Generalized linear models

Regression Models

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Linear models

- Linear models are the most useful applied statistical technique. However, they are not without their limitations.
  - Additive response models don't make much sense if the response is discrete, or stricly positive.
  - Additive error models often don't make sense, for example if the outcome has to be positive.
  - Transformations are often hard to interpret.
  - There's value in modeling the data on the scale that it was collected.
  - Particularly interpetable transformations, natural logarithms in specific, aren't applicable for negative or zero values.

Generalized linear models

- Introduced in a 1972 RSSB paper by Nelder and Wedderburn.
- Involves three components
  - An *exponential family* model for the response.
  - A systematic component via a linear predictor.
  - A link function that connects the means of the response to the linear predictor.

Example, linear models

- Assume that $Y_i \sim N(\mu_i, \sigma^2)$ (the Gaussian distribution is an exponential family distribution.)
- Define the linear predictor to be $\eta_i = \sum_{k=1}^{p} X_{ik} \beta_k$.
- The link function as g so that $g(\mu) = \eta$.
  - For linear models $g(\mu) = \mu$ so that $\mu_i = \eta_i$
- This yields the same likelihood model as our additive error Gaussian linear model

$$Y_i = \sum_{k=1}^{p} X_{ik} \beta_k + \epsilon_i$$

where $\epsilon_i \overset{iid}{\sim} N(0, \sigma^2)$

Example, logistic regression

- Assume that $Y_i \sim \text{Bernoulli}(\mu_i)$ so that $E[Y_i] = \mu_i$ where $0 \leq \mu_i \leq 1$.
- Linear predictor $\eta_i = \sum_{k=1}^{p} X_{ik} \beta_k$
- Link function $g(\mu) = \eta = \log\left(\frac{\mu}{1-\mu}\right)$ g is the (natural) log odds, referred to as the **logit**.
- Note then we can invert the logit function as

$$\mu_i = \frac{\exp(\eta_i)}{1 + \exp(\eta_i)} \quad \text{and} \quad 1 - \mu_i = \frac{1}{1 + \exp(\eta_i)}$$

Thus the likelihood is

$$\prod_{i=1}^{n} \mu_i^{y_i} (1 - \mu_i)^{1-y_i} = \exp\left(\sum_{i=1}^{n} y_i \eta_i\right) \prod_{i=1}^{n} (1 + \eta_i)^{-1}$$

Example, Poisson regression

- Assume that $Y_i \sim \text{Poisson}(\mu_i)$ so that $E[Y_i] = \mu_i$ where $0 \leq \mu_i$
- Linear predictor $\eta_i = \sum_{k=1}^{p} X_{ik} \beta_k$
- Link function $g(\mu) = \eta = \log(\mu)$
- Recall that e^x is the inverse of $\log(x)$ so that

$$\mu_i = e^{\eta_i}$$

Thus, the likelihood is

$$\prod_{i=1}^{n} (y_i!)^{-1} \mu_i^{y_i} e^{-\mu_i} \propto \exp\left(\sum_{i=1}^{n} y_i \eta_i - \sum_{i=1}^{n} \mu_i \right)$$

Some things to note

- In each case, the only way in which the likelihood depends on the data is through

$$\sum_{i=1}^{n} y_i \eta_i = \sum_{i=1}^{n} y_i \sum_{k=1}^{p} X_{ik} \beta_k = \sum_{k=1}^{p} \beta_k \sum_{i=1}^{n} X_{ik} y_i$$

  Thus if we don't need the full data, only $\sum_{i=1}^{n} X_{ik} y_i$. This simplification is a consequence of chosing so-called 'canonical' link functions.

- (This has to be derived). All models acheive their maximum at the root of the so called normal equations

$$0 = \sum_{i=1}^{n} \frac{(Y_i - \mu_i)}{\mathrm{Var}(Y_i)} W_i$$

  where W_i are the derivative of the inverse of the link function.

About variances

$$0 = \sum_{i=1}^{n} \frac{(Y_i - \mu_i)}{\mathrm{Var}(Y_i)} W_i$$

- For the linear model $\mathrm{Var}(Y_i) = \sigma^2$ is constant.
- For Bernoulli case $\mathrm{Var}(Y_i) = \mu_i(1 - \mu_i)$
- For the Poisson case $\mathrm{Var}(Y_i) = \mu_i$.
- In the latter cases, it is often relevant to have a more flexible variance model, even if it doesn't correspond to an actual likelihood

$$0 = \sum_{i=1}^{n} \frac{(Y_i - \mu_i)}{\phi\mu_i(1 - \mu_i)} W_i \quad \text{and} \quad 0 = \sum_{i=1}^{n} \frac{(Y_i - \mu_i)}{\phi\mu_i} W_i$$

- These are called 'quasi-likelihood' normal equations

Odds and ends

- The normal equations have to be solved iteratively. Resulting in $\hat{\beta}_k$ and, if included, $\hat{\phi}$.
- Predicted linear predictor responses can be obtained as $\hat{\eta} = \sum_{k=1}^{p} X_k \hat{\beta}_k$
- Predicted mean responses as $\hat{\mu} = g^{-1}(\hat{\eta})$
- Coefficients are interpretted as

$$g(E[Y|X_k = x_k + 1, X_{\sim k} = x_{\sim k}]) - g(E[Y|X_k = x_k, X_{\sim k} = x_{\sim k}]) = \beta_k$$

  or the change in the link function of the expected response per unit change in X_k holding other regressors constant.
- Variations on Newon/Raphson's algorithm are used to do it.
- Asymptotics are used for inference usually.
- Many of the ideas from linear models can be brought over to GLMs.

