

# Additive, abelian and exact categories

## 1. Additive categories

Def: An additive category is a category  $\mathcal{A}$  such that

(Add1)  $\mathcal{A}$  has a zero object  $0$  (i.e. an object that is both initial and terminal)

(Add2)  $\text{Hom}_{\mathcal{A}}(X, Y)$  has an abelian group structure such that composition is biadditive.

(Add3)  $\mathcal{A}$  has biproducts, i.e.  $\forall X_1, X_2 \in \text{Ob}(\mathcal{A}) \exists$  object  $X = X_1 \oplus X_2$  and morphisms

$$\begin{array}{ccccc} X_1 & \xleftarrow{\pi_1} & X & \xrightarrow{\pi_2} & X_2 \\ & \xrightarrow{\sigma_1} & & \xleftarrow{\sigma_2} & \\ & & X & & \end{array}$$

such that

$$\pi_i \sigma_j = \delta_{ij} \text{id}_{X_i}$$

$$\sigma_1 \pi_1 + \sigma_2 \pi_2 = \text{id}_X$$

Link (1) The zero element in the abelian group  $\text{Hom}_{\mathcal{A}}(X, Y)$  equals the unique morphism  $X \xrightarrow{0} Y$

(2)  $(X, \sigma_1, \sigma_2)$  is a coproduct and  $(X, \pi_1, \pi_2)$  is a product of  $X_1$  and  $X_2$

(3) The group structures on  $\text{Hom}_{\mathcal{A}}(X, Y)$  are intrinsic and no additional data!

Given  $X \xrightarrow[f]{g} Y$

$$\begin{array}{ccc} X & \xrightarrow{f+g} & Y \\ \downarrow \begin{pmatrix} \text{id}_X \\ \text{id}_X \end{pmatrix} = \begin{pmatrix} \sigma_1^X + \sigma_2^X \\ \text{id}_X \end{pmatrix} & & \uparrow \begin{pmatrix} \pi_1^Y + \pi_2^Y \\ \text{id}_Y \end{pmatrix} = \begin{pmatrix} \text{id}_Y \\ \text{id}_Y \end{pmatrix} \\ X \oplus X & \xrightarrow[\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}]{\sigma_1^Y f \pi_1^X + \sigma_2^Y g \pi_2^X} & Y \oplus Y \end{array}$$

The black morphisms can be constructed using only universal properties of (co)products and the fact that we have a zero object (and hence zero morphisms)

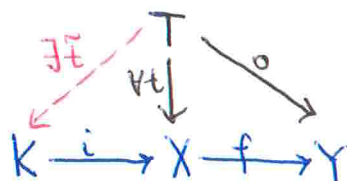
## 2. Abelian categories

Def Let  $\mathcal{A}$  be a category with a zero object and  $X \xrightarrow{f} Y$  a morphism.

- $(K, K \xrightarrow{i} X)$  is a kernel of  $f$

if  $\exists i=0 \quad \forall T \xrightarrow{t} X$  with  $ft=0$

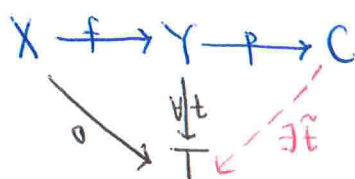
$\exists! T \xrightarrow{\tilde{t}} K$  with  $i\tilde{t}=t$



- $(C, Y \xrightarrow{p} C)$  is a cokernel of  $f$  if

$pf=0$  and  $\forall Y \xrightarrow{p} T$  with  $tf=0$

$\exists! C \xrightarrow{\tilde{p}} T$  with  $\tilde{p}p=t$

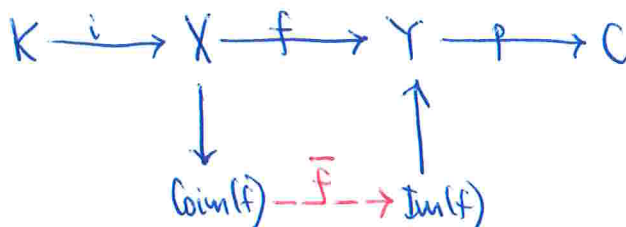


- The image of  $f$  is  $\text{Im}(f) := \ker(Y \rightarrow \text{Cok}(f))$

The coimage of  $f$  is  $\text{Coim}(f) := \text{Cok}(\ker(f) \rightarrow X)$

$\Rightarrow$  We get an induced morphism

$\bar{f}: \text{Coim}(f) \rightarrow \text{Im}(f)$



Def: An abelian category is an additive category  $\mathcal{A}$  such that every morphism has a kernel and cokernel and for every morphism  $f: X \rightarrow Y$  the induced morphism  $\bar{f}: \text{Coim}(f) \rightarrow \text{Im}(f)$  is an isomorphism.

