# SDP relaxation formulation of the OPF

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February 2024

$$
\begin{aligned}
\min_{v,S} \quad & \sum_{g\in G} c_g Re(S_g) + k_g \\
s.c. \quad & S_n = S_n^d + \sum_{l=(n,d)} \left(+\frac{\overline{y_l} - \mathbf{j} b_l}{\tau_l^2}|v_n|^2 - \overline{y_l}\frac{e^{\mathbf{j}\theta_l}}{\tau_l} v_n \overline{v_d}\right) \\
& - \sum_{l=(o,n)} \left(+(\overline{y_l} - \mathbf{j} b_l)|v_n|^2 - \overline{y_l}\frac{e^{-\mathbf{j}\theta_l}}{\tau_l} v_n \overline{v_o}\right) && \forall n \in N \\
& P_n^{min} \le Re(S_n) \le P_n^{max} && \forall n \in G \\
& Q_n^{min} \le Im(S_n) \le Q_n^{max} && \forall n \in G \\
& S_n = 0 && \forall n \notin G \\
& (v_n^{min})^2 \le |v_n|^2 \le (v_n^{max})^2 && \forall n \in N \\
& |+((y_l + j\tfrac{b_l}{2})\tfrac{1}{\tau_l^2})v_{o_l} - (y_l \tfrac{1}{\tau_l e^{-j\theta_l}})v_{d_l}|^2 \le (i_l^{max})^2 && \forall l \in B \\
& |-(y_l \tfrac{1}{\tau_l e^{j\theta_l}})v_{o_l} + (y_l + j\tfrac{b_l}{2})v_{d_l}|^2 \le (i_l^{max})^2 && \forall l \in B \\
& v_n \in \mathbb{C}, S_n \in \mathbb{C} && \forall n \in N.
\end{aligned}
\tag{1}
$$

The variables of this problem are:

- the complexe power generated at all the buses
  $S_n, \forall n \in N$ $(S_n = Re(S_n) + \mathbf{j} Im(S_n))$
- the complexe voltage at all the buses
  $v_n, \forall n \in N$ $(v_n = Re(v_n) + \mathbf{j} Im(v_n))$

First let's define some reformulate this problem using constant

$$
\begin{aligned}
\min_{v,S} \quad & \sum_{g\in G} c_g \mathbf{Re}(\mathbf{S_g}) + k_g \\
s.c. \quad & \mathbf{S_n} = S_n^d + \sum_{l=(n,d)} (S_l^{orig(1)} |\mathbf{v_n}|^{\mathbf{2}} + S_l^{orig(2)} \mathbf{v_n}\overline{\mathbf{v_d}}) \\
& - \sum_{l=(o,n)} (S_l^{dest(1)} |\mathbf{v_n}|^{\mathbf{2}} + S_l^{dest(2)} \mathbf{v_n}\overline{\mathbf{v_o}}) \\
& P_n^{min} \le \mathbf{Re}(\mathbf{S_n}) \le P_n^{max} & \forall n \in G \\
& Q_n^{min} \le \mathbf{Im}(\mathbf{S_n}) \le Q_n^{max} & \forall n \in G \\
& \mathbf{S_n} = 0 & \forall n \notin G \\
& (v_n^{min})^2 \le |\mathbf{v_n}|^{\mathbf{2}} \le (v_n^{max})^2 & \forall n \in N \\
& |i^{orig(1)}\mathbf{v_{ol}} + i^{orig(2)}\mathbf{v_{dl}}|^2 \le (i_l^{max})^2 & \forall l \in B \\
& |i^{dest(1)}\mathbf{v_{ol}} + i^{dest(2)}\mathbf{v_{dl}}|^2 \le (i_l^{max})^2 & \forall l \in B \\
& \mathbf{v_n} \in \mathbb{C}, \mathbf{S_n} \in \mathbb{C} & \forall n \in N.
\end{aligned} \tag{2}
$$

with

$$
\begin{aligned}
S_l^{orig(1)} &= \frac{\overline{y_l} - \mathbf{j}b_l}{\tau_l^2} \\
S_l^{orig(2)} &= -\overline{y_l}\frac{e^{\mathbf{j}\theta_l}}{\tau_l} \\
S_l^{dest(1)} &= \overline{y_l} - \mathbf{j}b_l \\
S_l^{dest(2)} &= -\overline{y_l}\frac{e^{-\mathbf{j}\theta_l}}{\tau_l} \\
i^{orig(1)} &= \frac{\overline{y_l} + \mathbf{j}b_l}{\tau_l^2} \\
i^{orig(2)} &= -\overline{y_l}\frac{e^{-\mathbf{j}\theta_l}}{\tau_l} \\
i^{dest(1)} &= -\overline{y_l}\frac{e^{\mathbf{j}\theta_l}}{\tau_l} \\
i^{dest(2)} &= \overline{y_l} + \mathbf{j}b_l
\end{aligned}
$$

