Languages and machines

The meaning of language and limits of computation

Samuel Grayson

April 11, 2019

Goals

1. Give you an overview of language classes and machine types
2. Don't just think about computable/non-computable; Think about computable *with what*
3. Subsets of Turing Complete; next week, supersets

Mathy Definitions

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Kleene star $\Sigma^* := \{\varepsilon\} \cup \Sigma \cup \Sigma^2 \cup \cdots$
where $\Sigma^2 = \Sigma\Sigma$ (Star means arbitrary repetitions)
Language over alphabet Σ := subset of Σ^*
The strings are considered 'valid words' of the language.

Definitions

Formal Grammar :=

- finite set of terminals (convention: lowercase)
- finite set of nonterminals (convention: uppercase)
- start symbol (convention: S)
- finite set of rules (pair of strings)

Grammars generate languages.
Abstract Machine := ?
Machines can decide if a string is in a language.

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  1. S $\rightarrow$ aSb
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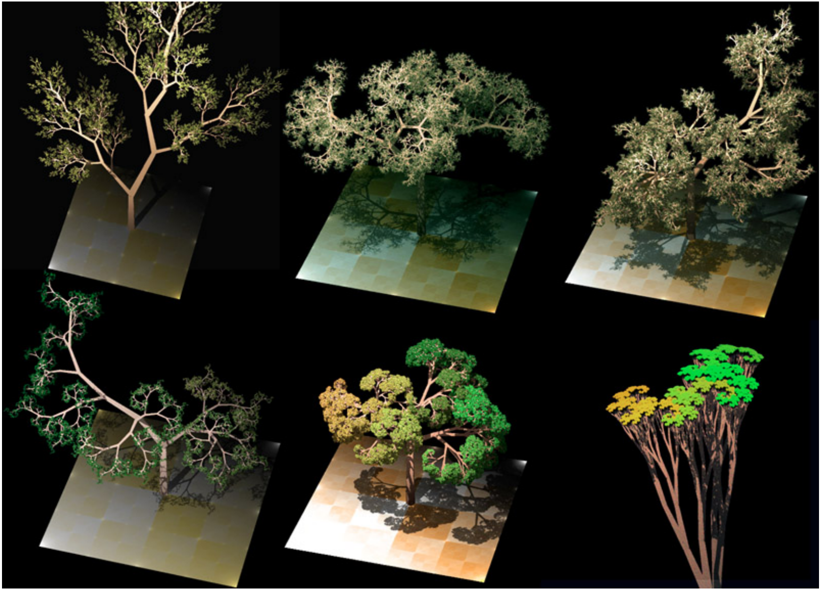
- The language generated by G is $\{a^n b^n | n \in \mathbb{N}\}$.

Lindenmayer system



Lindenmayer system

Grammar $\iff$ machine

Grammar generates string in the language
Machine recognizes a string that is in the language

Language relates to computing

- Consider language L consisting of factorials $\{$"1", "1", "2", "6", "24", $\ldots\}$

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- Consider language L consisting of factorials $\{$"1", "1", "2", "6", "24", $\ldots\}$
- A generator/recognizer would have to compute the factorial.

Language hierarchy

simple languages, quick machines $\rightleftharpoons$ complex languages, slow machines

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Chomsky Hierarchy (1956):

Language class	Grammar	Type of machine
Regular	A $\rightarrow$ Yz	deterministic finite-state automata
Context-free	A $\rightarrow \gamma$	deterministic pushdown automata
Context-sensitive	$\alpha \mathrm{A} \beta \rightarrow \alpha\gamma\beta$	linear-bounded non-deterministic Turing machine
Recursively Enumerable	$\alpha \rightarrow \beta$	Turing machine

where α, β, γ: strings of terminals and non-terminals

Language hierarchy

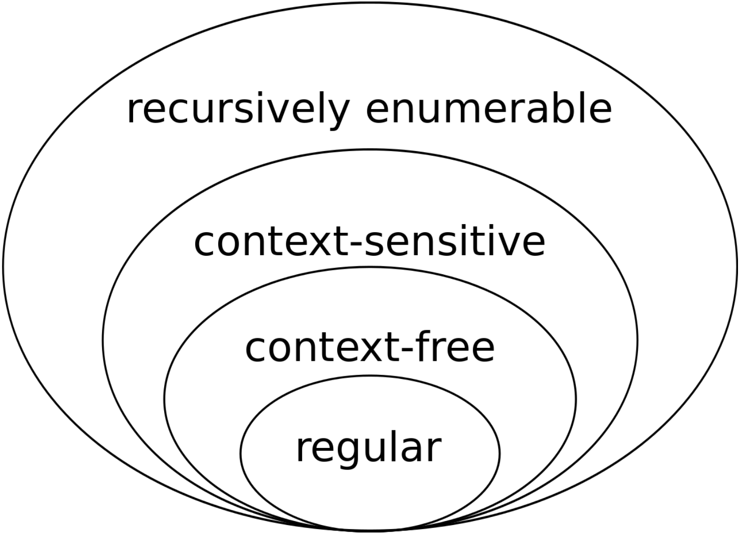

Figure from WikiMedia

Deterministic finite-state automata

Example language: Multiples of 3 in binary notation

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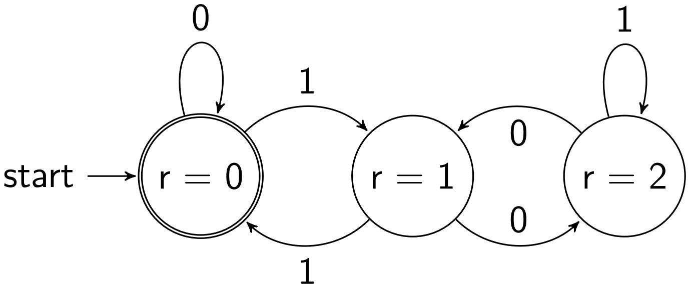

