Additive, abelian and exact categories

1. Additive categories

Def: An *additive category* is a category $\mathcal{A}$ such that

(Add1) $\mathcal{A}$ has a zero object 0 (i.e. an object that is both initial and terminal)

(Add2) $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ has an abelian group structure such that composition is biadditive.

(Add3) $\mathcal{A}$ has biproducts, i.e. $\forall X_1, X_2 \in \mathrm{Ob}(\mathcal{A})$ $\exists$ object $X =: X_1 \oplus X_2$ and morphisms

$$\begin{tikzcd} X_1 \arrow[r, shift left, "\sigma_1"'] & X \arrow[l, shift left, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & X_2 \arrow[l, shift left, "\sigma_2"] \end{tikzcd}$$

such that
- $\pi_i \sigma_j = \delta_{ij}\, \mathrm{id}_{X_i}$
- $\sigma_1 \pi_1 + \sigma_2 \pi_2 = \mathrm{id}_X$

Rmk (1) The zero element in the abelian group $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ equals the unique morphism

$$\begin{tikzcd} X \arrow[rr, "0"] \arrow[dr] & & Y \\ & 0 \arrow[ur] & \end{tikzcd}$$

(2) (X, σ_1, σ_2) is a coproduct and (X, π_1, π_2) is a product of X_1 and X_2

(3) The group structures on $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ are intrinsic and no additional data!

Given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}[column sep=huge] X \arrow[r, "f+g"] \arrow[d, "\begin{pmatrix} \mathrm{id}_X \\ \mathrm{id}_X \end{pmatrix} = \sigma_1^X + \sigma_2^X"'] & Y \\ X \oplus X \arrow[r, "\sigma_1^Y f \pi_1^X + \sigma_2^Y g \pi_2^X"', "= \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}"] & Y \oplus Y \arrow[u, "\pi_1^Y + \pi_2^Y = (\mathrm{id}_Y \ \mathrm{id}_Y)"'] \end{tikzcd}$$

The black morphisms can be constructed using only universal properties of (co)products and the fact that we have a zero object (and hence zero morphisms)

2. Abelian categories

<u>Def</u> Let $\mathcal{A}$ be a category with a zero object and $X \xrightarrow{f} Y$ a morphism.

- $(K, K \xrightarrow{i} X)$ is a *kernel* of f if $(fi=0)$ $\forall\ T \xrightarrow{t} X$ with $ft=0$ $\exists!\ T \xrightarrow{\tilde{t}} K$ with $i\tilde{t}=t$

$$\begin{tikzcd} & T \arrow[dl, dashed, "\exists \tilde{t}"'] \arrow[d, "\forall t"] \arrow[dr, "0"] & \\ K \arrow[r, "i"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

- $(C, Y \xrightarrow{p} C)$ is a *cokernel* of f if $pf=0$ and $\forall\ Y \xrightarrow{t} T$ with $tf=0$ $\exists!\ C \xrightarrow{\tilde{t}} T$ with $\tilde{t}p=t$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "0"'] & Y \arrow[r, "p"] \arrow[d, "\forall t"] & C \arrow[dl, dashed, "\exists \tilde{t}"] \\ & T & \end{tikzcd}$$

- The *image* of f is $\mathrm{Im}(f) := \ker\left(Y \longrightarrow \mathrm{Cok}(f)\right)$

  The *coimage* of f is $\mathrm{Coim}(f) := \mathrm{Cok}\left(\ker(f) \longrightarrow X\right)$

⇝ We get an induced morphism $\bar{f}: \mathrm{Coim}(f) \longrightarrow \mathrm{Im}(f)$

$$\begin{tikzcd} K \arrow[r, "i"] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "p"] & C \\ & \mathrm{Coim}(f) \arrow[r, dashed, "\bar{f}"] & \mathrm{Im}(f) \arrow[u] & \end{tikzcd}$$

<u>Def:</u> An *abelian category* is an additive category $\mathcal{A}$ such that every morphism has a kernel and cokernel and for every morphism $f: X \longrightarrow Y$ the induced morphism $\bar{f}: \mathrm{Coim}(f) \longrightarrow \mathrm{Im}(f)$ is an isomorphism.

