SDP relaxation formulation of the OPF

Charly Alizadeh

February 2024

$$
\begin{array}{rrl}
\min & \sum_{g\in\mathcal{G}} c_{g1} * \mathcal{S}^r_g + c_{g0} & \\
\forall g \in \mathcal{G} & \underline{\mathcal{S}_g} \leq \mathcal{S}_g & \leq \overline{\mathcal{S}_g} \\
\forall b \in \mathcal{B} & \tilde{S}_b + Tr(\mathbf{\Psi}_b W) + iTr(\hat{\mathbf{\Psi}}_b) & = \sum_{g\in\mathcal{G}_b} \mathcal{S}_g \\
\forall (b,a,h) \in \mathcal{L} & Tr(\mathbf{\Phi}_{bah} W)^2 + Tr(\hat{\mathbf{\Phi}}_{bah} W)^2 & \leq \overline{S}^2_{bah} \\
\forall (b,a,h) \in \mathcal{L}_0 & [\tan(\underline{\eta}_{bah}), \tan(\overline{\eta}_{bah})] Tr(\mathbf{\Theta}_{ba} W) & \ni Tr(\hat{\mathbf{\Theta}}_{ba} W) \\
\forall (b,a,h) \in \mathcal{L}_0 & Tr(\mathbf{\Theta}_{ba} W) & \geq 0 \\
 & e_r^T V^c = 0 \wedge e_r^T V^r & \geq 0 \\
\forall b \in \mathcal{B} & \underline{V}^2_b \leq Tr(\mathbf{\Theta}_{bb} W) & \leq \overline{V}^2_b \\
 & W & \succeq 0
\end{array} \tag{1}
$$

Let's reformulate this model using matrix indexes.

We define the admittance matrix $\mathcal{Y}$ as follows:

$$
\begin{array}{lll}
\forall b \in \mathcal{B} & \mathcal{Y}_{bb} & = A_b + \sum_{(b,a,h)\in\mathcal{L}_0} Y^{ff}_{bah} + \sum_{(b,a,h)\in\mathcal{L}_1} Y^{tt}_{abh} \\
\forall b, a \in \mathcal{B}, b \neq a & \mathcal{Y}_{ba} & = \sum_{(b,a,h)\in\mathcal{L}_0} Y^{ft}_{bah} + \sum_{(b,a,h)\in\mathcal{L}_1} Y^{tf}_{abh}
\end{array}
$$

1 Power Balance

Let $\mathcal{Y}_b$ the matrix $\mathcal{Y}$ with only zeros except for the row b

We have:

$$
\mathbf{\Psi}_b = \frac{1}{2} \begin{bmatrix} \mathcal{Y}^r_b + \mathcal{Y}^{rT}_b & \mathcal{Y}^{cT}_b - \mathcal{Y}^c_b \\ \mathcal{Y}^c_b - \mathcal{Y}^{cT}_b & \mathcal{Y}^r_b + \mathcal{Y}^{rT}_b \end{bmatrix}
$$

and

$$
\hat{\mathbf{\Psi}}_b = -\frac{1}{2} \begin{bmatrix} \mathcal{Y}^c_b + \mathcal{Y}^{cT}_b & \mathcal{Y}^r_b - \mathcal{Y}^{rT}_b \\ \mathcal{Y}^{rT}_b - \mathcal{Y}^r_b & \mathcal{Y}^c_b + \mathcal{Y}^{cT}_b \end{bmatrix}
$$

Which gives us

$$\boldsymbol{\Psi}_b = \frac{1}{2}\begin{bmatrix} \begin{bmatrix} & & & Y_b^r[1] & & \\ & & & Y_b^r[2] & & \\ & & & \vdots & & \\ Y_b^r[1] & Y_b^r[2] & \cdots & 2Y_b^r[b] & \cdots & Y_b^r[n] \\ & & & \vdots & & \\ & & & Y_b^r[n] & & \end{bmatrix} & \begin{bmatrix} & & & Y_b^c[1] & & \\ & & & Y_b^c[2] & & \\ & & & \vdots & & \\ -Y_b^c[1] & -Y_b^c[2] & \cdots & 0 & \cdots & -Y_b^c[n] \\ & & & \vdots & & \\ & & & Y_b^c[n] & & \end{bmatrix} \\ \begin{bmatrix} & & & -Y_b^c[1] & & \\ & & & -Y_b^c[2] & & \\ & & & \vdots & & \\ Y_b^c[1] & Y_b^c[2] & \cdots & 0 & \cdots & Y_b^c[n] \\ & & & \vdots & & \\ & & & -Y_b^c[n] & & \end{bmatrix} & \begin{bmatrix} & & & Y_b^r[1] & & \\ & & & Y_b^r[2] & & \\ & & & \vdots & & \\ Y_b^r[1] & Y_b^r[2] & \cdots & 2Y_b^r[b] & \cdots & Y_b^r[n] \\ & & & \vdots & & \\ & & & Y_b^r[n] & & \end{bmatrix} \end{bmatrix}$$