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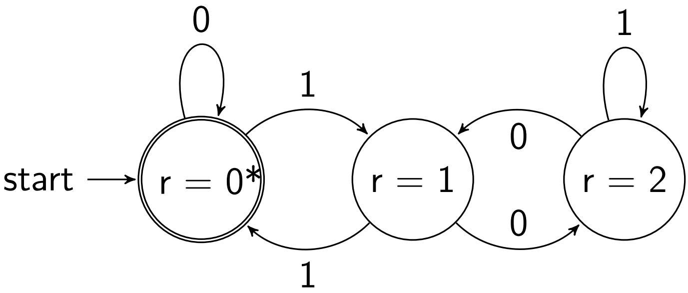

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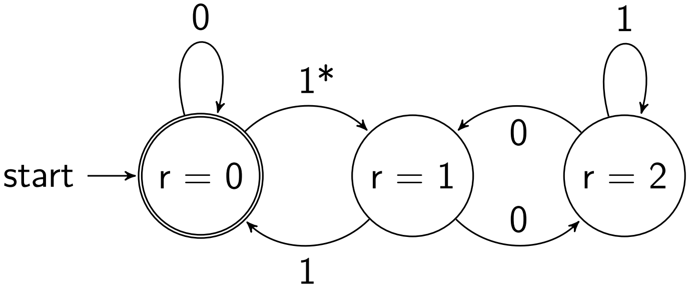

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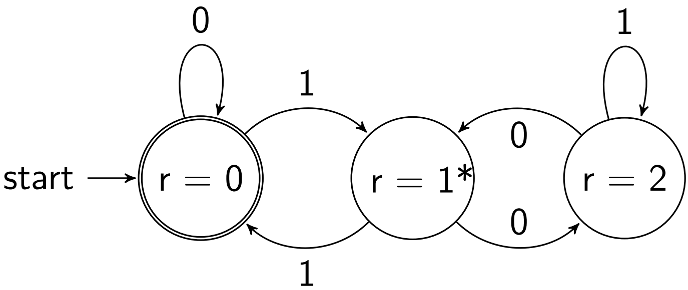

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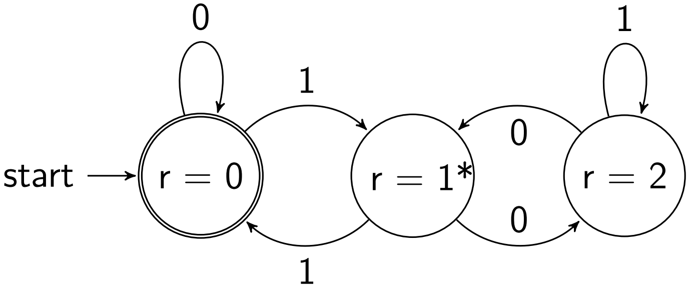

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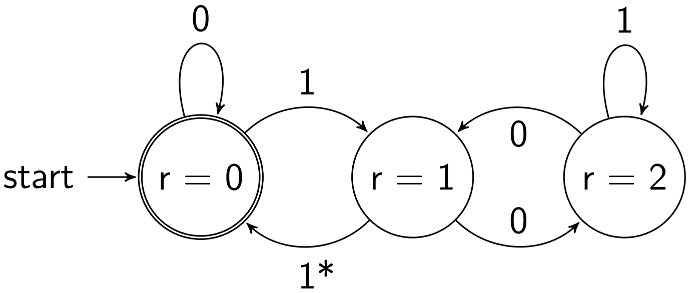

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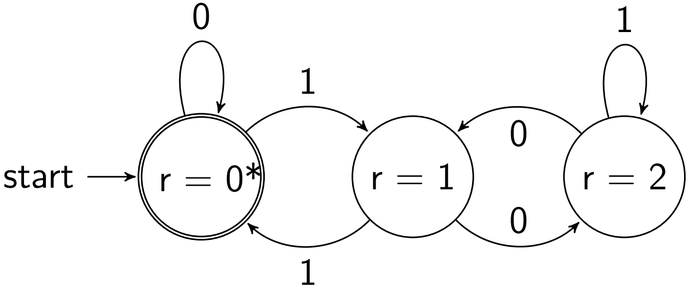

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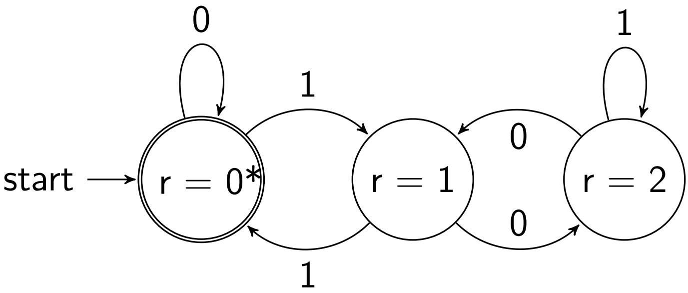

