1

ESTIMATING THE YIELD CURVE

Questions

1. Plot bond prices versus maturities
2. Plot the empirical forward rates as computed in the equation below versus maturities

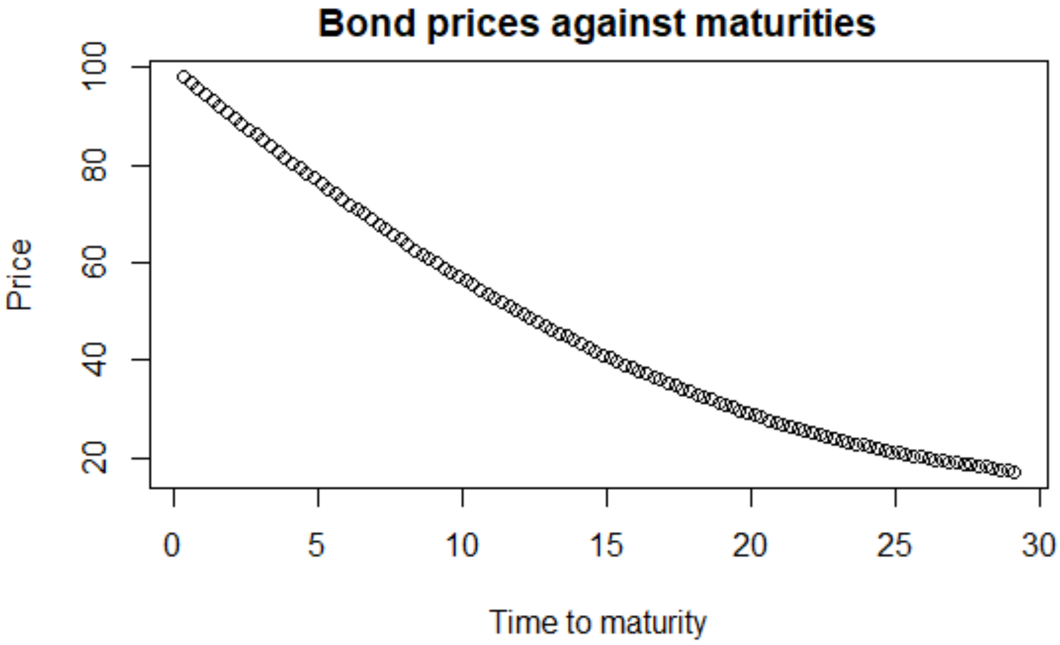
$$f_j = -\frac{\log P_{j+1} - \log P_j}{t_{j+1} - t_j}, \qquad \text{for } j = 1, \cdots n-1.$$

1. Smooth the empirical forward rates using second order and third order polynomials. Superimpose the smoothed curves versus the empirical forward rates.
2. Estimate the empirical spot rates for t in (t_1, t_n) using the equation below

$$r(t) = \frac{1}{t}\left[\sum_{i=1}^{j} f_{i-1}(t_i - t_{i-1}) + f_j(t - t_j)\right], \qquad \text{for } t_j < t \le t_{j+1};\ j = 1, \cdots, n-1,$$

1. Smooth the empirical spot rates using second order and third order polynomials. Superimpose the smoothed curves versus the empirical spot rates.
2. Comment your results

Question 1



Bond prices against maturities
Price
100
80
60
40
20
0
5
10
15
20
25
30
Time to maturity

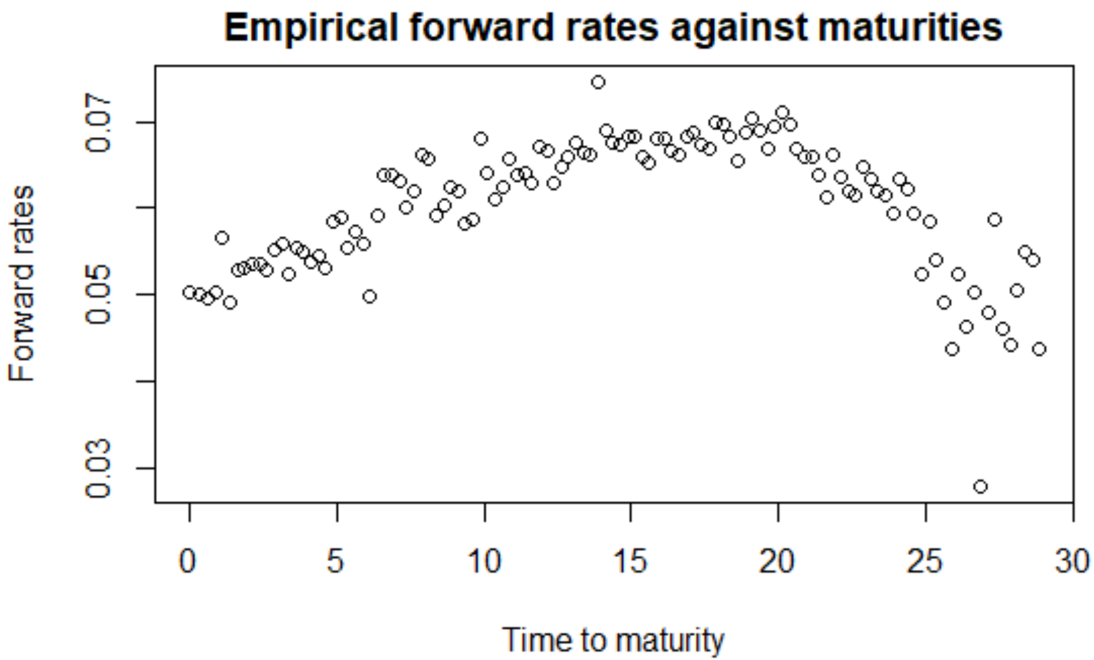
Question 2

$$f_j = -\frac{\log P_{j+1} - \log P_j}{t_{j+1} - t_j}, \qquad \text{for } j = 1, \cdots n-1.$$

