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Université  
de Paris



# Eureka moments in the acquisition of mathematical concepts

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# Introduction

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How does it feel to understand a mathematical concept ?

# Context and motivation

- Famous scientists experience sudden “Eureka moments”, notably in mathematics (Poincaré 1908, Hadamard 1959)
  - Sudden and unexpected understanding
  - Feeling of certainty

## Context and motivation

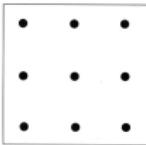
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  - No awareness that it was about to come
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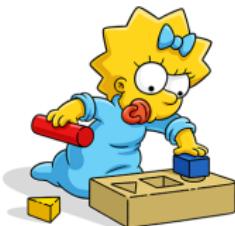
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The nine dot problem, Lung and Dominowski 1985

So far, no evidence of insights while learning a new concept



## Methodological difficulties

Learning science concepts is difficult and protracted (Carey 2009, Weber 2002, Asmuth and Rips 2006, Vosniadou 2019)

- Conducted in classrooms (Cross sectional designs or longitudinal designs with long delays between sessions)
- Lab experiment with simple learning targets (Ohlsson 2009) for example inferring a rule for categorizing images

# Goals

- Conceive a one session paradigm of conceptual learning, to access participants objective and subjective progress and modulate level of understanding through conditions
- See if participants report experiencing insights while learning a new concept and if these insights are related to performance
- Evaluate the relation between experiences of insights and other forms of introspection in a concept learning context

## Material and Methods

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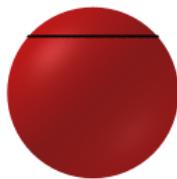
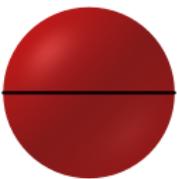
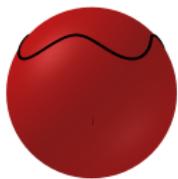
## The new concept : geodesic

Geodesic is the generalization of straight line to curved surfaces

Starting from a given point on a surface, it is a path which has a constant direction and never turns

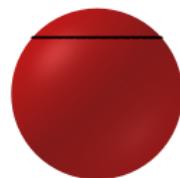
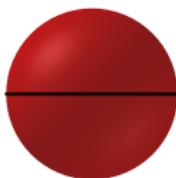
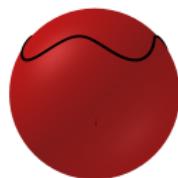
# Learning situation : geodesic on the sphere

"Is it straight?"



# Learning situation : geodesic on the sphere

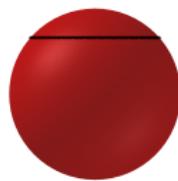
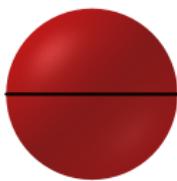
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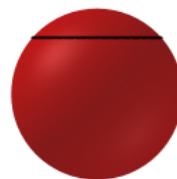
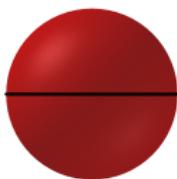
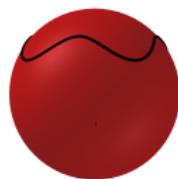
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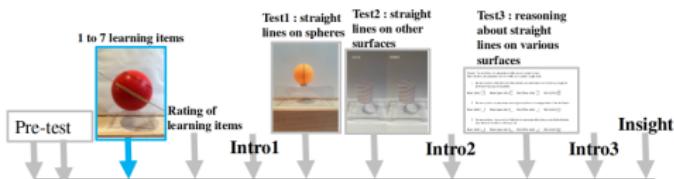


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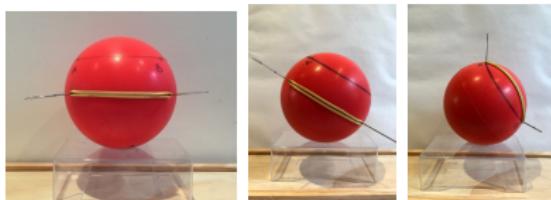
"Is it straight?"