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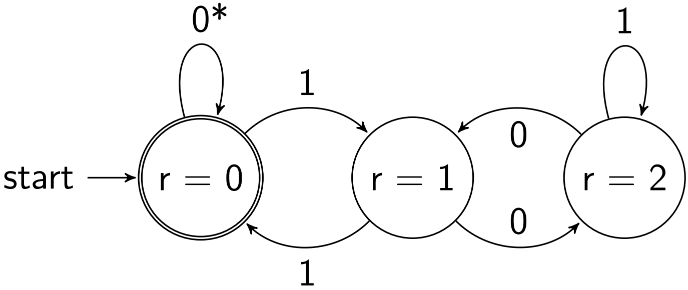

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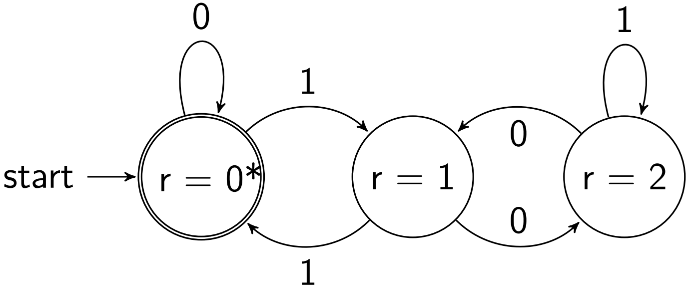

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NFA

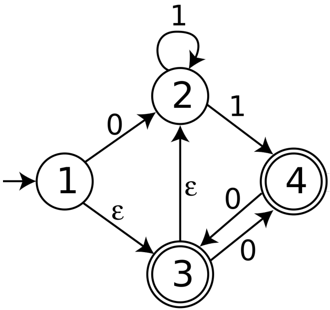
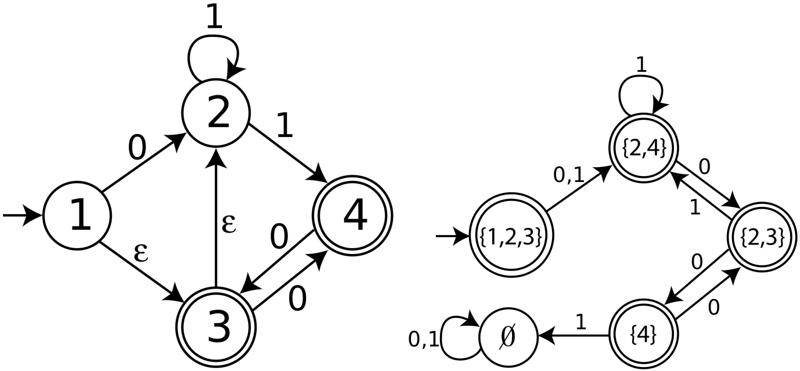


Figure from WikiMedia

NFA $\iff$ DFA



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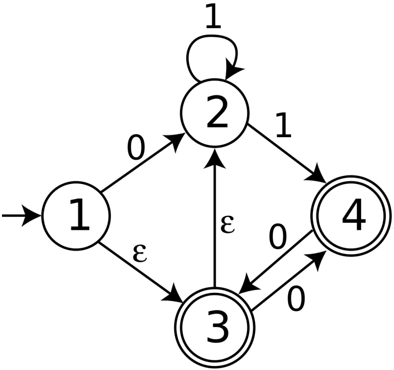

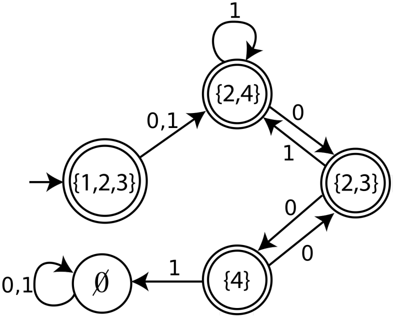

possibly n states $\rightarrow$ 2^n states, worst-case.

Figure 1 from WikiMedia

Figure 2 from WikiMedia

Regular grammar

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Likewise $A \to a_1 a_2 \cdots a_n B$ by $A \to a_1 A_1$, $A_1 \to a_2 A_2$, $\cdots$, $A_n \to a_n B$.

Regular Grammar example

Example language: Multiples of 3 in binary notation

remainder $x \pmod 3$	after appending 0 $x.0 \pmod 3$	after appending 1 $x.1 \pmod 3$
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A → 0A
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C → 0B
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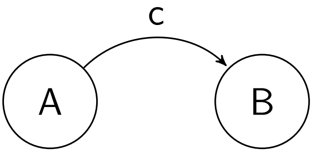
A → ε
S → A
TODO: redraw DFA

NFA $\iff$ Regular Grammar

Transition, , corresponds to rule, A $\rightarrow$ cB.

Start state = starting symbol, ending states $\rightarrow \varepsilon$

Useful for proofs.

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Algorithm to minimize (1979)
always halts; linear time

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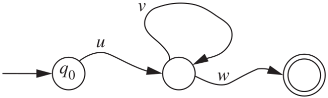

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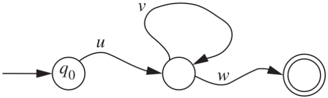

Now x can be decomposed into uvw where v can be repeated, so $uv^iw \in L$ also.
Note that v is non-empty and $|uv| \leq n$.