<u>Examples</u> of abelian categories:

- $\mathrm{Mod}\ R$ for a ring R
- $\mathrm{Coh}(X)$ for a scheme X
- $\mathrm{Fun}(\mathcal{C}, \mathcal{A})$ for a small cat. $\mathcal{C}$ and an abelian cat. $\mathcal{A}$

<u>Non-example:</u> $\mathcal{A}$ = category of f.g. free abelian groups, then $\mathcal{A}$ has all kernels and cokernels and $\mathrm{Im}(f) \cong \mathrm{Coim}(f)$ for every morphism f, but $\mathcal{A}$ is not abelian.

3. Exact categories

Def. An *exact category* is an additive category $\mathcal{A}$ endowed with a class of kernel-cokernel pairs $(\underset{\text{inflation}}{i}, \underset{\text{deflation}}{p})$, called conflations, s.t.

(Ex0) id_0 is a deflation

(Ex1) The composition of two deflations is a deflation

(Ex2) For every deflation p and morphism f the pullback exists and p' is again a deflation

$$\begin{tikzcd} Y' \arrow[r, two heads, "p'"] \arrow[d] & Z' \arrow[d, "f"] \\ Y \arrow[r, two heads, "p"] & Z \end{tikzcd}$$

($\mathrm{Ex}2^{op}$) For every inflation i and morphism f the pushout exists and i' is an inflation.

$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d, "f"'] & Y \arrow[d, "f'"] \\ X' \arrow[r, tail, "i'"] & Y' \end{tikzcd}$$

Rmk (1) The dual statements ($\mathrm{Ex}0^{op}$) and ($\mathrm{Ex}1^{op}$) can be derived from the above axioms.

(2) For every isomorphism φ the square

$$\begin{tikzcd} X \arrow[r, "\varphi", "\cong"'] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r, "\mathrm{id}_0"] & 0 \end{tikzcd}$$

is a pullback diagram

$\Rightarrow$ isomorphisms are deflations (and also inflations by the dual argument)

(3) For all $X, Z \in \mathrm{Ob}(\mathcal{A})$, $X \xrightarrow{0} 0$ is a deflation and

$$\begin{tikzcd} X \oplus Z \arrow[r, "(0 \ \mathrm{id}_Z)"] \arrow[d, "(\mathrm{id}_X \ 0)"'] & Z \arrow[d] \\ X \arrow[r] & 0 \end{tikzcd}$$

is a pullback square $\Rightarrow$ $X \xrightarrow{\binom{\mathrm{id}_X}{0}} X \oplus Z \xrightarrow{(0 \ \mathrm{id}_Z)} Z$ is a conflation

Examples • An abelian category can have different exact structures (e.g. conflations = all s.e.s. or conflations = split s.e.s., ...)

• An extension closed subcategory of an abelian cat. is exact.

4. The derived category of an exact category

Let $\mathcal{A}$ be an exact category.

We write $C(\mathcal{A})$ for the category of *chain complexes*

$$\cdots \longrightarrow X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} \cdots$$

A morphism $f^\bullet : X^\bullet \longrightarrow Y^\bullet$ of chain complexes is *nullhomotopic* if there are morphisms $s_i : X^i \longrightarrow Y^{i-1}$ in $\mathcal{A}$

$$\begin{tikzcd}
\cdots \arrow[r] & X^{i-1} \arrow[r, "d_X^{i-1}"] \arrow[d, "f^{i-1}"'] & X^i \arrow[r, "d_X^i"] \arrow[dl, "s^i"'] \arrow[d, "f^i"] & X^{i+1} \arrow[r] \arrow[dl, "s^{i+1}"'] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & Y^{i-1} \arrow[r, "d_Y^{i-1}"] & Y^i \arrow[r, "d_Y^i"] & Y^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

such that

$$f^i = s^{i+1} d_X^i + d_Y^{i-1} s^i$$

for all $i \in \mathbb{Z}$.