Generalized linear models, binary data

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Key ideas

- Frequently we care about outcomes that have two values
  - Alive/dead
  - Win/loss
  - Success/Failure
  - etc
- Called binary, Bernoulli or 0/1 outcomes
- Collection of exchangeable binary outcomes for the same covariate data are called binomial outcomes.

Example Baltimore Ravens win/loss

Ravens Data

```
download.file("https://dl.dropboxusercontent.com/u/7710864/data/ravensData.rda"
              , destfile="./data/ravensData.rda",method="curl")
load("./data/ravensData.rda")
head(ravensData)
```

```
  ravenWinNum ravenWin ravenScore opponentScore
1           1        W         24             9
2           1        W         38            35
3           1        W         28            13
4           1        W         34            31
5           1        W         44            13
6           0        L         23            24
```

Linear regression

$$RW_i = b_0 + b_1 RS_i + e_i$$

RW_i - 1 if a Ravens win, 0 if not

RS_i - Number of points Ravens scored

b_0 - probability of a Ravens win if they score 0 points

b_1 - increase in probability of a Ravens win for each additional point

e_i - residual variation due

Linear regression in R

```
lmRavens <- lm(ravensData$ravenWinNum ~ ravensData$ravenScore)
summary(lmRavens)$coef
```

```
                    Estimate Std. Error t value Pr(>|t|)
(Intercept)           0.2850   0.256643   1.111  0.28135
ravensData$ravenScore 0.0159   0.009059   1.755  0.09625
```

Odds

Binary Outcome 0/1

$$RW_i$$

Probability (0,1)

$$\Pr(RW_i|RS_i, b_0, b_1)$$

Odds $(0, \infty)$

$$\frac{\Pr(RW_i|RS_i, b_0, b_1)}{1 - \Pr(RW_i|RS_i, b_0, b_1)}$$

Log odds $(-\infty, \infty)$

$$\log\left(\frac{\Pr(RW_i|RS_i, b_0, b_1)}{1 - \Pr(RW_i|RS_i, b_0, b_1)}\right)$$

Linear vs. logistic regression

Linear

$$RW_i = b_0 + b_1 RS_i + e_i$$

or

$$E[RW_i | RS_i, b_0, b_1] = b_0 + b_1 RS_i$$

Logistic

$$Pr(RW_i | RS_i, b_0, b_1) = \frac{\exp(b_0 + b_1 RS_i)}{1 + \exp(b_0 + b_1 RS_i)}$$

or

$$\log\left(\frac{Pr(RW_i | RS_i, b_0, b_1)}{1 - Pr(RW_i | RS_i, b_0, b_1)} \right) = b_0 + b_1 RS_i$$

Interpreting Logistic Regression

$$\log\left(\frac{\Pr(RW_i|RS_i, b_0, b_1)}{1-\Pr(RW_i|RS_i, b_0, b_1)}\right) = b_0 + b_1 RS_i$$

b_0 - Log odds of a Ravens win if they score zero points

b_1 - Log odds ratio of win probability for each point scored (compared to zero points)

$\exp(b_1)$ - Odds ratio of win probability for each point scored (compared to zero points)

Odds

- Imagine that you are playing a game where you flip a coin with success probability p.
- If it comes up heads, you win X. If it comes up tails, you lose Y.
- What should we set X and Y for the game to be fair?

$$E[\text{earnings}] = Xp - Y(1-p) = 0$$

- Implies

$$\frac{Y}{X} = \frac{p}{1-p}$$

- The odds can be said as "How much should you be willing to pay for a p probability of winning a dollar?"
  - (If $p > 0.5$ you have to pay more if you lose than you get if you win.)
  - (If $p < 0.5$ you have to pay less if you lose than you get if you win.)

