-0.382 +0.459

Visualizing functions of multiple variables

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Abstract

Visualizing the concepts of multivariable calculus can be challenging. The students created several computer-generated visual models utilizing the Mathematica language. These models can be used as teaching aids for educators or extra instruction for students.

Acknowledgements

Contents

1   Introduction

As students, we know that visualizing these topics can sometimes be pretty difficult. I may or may not be guilty, on occasion, of simply using these formulas without an understanding of what they really do. If only there were some means of playing around with these topics without having to imagine clear mountains with magical rings or bleeding, toilet paper pasted shaved faces (inside joke). We believe that we have created the solution!

The magic of Multivariable can be seen through the "Sam and Wilson's 3D Multivariable Visualizer for Glazed-eyed LASA students" for a small fee of appreciation. The goals of this Visualizer are to allow students to interact with the topics discussed below and instead of telling the students what the topics are, the Visualizer simply shows them. The topics would reveal themselves to even the most distant of students simply nodding their heads as asked (inside joke once more). We hope that you incorporate this Visualizer into your lessons and take advantage of the many opportunities that this opens up for students.

2   Design Principles

- **Flexibility:** The same concept can be visualized on multiple examples, and we tried to make it so that the models can adjust at runtime to any user inputted example.

- **Interactivity:** The concepts are best illustrated when the user can tweak the values and adjust the parameters in a dynamic way that gives real-time feedback.

- **Maintainability:** The models *look good.* They are designed to be very clean and adjustable. This means that the code may be less aligned with the mathematics, but it is worth it for ease of maintaining. This way, when we graduate the models may still be adapted to the curriculum.

- **Instructiveness:** The models are not by themselves. They are accompanied by text explaining how they work and what they do.

3 Frenet-Serret frames

The Frenet frame is a tool for understanding the local motion of a curve. The Frenet frame gives you an idea of where the function would be going if it were a straight line (tangent vector), which way the function would be bending if it were a circle (normal vector), and which way the function is twisting (binormal vector). Additional quantities like the curvature (related to the radius of the function if it were a circle) and the torsion (related to the angular momentum in the direction of the binormal vector).

The input should be a regular path in real space. $c(t) : [a, b] \rightarrow \mathrm{R}^3$.

1. Calculate the unit tangent vector. Taylor's Theorem gives us $c(t_0) + c'(t) \cdot (t - t_0) \approx c(t)$ for small δt. A straight line starting at $\vec{u}$ in the direction of $\vec{v}$ is given by $\vec{u} + \vec{v} \cdot t$. Therefore the line that locally approximates the function f is the first-order Taylor's Theorem. Since the tangent vector is relative to the point $\vec{u} = \vec{0}$, and relative to the time $t = 0$, and unit normalized, it is given by $T(t) = \frac{c'(t)}{\|c'(t)\|}$.

2. Calculate the unit normal vector. Let $c(t)$ be approximated by a circle of radius r lying in the plane spanned by $T(0)$ and $N(0)$ (this vector is not yet known). $c(t) \approx rT(0) \sin t + rN(0) \cos t$. Then $c'(t) = rT(0) \cos t - rN(0) \sin t$ (the reader can verify that at $t = 0$, the tangent vector is indeed $T(0)$). Then $c''(t) = T'(t) = -rT(0) \sin t - rN(0) \cos t$. Therefore $T'(0) = -rN(0)$ which points in the direction of the normal vector. Therefore the unit-normalized normal vector is given by $N(t) = \frac{T'(t)}{\|T'(t)\|}$.

3. Calculate the curvature. $\kappa(t) = \frac{1}{r}$ in the circle described by $c(t) \approx rT(0) \sin t + rN(0) \cos t$. As previously shown, $T'(0) = -rN(0)$. Therefore

4. Calculate the binormal vector. In order to form an orthnormal basis, $B(t) = T(t) \times N(t)$.