# Learning phase



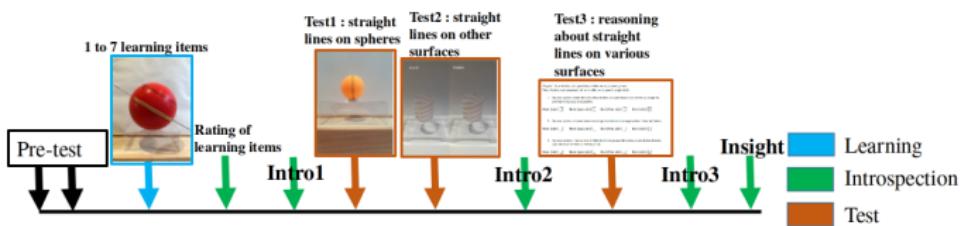
## The rubber band lesson



The elastic is straight and follows greater circles on a sphere, but when applied to a small circle, the elastic does not follow the line

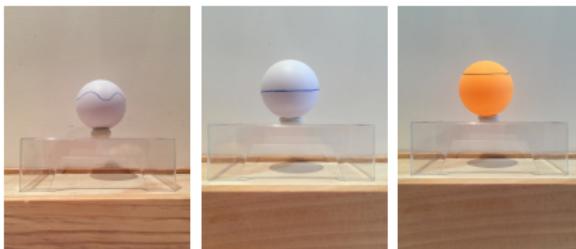
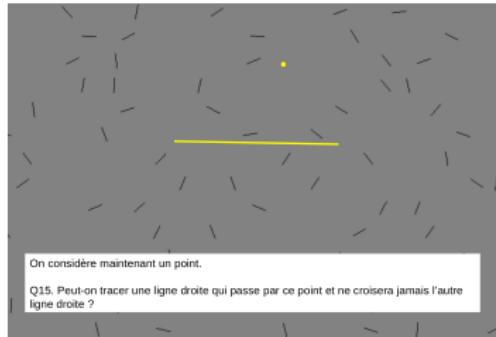
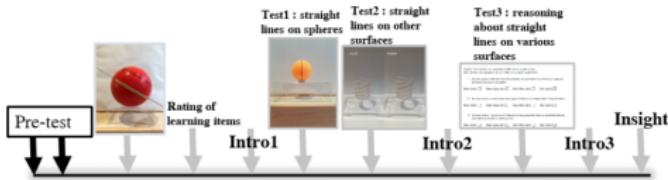
Four conditions : 1, 3, 5 or 7 different lessons