Figure from "Introduction to Languages and the Theory of Computation" by John C. Martin

Pumping lemma (1969)

Given a regular language, L,
There exists $n \in \mathbb{N}$ such that

- For all $x \in L$ and $|x| > n$
  - There exists u, v, w where
    1. $x = uvw$
    2. $v \neq \varepsilon$
    3. $|uv| \leq n$
    4. $uv^i w \in L$ for all $i \in N$

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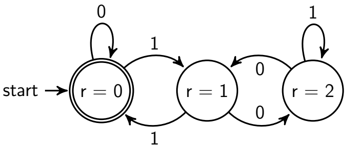

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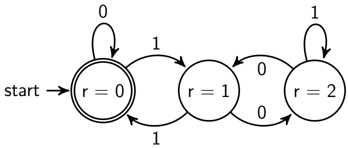

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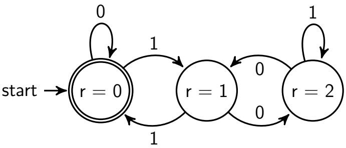

- Let's try divisble-by-3 DFA for $9 = 1001$.
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- Indeed $1 + 2^{1+2n} \pmod 3 = 1 + (-1) = 0$.

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  For every n, no less than n-long substring of $a^n b^n$ can be pumped up.
- Language where length of strings is not "eventually linear"
- Challenge: L is not regular, but L^2 is.

Pushdown Automata

Definition

Non-/Deterministic pushdown automata $:=$

- S: a finite set of states
- $A \subseteq S$: a set of accepting states
- $S_0 \in S$: an initial state
- Σ: a finite alphabet for words
- Γ: a stack alphabet
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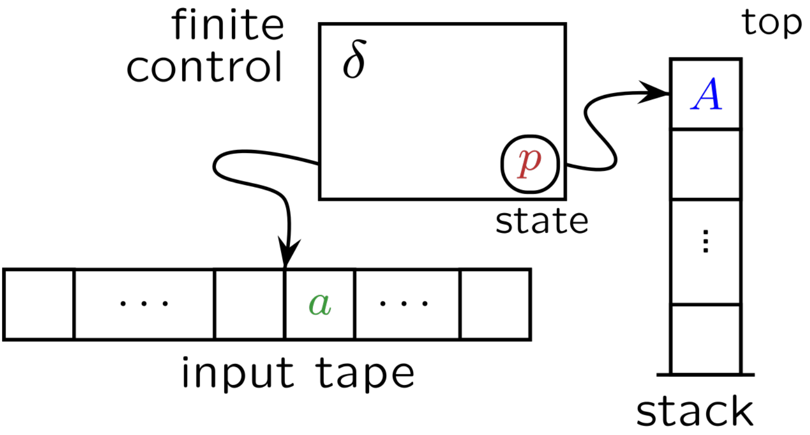
Accept by empty stack

Pushdown Automata



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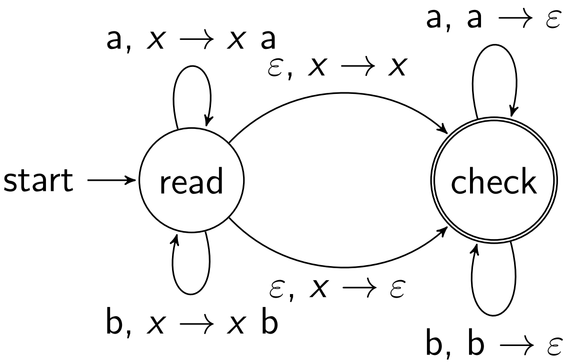

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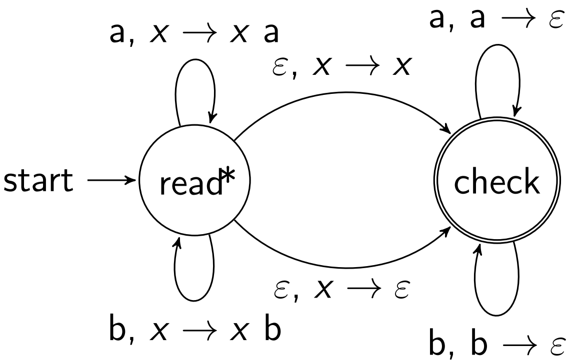

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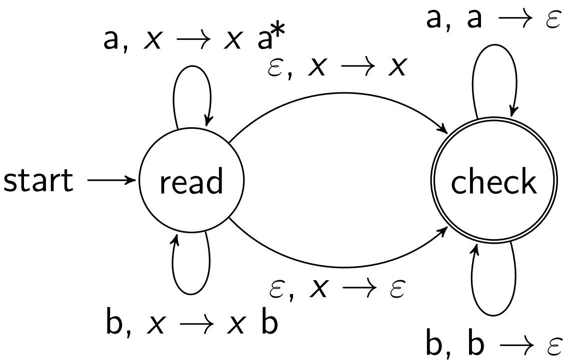

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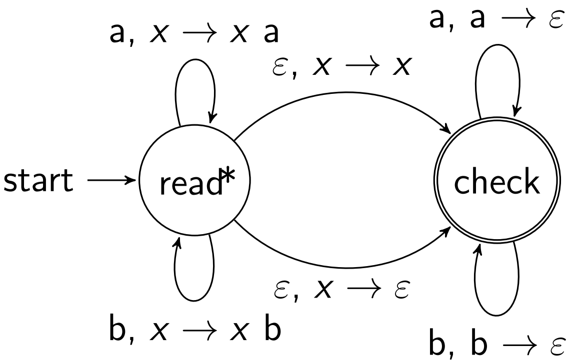