Examples of abelian categories:

- $\text{Mod } R$  for a ring  $R$
- $\text{Coh}(X)$  for a scheme  $X$
- $\text{Fun}(\mathcal{C}, \mathcal{A})$  for a small cat.  $\mathcal{C}$  and an abelian cat.  $\mathcal{A}$

Non-example:  $\mathcal{A}$  = category of f.g. free abelian groups, then  $\mathcal{A}$  has all kernels and cokernels and  $\text{Im}(f) \cong \text{Coim}(f)$  for every morphism  $f$ , but  $\mathcal{A}$  is not abelian.

### 3. Exact categories

Def. An exact category is an additive category  $\mathcal{A}$  endowed with a class of kernel-cokernel pairs  $(i, p)$ , called conflations, s.t.

inflation  $\nearrow$   $\nwarrow$  deflation

(Ex0)  $\text{id}_0$  is a deflation

(Ex1) The composition of two deflations is a deflation

(Ex2) For every deflation  $p$  and morphism  $f$  the pullback exists and  $p'$  is again a deflation

$$\begin{array}{ccc} Y' & \xrightarrow{p'} & Z' \\ \downarrow & & \downarrow f \\ Y & \xrightarrow{p} & Z \end{array}$$

(Ex2<sup>op</sup>) For every inflation  $i$  and morphism  $f$  the pushout exists and  $i'$  is an inflation.

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ f \downarrow & & \downarrow f' \\ X' & \xrightarrow{i'} & Y' \end{array}$$

Hint (1) The dual statements (Ex0<sup>op</sup>) and (Ex1<sup>op</sup>) can be derived from the above axioms.

(2) for every isomorphism  $\varphi$  the square

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow & & \downarrow \\ 0 & \xrightarrow{\text{id}_0} & 0 \end{array}$$

$\Rightarrow$  isomorphisms are deflations

(and also inflations by the dual argument)

(3) For all  $X, Z \in \text{Ob}(\mathcal{A})$ ,  $X \xrightarrow{0} 0$  is a deflation and

$$\begin{array}{ccc} X \oplus Z & \xrightarrow{(0 \text{ id}_Z)} & Z \\ (id_X \ 0) \downarrow & & \downarrow \\ X & \longrightarrow & 0 \end{array}$$

is a pullback square

$$\Rightarrow X \xrightarrow{\begin{pmatrix} id_X \\ 0 \end{pmatrix}} X \oplus Z \xrightarrow{(0 \text{ id}_Z)} Z$$

is a conflation

Examples • An abelian category can have different exact structures (e.g. conflations = all s.e.s. or conflations = split s.e.s., ...)

• An extension closed subcategory of an abelian cat. is exact.

#### 4. The derived category of an exact category

Let  $\mathcal{A}$  be an exact category.

We write  $C(\mathcal{A})$  for the category of chain complexes

$$\cdots \longrightarrow X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} \cdots$$

A morphism  $f^\bullet: X^\bullet \longrightarrow Y^\bullet$  of chain complexes is nullhomotopic if there are morphisms  $s_i: X^i \longrightarrow Y^{i-1}$  in  $\mathcal{A}$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X^{i-1} & \xrightarrow{d_X^{i-1}} & X^i & \xrightarrow{d_X^i} & X^{i+1} \longrightarrow \cdots \\ & & \downarrow f^{i-1} & \swarrow s_i & \downarrow f^i & \swarrow s_{i+1} & \downarrow f^{i+1} \\ \cdots & \longrightarrow & Y^{i-1} & \xrightarrow{d_Y^{i-1}} & Y^i & \xrightarrow{d_Y^i} & Y^{i+1} \longrightarrow \cdots \end{array}$$

such that

$$f^i = s_{i+1}^i d_X^i + d_Y^{i-1} s_i$$

for all  $i \in \mathbb{Z}$ .

Let  $H(\mathcal{A}) := C(\mathcal{A}) / \text{(nullhomotopic morphism)}$

be the homotopy category of (the underlying add. cat. of)  $\mathcal{A}$

Def: A complex  $X^\bullet \in C(\mathcal{A})$  is acyclic if there are factorizations

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X^{i-1} & \longrightarrow & X^i & \longrightarrow & X^{i+1} \longrightarrow \cdots \\ & & \searrow & & \swarrow & & \searrow \\ & & Z^i & & Z^{i+1} & & \end{array}$$

such that for every  $i \in \mathbb{Z}$

$$Z^i \longrightarrow X^i \longrightarrow Z^{i+1}$$

is a conflation.

Def: A morphism  $f^\bullet: X^\bullet \longrightarrow Y^\bullet$  in  $C(\mathcal{A})$  is a quasi-isomorphism if the mapping cone

$$\text{cone}(f^\bullet) := \cdots \longrightarrow Y^i \oplus X^{i+1} \xrightarrow{\begin{pmatrix} d_Y^i & f^{i+1} \\ 0 & -d_X^{i+1} \end{pmatrix}} Y^{i+1} \oplus X^{i+2} \longrightarrow \cdots$$

is isomorphic in  $H(\mathcal{A})$  to an acyclic complex.