Let $H(\mathcal{A}) := C(\mathcal{A}) / \left(\begin{smallmatrix}\text{nullhomotopic}\\ \text{morphism}\end{smallmatrix}\right)$ be the *homotopy category* of (the underlying add. cat. of) $\mathcal{A}$

Def: A complex $X^\bullet \in C(\mathcal{A})$ is *acyclic* if there are factorizations

$$\begin{tikzcd}
\cdots \arrow[r] & X^{i-1} \arrow[r] \arrow[dr] & X^i \arrow[r] \arrow[dr] & X^{i+1} \arrow[r] \arrow[dr] & \cdots \\
\cdots \arrow[ur] & & Z^i \arrow[ur] & & Z^{i+1} \arrow[ur] & \cdots
\end{tikzcd}$$

such that for every $i \in \mathbb{Z}$

$$Z^i \longrightarrow X^i \longrightarrow Z^{i+1}$$

is a conflation.

Def: A morphism $f^\bullet X^\bullet \longrightarrow Y^\bullet$ in $C(\mathcal{A})$ is a *quasi-isomorphism* if the *mapping cone*

$$\operatorname{cone}(f^\bullet) := \quad \cdots \longrightarrow Y^i \oplus X^{i+1} \xrightarrow{\begin{pmatrix} d_Y^i & f^{i+1} \\ 0 & -d_X^{i+1} \end{pmatrix}} Y^{i+1} \oplus X^{i+2} \longrightarrow \cdots$$

is isomorphic in $H(\mathcal{A})$ to an acyclic complex.

__Example__ Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a conflation in $\mathcal{A}$, then the following morphisms of chain complexes are quasi-isomorphisms:

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & Z \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd} \qquad \begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

In particular

$$\mathrm{cone}\left(\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & Y \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd} \right)$$

$$= \left(\cdots 0 \longrightarrow X \xrightarrow{f} Y \longrightarrow 0 \longrightarrow \cdots \right)$$

is quasi-isomorphic to $\left(\cdots \longrightarrow 0 \longrightarrow Z \longrightarrow 0 \longrightarrow \cdots \right)$

("conflation in $\mathcal{A}$ give rise to distinguished triangles in $\mathcal{D}(\mathcal{A})$")

__Definition__ The derived category of the exact category $\mathcal{A}$ is

$$\mathcal{D}(\mathcal{A}) := H(\mathcal{A})[(\text{classes of quasi-isomorphisms})^{-1}]$$
$$= C(\mathcal{A})[(\text{quasi-isomorphisms})^{-1}]$$

__Recall__ (Universal property of localization of a category)

Let $\mathcal{C}$ be a category, $W \subseteq \mathrm{Mor}(\mathcal{C})$ a class of morphisms. Then $\mathcal{C} \xrightarrow{\gamma} \mathcal{C}[W^{-1}]$ is a localization of $\mathcal{C}$ at W if

for every functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ which maps W to isomorphisms, there exists a unique functor $\tilde{F} : \mathcal{C}[W^{-1}] \longrightarrow \mathcal{D}$ such that $\tilde{F} \circ \gamma = F$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[d, "\forall F"'] & \mathcal{C}[W^{-1}] \arrow[dl, dashed, "\exists ! \tilde{F}"] \arrow[dl, phantom, "\circlearrowleft", near start, shift right=4] \\ \mathcal{D} & \end{tikzcd}$$

Rmk:

"hard" (no calculus of fractions)

$$\begin{tikzcd}
\mathcal{C}(\mathcal{A}) \arrow[rr, "\text{localization at quasi-isomorphisms}"] \arrow[dr, swap, "\text{factor out null homotopic morphisms}"] & & \mathcal{D}(\mathcal{A}) \\
& H(\mathcal{A}) \arrow[ur, swap, "\text{localization at (classes of) quasi-isomorphisms}"] &
\end{tikzcd}$$

"easy"

can be made explicit!

I.e. we have a calculus of fractions, because we benefit from the structure of a triangulated category on $H(\mathcal{A})$.