- We calculated the negative difference in log prices at each time period divided by difference in time period to give us the forward rates from time 1.
- For time 0 we calculated forward rate with the formula $-\log(0.01*P_{t1})/t_1$.

	time <dbl>	forward_rates <dbl>
1	0.0000	0.05034
2	0.3699	0.05008
3	0.6219	0.04971
4	0.8740	0.05032
5	1.1260	0.05661
6	1.3699	0.04905

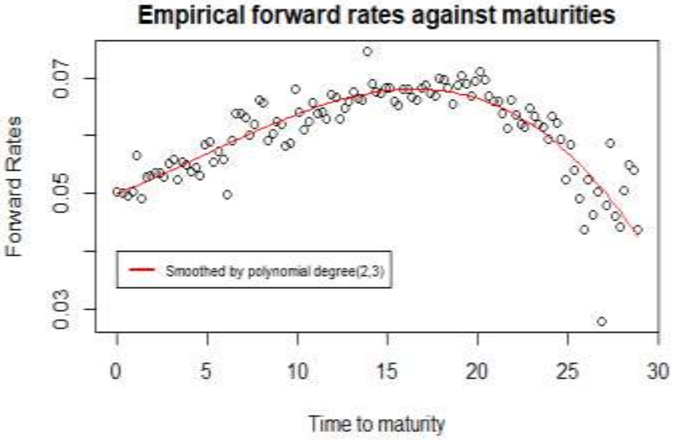
6 rows



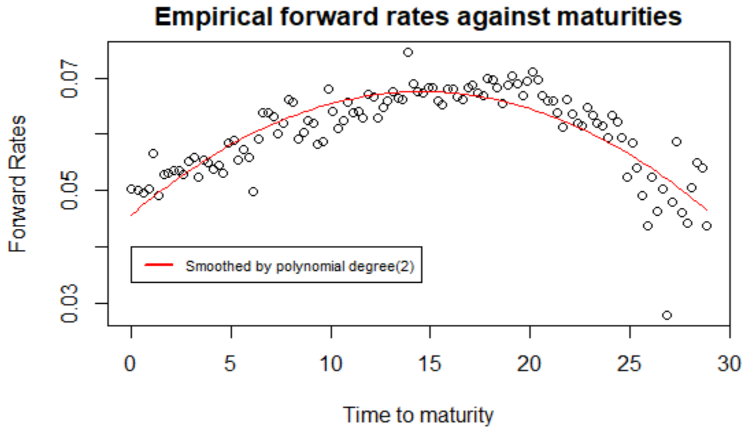
- These forward rates were then plotted against the maturities

Question 3

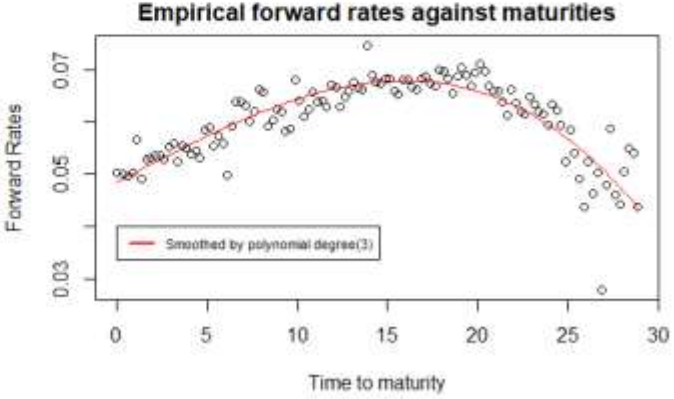
- To smooth the empirical forward rates. We fitted three different models using the lm() function. The model specifications are:
  - Model 1: $\widehat{forward\_rate} = \hat{\beta}_0 + \hat{\beta}_1 time + \hat{\beta}_2 time^2 + \hat{\beta}_3 time^3$
  - Model 2: $\widehat{forward\_rate} = \hat{\beta}_0 + \hat{\beta}_1 time + \hat{\beta}_2 time^2$
  - Model 3: $\widehat{forward\_rate} = \hat{\beta}_0 + \hat{\beta}_1 time + \hat{\beta}_3 time^3$



$$\widehat{forward\_rate} = \hat{\beta}_0 + \hat{\beta}_1 time + \hat{\beta}_2 time^2 + \hat{\beta}_3 time^3$$



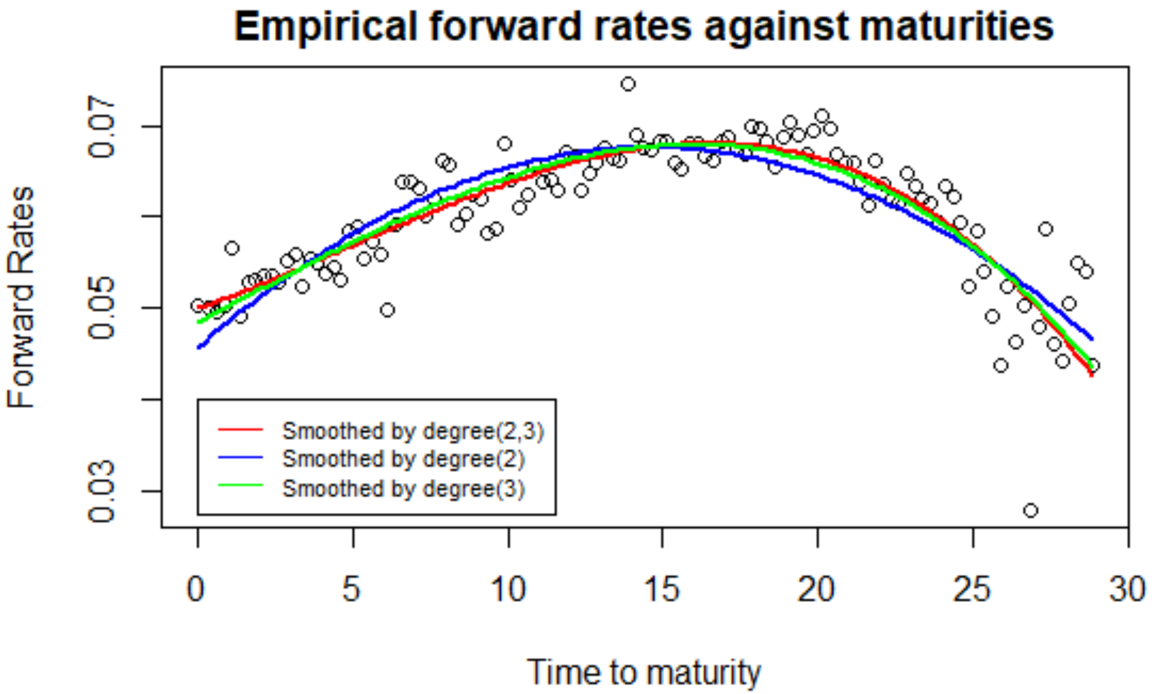
$$\widehat{forward\_rate} = \hat{\beta}_0 + \hat{\beta}_1 time + \hat{\beta}_2 time^2$$