and

$$\hat{\boldsymbol{\Psi}}_b = -\frac{1}{2}\begin{bmatrix} \begin{bmatrix} & & & Y_b^c[1] & & \\ & & & Y_b^c[2] & & \\ & & & \vdots & & \\ Y_b^c[1] & Y_b^c[2] & \cdots & 2Y_b^c[b] & \cdots & Y_b^c[n] \\ & & & \vdots & & \\ & & & Y_b^c[n] & & \end{bmatrix} & \begin{bmatrix} & & & -Y_b^r[1] & & \\ & & & -Y_b^r[2] & & \\ & & & \vdots & & \\ Y_b^r[1] & Y_b^r[2] & \cdots & 0 & \cdots & Y_b^r[n] \\ & & & \vdots & & \\ & & & -Y_b^r[n] & & \end{bmatrix} \\ \begin{bmatrix} & & & Y_b^r[1] & & \\ & & & Y_b^r[2] & & \\ & & & \vdots & & \\ -Y_b^r[1] & -Y_b^r[2] & \cdots & 0 & \cdots & -Y_b^r[n] \\ & & & \vdots & & \\ & & & Y_b^r[n] & & \end{bmatrix} & \begin{bmatrix} & & & Y_b^c[1] & & \\ & & & Y_b^c[2] & & \\ & & & \vdots & & \\ Y_b^c[1] & Y_b^c[2] & \cdots & 2Y_b^c[b] & \cdots & Y_b^c[n] \\ & & & \vdots & & \\ & & & Y_b^c[n] & & \end{bmatrix} \end{bmatrix}$$

$$\begin{aligned}
&\tilde{S}_b + Tr(\boldsymbol{\Psi}_b W) + iTr(\hat{\boldsymbol{\Psi}}_b W) = \sum_{g \in \mathcal{G}_b} \mathcal{S}_g \\
&\Leftrightarrow \\
&\tilde{S}_b \\
&+ \sum_{i=1}^{n} \mathcal{Y}[b,i]^r (W[b,i] + W[b+n, n+i]) + \sum_{i=1, i\neq b}^{n} \mathcal{Y}[b,i]^c (W[n+b,i] - W[n+i,b]) \\
&-i\left(\sum_{i=1}^{n} \mathcal{Y}[b,i]^c (W[b,i] + W[b+n,n+i]) + \sum_{i=1,i\neq b}^{n} \mathcal{Y}[b,i]^r (-W[n+b,i] + W[n+i,b])\right) \\
&= \sum_{g \in \mathcal{G}_b} \mathcal{S}_g
\end{aligned}$$

2 Power limits

We have

$$\Phi_{bah} := \left\{ \begin{array}{ll} Y_{bah}^{ff} e_b e_b^T + Y_{bah}^{ft} e_b e_a^T, & if\ (b,a,h) \in \mathcal{L}_0 \\ Y_{abh}^{tt} e_b e_b^T + Y_{abh}^{tf} e_b e_a^T, & if\ (b,a,h) \in \mathcal{L}_1 \end{array} \right\}$$

$$\Phi_{bah}{}^{\mathcal{L}_0} = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & Y_{bah}^{ff} & \cdots & Y_{bah}^{ft} & \cdots \\ & \vdots & & \vdots & \\ & \vdots & & \vdots & \\ & \vdots & & \vdots & \end{bmatrix}$$

$$\Phi_{bah}{}^{\mathcal{L}_1} = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & Y_{bah}^{tt} & \cdots & Y_{bah}^{tf} & \cdots \\ & \vdots & & \vdots & \\ & \vdots & & \vdots & \\ & \vdots & & \vdots & \end{bmatrix}$$

$$\mathbf{\Phi}_{bah} := \frac{1}{2} \begin{bmatrix} \Phi_{bah}^{r} + \Phi_{bah}^{rT} & \Phi_{bah}^{cT} - \Phi_{bah}^{c} \\ \Phi_{bah}^{c} - \Phi_{bah}^{cT} & \Phi_{bah}^{r} + \Phi_{bah}^{rT} \end{bmatrix}$$