Example Let  $X \xrightarrow{f} Y \xrightarrow{g} Z$  be a conflation in  $\mathcal{A}$ ,  
then the following morphisms of chain complexes are quasi-isomorphisms:

$$\begin{array}{ccccccc} \cdots & \rightarrow & 0 & \rightarrow & X & \xrightarrow{f} & Y & \rightarrow & 0 & \rightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow g & & \downarrow & \\ \cdots & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & Z & \rightarrow & 0 & \rightarrow \cdots \end{array} \quad \left| \quad \begin{array}{ccccccc} \cdots & \rightarrow & 0 & \rightarrow & X & \rightarrow & 0 & \rightarrow & 0 & \rightarrow \cdots \\ & & \downarrow & & \downarrow f & & \downarrow & & \downarrow & \\ \cdots & \rightarrow & 0 & \rightarrow & Y & \xrightarrow{g} & Z & \rightarrow & 0 & \rightarrow \cdots \end{array} \right.$$

In particular

$$\text{cone} \left( \begin{array}{ccccccc} \cdots & \rightarrow & 0 & \rightarrow & X & \rightarrow & 0 & \rightarrow \cdots \\ & & \downarrow & & \downarrow f & & \downarrow & \\ \cdots & \rightarrow & 0 & \rightarrow & Y & \rightarrow & 0 & \rightarrow \cdots \end{array} \right)$$

$$= \left( \cdots \rightarrow X \xrightarrow{f} Y \rightarrow 0 \rightarrow \cdots \right)$$

is quasi-isomorphic to  $\left( \cdots \rightarrow 0 \rightarrow Z \rightarrow 0 \rightarrow \cdots \right)$

("conflation in  $\mathcal{A}$  give rise to distinguished triangles in  $\mathcal{D}(\mathcal{A})$ ")

Definition The derived category of the exact category  $\mathcal{A}$  is

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &:= H(\mathcal{A})[(\text{classes of quasi-isomorphisms})^{-1}] \\ &= C(\mathcal{A})[(\text{quasi-isomorphisms})^{-1}] \end{aligned}$$

Recall (Universal property of localization of a category)

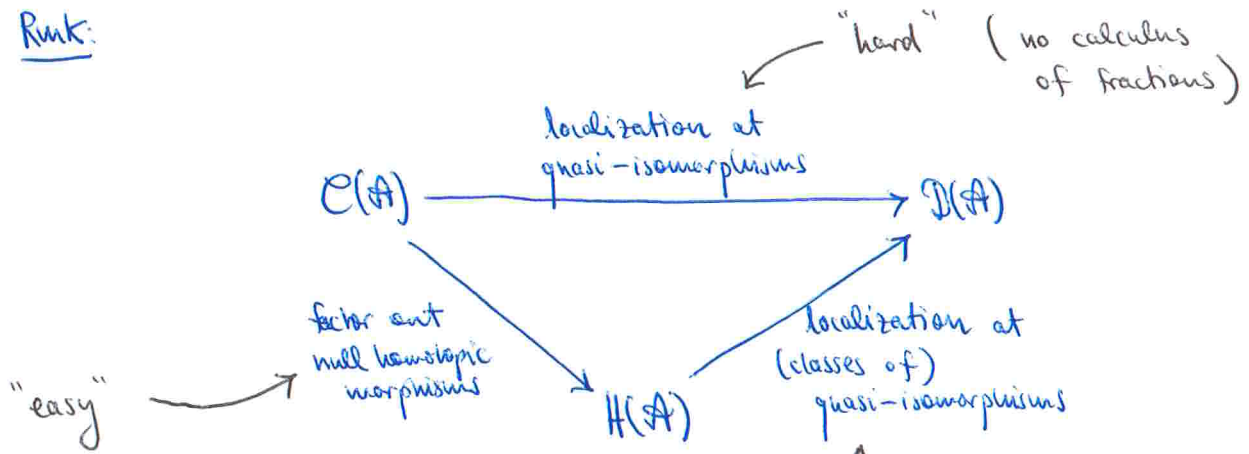
Let  $\mathcal{C}$  be a category,  $W \subseteq \text{Mor}(\mathcal{C})$  a class of morphisms.

Then  $\mathcal{C} \xrightarrow{\gamma} \mathcal{C}[W^{-1}]$  is a localization of  $\mathcal{C}$  at  $W$  if

for every functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  which maps  
 $W$  to isomorphisms, there exists a  
unique functor  $\tilde{F}: \mathcal{C}[W^{-1}] \rightarrow \mathcal{D}$   
such that  $\tilde{F} \circ \gamma = F$ .

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\gamma} & \mathcal{C}[W^{-1}] \\ \downarrow F & \searrow \exists! \tilde{F} & \\ \mathcal{D} & & \end{array}$$

Hint:



I.e. we have a calculus of fractions,  
because we benefit from the structure  
of a triangulated category on  $H(A)$ .