Homework 1

Part 1

A. Qubit states and their representation in the Bloch sphere

> **Exercise 1**
> Construct 2×2 matrix $\hat{Z}$, the eigenvectors of which are $|0\rangle$ and $|1\rangle$ and the corresponding eigenvalues are $+1$ and -1. That is $\hat{Z}|0\rangle = +1|0\rangle$ and $\hat{Z}|1\rangle = -1|1\rangle$.

Let $\hat{Z} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then:

$$\begin{aligned} \hat{Z}|0\rangle = +1|0\rangle &\Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\Leftrightarrow a = 1 \land c = 0 \end{aligned}$$

So $\hat{Z} = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$. In the same way, we find:

$$\begin{aligned} \hat{Z}|0\rangle = -1|1\rangle &\Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ &\Leftrightarrow b = 0 \land d = -1 \end{aligned}$$

Therefore, we can build $\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

> **Exercise 2**
> Find the matrix for a linear operator $\hat{X}$ which turns $|0\rangle$ into $|1\rangle$ and $|1\rangle$ into $|0\rangle$.

Let $\hat{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then:

$$\begin{aligned} \hat{X}|0\rangle = |1\rangle &\Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &\Leftrightarrow a = 0 \land c = 1 \end{aligned}$$

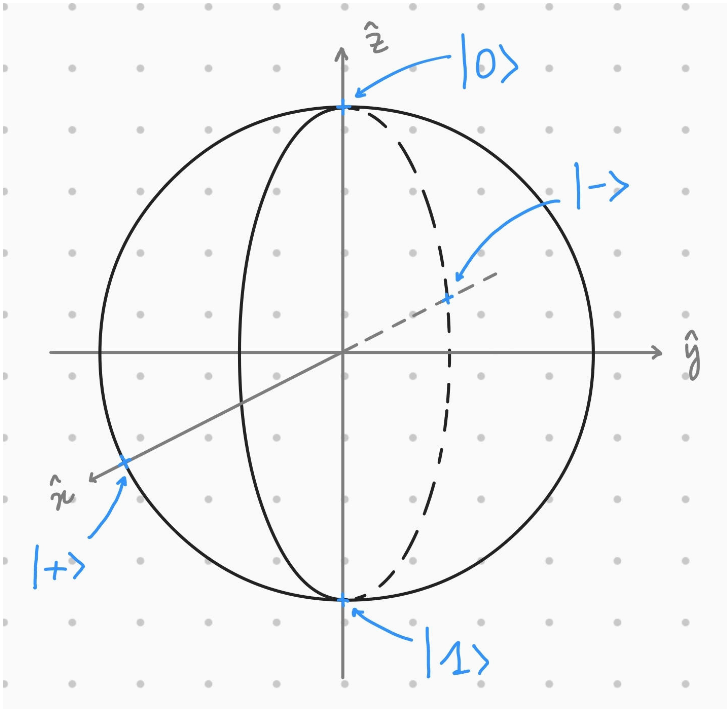
and:

$$\begin{aligned} \hat{X}|1\rangle = |0\rangle &\Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\Leftrightarrow b = 1 \land d = 0 \end{aligned}$$

Therefore, we can build $\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

> **Exercise 3**
> Mark the states $|+\rangle$ and $|-\rangle$ together with states $|0\rangle$ and $|1\rangle$ on the Bloch sphere.

The states $|+\rangle$ and $|-\rangle$ are at the intersections of the sphere with the x-axis, and $|0\rangle$ and $|1\rangle$ are at the intersections of the sphere with the z-axis:



Exercise 4

Show that $|+\rangle$ and $|-\rangle$ also form a basis. What would be $|0\rangle$ and $|1\rangle$ in this new basis?

We have:

$$\langle +|+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} + \frac{1}{2} = 1$$

and:

$$\langle -|-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} + \frac{1}{2} = 1$$

So $|+\rangle$ and $|-\rangle$ are normal.

We also have:

$$\langle +|-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} - \frac{1}{2} = 0$$

and:

$$\langle -|+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = -\frac{1}{2} + \frac{1}{2} = 0$$

So $|+\rangle$ and $|-\rangle$ are orthogonal.

Therefore, they form a basis.

Now, we want to express $|0\rangle$ and $|1\rangle$ in this new basis. Looking at the states on the Bloch sphere, we intuit that we might find $|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$. We check this by substitution:

$$\begin{aligned}\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle &= \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle \\ &= |0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}|1\rangle \\ &= |0\rangle\end{aligned}$$

Similarly with $|1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$:

$$\begin{aligned}\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle &= \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle \\ &= \frac{1}{2}|0\rangle - \frac{1}{2}|0\rangle + |1\rangle \\ &= |1\rangle\end{aligned}$$

> **Exercise 5**
> Find the matrix for a linear operator $\hat{X}$ (int the computational basis), the eigenvectors of which are $|+\rangle$ and $|-\rangle$ and eigenvalues $+1$ and -1, respectively.

We want to find $\hat{X}$ such that:
- $\hat{X}|+\rangle = +1|+\rangle$
- $\hat{X}|-\rangle = -1|-\rangle$

Let $\hat{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then:

$$\begin{aligned}\hat{X}|+\rangle = +1|+\rangle &\Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} \frac{a+b}{\sqrt{2}} \\ \frac{c+d}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} a+b \\ c+d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &\Leftrightarrow a+b = 1 \land c+d = 1\end{aligned}$$

In the same way:

$$\begin{aligned}\hat{X}|-\rangle = -1|-\rangle &\Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} \frac{a-b}{\sqrt{2}} \\ \frac{c-d}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} a-b \\ c-d \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &\Leftrightarrow a-b = -1 \land c-d = 1\end{aligned}$$

Therefor we have:

$$\begin{cases} a+b=1 \\ a-b=-1 \\ c+d=1 \\ c-d=1 \end{cases} \Leftrightarrow \begin{cases} a=0 \\ b=1 \\ c=1 \\ d=0 \end{cases}$$

And we get: $\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is a linear operator since it's a matrix.

