Boundary of unchanging region

Sam 'Swag' Grayson

July 25, 2016

Abstract

I prove the existence and provide an upper limit for the a boundary above which the k-ary divisors are the same as the unitary (for odd k) or biunitary (for even k).

1 Words that mean things

Definition 1. *Let* $D_0(n) = \{a \in \mathbb{Z} : a|b\}$ *and* $D_k(n) = \{a \in \mathbb{Z} : a|b \wedge D_k(a) \cap D_k(n/a) = \{1\}\}$ *for* $k > 0$. *(the set of k-ary divisors)*

Definition 2. *a and k (where $k > 2$) are called unchanged iff* $D_k(p^a) = D_{k-2}(p^a)$.

Definition 3. *a and k (where $k > 2$) are called completely unchanged iff* $D_k(p^a) = D_2(p^a)$ *when k is even or* $D_k(p^a) = D_1(p^a)$ *when k is odd.*

2 Boring stuff

The following are proved in my previous report on k-ary convolutions

Theorem 1. *For all k,* $\{1, n\} \subseteq A_k(n)$.

Theorem 2. *For* $p \in \mathbb{P}$, $D_0(p^a) = \{1, p, p^2, \ldots, p^a\}$

Theorem 3. *For* $p \in \mathbb{P}$, $D_1(p^a) = \{1, p^a\}$

Theorem 4. *For* $p \in \mathbb{P}$, $D_2(p^a) = \begin{cases} D_0(p^a) \setminus \{p^{a/2}\} & 2 \mid a \\ D_0(p^a) & 2 \nmid a \end{cases}$

Theorem 5. *For* $p \in \mathbb{P}$, $D_3(p^a) = \begin{cases} \{1, p, p^2, p^3\} & a = 3 \\ \{1, p^2, p^4, p^6\} & a = 6 \\ A_1(p^a) & \textit{otherwise} \end{cases}$

3 What you've all been waiting for

Theorem 6. *For every k, I give a prime power a_0, such that for all $a > a_0$, a_0 under k-ary convolution is completely unchanged (and hence, unchanged).*

Proof. *(by induction on k)*

Base case: For $a > 6$, $A_3(p^a) = A_1(p^a)$ by Thm 5, and hence p^a is completely unchanged for 3-ary convolution.

Inductive Hypothesis: Assume the Thm holds for k_0. Let a_0 be the boundary above which all prime-powers are unchanged for k_0. I will split the inductive step into one step for even k_0 and one step for odd k_0, and in each one show the Thm holds for $k_0 + 1$.

Even inductive step: For $a > 2a_0$, p^a, all factors are in the following form: p^b and p^{a-b} for $b \le \lfloor a/2 \rfloor$. Thus $a_0 < \lfloor a/2 \rfloor \le a - b$, so at least one factor (namely p^{a-b}) is in the completely unchanged region of k_0. Hence $D_{k_0}(p^{a-b}) \cap D_{k_0}(p^b) = \{1\}$ is equivalent to (by the inductive hypothesis) $D_1(p^{a-b}) \cap D_{k_0}(p^b) = \{1\}$ is equivalent to (by Thm 3) $\{1, p^{a-b}\} \cap D_{k_0}(p^b) = \{1\}$ which is equivalent to (ignoring the case where $b = 0$) $a - b \ne b$ which is equivalent to $b \ne a/2$. Therefore $D_{k_0+1}(p^a) = \{p^b | b \le a \wedge D_{k_0}(p^{a-b}) \cap D_{k_0}(p^b) = \{1\}\}$ is equivalent to $\{p^b | b \le a \wedge b \ne a/2\}$

Odd inductive step: By the same reasoning, for $a > 3a_0$ and $a > 3$ we can split p^a into two factors, p^b and p^{a-b} where p^{a-b} is in the completely unchanged region of k_0. Then $D_{k_0}(p^{a-b}) \cap D_{k_0}(p^b) = \{1\}$ is equivalent to (by the inductive hypothesis) $D_2(p^{a-b}) \cap D_{k_0}(p^b) = \{1\}$. When $a-b \ne 2b$, we know that $p^b \in D_{k_0}(p^b)$ by Thm 1 and $p^b \in p^{a-b}$ by Them 4 since $b < a-b$ and $2b \ne a-b$. When $a-b = 2b$, then $b = a/3 > a_0$ and so **both** factors fall into the completely unchanged region of k_0. In that case $p^{b-1} \in D_2(p^b)$ and $p^{b-1} \in p^{a-b}$ by Thm 4. Therefore $D_2(p^{a-b}) \cap D_{k_0}(p^b) = \{1\}$ is never met except for when $b = 0$. Thus $D_{k_0+1}(p^a) = \{p^b | b \le a \wedge D_{k_0}(p^{a-b}) \cap D_{k_0}(p^b) = \{1\}\}$ is equivalent to $D_{k_0+1}(p^a) = \{p^b | b = 0 \vee b = a\}$. □

4 Looking forward

We should find out how to prove non-associativity for odd k now that we have it for even k.