Visualizing fitting logistic regression curves

```
x <- seq(-10, 10, length = 1000)
manipulate(
    plot(x, exp(beta0 + beta1 * x) / (1 + exp(beta0 + beta1 * x)),
         type = "l", lwd = 3, frame = FALSE),
    beta1 = slider(-2, 2, step = .1, initial = 2),
    beta0 = slider(-2, 2, step = .1, initial = 0)
    )
```

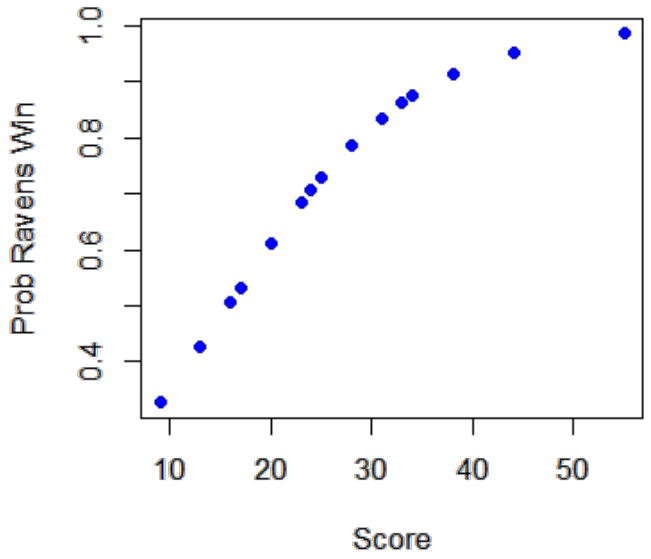
Ravens logistic regression

```
logRegRavens <- glm(ravensData$ravenWinNum ~ ravensData$ravenScore,family="binomial")
summary(logRegRavens)
```

```

Call:
glm(formula = ravensData$ravenWinNum ~ ravensData$ravenScore,
    family = "binomial")

Deviance Residuals:
   Min      1Q  Median      3Q     Max
-1.758  -1.100   0.530   0.806   1.495

Coefficients:
                     Estimate Std. Error z value Pr(>|z|)
(Intercept)           -1.6800     1.5541   -1.08     0.28
ravensData$ravenScore  0.1066     0.0667    1.60     0.11

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 24.435  on 19  degrees of freedom
Residual deviance: 20.895  on 18  degrees of freedom
AIC: 24.89
```

Ravens fitted values

```
plot(ravensData$ravenScore,logRegRavens$fitted,pch=19,col="blue",xlab="Score",ylab="Prob Ravens Win")
```



Odds ratios and confidence intervals

```
exp(logRegRavens$coeff)
```

```
          (Intercept) ravensData$ravenScore
               0.1864                1.1125
```

```
exp(confint(logRegRavens))
```

```
                        2.5 % 97.5 %
(Intercept)           0.005675  3.106
ravensData$ravenScore 0.996230  1.303
```

ANOVA for logistic regression

```
anova(logRegRavens,test="Chisq")
```

```
Analysis of Deviance Table

Model: binomial, link: logit

Response: ravensData$ravenWinNum

Terms added sequentially (first to last)

                     Df Deviance Resid. Df Resid. Dev Pr(>Chi)
NULL                                    19       24.4
ravensData$ravenScore  1     3.54        18       20.9     0.06 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Interpreting Odds Ratios

- Not probabilities
- Odds ratio of 1 = no difference in odds
- Log odds ratio of 0 = no difference in odds
- Odds ratio < 0.5 or > 2 commonly a "moderate effect"
- Relative risk $\frac{\Pr(RW_i|RS_i=10)}{\Pr(RW_i|RS_i=0)}$ often easier to interpret, harder to estimate
- For small probabilities RR ≈ OR but **they are not the same**!

Wikipedia on Odds Ratio

Further resources

- Wikipedia on Logistic Regression
- Logistic regression and glms in R
- Brian Caffo's lecture notes on: Simpson's paradox, Case-control studies
- Open Intro Chapter on Logistic Regression

Count outcomes, Poisson GLMs

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Key ideas

- Many data take the form of counts
  - Calls to a call center
  - Number of flu cases in an area
  - Number of cars that cross a bridge
- Data may also be in the form of rates
  - Percent of children passing a test
  - Percent of hits to a website from a country
- Linear regression with transformation is an option

Poisson distribution

- The Poisson distribution is a useful model for counts and rates
- Here a rate is count per some monitoring time
- Some examples uses of the Poisson distribution
  - Modeling web traffic hits
  - Incidence rates
  - Approximating binomial probabilities with small p and large n
  - Analyzing contigency table data

The Poisson mass function

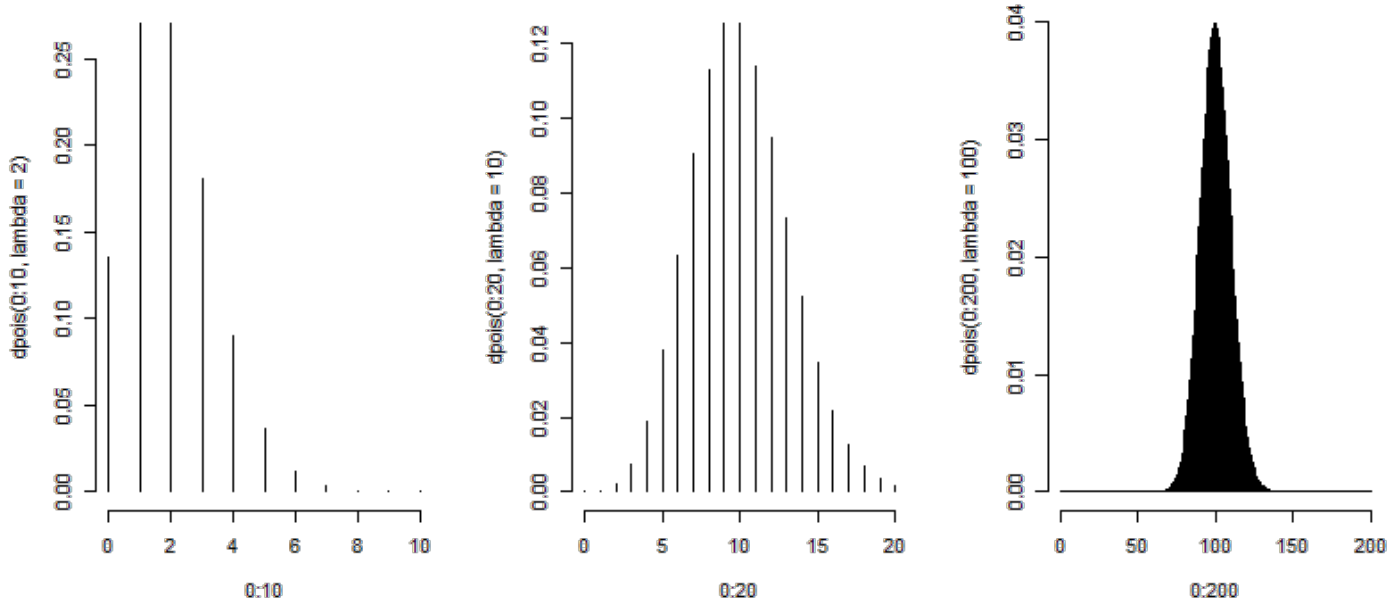
- $X \sim \text{Poisson}(t\lambda)$ if

$$P(X = x) = \frac{(t\lambda)^x e^{-t\lambda}}{x!}$$

  For $x = 0, 1, \ldots$.

- The mean of the Poisson is $E[X] = t\lambda$, thus $E[X/t] = \lambda$
- The variance of the Poisson is $\text{Var}(X) = t\lambda$.
- The Poisson tends to a normal as $t\lambda$ gets large.