$$\widehat{forward\_rate} = \hat{\beta}_0 + \hat{\beta}_1 time + \hat{\beta}_3 time^3$$

Model	Adj R Squared
Model1	0.748
Model2	0.704
Model3	0.746

Comparing the three models



Question 4

$$r(t) = \frac{1}{t}\left[\sum_{i=1}^{j} f_{i-1}(t_i - t_{i-1}) + f_j(t - t_j)\right], \qquad \text{for } t_j < t \leq t_{j+1};\ j = 1, \cdots, n-1,$$

- We first create 29 years/intervals using the cut() function. We aim to find the spot rates per annum.
- The spot rate for time to maturity t is given as the sum of cumulative product of forward rates * (time interval between forward rate and the next forward rate) and the final forward rate * (time interval between forward rate and t)

- We use cumsum() to calculate all the cumulative sum products
- We find the forward rate just before interval t, which is given by index i and extract the value at i-1
- We then take the sum of the i-1 index value with the product of forward rate at index i and the difference between t and time at index i and divide by time t

	time	forward_rates	year	cumsum
1	0.0000	0.05034	1	0.01862
2	0.3699	0.05008	1	0.03124
3	0.6219	0.04971	1	0.04377
4	0.8740	0.05032	1	0.05645
5	1.1260	0.05661	2	0.07026

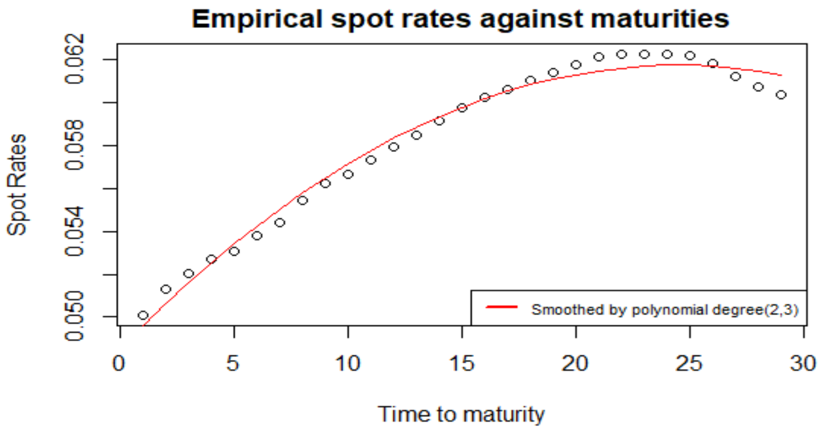
$$Spot\_rate_1 = \frac{0.04377 + (0.05032 \times (1 - 0.8740))}{1}$$

- We repeat this step for each interval, i.e. 29 times

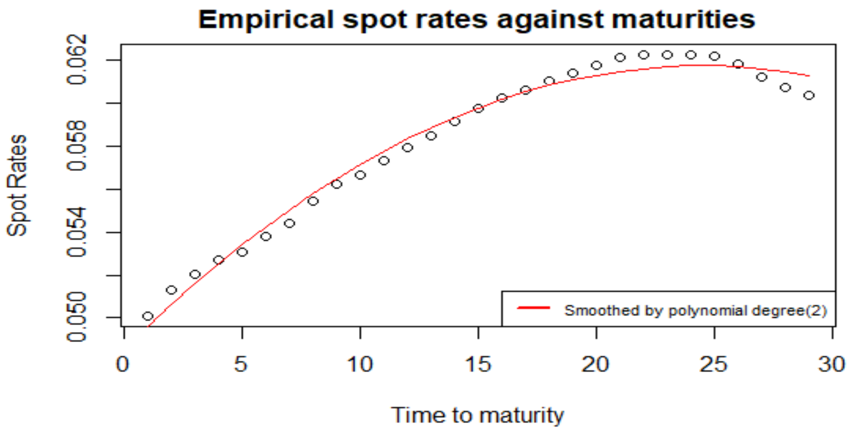
Spotrates	Year
0.05011	1
0.05132	2
0.05205	3
0.05272	4
0.05310	5
0.05379	6
0.05443	7
0.05544	8
0.05620	9
0.05669	10
0.05732	11
0.05790	12
0.05846	13
0.05912	14
0.05977	15
0.06022	16
0.06063	17
0.06104	18
0.06142	19
0.06179	20
0.06212	21
0.06222	22
0.06226	23
0.06226	24
0.06218	25
0.06180	26
0.06122	27
0.06072	28
0.06038	29