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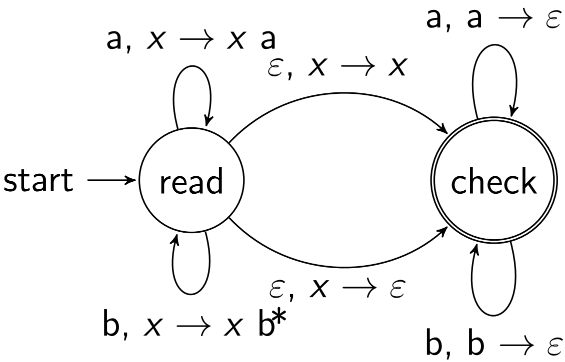

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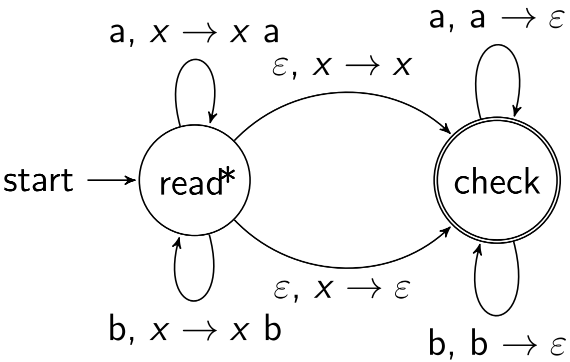

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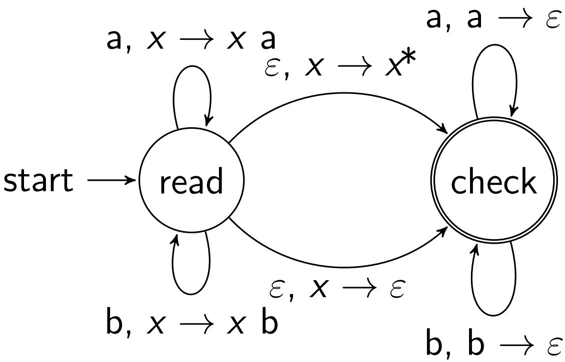

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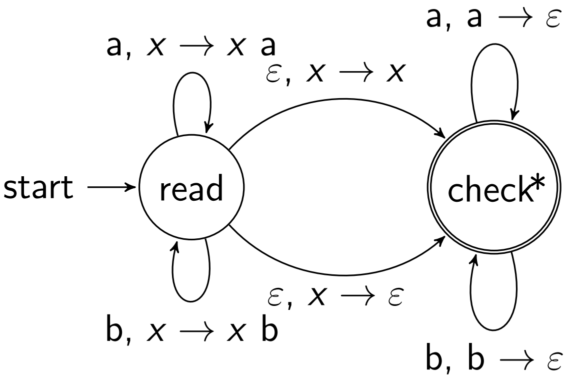

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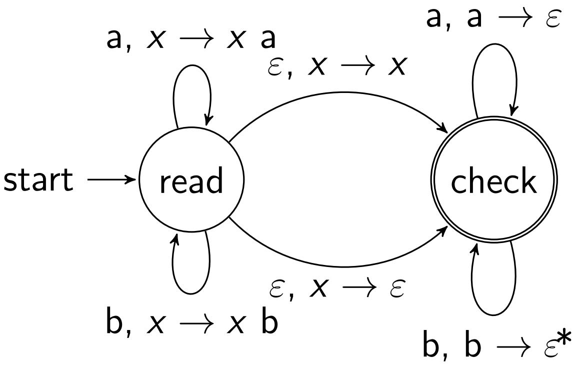

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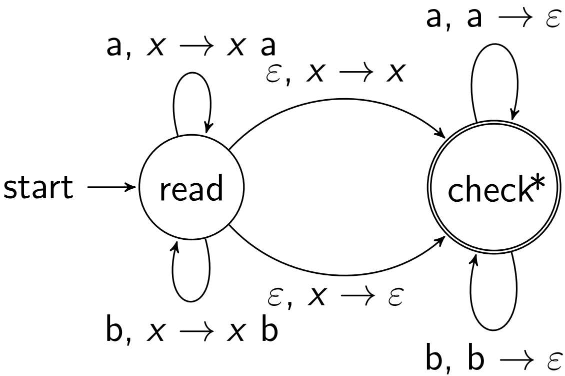

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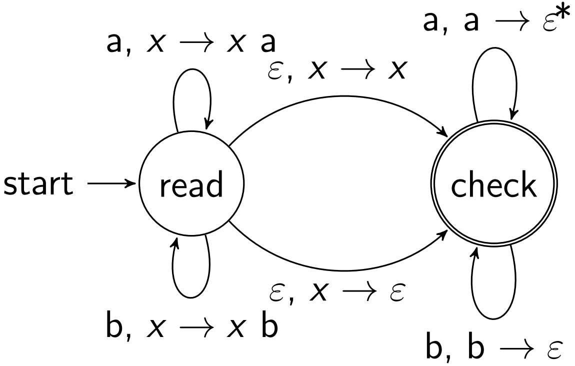
Stack: Ha

Pushdown Automata Example



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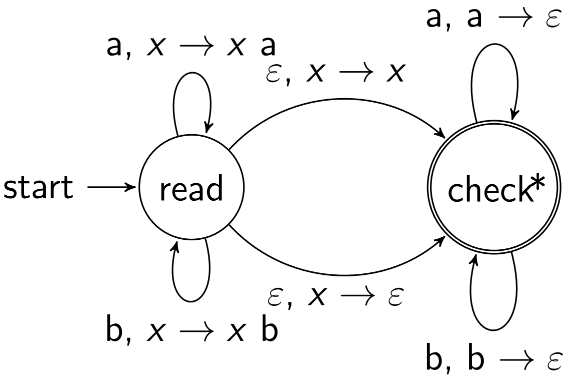

Input: abba
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Examples

Matched named brackets

Examples

Matched named brackets

State: If left bracket, push bracket-type to stack. If right bracket and correct bracket-type on stack, pop from stack.