```r
par(mfrow = c(1, 3))
plot(0 : 10, dpois(0 : 10, lambda = 2), type = "h", frame = FALSE)
plot(0 : 20, dpois(0 : 20, lambda = 10), type = "h", frame = FALSE)
plot(0 : 200, dpois(0 : 200, lambda = 100), type = "h", frame = FALSE)
```



Poisson distribution

Sort of, showing that the mean and variance are equal

```
x <- 0 : 10000; lambda = 3
mu <- sum(x * dpois(x, lambda = lambda))
sigmasq <- sum((x - mu)^2 * dpois(x, lambda = lambda))
c(mu, sigmasq)
```

```
[1] 3 3
```

Example: Leek Group Website Traffic

- Consider the daily counts to Jeff Leek's web site

http://biostat.jhsph.edu/~jleek/

- Since the unit of time is always one day, set $t = 1$ and then the Poisson mean is interpretted as web hits per day. (If we set $t = 24$, it would be web hits per hour).

Website data

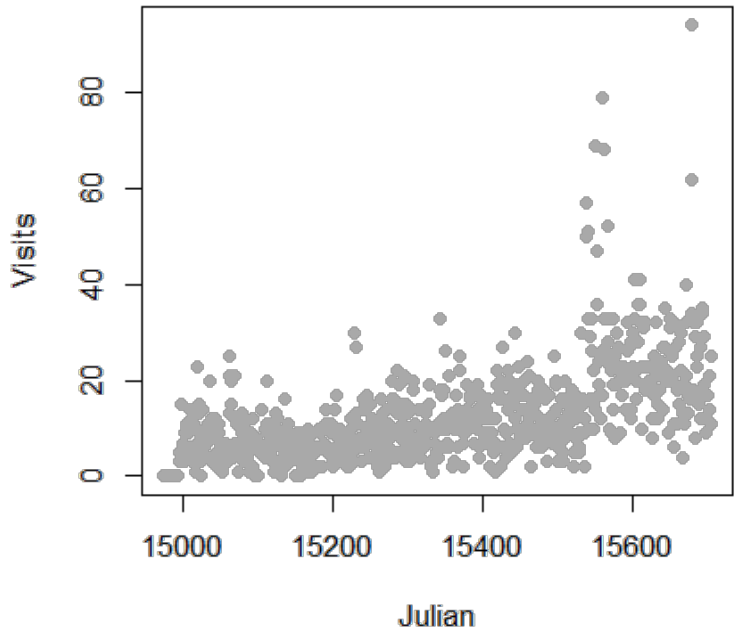
```
download.file("https://dl.dropboxusercontent.com/u/7710864/data/gaData.rda",destfile="./data/gaData.rda
load("./data/gaData.rda")
gaData$julian <- julian(gaData$date)
head(gaData)
```

```
        date visits simplystats julian
1 2011-01-01      0           0  14975
2 2011-01-02      0           0  14976
3 2011-01-03      0           0  14977
4 2011-01-04      0           0  14978
5 2011-01-05      0           0  14979
6 2011-01-06      0           0  14980
```

http://skardhamar.github.com/rga/

Plot data