# Experimental protocol

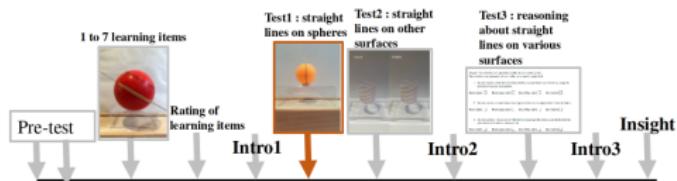


N=56, 18-43 years ( $M=25.5$ ), time  $\approx 1h30$

# Preliminary tests



# Performance measures : Test1



Not straight non planar lines

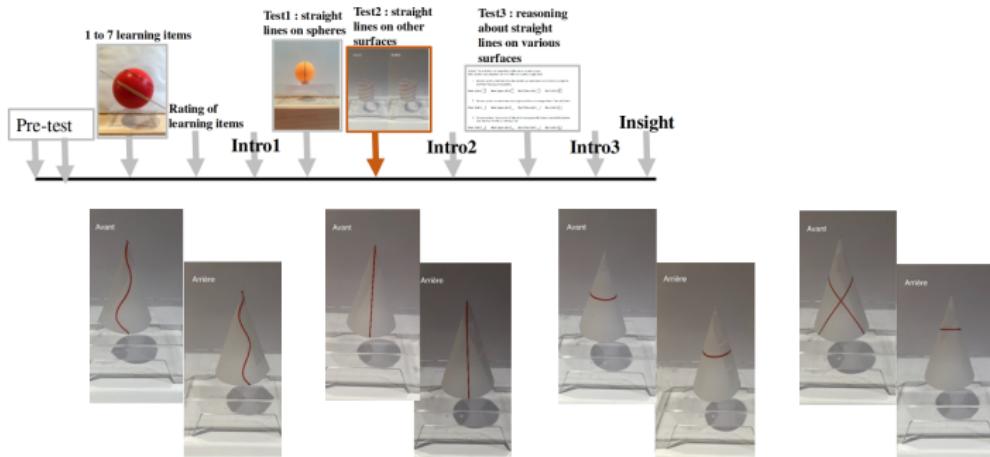


Straight planar lines

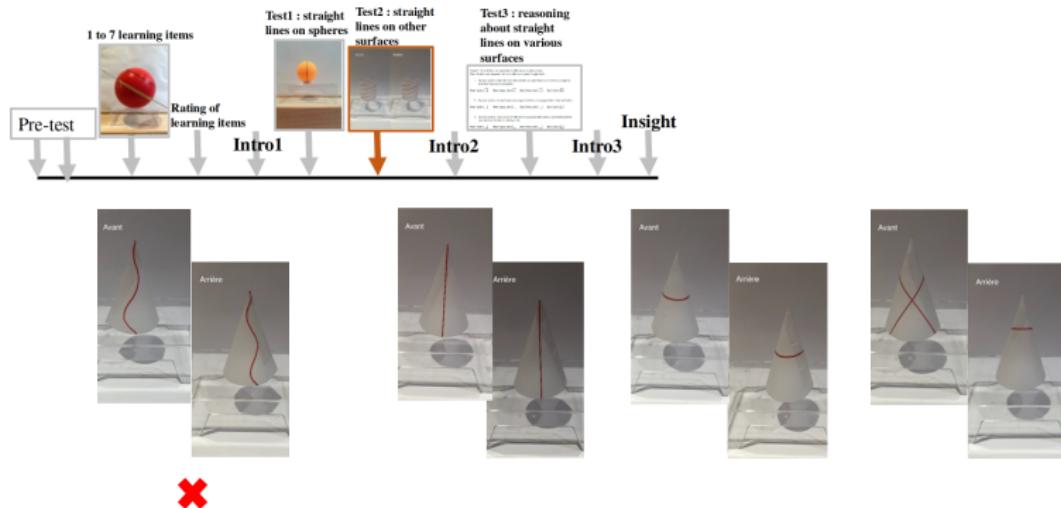


