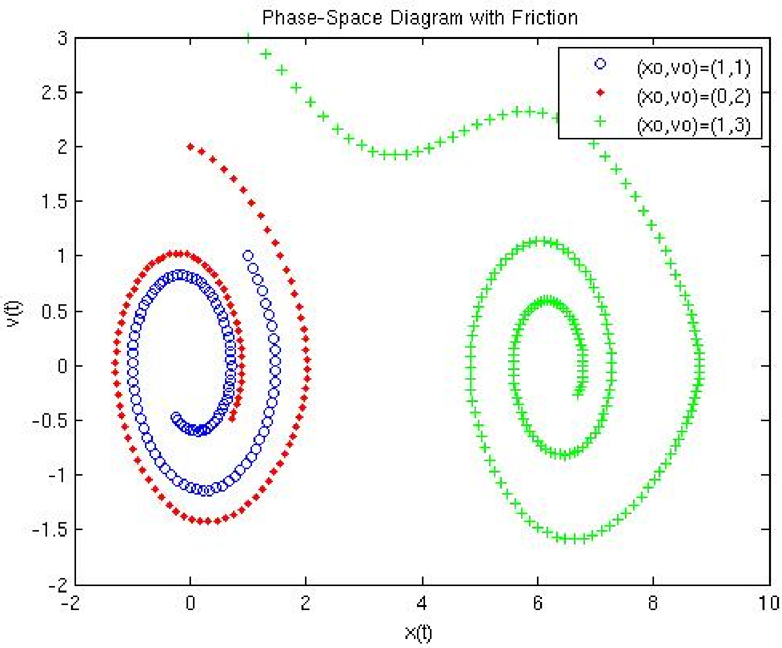

Figure 3.2: Phase-space plot of velocity vs position with friction. The plot is generated using Matlab. The data points in blue are for the case $(x_0, v_0) =$ (1,1), the data points in red are for the case $(x_0, v_0) =$ (0,2), and the data points in green are for the case $(x_0, v_0) =$ (1,3). All of these data points are generated by solving the pendulum equation by using the 4th-order Runge Kutta method

We examined three different initial condition cases, using $(x_0, v_0) =$ (1,1), (0,2), and (1,3). In all three cases, we used the same parameters as in the previous section. The phase-space diagrams for these cases can be seen in Figure 3.2

3.3 With External Torque

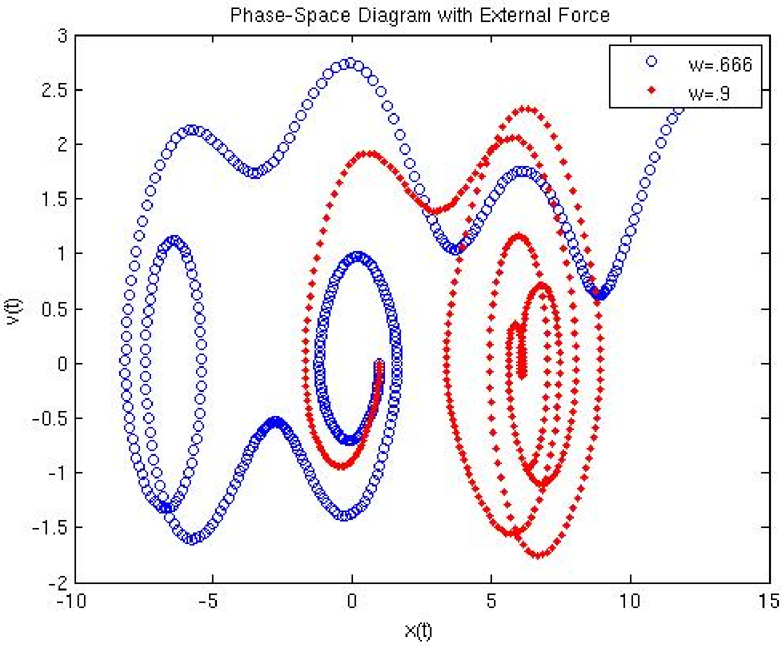

Figure 3.3: Phase-space plot of velocity vs position when an external torque is present. The plot is created using Matlab. The data points in blue are for the case $\omega = .666$ and the data points in red are generated when $\omega = 0.9$. All of these data points are generated by solving the pendulum equation by using the 4th-order Runge Kutta method

We examined two different cases, using a driving frequency of $\omega = .666$ and .9. In both cases, we used the same parameters as before, except the initial conditions are (x_0, v_0) = (1,0) for both cases and the timestep of 0.1. The phase-space diagrams can be observed in Figure 3.3

3.4 With Friction and External Torque

We examined three different initial condition cases, using (x_0, v_0) = (-.0883, .8), (-.0885, .8) and (-.0888, .8). In all three cases, we used the same parameters as in the previous section except with timesteps of 0.25. The visualization of the phase-space diagram is in Figure 3.4