Question 5

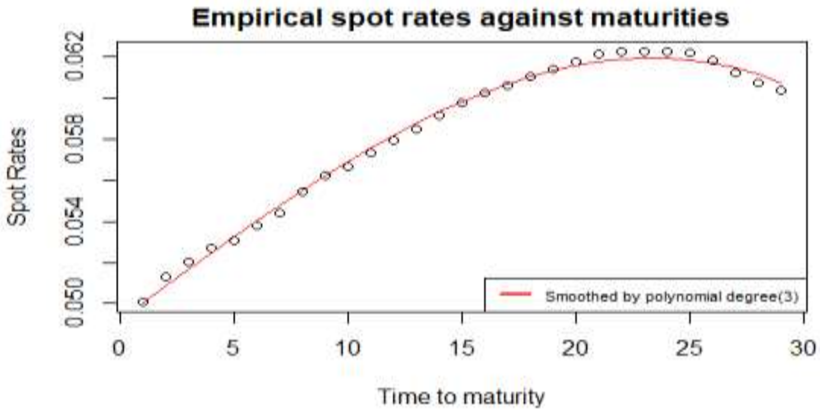
- To smooth the empirical spot rates. We fitted three different models using the lm() function. The model specifications are:
  - Model 1: $\hat{spot\_rate} = \hat{\beta}_0 + \hat{\beta}_1 time + \hat{\beta}_2 time^2 + \hat{\beta}_3 time^3$
  - Model 2: $\hat{spot\_rate} = \hat{\beta}_0 + \hat{\beta}_1 time + \hat{\beta}_2 time^2$
  - Model 3: $\hat{spot\_rate} = \hat{\beta}_0 + \hat{\beta}_1 time + \hat{\beta}_2 time^3$



$$\hat{spot\_rate} = \hat{\beta}_0 + \hat{\beta}_1 time + \hat{\beta}_2 time^2 + \hat{\beta}_3 time^3$$



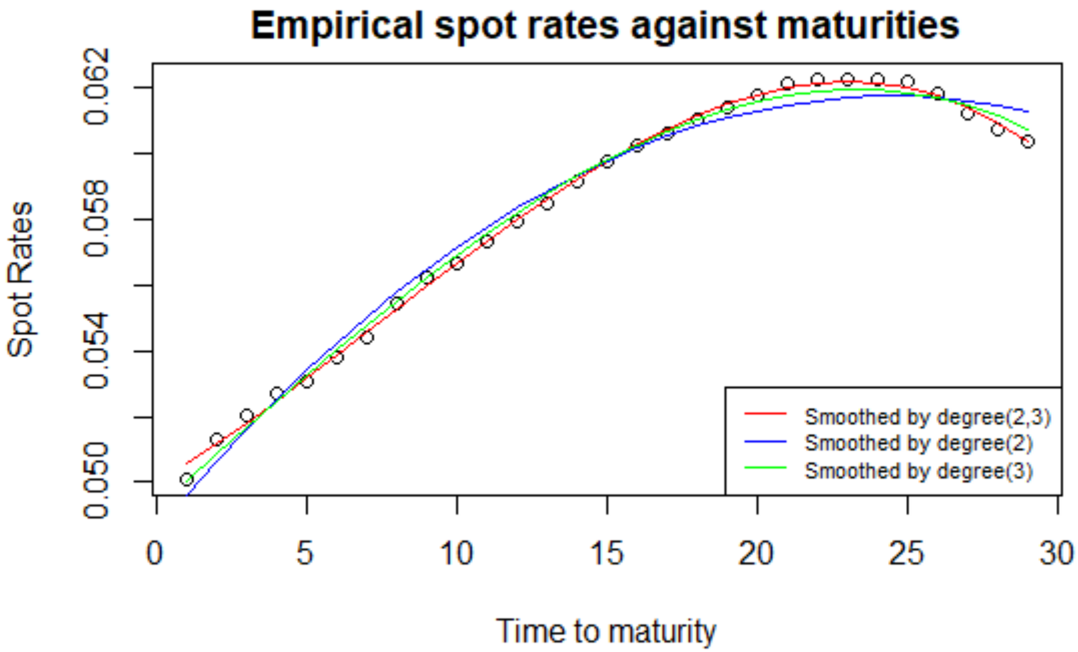
$$\hat{spot\_rate} = \hat{\beta}_0 + \hat{\beta}_1 time + \hat{\beta}_2 time^2$$



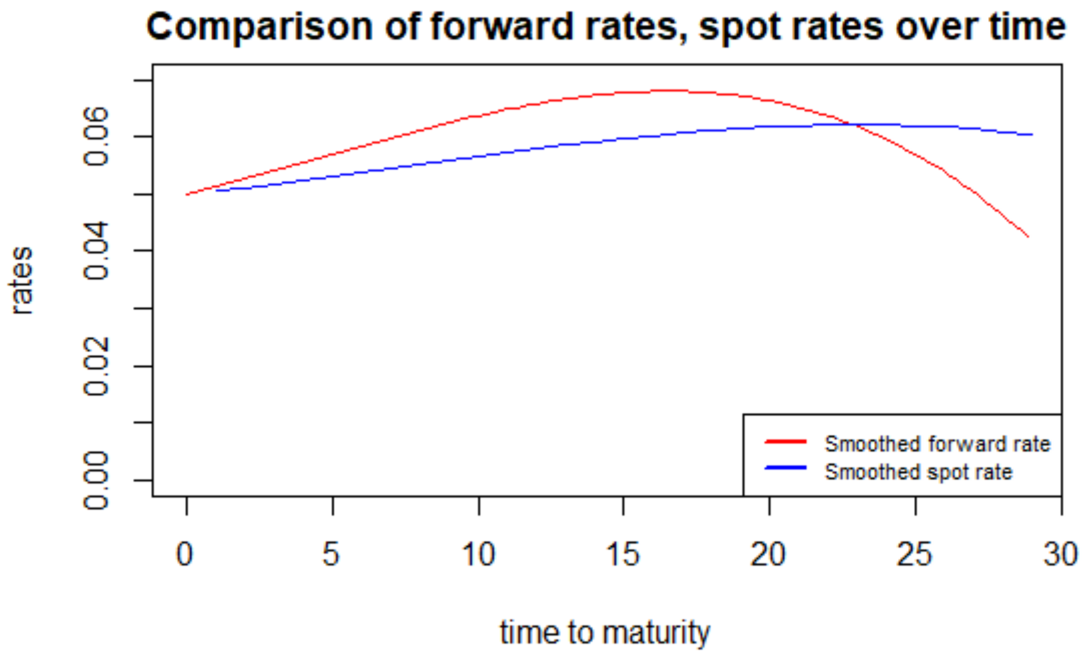
$$\hat{spot\_rate} = \hat{\beta}_0 + \hat{\beta}_1 time + \hat{\beta}_2 time^3$$