$$
\begin{aligned}
\min_{v,S} \quad & \sum_{g\in G} c_g \mathbf{Re}(\mathbf{S_g}) + k_g \\
s.c. \quad & \mathbf{Re}(\mathbf{S_n}) = Re(S_n^d) + \sum_{l=(n,d)} Re(S_l^{orig(1)}|\mathbf{v_n}|^{\mathbf{2}} + S_l^{orig(2)}\mathbf{v_n}\overline{\mathbf{v_d}}) \\
& - \sum_{l=(o,n)} Re(S_l^{dest(1)}|\mathbf{v_n}|^{\mathbf{2}} + S_l^{dest(2)}\mathbf{v_n}\overline{\mathbf{v_o}}) \\
& \\
& \mathbf{Im}(\mathbf{S_n}) = Im(S_n^d) + \sum_{l=(n,d)} Im(S_l^{orig(1)}|\mathbf{v_n}|^{\mathbf{2}} + S_l^{orig(2)}\mathbf{v_n}\overline{\mathbf{v_d}}) \\
& - \sum_{l=(o,n)} Im(S_l^{dest(1)}|\mathbf{v_n}|^{\mathbf{2}} + S_l^{dest(2)}\mathbf{v_n}\overline{\mathbf{v_o}}) \\
& P_n^{min} \leq \mathbf{Re}(\mathbf{S_n}) \leq P_n^{max} && \forall n \in G \\
& Q_n^{min} \leq \mathbf{Im}(\mathbf{S_n}) \leq Q_n^{max} && \forall n \in G \\
& \mathbf{Re}(\mathbf{S_n}) = 0 && \forall n \notin G \\
& \mathbf{Im}(\mathbf{S_n}) = 0 && \forall n \notin G \\
& (v_n^{min})^2 \leq |\mathbf{v_n}|^{\mathbf{2}} \leq (v_n^{max})^2 && \forall n \in N \\
& |i^{orig(1)}\mathbf{v_{ol}} + i^{orig(2)}\mathbf{v_{dl}}|^2 \leq (i_l^{max})^2 && \forall l \in B \\
& |i^{dest(1)}\mathbf{v_{ol}} + i^{dest(2)}\mathbf{v_{dl}}|^2 \leq (i_l^{max})^2 && \forall l \in B \\
& \mathbf{v_n} \in \mathbb{C}, \mathbf{S_n} \in \mathbb{C} && \forall n \in N.
\end{aligned} \tag{3}
$$

Using the matrix:

$$V = \begin{pmatrix} Re(v_1)^2 & Re(v_1)Im(v_1) & Re(v_1)Re(v_2) & Re(v_1)Im(v_2) & \dots \\ Re(v_1)Im(v_1) & Im(v_1)^2 & Im(v_1)Re(v_2) & Im(v_1)Im(v_2) & \dots \\ Re(v_1)Re(v_2) & Im(v_1)Re(v_2) & Re(v_2)^2 & Re(v_2)Im(v_2) & \dots \\ Re(v_1)Im(v_2) & Im(v_1)Im(v_2) & Re(v_2)Im(v_2) & Im(v_2)^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and the equalities:

$$\begin{aligned} |v_n|^2 &= Re(v_n)^2 + Im(v_n)^2 \\ v_n\overline{v_d} &= Im(v_n)Im(v_d) + Re(v_n)Re(v_d) + \mathbf{j}(Im(v_n)Re(v_d) - Re(v_n)Im(v_d)) \\ v_n\overline{v_o} &= Im(v_n)Im(v_o) + Re(v_n)Re(v_o) + \mathbf{j}(Im(v_n)Re(v_o) - Re(v_n)Im(v_o)) \end{aligned}$$

We can then express the model using the power variables $S_n$ and the coefficient in the matrix $V$.

$$Re(S_l^{orig(1)}|\mathbf{v_n}|^\mathbf{2} + S_l^{orig(2)}\mathbf{v_n}\overline{\mathbf{v_d}})$$
$$=$$
$$Re(S_l^{orig(1)})(\mathbf{Re(v_n)^2} + \mathbf{Im(v_n)^2}) + Re(S_l^{orig(2)})(\mathbf{Im(v_n)Im(v_d)} + \mathbf{Re(v_n)Re(v_d)})$$

$$Im(S_l^{orig(1)}|\mathbf{v_n}|^\mathbf{2} + S_l^{orig(2)}\mathbf{v_n}\overline{\mathbf{v_d}})$$
$$=$$
$$Im(S_l^{orig(1)})(\mathbf{Re(v_n)^2} + \mathbf{Im(v_n)^2}) + Im(S_l^{orig(2)})(\mathbf{Im(v_n)Re(v_d)} + \mathbf{Re(v_n)Im(v_d)})$$

$$Re(S_l^{dest(1)}|\mathbf{v_n}|^\mathbf{2} + S_l^{dest(2)}\mathbf{v_n}\overline{\mathbf{v_o}})$$
$$=$$
$$Re(S_l^{dest(1)})(\mathbf{Re(v_n)^2} + \mathbf{Im(v_n)^2}) + Re(S_l^{dest(2)})(\mathbf{Im(v_n)Im(v_o)} + \mathbf{Re(v_n)Re(v_o)})$$

$$Im(S_l^{dest(1)}|\mathbf{v_n}|^\mathbf{2} + S_l^{dest(2)}\mathbf{v_n}\overline{\mathbf{v_o}})$$
$$=$$
$$Im(S_l^{dest(1)})(\mathbf{Re(v_n)^2} + \mathbf{Im(v_n)^2}) + Im(S_l^{dest(2)})(\mathbf{Im(v_n)Re(v_o)} - \mathbf{Re(v_n)Im(v_o)})$$

$$Re(i^{orig(1)}\mathbf{v_{ol}} + i^{orig(2)}\mathbf{v_{dl}})$$
$$=$$
$$Re(i^{orig(1)})\mathbf{Re(v_{ol})} + Im(i^{orig(1)})\mathbf{Im(v_{ol})} + Re(i^{orig(2)})\mathbf{Re(v_{dl})} + Im(i^{orig(2)})\mathbf{Im(v_{dl})}$$

$$Im(i^{orig(1)}\mathbf{v_{ol}} + i^{orig(2)}\mathbf{v_{dl}})$$
$$=$$
$$Re(i^{orig(1)})\mathbf{Im(v_{ol})} + Im(i^{orig(1)})\mathbf{Re(v_{ol})} + Re(i^{orig(2)})\mathbf{Im(v_{dl})} + Im(i^{orig(2)})\mathbf{Re(v_{dl})}$$