Not straight planar lines

# Performance measures: Test2



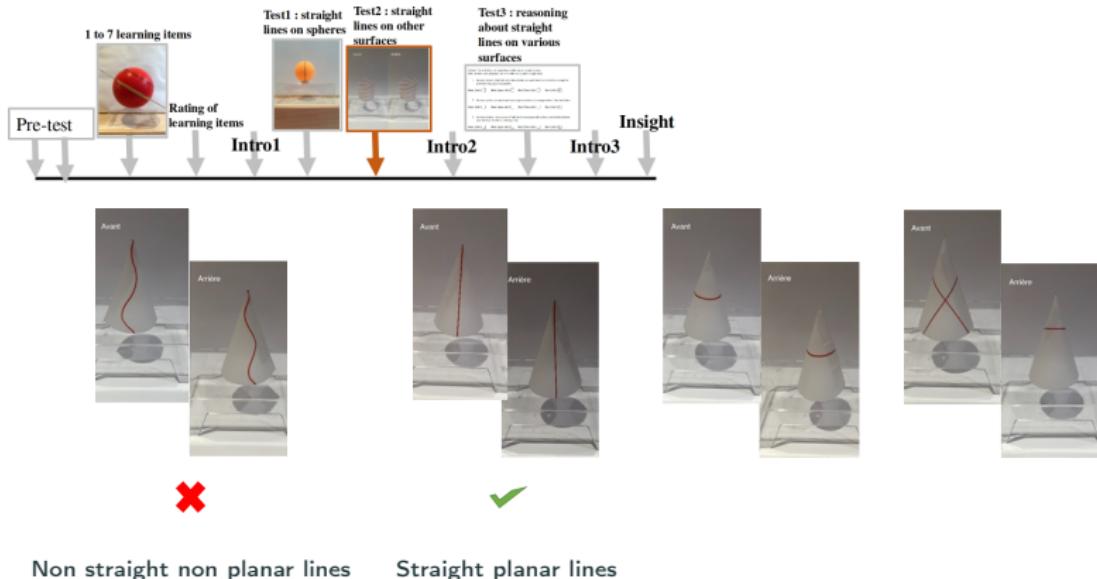
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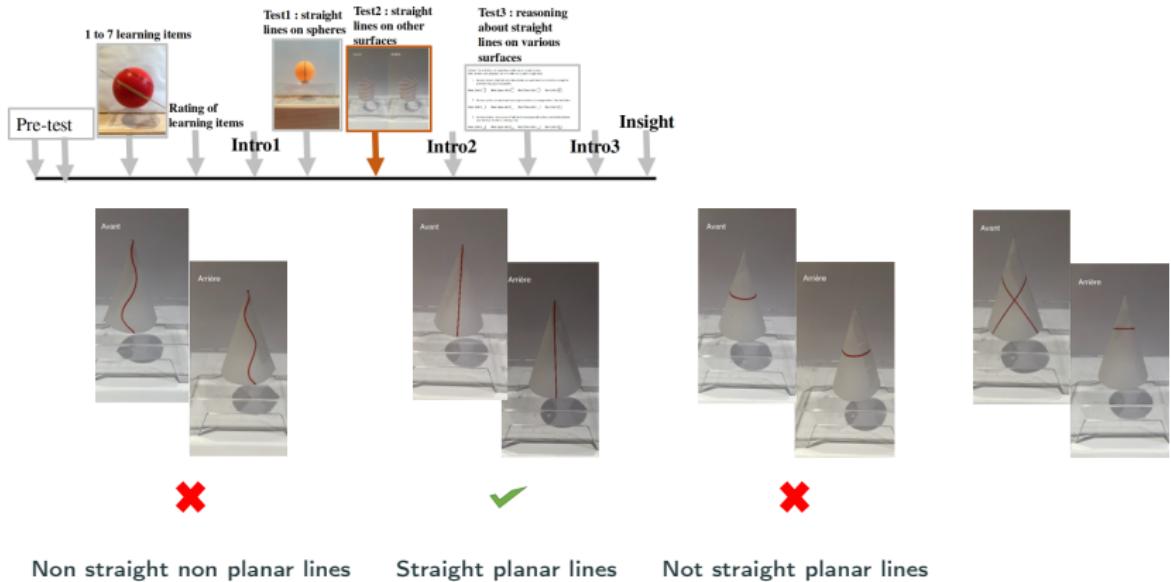
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Non straight non planar lines