CFG

Definition

Context-free grammar := Grammar where all rules are of the form $A \to \alpha$, where $\alpha \in (\Sigma \cup \Gamma)^*$.

Encodes everything a regular grammar can (repetition, optionals) + "arbitrary nestedness"

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  1. S $\to$ aSb
  2. S $\to \varepsilon$
- Palindromes similar

CFGs in the wild

Named brackets

CFGs in the wild

Named brackets

```
<html>
  <body>
    <h1>This is the tile</h1>
    <p>This is a paragraph.</p>
  </body>
</html>
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CFGs in the wild

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Programming languages

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Programming languages
95% of natural languages

Parse trees

1. S $\rightarrow$ [S]
2. S $\rightarrow$ S S
3. S $\rightarrow$ ε

[[]][]

Parse trees

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2. S → S S
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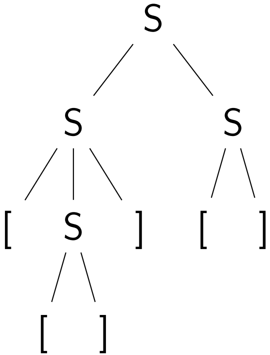

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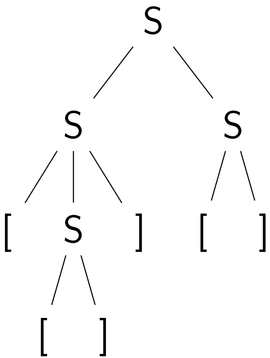
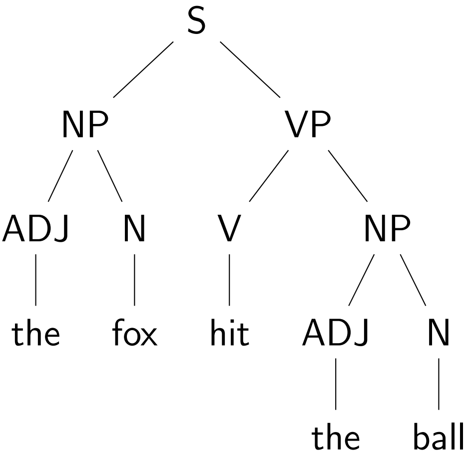


Parse trees for regular languages?

Ex. Natural language

- S $\rightarrow$ NP VP
- NP $\rightarrow$ ADJ N
- VP $\rightarrow$ V
- VP $\rightarrow$ V NP



Ambiguity

Example

$$\text{EXPR} \rightarrow \text{EXPR} + \text{EXPR}$$
$$\text{EXPR} \rightarrow \text{EXPR} * \text{EXPR}$$
$$\text{EXPR} \rightarrow \text{NUMBER}$$

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$$1 + 3 * 4$$

Ambiguity

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EXPR $\rightarrow$ NUMBER

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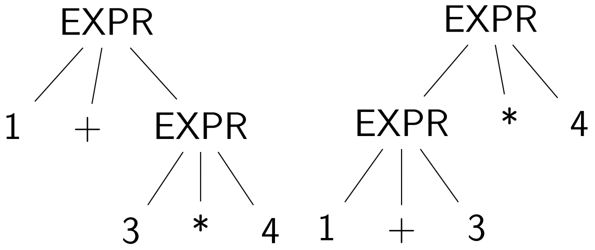

Resolution: order rules,

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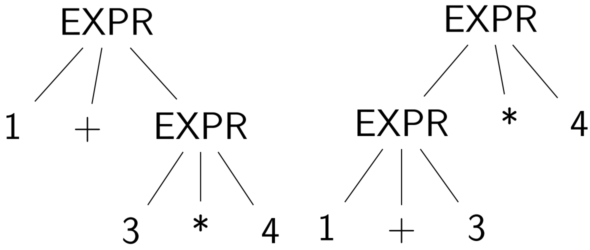

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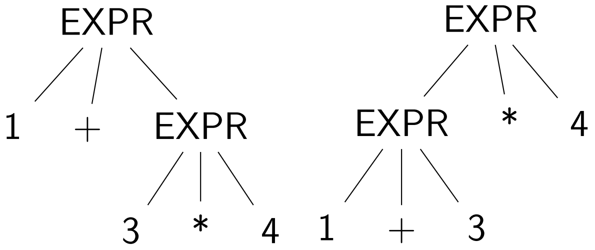


Resolution: order rules, order tokens, **don't**.

Ambiguity

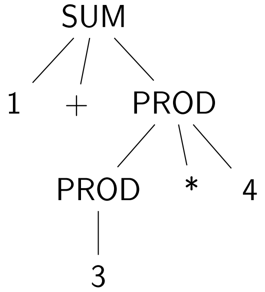
Example

SUM $\rightarrow$ SUM $+$ PROD
SUM $\rightarrow$ PROD
PROD $\rightarrow$ PROD * NUMBER
PROD $\rightarrow$ NUMBER

Ambiguity

Example

SUM → SUM + PROD
SUM → PROD
PROD → PROD * NUMBER
PROD → NUMBER



Ambiguity

Is a CFG ambiguous?

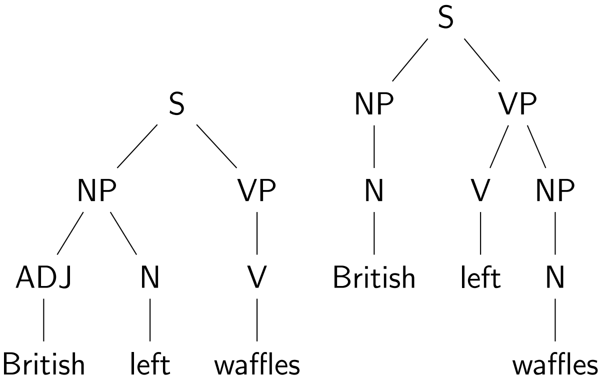
Ambiguity

Is a CFG ambiguous? Undecidable

It happens in the wild all the time!

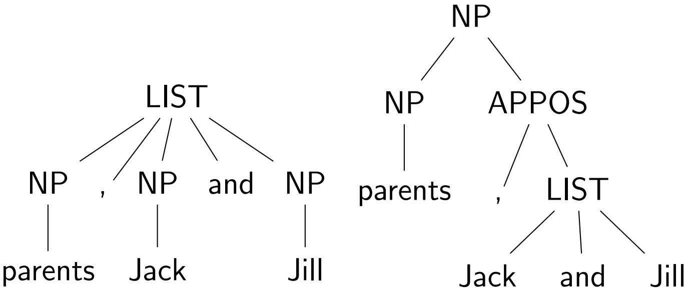
Ambiguity in natural language

British left waffles on Falklands



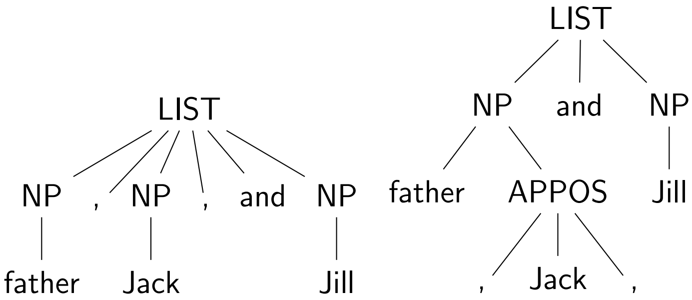
Ambiguity in natural language

"My parents, Jack and Jill" refers to 2 or 4 people.



Ambiguity in natural language

“My father, Jack, and Jill” refers to 2 or 3 people.



Lojban

Ambiguity in constructed languages

