Part 1 triangulated categories

roughly: exact category / homotopy
→ want notion of long exact sequence

Definition A triangulated category consists of
- an additive category $\mathcal{C}$
- an autoequivalence $\Sigma : \mathcal{C} \to \mathcal{C}$ called suspension
- a class of ~~exact~~ triangles Δ called exact triangles

A triangle is a diagram of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

They must satisfy

(TR1)
- Δ is closed under isomorphisms
- Every morphism $A \xrightarrow{f} B$ in $\mathcal{C}$ can be completed to ~~an~~ exact triangle $A \xrightarrow{f} B \to C \to \Sigma A$

(TR2) (Rotation)

If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is exact

then $B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{\Sigma f} \Sigma B$ is exact

Note: iterated rotation yields a diagram of the form

$$A \to B \to C \to \Sigma A \to \Sigma B \to \Sigma C \to \Sigma^2 A \cdots$$

This allows construction of long exact sequences from triangles

(TR3) Given a diagram with exact rows

$$\begin{array}{ccccccc} A & \to & B & \to & C & \to & \Sigma A \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \vdots{\scriptstyle \exists \varphi} & & \downarrow{\scriptstyle \Sigma f} \\ A' & \to & B' & \to & C' & \to & \Sigma A' \end{array}$$

there is a morphism φ making everything commute.

(TR4) The octahedral axiom

roughly: $A/C \big/ B/C \cong A/C$

Examples

Derived categories ↗ other talks

Stable category of Frobenius category ↗ this talk

Part 2 Frobenius categories

Running example: Let A be f.d. algebra and consider exact structures on mod-A:

S_{split}: conflations are split exact sequences

S_{short}: conflations are short exact sequences

Definition Let $(\mathcal{F}, S)$ be an exact category.
$I \in \mathcal{F}$ is injective if $\mathcal{F}(-, I)$ maps conflations to ~~ss~~ s.e.s.

$\mathcal{F}$ has enough ~~projectives~~ injectives if every $A \in \mathcal{F}$ fits into a conflation $A \rightarrowtail I \twoheadrightarrow \Sigma A$ with I injective

Dually: Projective, enough projectives

Example

For $(\text{mod-}A, S_{split})$ all objects are projective and injective

For $(\text{mod-}A, S_{short})$ projectives and injectives are the usual ones.

Definition $(\mathcal{F}, S)$ is a Frobenius category if

- $\mathcal{F}$ has enough projectives and injectives
- The projectives and injectives coincide

Example $(\text{mod-}A, \mathcal{S}_{split})$ is Frobenius

$(\text{mod-}A, \mathcal{S}_{short})$ is Frobenius iff A is self-injective

Definition Let $(\mathcal{F}, \mathcal{S})$ be a Frobenius category.
The stable category $\underline{\mathcal{F}}$ has

- objects : same as $\mathcal{F}$
- morphisms : $\underline{\mathcal{F}}(A,B) := \mathcal{F}(A,B) / \left(\begin{matrix}\text{morphisms that} \\ \text{factor through} \\ \text{projectives}\end{matrix} \right)$

A notion of homotopy

$\rightsquigarrow$ expect $\underline{\mathcal{F}}$ triangulated

Note: Let $P \in \mathcal{F}$ be projective then $0_P = id_P$ in $\underline{\mathcal{F}}$
$\Rightarrow$ $P \cong 0$ in $\underline{\mathcal{F}}$

$\rightsquigarrow$ $\underline{(\text{mod-}A, \mathcal{S}_{split})}$ is trivial
$\underline{(\text{mod-}A, \mathcal{S}_{short})}$ is more interesting

Theorem Let $(\mathcal{F}, \mathcal{S})$ be a Frobenius category
then $\underline{\mathcal{F}}$ is a triangulated category.

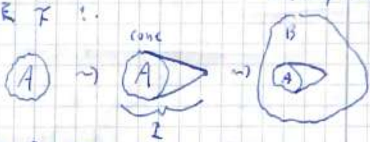
Construction: need Σ and Δ

Σ: for every $A \in \mathcal{F}$ pick a conflation
$A \rightarrowtail I \twoheadrightarrow \Sigma A$ with I injective.

Schanuel's Lemma $\Rightarrow$ ΣA is unique up to injective direct summands $\rightsquigarrow$ ΣA is unique up to iso in $\underline{\mathcal{F}}$
$\rightsquigarrow$ can turn Σ into a functor
Pick a second conflation $\Omega(A) \rightarrowtail P \twoheadrightarrow A$ with P projective.
$\rightsquigarrow$ can turn Ω into a quasi-inverse of Σ

<u>Note</u>: The conflation $A \rightarrowtail I \twoheadrightarrow \Sigma A$ shows $A \simeq \Omega \Sigma A$ in $\underline{\mathcal{F}}$ since I is projective

Δ: Let $f: A \to B \in \mathcal{F}$:.

topological motivation : 

Construction: Let $A \overset{\iota}{\rightarrowtail} I \overset{p}{\twoheadrightarrow} \Sigma A$ be a conflation with I injective. Consider $B \oplus_A I$.

obtain exact triangle

$$A \xrightarrow{f} B \xrightarrow{(\mathrm{id},0)} B \oplus_A I \xrightarrow{(0,p)} \Sigma A \qquad \square$$

<u>Part 3</u> Let $\mathcal{A}$ be an abelian category.

<u>Definition</u> A full subcategory W of $\mathcal{A}$ is <u>wide</u> if

- it is closed under summands
- if two objects in an s.e.s. are in W then so is the third

<u>Definition</u> Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$ then

$\mathcal{X}^{\perp} := \{Y \in \mathcal{A} \mid \mathrm{Ext}^1(\mathcal{X}, Y) = 0\}$

${}^{\perp}\mathcal{X} := \{Y \in \mathcal{A} \mid \mathrm{Ext}^1(Y, \mathcal{X}) = 0\}$

A pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories is a <u>cotorsion pair</u> if $\mathcal{X}^{\perp} = \mathcal{Y}$ and $\mathcal{X} = {}^{\perp}\mathcal{Y}$

it is <u>functorially complete</u> if every $A \in \mathcal{A}$ admits functorial s.e.s. $0 \to Y \to X \to A \to 0$ and $0 \to A \to Y' \to X' \to 0$ with $X, X' \in \mathcal{X}$ and $Y, Y' \in \mathcal{Y}$

it is <u>hereditary</u> if $\mathrm{Ext}^{\geq 1}(\mathcal{X}, \mathcal{Y}) = 0$