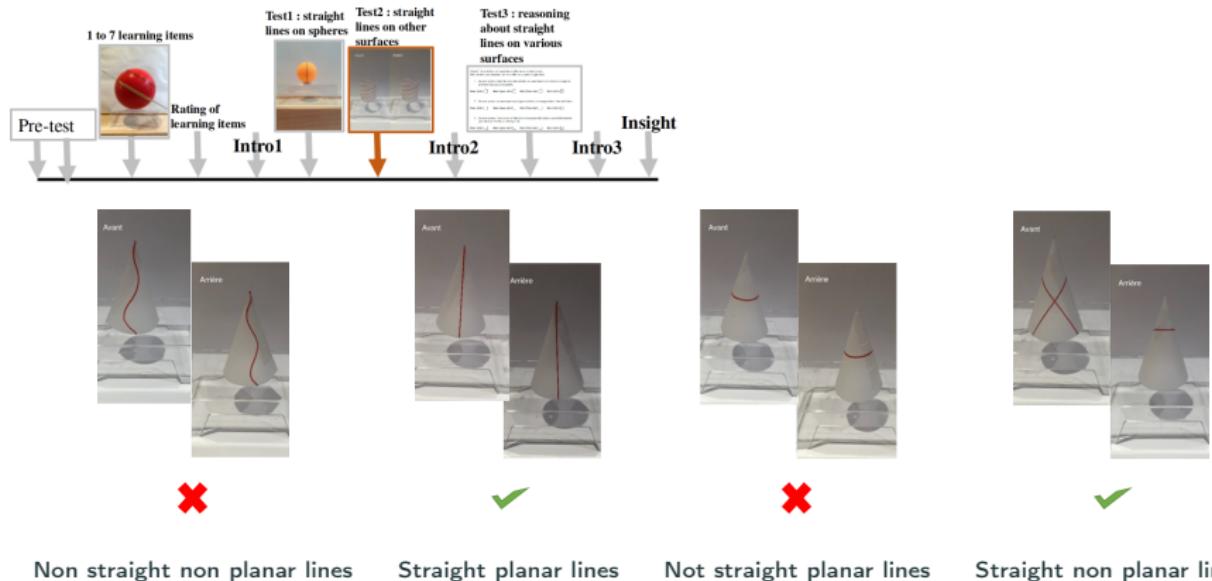
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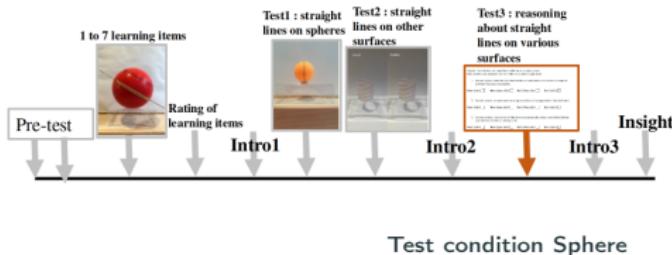
# Performance measures: Test2



# Performance measures: Test2



# Performance measures: Test3



4. Sur une sphère, on peut tracer deux lignes droites qui ne se coupent jamais.

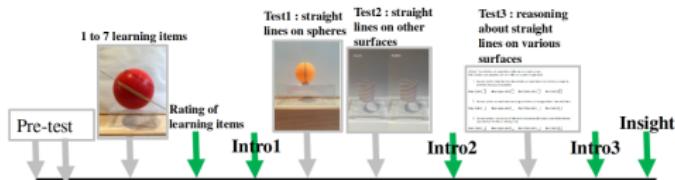
Non (sûr)  Non (pas sûr)  Oui (Pas sûr)  Oui (sûr)

## Test condition All surfaces

10. Il existe une surface sur laquelle par un point donné ne passe aucune ligne droite parallèle à une autre droite donnée.

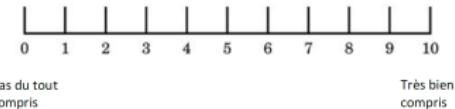
Non (sûr)  Non (pas sûr)  Oui (Pas sûr)  Oui (sûr)

# Introspection measures



- Feeling Of Understading (FOU) rating from 0 to 10

Pensez-vous avoir compris la notion de ligne droite ?



- Insight reports “Did you experienced an insight ?”