Model	**Adj R Squared**
Model1	0.998
Model2	0.984
Model3	0.995

Comparing all 3 models



Question 6


Comparison of forward rates, spot rates over time
rates
0.00
0.02
0.04
0.06
0
5
10
15
20
25
30
time to maturity
Smoothed forward rate
Smoothed spot rate

2

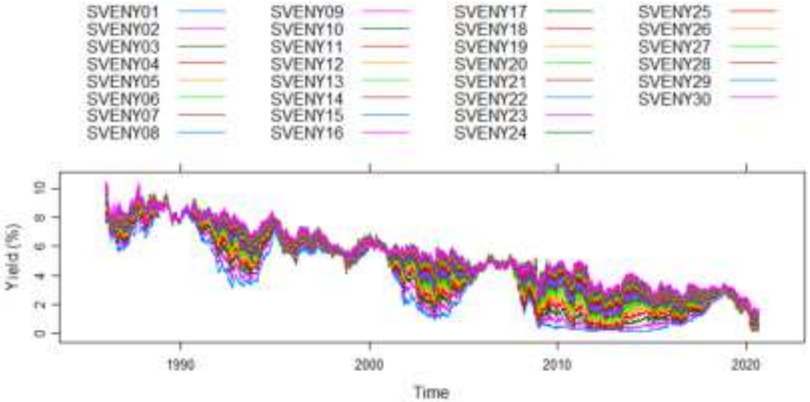
FITTING THE YIELD CURVE

Questions

1. Present the data in a suitable way
2. Fit the NS and NSS models to the yield data by minimizing sum of squared errors. Compare the two models using suitable diagnostics and model selection criteria
3. The changing patterns of the yield curve can be studied through the parameters $\boldsymbol{\theta}$. What information can we extract from the estimates of $\boldsymbol{\theta}$?
4. How may the data tell you about the response of the spot rates at the long end with respect to the spot rates at the short end?
5. Fitting the NS and NSS Models on each day is a static exercise. Can the NS and NSS Models be used to predict future interest rate movements?

Question 1

- We first attempted to plot all the variables on a single graph



- As we can see there are too many variables to make any meaning of the graph except for that fact that rates have been trending downwards

```
> summary(prcomp(ZCBData[,-1], center = TRUE, scale. = TRUE))
Importance of components:
                          PC1     PC2     PC3     PC4     PC5     PC6      PC7      PC8       PC9
Standard deviation     5.4037 0.85171 0.22664 0.14290 0.05352 0.01591 0.003818 0.001292 0.0003541
Proportion of Variance 0.9733 0.02418 0.00171 0.00068 0.00010 0.00001 0.000000 0.000000 0.0000000
Cumulative Proportion  0.9733 0.99750 0.99921 0.99990 0.99999 1.00000 1.000000 1.000000 1.0000000
                            PC10      PC11      PC12      PC13      PC14      PC15      PC16
Standard deviation     9.136e-05 2.437e-05 1.462e-05 1.428e-05 1.421e-05 1.406e-05 1.401e-05
Proportion of Variance 0.000e+00 0.000e+00 0.000e+00 0.000e+00 0.000e+00 0.000e+00 0.000e+00
Cumulative Proportion  1.000e+00 1.000e+00 1.000e+00 1.000e+00 1.000e+00 1.000e+00 1.000e+00
                           PC17     PC18      PC19      PC20      PC21      PC22      PC23     PC24
Standard deviation     1.39e-05 1.37e-05 1.363e-05 1.345e-05 1.339e-05 1.331e-05 1.313e-05 1.31e-05
Proportion of Variance 0.00e+00 0.00e+00 0.000e+00 0.000e+00 0.000e+00 0.000e+00 0.000e+00 0.00e+00
Cumulative Proportion  1.00e+00 1.00e+00 1.000e+00 1.000e+00 1.000e+00 1.000e+00 1.000e+00 1.00e+00
                            PC25      PC26      PC27      PC28      PC29      PC30
Standard deviation     1.289e-05 1.286e-05 1.269e-05 1.248e-05 1.225e-05 1.189e-05
Proportion of Variance 0.000e+00 0.000e+00 0.000e+00 0.000e+00 0.000e+00 0.000e+00
Cumulative Proportion  1.000e+00 1.000e+00 1.000e+00 1.000e+00 1.000e+00 1.000e+00
```

- Thus we performed a Principal Component Analysis (PCA) to find out which variables have a larger weight
- We can see that PC1 accounts for the largest proportion of variance and as maturity increases, the proportion of variance decreases.
- PC1 and PC2 make up 99.75% of the variance.