```
class1 obj (class2 ());
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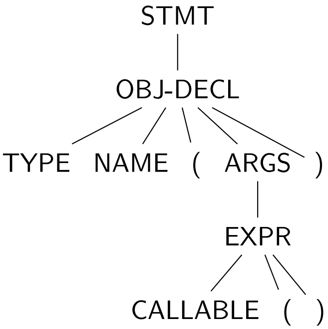

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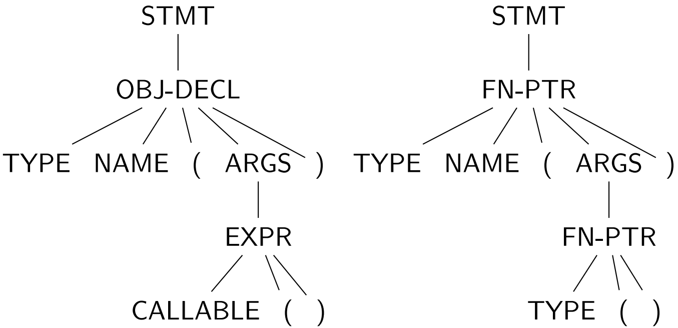

Public Service Announcement

```
// class1 constructed from class2
class1 obj ((class2 ()));
class1 obj {class2 {}}; // C++11

// Function pointer
class1 obj (class2 name ());
```

Bottom-up parsing

$S \rightarrow (\; S \;)$
$S \rightarrow \varepsilon$
Stack:
Input: (())()

Bottom-up parsing

$S \to (\ S\ )$
$S \to \varepsilon$
Stack: (
Input: ())()

Bottom-up parsing

$S \to (\; S \;)$
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Input:)

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Input:

Bottom-up parsing

$S \to (\ S \ )$
$S \to \varepsilon$
Stack: SS
Input:

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Input:

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- Chomsky–Schützenberger Theorem (1963): CFG $=$ named-bracket $\cap$ regular

Pumping lemma for CFG

For all $s \in L$ and $|s| \geq n$,

- $s = uvwxy$
- $uv^iwx^iy \in L$
- $|vx| \geq 1$
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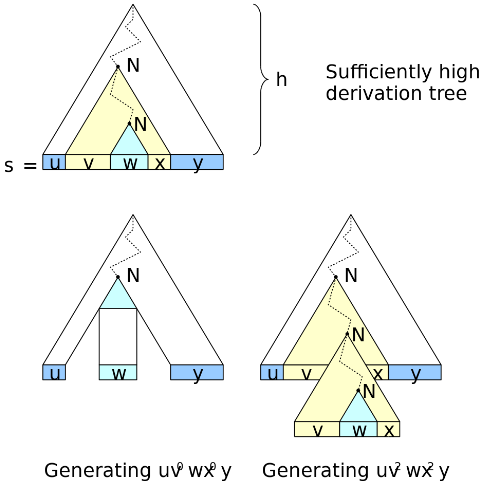