```
plot(gaData$julian,gaData$visits,pch=19,col="darkgrey",xlab="Julian",ylab="Visits")
```



Linear regression

$$NH_i = b_0 + b_1 JD_i + e_i$$

NH_i - number of hits to the website

JD_i - day of the year (Julian day)

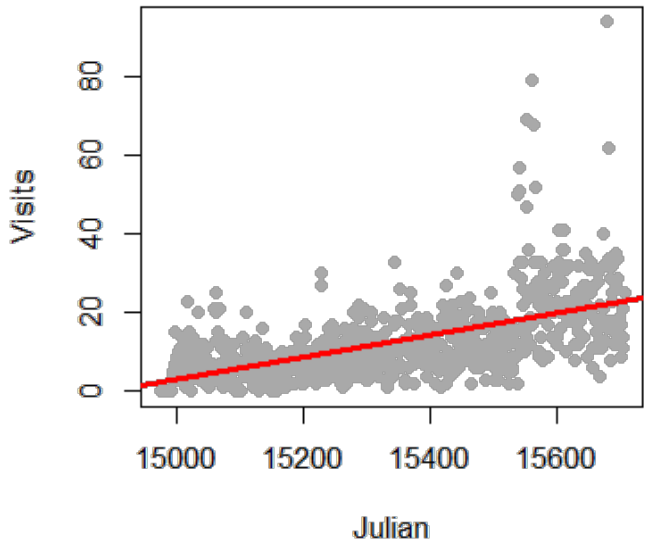
b_0 - number of hits on Julian day 0 (1970-01-01)

b_1 - increase in number of hits per unit day

e_i - variation due to everything we didn't measure

Linear regression line

```
plot(gaData$julian,gaData$visits,pch=19,col="darkgrey",xlab="Julian",ylab="Visits")
lm1 <- lm(gaData$visits ~ gaData$julian)
abline(lm1,col="red",lwd=3)
```



Aside, taking the log of the outcome

- Taking the natural log of the outcome has a specific interpretation.
- Consider the model

$$\log(NH_i) = b_0 + b_1 JD_i + e_i$$

NH_i - number of hits to the website

JD_i - day of the year (Julian day)

b_0 - log number of hits on Julian day 0 (1970-01-01)

b_1 - increase in log number of hits per unit day

e_i - variation due to everything we didn't measure

Exponentiating coefficients

- $e^{E[\log(Y)]}$ geometric mean of Y.
  - With no covariates, this is estimated by $e^{\frac{1}{n}\sum_{i=1}^{n}\log(y_i)} = (\prod_{i=1}^{n} y_i)^{1/n}$
- When you take the natural log of outcomes and fit a regression model, your exponentiated coefficients estimate things about geometric means.
- e^{β_0} estimated geometric mean hits on day 0
- e^{β_1} estimated relative increase or decrease in geometric mean hits per day
- There's a problem with logs with you have zero counts, adding a constant works