Post-Presentation Clarification

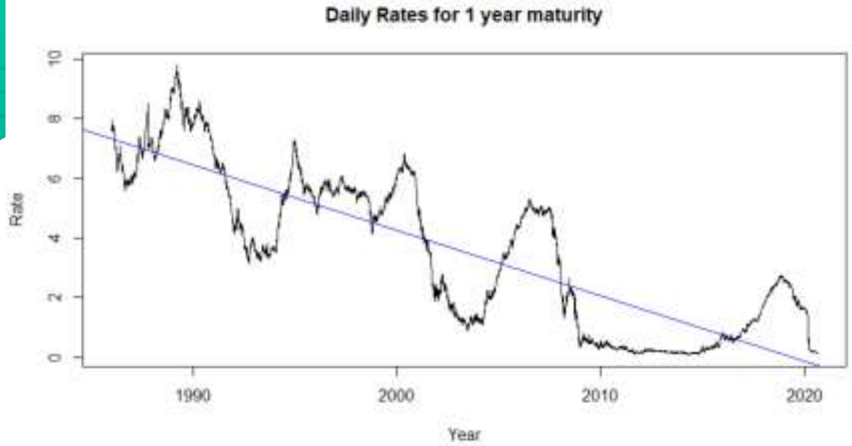
- Clarification on weights of maturities

	PC1
SVENY01	0.16676
SVENY02	0.17247
SVENY03	0.17648
SVENY04	0.17937
SVENY05	0.18142
SVENY06	0.18284
SVENY07	0.18377
SVENY08	0.18435
SVENY09	0.18466
SVENY10	0.18479
SVENY11	0.18481
SVENY12	0.18475
SVENY13	0.18466
SVENY14	0.18455
SVENY15	0.18444
SVENY16	0.18434
SVENY17	0.18424
SVENY18	0.18416
SVENY19	0.18408
SVENY20	0.18401
SVENY21	0.18395
SVENY22	0.18388
SVENY23	0.18381
SVENY24	0.18373
SVENY25	0.18364
SVENY26	0.18352
SVENY27	0.18339
SVENY28	0.18323
SVENY29	0.18304
SVENY30	0.18282

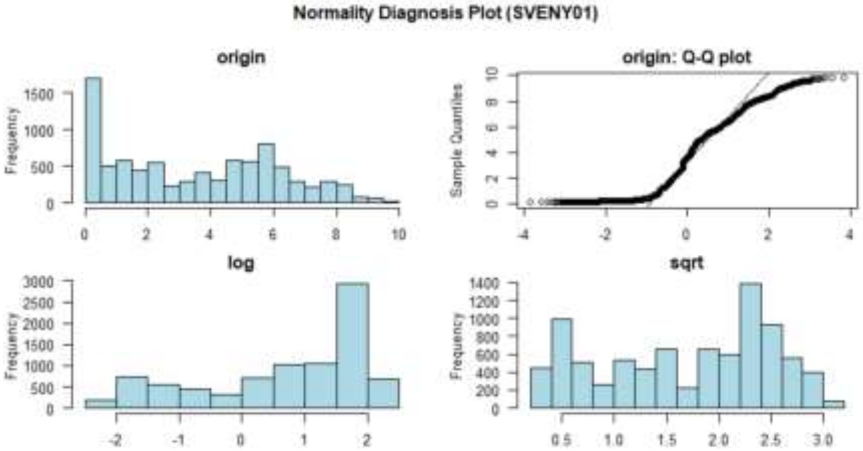
Post-Presentation Clarification

- Clarification on eigenvectors

```
Standard deviations:
     Comp.1      Comp.2      Comp.3      Comp.4      Comp.5      Comp.6      Comp.7      Comp.8 
1.2017e+01  2.0624e+00  4.8970e-01  3.3827e-01  1.2567e-01  3.7733e-02  8.8613e-03  2.9861e-03 
     Comp.9     Comp.10     Comp.11     Comp.12     Comp.13     Comp.14     Comp.15     Comp.16 
8.1797e-04  2.0743e-04  5.5094e-05  3.0850e-05  2.9939e-05  2.9831e-05  2.9671e-05  2.9588e-05 
    Comp.17     Comp.18     Comp.19     Comp.20     Comp.21     Comp.22     Comp.23     Comp.24 
2.9416e-05  2.9312e-05  2.9209e-05  2.8914e-05  2.8848e-05  2.8793e-05  2.8658e-05  2.8446e-05 
    Comp.25     Comp.26     Comp.27     Comp.28     Comp.29     Comp.30 
2.8363e-05  2.8207e-05  2.7950e-05  2.7843e-05  2.7801e-05  2.7671e-05 
```



Daily rates for 1 year maturity



Normality diagnosis plot for 1 year maturity

Background on NS and NSS Model

- Nelson and Siegel in 1985 proposed a model to capture the shape of the yield curve by using the spot rate as a function of maturity

$$r(t) = \theta_0 + \left(\theta_1 + \frac{\theta_2}{\theta_3}\right)\left(\frac{1 - e^{-\theta_3 t}}{\theta_3 t}\right) - \frac{\theta_2}{\theta_3} e^{-\theta_3 t}, \qquad \theta_0,\ \theta_3 > 0.$$

- This was furthered by Svensson in 1994 to add an additional term to the existing NS model

$$r(t) = \theta_0 + \left(\theta_1 + \frac{\theta_2}{\theta_3}\right)\left(\frac{1 - e^{-\theta_3 t}}{\theta_3 t}\right) - \frac{\theta_2}{\theta_3} e^{-\theta_3 t} + \frac{\theta_4}{\theta_5}\left[\frac{1 - e^{-\theta_5 t}}{\theta_5 t} - e^{-\theta_5 t}\right], \quad \theta_0,\ \theta_3,\ \theta_5 > 0.$$

- Given a set of spot rates r(t) at various maturities, the parameters θ can be estimated minimizing the sum of squares of the nonlinear regression model

Background on NS and NSS Model

- The official Nelson-Siegel (NS) Model from YieldCurve package:

