Induction

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Cool school

February 7, 2017

Climbing the ladder

I'm at the first rung.

Climbing the ladder

I'm at the first rung.
If I'm at a rung on the ladder, I can get to the next one.

Climbing the ladder

I'm at the first rung.
If I'm at a rung on the ladder, I can get to the next one.
$\therefore$ I can get to any rung of the ladder

Climbing the ladder (with symbols)

$P(0)$

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$$P(0)$$

$$P(k) \implies P(k+1), \text{ for } k \in \mathbb{N}$$

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$\forall n \geq 0, P(n)$ by the Principle of Mathematical Induction

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$\forall n \geq 0, P(n)$ by the Principle of Mathematical Induction
Reasoning: $P(0)$ and $P(0) \implies P(1)$ therefore $P(1)$. $P(1)$ and $P(1) \implies P(2)$ therefore $P(2)$.
Note that mathematical induction is a very powerful proof technique, because proving the basis step is usually very easy. Proving the inductive step isn't that hard either, because you get to assume the inductive hypothesis for free.
However, you already need to know what you are trying to prove to use induction. You already have to know what $P(n)$ is. You can prove existing theorems, but making new ones is not as easy.

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Example: Geometric series

Prove: $1 + r + r^2 + \cdots + r^n = (r^{n+1} - 1)/(r - 1)$
Setup: Let $P(n) := \sum_{i=0}^{n} r^i = (r^{n+1} - 1)/(r - 1)$. We want to show $\forall n \geq$

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   $(r^2 - 1)/(r - 1) = (r + 1)(r - 1)/(r - 1) = (r + 1)$, algebra

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We have shown $P(0)$, concluding the basis step

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3. $(r^{k+1} - 1)/(r - 1) + r^{k+1}(r - 1)/(r - 1)$

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Algebra

We have shown $P(k) \implies P(k+1)$ **Conclusion:** We have shown $P(0)$ and $P(k) \implies P(k+1)$ for all $k \geq 1$, therefore by the Principle of Mathematical Induction $\forall n \geq 1, P(n)$

Alternative fact:

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yeah, I went there

Alternative fact: All horses are the same color

- **Setup**
  - $P(n) :=$ "In any set of n horses, all the horses are the same color"
  - $\forall n \geq 1, P(n)$ 0 would also work here, 1 is easier to understand
  - Proof by induction.

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  - Proof by induction.
- **Basis Step**
  - $P(1)$ says "In a set of 1 horses, all the horses are the same color"
  - 1. Let $S = x$ be a set of horses, assumption
    2. x is the same color as x, transitivity
  - We have proved that $P(0)$ is true, completing the basis step

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- **Inductive Hypothesis**
  - Let k be an arbitrary integer
  - $P(k)$ says that "In a set of k horses, all the horses are the same color"

- **Inductive Step**
  - Assume the IH for some $k \geq 1$
  - $P(k+1)$ says that "In a set of $k+1$ horses, all the horses are the same color"
    1. Let $S = \{x_1, x_2, \ldots, x_{k+1}\}$ be a set of horses, assumption
    2. Let $S_1 = \{x_1, x_2, \ldots, x_k\}$, assumption
    3. Let $S_2 = \{x_2, x_3, \ldots, x_{k+1}\}$, assumption
    4. All horses in S_1 are the same color as x_2, IH
    5. All horses in S_2 are the same color as x_2, IH
    6. $S_1 \cup S_2 = S$, defn of S_1 and S_2
    7. All horses in S are the same color as x_2, transitivity
  - We have shown $P(k) \implies P(k+1)$, completing the inductive step.

- **Inductive Step**
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    2. Let $S_1 = \{x_1, x_2, \ldots, x_k\}$, assumption
    3. Let $S_2 = \{x_2, x_3, \ldots, x_{k+1}\}$, assumption
    4. All horses in S_1 are the same color as x_2, IH
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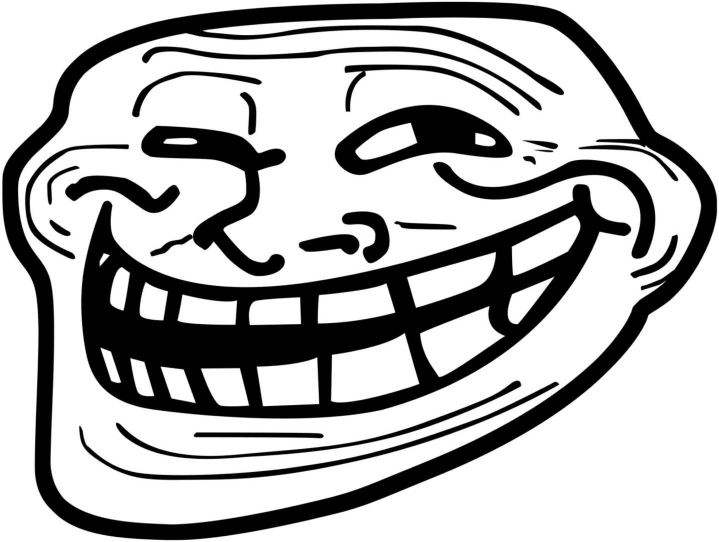
  - We have shown $P(k) \implies P(k+1)$, completing the inductive step.
- Conclusion omitted.



How would we get $P(3)$?

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$P(0)$

$P(0) \implies P(1)$

$\therefore P(1)$

$P(1) \implies P(2)$

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$P(2) \implies P(3)$

$\therefore P(3)$

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$P(2) \implies P(3)$

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But does $P(0) \implies P(1)$?

Strong Induction

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Bruh, do you even lift?

Strong Induction

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IH was $P(k)$, trying to prove $P(k+1)$
Now IH is $P(i)$ for all $b \leq i \leq k$
Also spelled, $\bigwedge_{i=b}^{k} P(i)$
Also spelled, $\forall i \in \mathbb{N}, (b \leq i \leq k \implies P(i))$

Strong Induction

IH was $P(k)$, trying to prove $P(k+1)$
Now IH is $P(i)$ for all $b \leq i \leq k$
Also spelled, $\bigwedge_{i=b}^{k} P(i)$
Also spelled, $\forall i \in \mathbb{N}, (b \leq i \leq k \implies P(i))$
Still trying to prove $P(k+1)$

Why?

Induction breaks a big problem into smaller pieces.
If you break one piece off at a time (reduce by one), use weak induction.
If you break your problem into smaller subproblems that are much smaller, use string induction.

Using induction to make money

Strong induction = weak induction

Structural induction

We are computer scientists here, right?
Recursively defined sets
Proofs on those sets

Structural induction

Handshaking
Infinity