Open problem: prove unchanged region is equal to completely unchanged region when k is not a power of 2

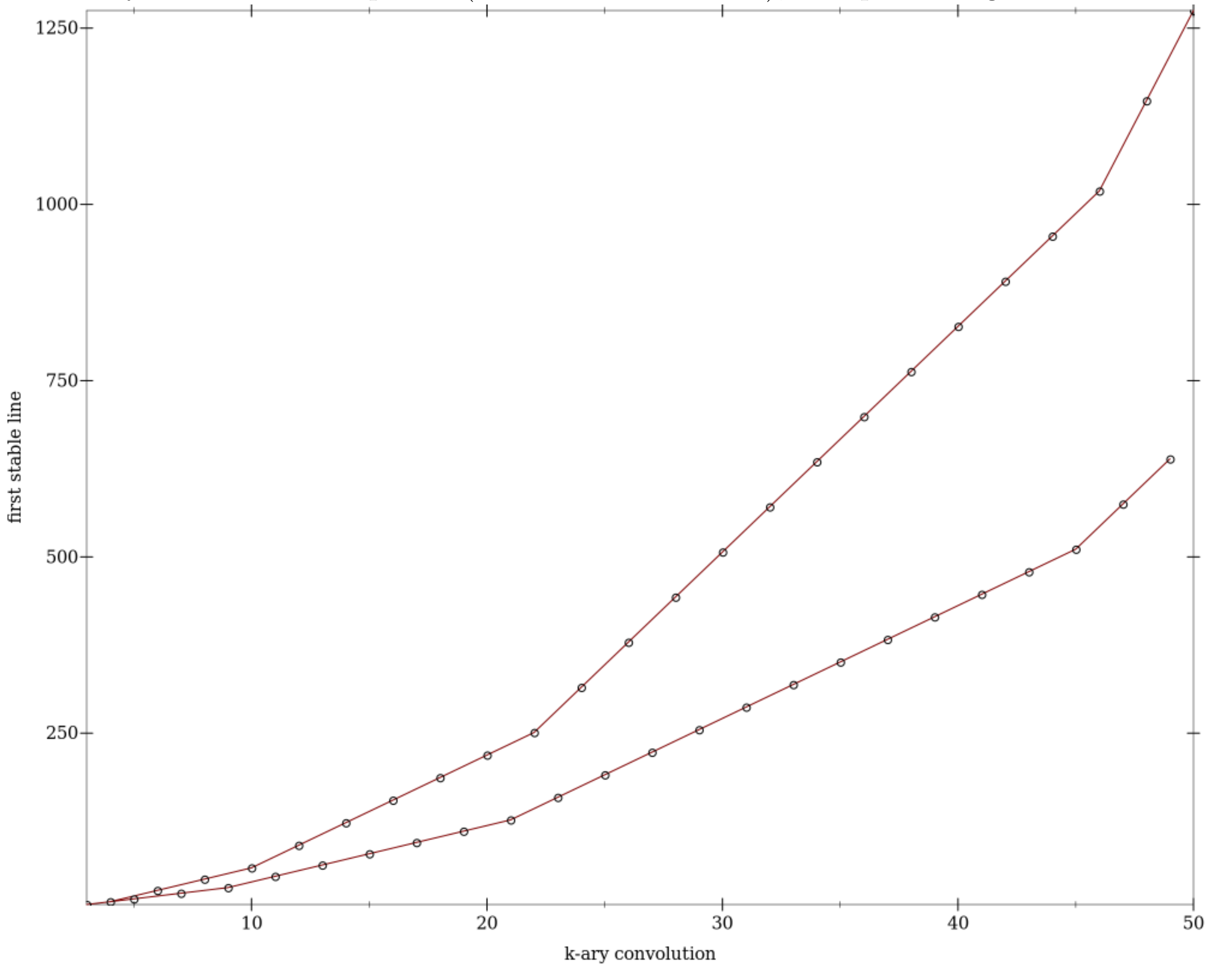
What we actually discussed in the meeting was slightly different conjecture: that for every k, there is a number, a_0, such that every prime-power above it is unchanged. I showed that every prime-power above it is completely unchanged. They imply each other, but they might have different bounds (different a_0).

However, I numerically computed the boundary of the **unchanging region** for every k from 1 to 50 and found that it is also the same as **completely unchanging** region (!) with exceptions at 2^n.

Open problem: tighten the upper bound on the boundary of the completely unchanged region.

My computations reveal that this upper bound is not very tight. The actual boundary is less. More precisely, its growth less. When k is odd, the boundary to the completely unchanged region increases by approximately a factor of 1.87 every 2 increases in k (that is the exponent increasing exponentially). When k is even, that number is 1.66. My upper bound grows by 6 in both cases (in both cases, one odd increase times one even increase).

Furthermore, there seems to be an obvious pattern. First, I noticed that the odds and evens bifurcate into different curves with the evens consistently larger. Then each curve seems to have its own pair of counters (m, n) which start at (3, 3) for evens and (2, 2) for odds. Each curve increases (arithmetically) by 2^n until you hit 2^m and then transitions to $n := n + 1$ and $m := m + 2$. This **perfectly** describes the boundary. When a simple pattern perfectly describes the boundary for over 25 datapoints (25 odds and 25 evens), I suspect things.



The raw data can be found in the TEXsource code at the end of the document in CSV form.