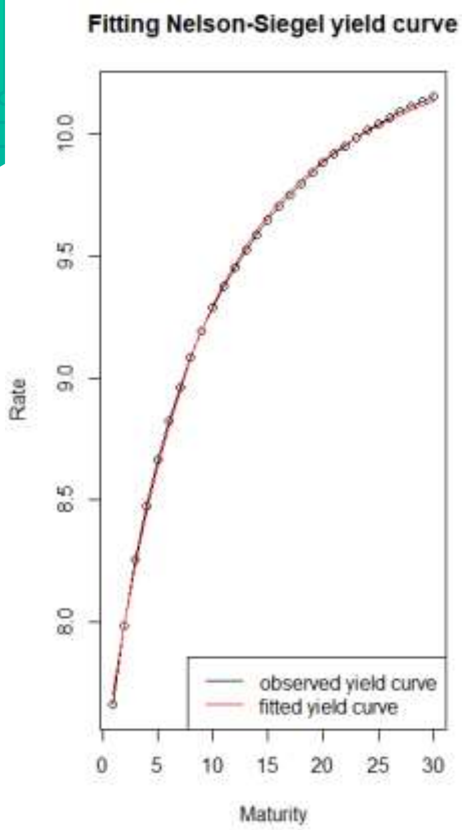
$$y_t(\tau) = \beta_{0t} + \beta_{1t}\frac{1-\exp(-\lambda\tau)}{\lambda\tau} + \beta_{2t}\left(\frac{1-\exp(-\lambda\tau)}{\lambda\tau} - \exp(-\lambda\tau)\right)$$

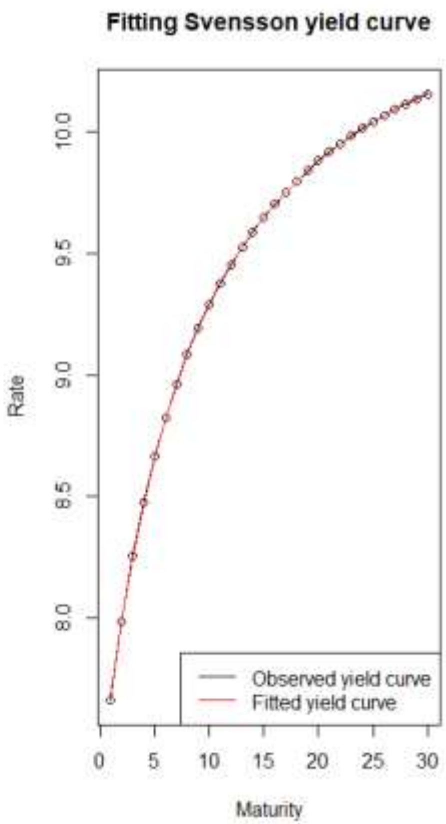
- The official Nelson-Siegel-Svensson (NSS) Model from YieldCurve package:

$$y_t(\tau) = \beta_0 + \beta_1\frac{1-\exp(-\frac{\tau}{\lambda_1})}{\frac{\tau}{\lambda_1}} + \beta_2\left[\frac{1-\exp(-\frac{\tau}{\lambda_1})}{\frac{\tau}{\lambda_1}} - \exp(-\frac{\tau}{\lambda_1})\right] + \beta_3\left[\frac{1-\exp(-\frac{\tau}{\lambda_2})}{\frac{\tau}{\lambda_2}} - \exp(-\frac{\tau}{\lambda_2})\right]$$

- Beta =/= theta

Question 2





- We check the RMSE of both models. The RMSE for the NS model is 0.016265, while the RMSE of the NSS model is 0.010138. Since the RMSE of the NSS model is smaller, the NSS model is better suited to the data.

Question 3

- θ_0: Long run level of interest.
- θ_1: Short term vs long term component. Starts high, decays to 0. Slower decay with a smaller θ_3.
- θ_2: Medium term vs long term component. Starts low, increases, then decays to 0. The first 'hump/trough'.
- θ_3: Inverse of the decay factor τ.
- θ_4: Secondary medium term component. The second 'hump/trough'.
- θ_5: Inverse of the second decay factor.

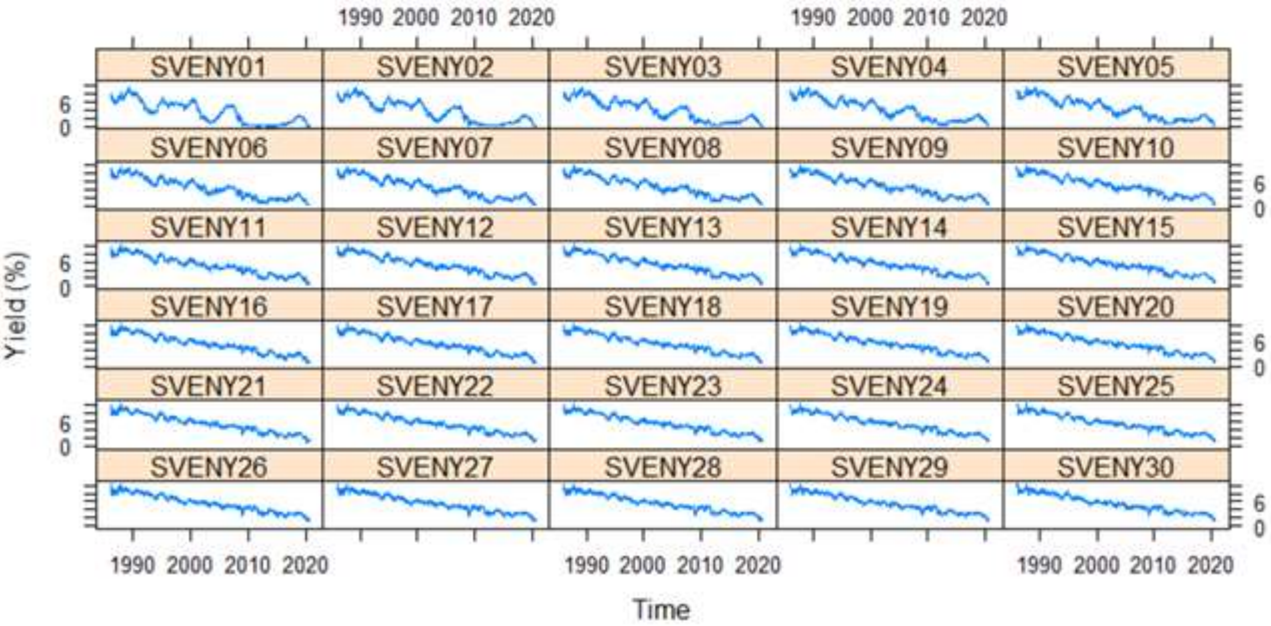
Question 3

Model	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5
NS Model	0.2897160	0.0085392	-0.0032782	-0.0215649		
NSS Model	11.211998	0.075750	0.124184	0.120698	0.052751	0.060285