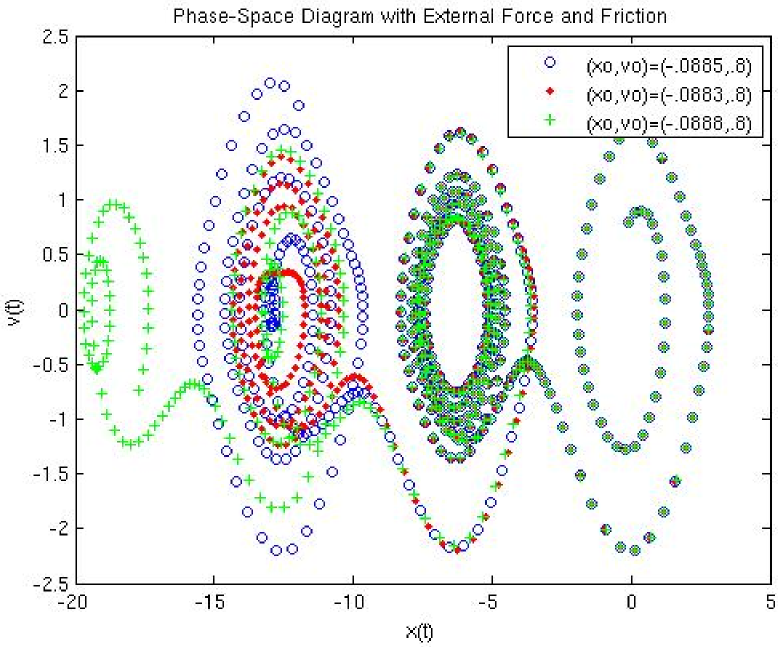

Figure 3.4: Phase-space plot of velocity vs position with friction and external torque present. The plot is generated using Matlab. The data points in blue are for the case (x_0, v_0) = (-.0885,.8), the data points in red are for the case (x_0, v_0) = (-.0883,.8), and the data points in green are for the case (x_0, v_0) = (-.0888,.8). All of these data points are generated by solving the pendulum equation by using the 4th-order Runge Kutta method

4 Discussion

We now interpret the results we have obtained from plotting the data points obtained by solving the pendulum equations using the 4th-order Runge Kutta method for the cases when no friction or external torque is present, when only friction is present, when only an external torque is present, and when both friction and the external torque is present. When friction and external torque are both absent, by looking at Figure 3.1, the phase-space diagrams for the three cases appear to be ellipses when (x_0, v_0) = (1,0) and (2,0), but an oval when (x_0, v_0) = (3,0). These results can be compared to the expected results if we had used a small-angle approximation instead of the 4th order Runge Kutta method. According to equation 2.7, we expect that the trajectories should be ellipses for all three initial conditions.

When friction is present, by looking at Figure 3.2, the phase-space diagrams for the three different initial conditions appear to spiral inwards and would eventually come to a halt. The (x_0, v_0) = (1,1), (0,2) cases both are shifted towards a lesser value for $x(t)$ than the (1,3) case. The inward spiraling still matches what we expect, regardless if we used the small-angle approximation instead of the 4th-order Runge Kutta method, because we expect the object to eventually reach a velocity of zero and stop oscillating if the only force if were subjected to was friction. Figure 3.3 displays the phase-space diagrams for the trajectories when an external force is present and we use $\omega = .666$ and $\omega = 0.9$. The resulting trajectories for both cases appear to be chaotic. However, the trajectory when $\omega = 0.9$ appears to resemble an ellipse for greater values of t after starting off from (x_0, v_0)=(1,0). This is what we expect because as ω approaches ω_o, we would expect a resonance in the oscillation, and thus an orbit resembling an ellipse as seen in the trajectories in Section 3.1 Without Friction or External Torque. Finally, as can be seen in Figure 3.4 when we include both friction and the external torque, the phase-space diagrams for the three cases appear take on periodic spiraling trajectories, spiraling outwards and then inwards, and then repeating the spiraling process by shifting towards negative values of $x(t)$. We expect that the trajectories should not just spiral inwards to a halt, as in the case with just Friction, because an external torque is also present to keep the object in motion. There is also an element of chaos to the trajectories, which we noticed occurred in the case without friction and only external torque.

5 Conclusion

We have described a physical pendulum problem and computed the positions and velocities that are necessary to construct phase-space trajectories. Our computations are based on the 4th order Runge-Kutta method. Our results demonstrate the phase-space trajectories can differ vastly or stay relatively unchanged depending on the initial conditions and whether friction and/or external torque are included or not. When there is an external torque present, the pendulum undergoes a mention that is considerably more chaotic than the case with only friction present, in which case the pendulum eventually comes to a halt. Understanding chaos helps us gain a better understanding of the physical world. We are able to learn that microscopic changes can lead to noticeable macroscopic effects. Chaotic systems and pendulums, together with their complexity, will be helpful to studying other subfields of physics in the future as well.

References

[1] Rubin Landau and Manuel Paez. *Computational Physics Problem Solving with Computers*. John Wiley & Sons, Inc., New York, New York, 1997.

[2] Wikipedia. Pendulum — Wikipedia, the free encyclopedia, 2014. [Online; accessed 14-December-2014].