(description adapted from Danek and Wiley 2017)

## Analyses

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- Is performance influenced by the number of lessons ?
- Do participants report insights ?
- Are insights related to a good performance ?
- What is the relation between insight reports and feelings of knowing ?

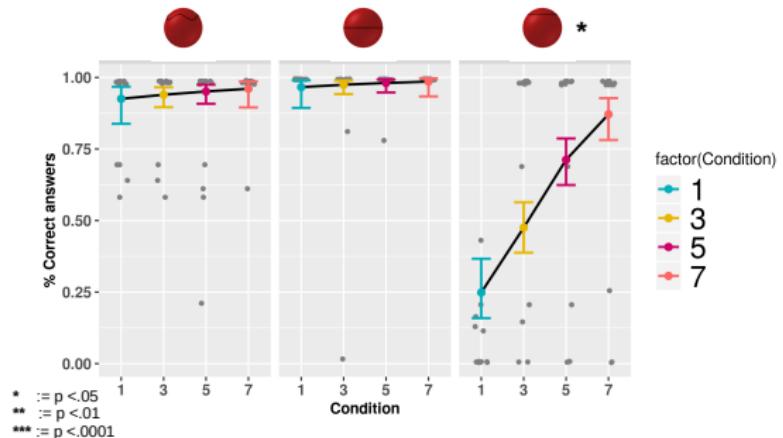
## Results

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**Is performance influenced by  
the number of lessons ?**

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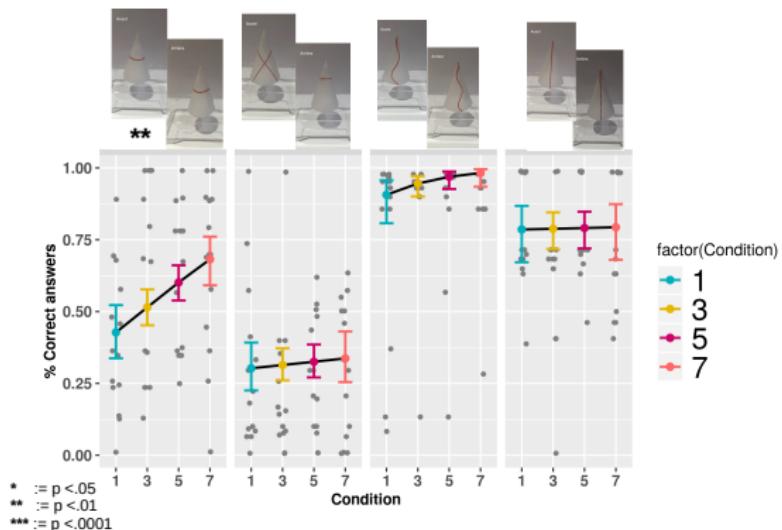
# Is performance influenced by the number of lessons ? first test



*Predictions of the logistic mixed model by test condition and number of lessons, with individual participants' performance, corrected for years of mathematic education after 10th grade*