$$\hat{\mathbf{\Phi}}_{bah} := -\frac{1}{2} \begin{bmatrix} \Phi_{bah}^{c} + \Phi_{bah}^{cT} & \Phi_{bah}^{r} - \Phi_{bah}^{rT} \\ \Phi_{bah}^{rT} - \Phi_{bah}^{r} & \Phi_{bah}^{c} + \Phi_{bah}^{cT} \end{bmatrix}$$

$$\mathbf{\Phi}_{bah}{}^{\mathcal{L}_0} = \frac{1}{2} \begin{bmatrix} \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 2Y_{bah}^{ff\ r} & \cdots & Y_{bah}^{ft\ r} & \cdots \\ & \vdots & & \vdots & \\ & Y_{bah}^{ft\ r} & & \vdots & \\ & \vdots & & \vdots & \end{bmatrix} & \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & -Y_{bah}^{ft\ c} & \cdots \\ & \vdots & & \vdots & \\ & Y_{bah}^{ft\ c} & & \vdots & \\ & \vdots & & \vdots & \end{bmatrix} \\ \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & Y_{bah}^{ft\ c} & \cdots \\ & \vdots & & \vdots & \\ & -Y_{bah}^{ft\ c} & & \vdots & \\ & \vdots & & \vdots & \end{bmatrix} & \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 2Y_{bah}^{ff\ r} & \cdots & Y_{bah}^{ft\ r} & \cdots \\ & \vdots & & \vdots & \\ & Y_{bah}^{ft\ r} & & \vdots & \\ & \vdots & & \vdots & \end{bmatrix} \end{bmatrix}$$

$$
\boldsymbol{\Phi}_{bah}{}^{\mathcal{L}_1} = -\frac{1}{2}
\begin{bmatrix}
\begin{bmatrix}
 & \vdots & & \vdots & \\
\cdots & 2Y_{bah}^{tt\ r} & \cdots & Y_{bah}^{tf\ r} & \cdots \\
 & \vdots & & \vdots & \\
 & Y_{bah}^{tf\ r} & & \vdots & \\
 & \vdots & & \vdots &
\end{bmatrix}
&
\begin{bmatrix}
 & \vdots & & \vdots & \\
\cdots & 0 & \cdots & -Y_{bah}^{tf\ c} & \cdots \\
 & \vdots & & \vdots & \\
 & Y_{bah}^{tf\ c} & & \vdots & \\
 & \vdots & & \vdots &
\end{bmatrix}
\\[2ex]
\begin{bmatrix}
 & \vdots & & \vdots & \\
\cdots & 0 & \cdots & Y_{bah}^{tf\ c} & \cdots \\
 & \vdots & & \vdots & \\
 & -Y_{bah}^{tf\ c} & & \vdots & \\
 & \vdots & & \vdots &
\end{bmatrix}
&
\begin{bmatrix}
 & \vdots & & \vdots & \\
\cdots & 2Y_{bah}^{tt\ r} & \cdots & Y_{bah}^{tf\ r} & \cdots \\
 & \vdots & & \vdots & \\
 & Y_{bah}^{tf\ r} & & \vdots & \\
 & \vdots & & \vdots &
\end{bmatrix}
\end{bmatrix}
$$

$$
\hat{\boldsymbol{\Phi}}_{bah}^{\mathcal{L}_0} = \frac{1}{2}
\begin{bmatrix}
\begin{bmatrix}
 & \vdots & & \vdots & \\
\cdots & 2Y_{bah}^{ff\ c} & \cdots & Y_{bah}^{ft\ c} & \cdots \\
 & \vdots & & \vdots & \\
 & Y_{bah}^{ft\ c} & & \vdots & \\
 & \vdots & & \vdots &
\end{bmatrix}
&
\begin{bmatrix}
 & \vdots & & \vdots & \\
\cdots & 0 & \cdots & Y_{bah}^{ft\ r} & \cdots \\
 & \vdots & & \vdots & \\
 & -Y_{bah}^{ft\ r} & & \vdots & \\
 & \vdots & & \vdots &
\end{bmatrix}
\\[2ex]
\begin{bmatrix}
 & \vdots & & \vdots & \\
\cdots & 0 & \cdots & -Y_{bah}^{ft\ r} & \cdots \\
 & \vdots & & \vdots & \\
 & Y_{bah}^{ft\ r} & & \vdots & \\
 & \vdots & & \vdots &
\end{bmatrix}
&
\begin{bmatrix}
 & \vdots & & \vdots & \\
\cdots & 2Y_{bah}^{ff\ c} & \cdots & Y_{bah}^{ft\ c} & \cdots \\
 & \vdots & & \vdots & \\
 & Y_{bah}^{ft\ c} & & \vdots & \\
 & \vdots & & \vdots &
\end{bmatrix}
\end{bmatrix}
$$