Figure from WikiMedia

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- Programming language with no undefined terms

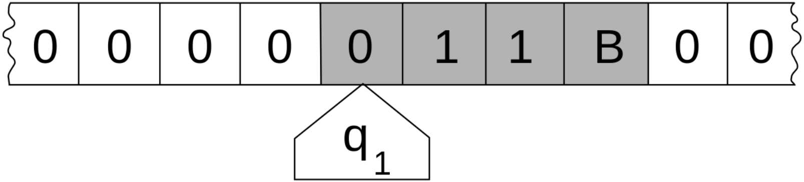
Turing Machine

Definition

Non-/Deterministic Turing Machine $:=$

- S: set of states
- $S_0 \in S$: initial state
- $S_r, S_a \in S$: halt reject, halt accept state
- Γ: tape alphabet
- $b \in \Gamma$: blank symbol
- $\Sigma \subseteq \Gamma \setminus \{b\}$: input alphabet
- $f : S \times \Sigma \to S \times \Sigma \times \{R, L\}$: deterministic transition function
- $f : S \times (\Sigma \cup \{\varepsilon\}) \to \mathcal{P}(S \times \Sigma \times \{R, L\})$: non-deterministic transition function

Turing Machine



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Nobody knows if LBNTM $\iff$ LB(D)TM

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Note: Non-decreasing

LBNTM $\iff$ CSL

Suppose $\alpha \to \beta$ in a CSL with terminals Σ and non-terminals Γ.

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CSL's are non-decreasing, so linear bounded.

LBNTM $\iff$ CSG

Construct CSG for given LBNTM:
Let configuration $:= x_0 x_1 \cdots x_{n-1} S_i x_n \cdots x_k$

LBNTM $\Longleftrightarrow$ CSG

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Suppose $(S_i', x_n', L) \in f(S_i, x_n)$. Then
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$x_0x_1 \cdots x_{n-1}S_ix_n \cdots x_k$
We can work 'backwards' to starting-tapes that the machine would accept!
We will start with a rule that generates arbitrary finishing-tapes: $b \to x_ib$ for $x_i \in \Gamma$, in an accepting state $S \to S_ab$.

CSL equivalents

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  $(AB \to CD, A \to BC, A \to B, A \to a)$
  (exception for $\varepsilon \in L$)

Turing Machines equivalents

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Turing Machines equivalents

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- FlooP

None of these are guaranteed to halt!

P = NP

Deterministic TM **P**olynomial-time algorithm $\iff$ **N**on-deterministic TM **P**olynomial-time algorithm?

P $\neq$ NP

Deterministic TM **P**olynomial-time algorithm $\not\Leftrightarrow$ **N**on-deterministic TM **P**olynomial-time algorithm?

Universal Turing Program

Turing Machine takes encoding of a machine and its input, separated by a marker symbol.
Decides if machine accepts given input.

Unrestricted Grammars

$\alpha \to \beta$
α and β can be anything in $(\Gamma \cup \Sigma)^*$!

Unrestricted Grammars $\iff$ NTM

Task: construct NTM that recognizes strings genreated by an unrestricted grammar

$x_0 x_1 \cdots x_n$ on tape.

Unrestricted Grammars $\Longleftrightarrow$ NTM

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Start simulating a derivation, replacing α with β according to grammar. Result is unbounded, but so is tape.

Unrestricted Grammars $\Longleftrightarrow$ NTM

Task: construct NTM that recognizes strings genreated by an unrestricted grammar

$x_0 x_1 \cdots x_n$ on tape.

Append $\#S$.

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If string before $\#$ matches that after, accept.

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Could take forever

Unrestricted Grammars $\Longleftrightarrow$ NTM

$x_0 x_1 \cdots x_n$

Unrestricted Grammars $\iff$ NTM

$x_0 x_1 \cdots x_n$

$x_0 x_1 \cdots x_n \# S$

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$x_0 x_1 \cdots x_n$
$x_0 x_1 \cdots x_n \# S$
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$x_0 x_1 \cdots x_n \# \beta\theta\gamma\alpha$

Unrestricted Grammars $\Longleftrightarrow$ NTM

$$x_0x_1\cdots x_n$$
$$x_0x_1\cdots x_n\#S$$
$$x_0x_1\cdots x_n\#\alpha\gamma$$
$$x_0x_1\cdots x_n\#\beta\theta\gamma\alpha$$
$$\vdots$$

Unrestricted Grammars $\iff$ NTM

$$x_0 x_1 \cdots x_n$$
$$x_0 x_1 \cdots x_n \# S$$
$$x_0 x_1 \cdots x_n \# \alpha\gamma$$
$$x_0 x_1 \cdots x_n \# \beta\theta\gamma\alpha$$
$$\vdots$$
$$x_0 x_1 \cdots x_n \# x_1 x_2 \cdots x_n$$

Unrestricted Grammars $\Longleftrightarrow$ NTM

$x_0 x_1 \cdots x_n$

$x_0 x_1 \cdots x_n \# S$

$x_0 x_1 \cdots x_n \# \alpha\gamma$

$x_0 x_1 \cdots x_n \# \beta\theta\gamma\alpha$

$\vdots$

$x_0 x_1 \cdots x_n \# x_1 x_2 \cdots x_n$

accept

Unrestricted Grammars $\iff$ NTM

Suppose $(S_i', x_n', L) \in f(S_i, x_n)$,
then $S_i' x_{n-1} x_n' \rightarrow x_{n-1} S_i x_n$.