Question 3

- θ_0: General constant that denotes the expected long term level of zero rates.
- θ_1: Small value -> Small effect
- θ_2: Downwards trough for NS Model, Upwards hump for NSS model
- θ_3: Decay factor
- θ_4: Second upwards hump for NSS model
- θ_5: Second decay factor for NSS model only

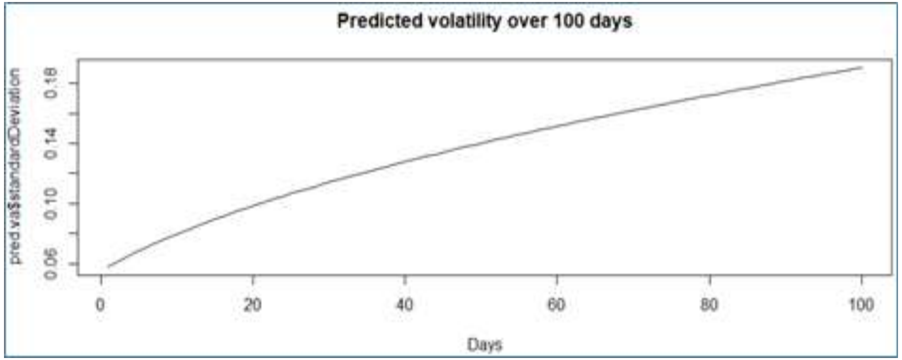
Question 4

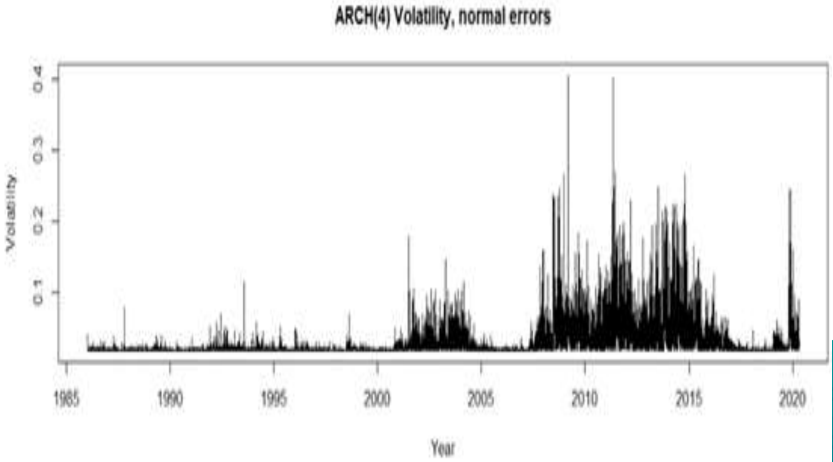


- For SVENY01, Quantile Kurtosis is 5.09, Moment Kurtosis is 17.89

Question 5

- NS NSS models easy to use
- Commonly used by central banks
- But parameters hard to fix
- ARCH(4) Model

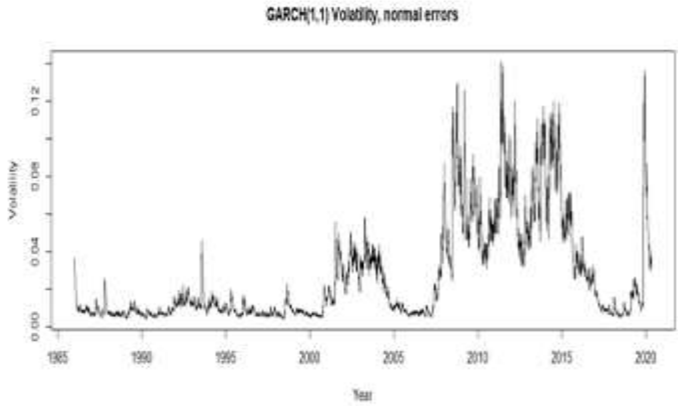


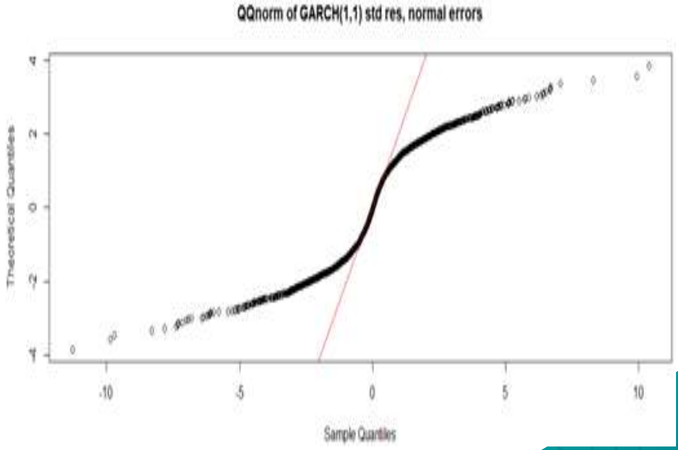


- GARCH(1,1) model

Interest rates have:

- Volatility clustering
- Exhibit heteroskedasticity
- Thick-tailed
- More parsimonious
- Well known to be better fit than ARCH





Radha, S. and Thenmozhi, M., Forecasting Short Term Interest Rates Using Arma, Arma-Garch and Arma-Egarch Models. Indian Institute of Capital Markets 9th Capital Markets Conference Paper, Available at SSRN: https://ssrn.com/abstract=87655 6 or http://dx.doi.org/10.2139/ssrn.876556

THANK YOU!