```
round(exp(coef(lm(I(log(gaData$visits + 1)) ~ gaData$julian))), 5)
```

```
  (Intercept) gaData$julian
        0.000         1.002
```

Linear vs. Poisson regression

Linear

$$NH_i = b_0 + b_1 JD_i + e_i$$

or

$$E[NH_i | JD_i, b_0, b_1] = b_0 + b_1 JD_i$$

Poisson/log-linear

$$\log\left(E[NH_i | JD_i, b_0, b_1]\right) = b_0 + b_1 JD_i$$

or

$$E[NH_i | JD_i, b_0, b_1] = \exp\left(b_0 + b_1 JD_i\right)$$

Multiplicative differences

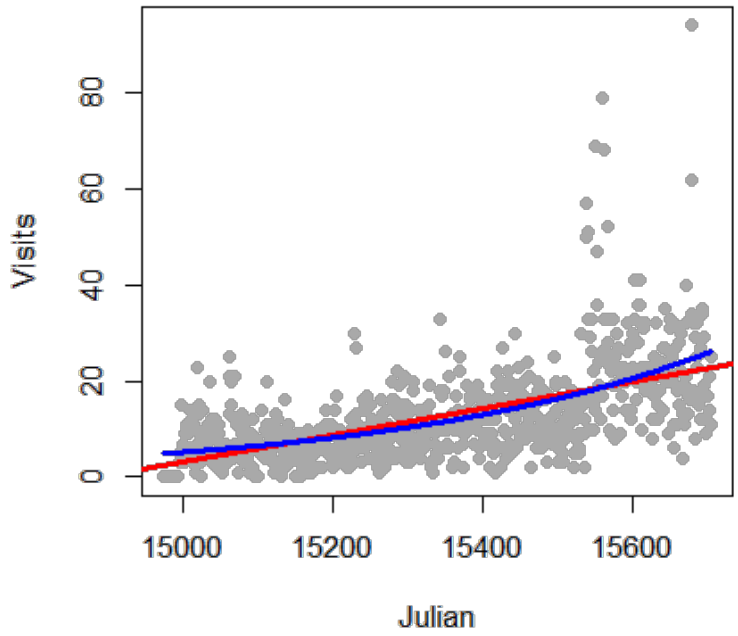
$$E[NH_i|JD_i, b_0, b_1] = \exp\left(b_0 + b_1 JD_i\right)$$

$$E[NH_i|JD_i, b_0, b_1] = \exp\left(b_0\right) \exp\left(b_1 JD_i\right)$$

If JD_i is increased by one unit, $E[NH_i|JD_i, b_0, b_1]$ is multiplied by $\exp\left(b_1\right)$

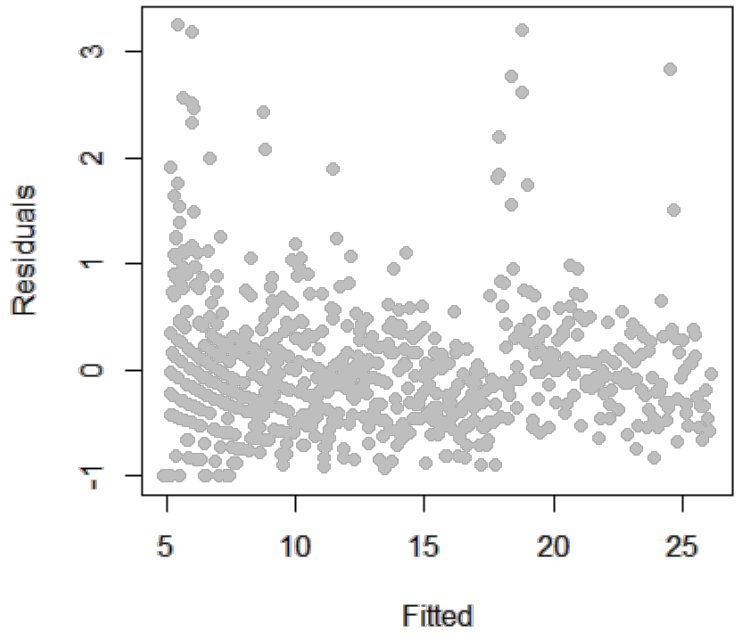
Poisson regression in R

```
plot(gaData$julian,gaData$visits,pch=19,col="darkgrey",xlab="Julian",ylab="Visits")
glm1 <- glm(gaData$visits ~ gaData$julian,family="poisson")
abline(lm1,col="red",lwd=3); lines(gaData$julian,glm1$fitted,col="blue",lwd=3)
```



Mean-variance relationship?

```
plot(glm1$fitted,glm1$residuals,pch=19,col="grey",ylab="Residuals",xlab="Fitted")
```



Model agnostic standard errors

```
library(sandwich)
confint.agnostic <- function (object, parm, level = 0.95, ...)
{
    cf <- coef(object); pnames <- names(cf)
    if (missing(parm))
        parm <- pnames
    else if (is.numeric(parm))
        parm <- pnames[parm]
    a <- (1 - level)/2; a <- c(a, 1 - a)
    pct <- stats:::format.perc(a, 3)
    fac <- qnorm(a)
    ci <- array(NA, dim = c(length(parm), 2L), dimnames = list(parm,
                                                               pct))
    ses <- sqrt(diag(sandwich::vcovHC(object)))[parm]
    ci[] <- cf[parm] + ses %o% fac
    ci
}
```

http://stackoverflow.com/questions/3817182/vcovhc-and-confidence-interval

Estimating confidence intervals

```
confint(glm1)
```

```
                  2.5 %     97.5 %
(Intercept)   -34.34658 -31.159716
gaData$julian   0.00219   0.002396
```