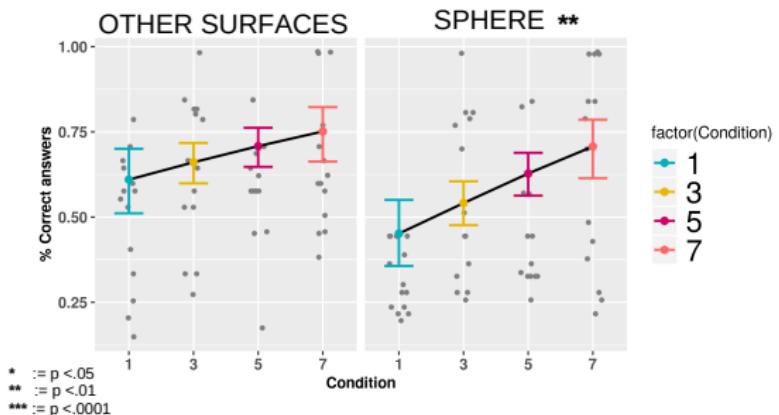
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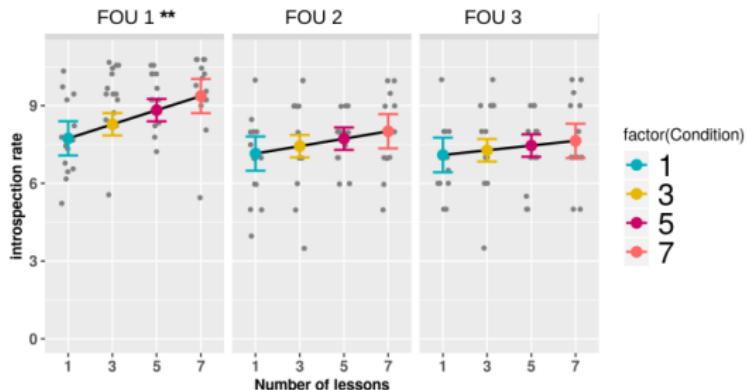
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# Is performance influenced by the number of lessons ? Third test



N=56, 18-43 years ( $M=25.5$ )

# Is FOU influenced by the number of lessons ?



*Predictions of the mixed model by measurement time and number of lessons, with individual participants feeling of knowing, corrected for years of mathematic education after 10th grade*

*Feeling of understanding 1: measured just after participants completed the teaching phase;  
Feeling of understanding 2: measured after the various surfaces straight lines test; Feeling of  
understanding 3: measured after the reasoning test*

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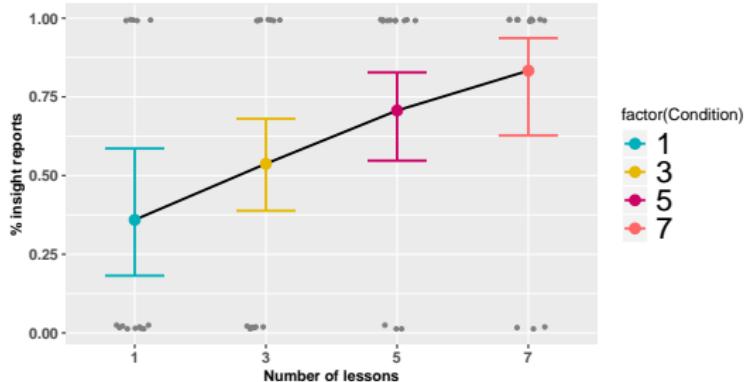
Did participants report insights  
and are these insights related  
to a good performance ?

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# Insight reports

60 % of insight reports, more insights reported in the conditions with more lessons (\*\*)

Number of lessons effect on insight reports



*Predictions of the binomial linear model by number of lessons, with individual participants' insight reports, corrected for years of mathematic education after 10th grade*

N=56, 18-43 years ( $M=25.5$ )

What is the relation between  
the different introspection  
measures ?