$$\hat{\mathbf{\Phi}}_{bah}^{\mathcal{L}_1} = -\frac{1}{2}\begin{bmatrix} \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 2{Y_{bah}^{tt}}^{c} & \cdots & {Y_{bah}^{tf}}^{c} & \cdots \\ & \vdots & & \vdots & \\ & {Y_{bah}^{tf}}^{c} & & \vdots & \\ & \vdots & & \vdots & \end{bmatrix} & \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & {Y_{bah}^{tf}}^{r} & \cdots \\ & \vdots & & \vdots & \\ & -{Y_{bah}^{tf}}^{r} & & \vdots & \\ & \vdots & & \vdots & \end{bmatrix} \\ \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & -{Y_{bah}^{tf}}^{r} & \cdots \\ & \vdots & & \vdots & \\ & {Y_{bah}^{tf}}^{r} & & \vdots & \\ & \vdots & & \vdots & \end{bmatrix} & \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 2{Y_{bah}^{tt}}^{c} & \cdots & {Y_{bah}^{tf}}^{c} & \cdots \\ & \vdots & & \vdots & \\ & {Y_{bah}^{tf}}^{c} & & \vdots & \\ & \vdots & & \vdots & \end{bmatrix} \end{bmatrix}$$

$$Tr(\mathbf{\Phi}_{bah}W)_{\mathcal{L}_0}$$
$$=$$
$${Y_{bah}^{ff}}^{r}(W[b,b]+W[n+b,n+b])+{Y_{bah}^{ft}}^{r}(W[b,a]+W[n+b,n+a])+{Y_{bah}^{ft}}^{c}(W[n+b,a])-W[b,n+a])$$

$$Tr(\mathbf{\Phi}_{bah}W)_{\mathcal{L}_1}$$
$$=$$
$$-({Y_{bah}^{tt}}^{r}(W[b,b]+W[n+b,n+b])+{Y_{bah}^{tf}}^{r}(W[b,a]+W[n+b,n+a])+{Y_{bah}^{tf}}^{c}(W[n+b,a])-W[b,n+a]))$$

$$Tr(\hat{\mathbf{\Phi}}_{bah}W)_{\mathcal{L}_0}$$
$$=$$
$${Y_{bah}^{ff}}^{c}(W[b,b]+W[n+b,n+b])+{Y_{bah}^{ft}}^{c}(W[b,a]+W[n+b,n+a])+{Y_{bah}^{ft}}^{r}(-W[n+b,a])+W[b,n+a])$$

$$Tr(\hat{\mathbf{\Phi}}_{bah}W)_{\mathcal{L}_1}$$
$$=$$
$$-({Y_{bah}^{tt}}^{c}(W[b,b]+W[n+b,n+b])+{Y_{bah}^{tf}}^{c}(W[b,a]+W[n+b,n+a])+{Y_{bah}^{tf}}^{r}(-W[n+b,a])+W[b,n+a]))$$

3 $\mathbf{\Theta}$ and $\hat{\mathbf{\Theta}}$ equations

$$\mathbf{\Theta} = \frac{1}{2}\begin{bmatrix} \begin{bmatrix} . & . & . & . \\ . & . & 1 & . \\ . & 1 & . & . \\ . & . & . & . \end{bmatrix} & \begin{bmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix} \\ \begin{bmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix} & \begin{bmatrix} . & . & . & . \\ . & . & 1 & . \\ . & 1 & . & . \\ . & . & . & . \end{bmatrix} \end{bmatrix}$$

$$\hat{\mathbf{\Theta}} = -\frac{1}{2}\begin{bmatrix} \begin{bmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix} & \begin{bmatrix} . & . & . & . \\ . & . & 1 & . \\ . & -1 & . & . \\ . & . & . & . \end{bmatrix} \\ \begin{bmatrix} . & . & . & . \\ . & . & -1 & . \\ . & 1 & . & . \\ . & . & . & . \end{bmatrix} & \begin{bmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix} \end{bmatrix}$$

$$Tr(\mathbf{\Theta_{ba}}W) = W[b,a] + W[n+b,n+a]$$

$$Tr(\hat{\mathbf{\Theta}}_{\mathbf{ba}}W) = -W[b,n+a] + W[n+b,a]$$

$$Tr(\mathbf{\Theta_{bb}}W) = W[b,b] + W[n+b,n+b]$$

$$Tr(\hat{\mathbf{\Theta}}_{\mathbf{bb}}W) = 0$$