```
confint.agnostic(glm1)
```

```
                  2.5 %     97.5 %
(Intercept)   -36.362675 -29.136997
gaData$julian   0.002058   0.002528
```

Rates

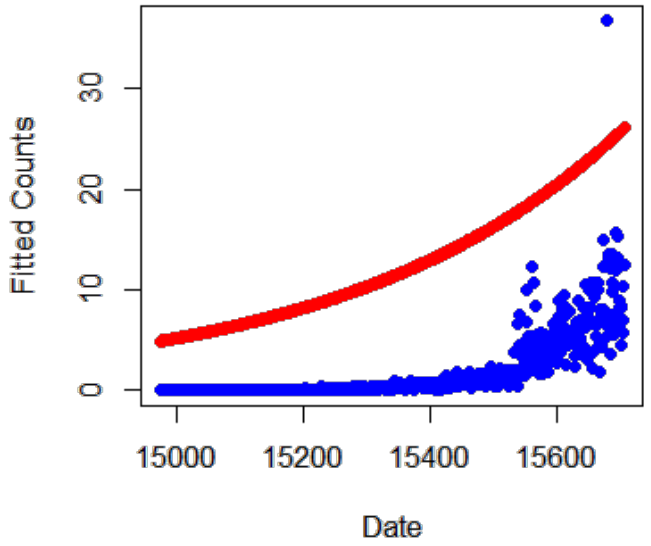
$$E[NHSS_i|JD_i, b_0, b_1]/NH_i = \exp\left(b_0 + b_1 JD_i\right)$$

$$\log\left(E[NHSS_i|JD_i, b_0, b_1]\right) - \log(NH_i) = b_0 + b_1 JD_i$$

$$\log\left(E[NHSS_i|JD_i, b_0, b_1]\right) = \log(NH_i) + b_0 + b_1 JD_i$$

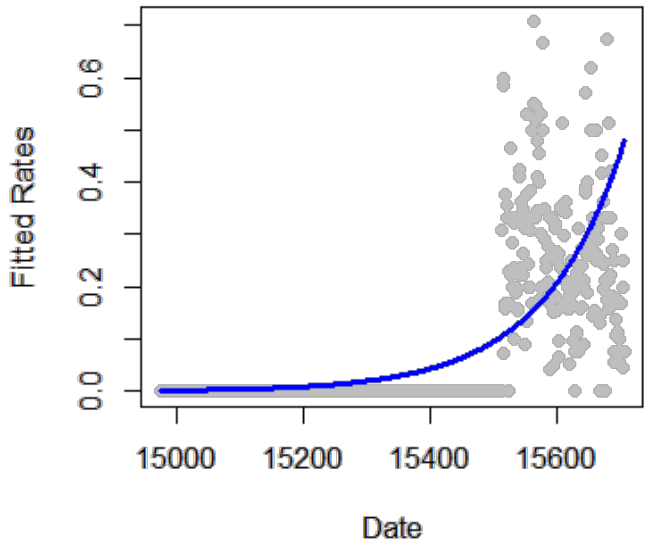
Fitting rates in R

```
glm2 <- glm(gaData$simplystats ~ julian(gaData$date),offset=log(visits+1),
            family="poisson",data=gaData)
plot(julian(gaData$date),glm2$fitted,col="blue",pch=19,xlab="Date",ylab="Fitted Counts")
points(julian(gaData$date),glm1$fitted,col="red",pch=19)
```



Fitting rates in R

```
glm2 <- glm(gaData$simplystats ~ julian(gaData$date),offset=log(visits+1),
            family="poisson",data=gaData)
plot(julian(gaData$date),gaData$simplystats/(gaData$visits+1),col="grey",xlab="Date",
     ylab="Fitted Rates",pch=19)
lines(julian(gaData$date),glm2$fitted/(gaData$visits+1),col="blue",lwd=3)
```



More information

- Log-linear models and multiway tables
- Wikipedia on Poisson regression, Wikipedia on overdispersion
- Regression models for count data in R
- pscl package - the function *zeroinfl* fits zero inflated models.

Hodgepodge

Regression models

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How to fit functions using linear models

- Consider a model $Y_i = f(X_i) + \epsilon$.
- How can we fit such a model using linear models (called scatterplot smoothing)
- Consider the model

$$Y_i = \beta_0 + \beta_1 X_i + \sum_{k=1}^{d} (x_i - \xi_k)_+ \gamma_k + \epsilon_i$$

where $(a)_+ = a$ if $a > 0$ and 0 otherwise and $\xi_1 \leq \ldots \leq \xi_d$ are known knot points.

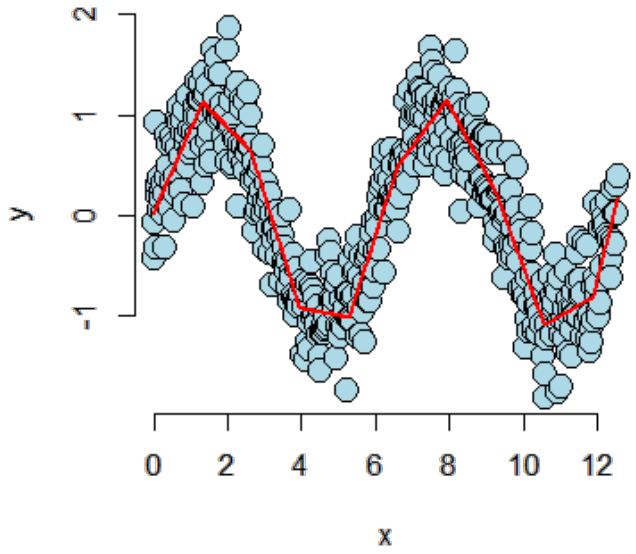
- Prove to yourelf that the mean function

$$\beta_0 + \beta_1 X_i + \sum_{k=1}^{d} (x_i - \xi_k)_+ \gamma_k$$

is continuous at the knot points.

Simulated example