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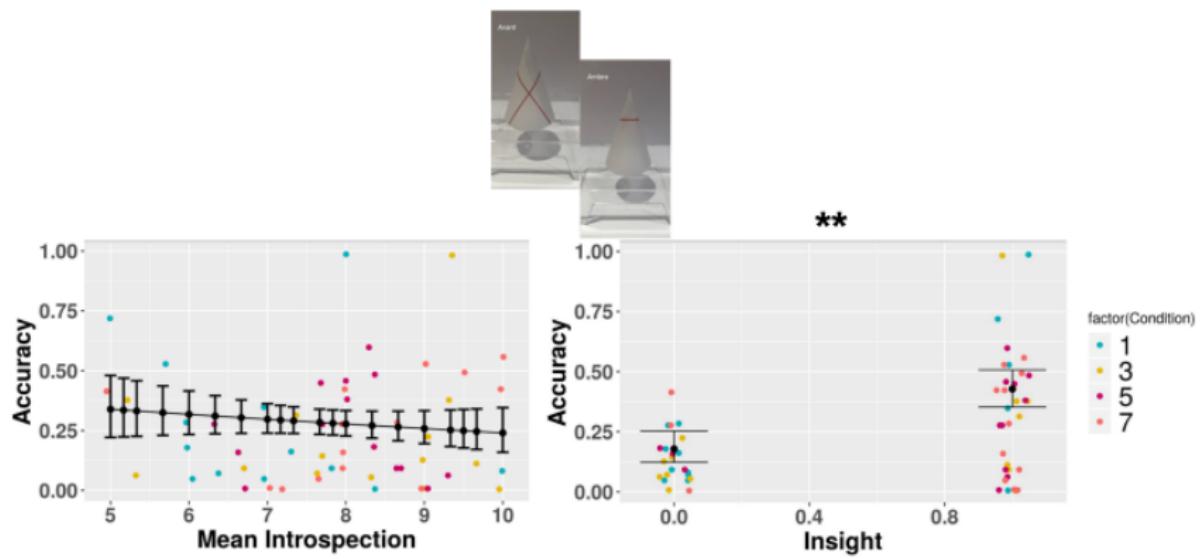
# Relations between introspection measures

Introspection measure	FOU 1	FOU 2	FOU 3	Insight
FOU 1	X	.6 (<.0001)	.6 (<.0001)	.27 (.13)
FOU 2	.6 (<.0001)	X	.85 (.<0001)	.19 (.33)
FOU 3	.6 (<.0001)	.85 (<.0001)	X	.17 (.33)
Insight	.15 (.81)	.14 (.81)	.15 (.81)	X

Spearman's rho coefficients and p values for pairwise correlation tests. Above diagonal : zero-order correlation, below: with number of lessons and education in mathematics as covariates. All p values were corrected for multiple comparisons using Holm's method.

Feeling of understanding 1: measured just after participants completed the teaching phase;  
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# Relations between generalization test condition and introspection measures



*Predictions of the logistic mixed model by number of lessons, mean introspection and insight reports, with individual participants' performance, corrected for years of mathematic education after 10th grade*

N=56, 18-43 years ( $M=25.5$ )

## Conclusion

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## Overview of results

Participants learned and they learned a concept

- Reading more lessons led to better performance in several post-teaching test conditions
- Two characteristic signatures of conceptual learning
  - First, learning was difficult (cf positive linear effects of the number of lessons on test performance, accounting that all lessons had the same maths content)
  - Content learned was inferentially rich (ability to draw inferences from this information)  
Non trivial inferences about the properties of straight lines in spherical geometry : realize that two straight lines drawn on a sphere can never be parallel

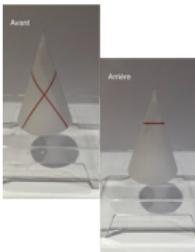
## Overview of results

Participants reported insights and these insights are related to the learning process

- After reading this description, a little over half of our participants reported experiencing insight episodes during the course of our experiment (60%)
- Modulated by our experimental manipulation and consistent with performance. (not simply reflect variations in the personalities, or in level in mathematics)
- Relation between insights and performance took the form of an interaction (insights related to better performance in some but not all test conditions), even when controlling for number of lessons, level in maths, and FOU

Insights and introspection are dissociated, and only insights are related to learning in a test of generalization for their new concept of straight line

- FOU measures strongly correlated but none was correlated to experiences of insight
- Insight experiences uniquely associated with performance in one of our test condition, even after controlling for FOU



- FOU was also related to post-teaching performance independently of the presence of insights (relation is difficult to decipher as the ratings of FOU did not map any test condition)

## Discussion

Do the insights experienced during concept learning and during problem solving reflect similar psychological mechanisms?

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- Continuous vs discrete

## Discussion

Is concept learning subtended by a cumulative process?

**Thanks for your attention !**