```
n <- 500; x <- seq(0, 4 * pi, length = n); y <- sin(x) + rnorm(n, sd = .3)
knots <- seq(0, 8 * pi, length = 20);
splineTerms <- sapply(knots, function(knot) (x > knot) * (x - knot))
xMat <- cbind(1, x, splineTerms)
yhat <- predict(lm(y ~ xMat - 1))
plot(x, y, frame = FALSE, pch = 21, bg = "lightblue", cex = 2)
lines(x, yhat, col = "red", lwd = 2)
```

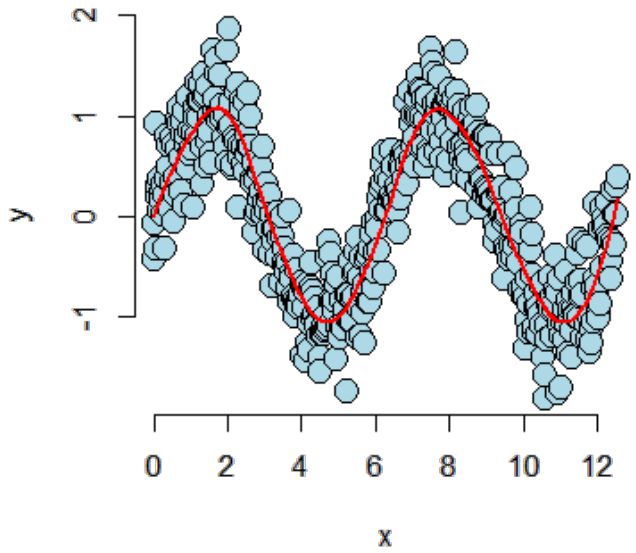


Adding squared terms

- Adding squared terms makes it continuously differentiable at the knot points.
- Adding cubic terms makes it twice continuously differentiable at the knot points; etcetera.

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \sum_{k=1}^{d} (x_i - \xi_k)_+^2 \gamma_k + \epsilon_i$$

```r
splineTerms <- sapply(knots, function(knot) (x > knot) * (x - knot)^2)
xMat <- cbind(1, x, x^2, splineTerms)
yhat <- predict(lm(y ~ xMat - 1))
plot(x, y, frame = FALSE, pch = 21, bg = "lightblue", cex = 2)
lines(x, yhat, col = "red", lwd = 2)
```

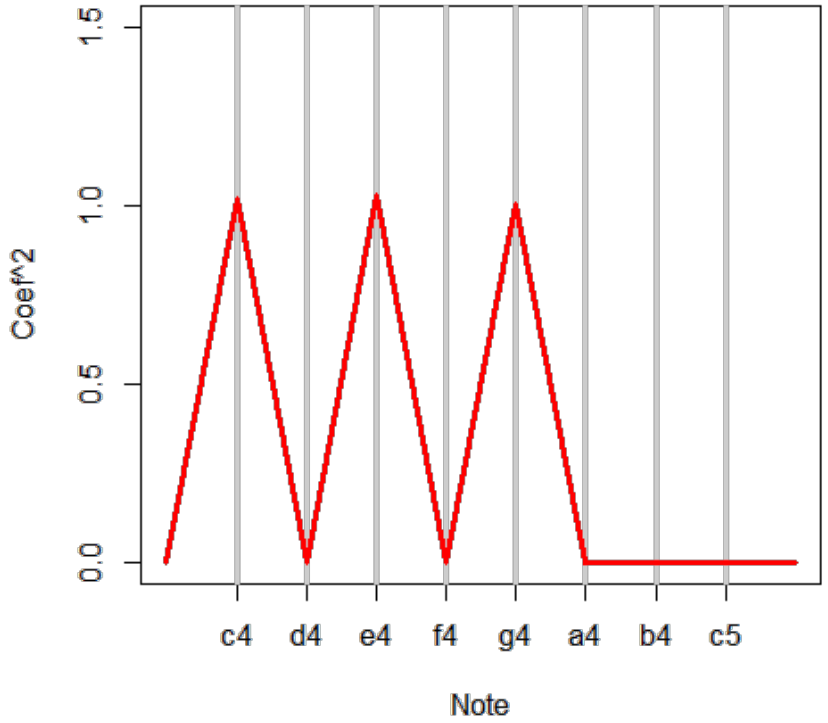


Notes

- The collection of regressors is called a basis.
  - People have spent **a lot** of time thinking about bases for this kind of problem. So, consider this as just a teaser.
- Single knot point terms can fit hockey stick like processes.
- These bases can be used in GLMs as well.
- An issue with these approaches is the large number of parameters introduced.
  - Requires some method of so called regularization.

Harmonics using linear models

```
##Chord finder, playing the white keys on a piano from octave c4 - c5
notes4 <- c(261.63, 293.66, 329.63, 349.23, 392.00, 440.00, 493.88, 523.25)
t <- seq(0, 2, by = .001); n <- length(t)
c4 <- sin(2 * pi * notes4[1] * t); e4 <- sin(2 * pi * notes4[3] * t);
g4 <- sin(2 * pi * notes4[5] * t)
chord <- c4 + e4 + g4 + rnorm(n, 0, 0.3)
x <- sapply(notes4, function(freq) sin(2 * pi * freq * t))
fit <- lm(chord ~ x - 1)
```


Coef^2
0.0
0.5
1.0
1.5
c4
d4
e4
f4
g4
a4
b4
c5
Note

```
##(How you would really do it)
a <- fft(chord); plot(Re(a)^2, type = "l")
```