Exercise 6
Find out states $\hat{X}|0\rangle, \hat{X}|1\rangle, \hat{Z}|+\rangle, \hat{Z}|-\rangle$.

$$\hat{X}|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{X}|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{Z}|+\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\hat{Z}|-\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Exercise 7
Apply $\hat{H}$ to states $|0\rangle, |1\rangle, |+\rangle, |-\rangle$. Check that $\hat{H}^2 = \hat{I}$

$$\hat{H}|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = |+\rangle$$

$$\hat{H}|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = |-\rangle$$

$$\hat{H}|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

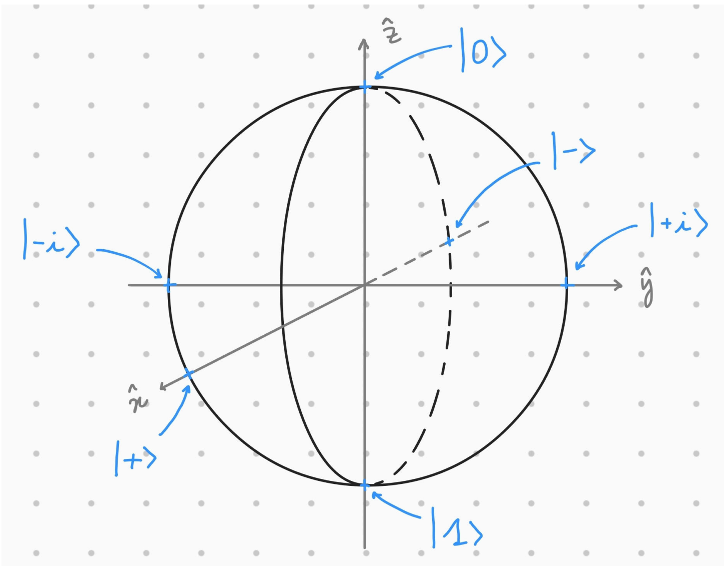
$$\hat{H}|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

What's more:

$$\begin{aligned} \hat{H}^2 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I} \end{aligned}$$

Exercise 8
Mark the states $|+i\rangle, |-i\rangle$ on the Bloch sphere with respect to the states $|0\rangle, |1\rangle, |+\rangle, |-\rangle$.

The new states $|+i\rangle$ and $|-i\rangle$ are at the intersections of the y-axis and the sphere:



Exercise 9

Write down the following vectors (as columns and rows): $|+i\rangle$, $\langle +i|$, $|-i\rangle$, $\langle -i|$.

We have:

$$|+i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{i}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \boxed{\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{i}{\sqrt{2}}\end{pmatrix}}$$

$$\langle +i| = |+i\rangle^\dagger = \boxed{\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix}}$$

And:

$$|-i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} - \frac{i}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \boxed{\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{i}{\sqrt{2}}\end{pmatrix}}$$

$$\langle -i| = |-i\rangle^\dagger = \boxed{\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}}$$

Exercise 10

Find the matrix for a linear operator $\hat{Y}$, the eigenvalues of which are $+1$ and -1, and the corresponding eigenvectors are $|+i\rangle$ and $|-i\rangle$.

We want to find $\hat{Y}$ such that:
- $\hat{Y}|+i\rangle = +1|+i\rangle$
- $\hat{Y}|-i\rangle = -1|-i\rangle$

Let $\hat{Y} = \begin{pmatrix}a & b\\c & d\end{pmatrix}$. Then:

$$\hat{Y}|{+i}\rangle = +1|{+i}\rangle \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} \frac{a+ib}{\sqrt{2}} \\ \frac{c+id}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} a+ib \\ c+id \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$
$$\Leftrightarrow a+ib = 1 \land c+id = i$$

In the same way:

$$\hat{Y}|{-i}\rangle = -1|{-i}\rangle \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} \frac{a-ib}{\sqrt{2}} \\ \frac{c-id}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} a-ib \\ c-id \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix}$$
$$\Leftrightarrow a-ib = -1 \land c-id = i$$

Therefore, we have:

$$\begin{cases} a+ib = 1 \\ a-ib = -1 \\ c+id = i \\ c-id = i \end{cases} \Leftrightarrow \begin{cases} a = 0 \\ b = -i \\ c = i \\ d = 0 \end{cases}$$

And we get: $\hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Exercise 11

Show that the pair of states $|{+i}\rangle$ and $|{-i}\rangle$ form a basis in the vector space of states of our qubit.

We have:

$$\langle{+i}|{+i}\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = 1$$

and:

$$\langle - | - \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = 1$$

So $|+\rangle$ and $|-\rangle$ are normal.

We also have:

$$\langle + | - \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = 0$$

and:

$$\langle - | + \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = 0$$

So $|+\rangle$ and $|-\rangle$ are orthogonal.

Therefore, they form a basis.

> **Exercise 12**
> Find the matrices (in the computational basis), which, by analogy with the Hadamard operator, would convert the basis states $|+i\rangle$ and $|-i\rangle$ into $|0\rangle$ and $|1\rangle$ and back.

Let our matrix be $\widehat{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then:

$$\begin{aligned} \widehat{M}|+i\rangle = |0\rangle &\Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} \frac{a+ib}{\sqrt{2}} \\ \frac{c+id}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \frac{a+ib}{\sqrt{2}} = 1 \land \frac{c+id}{\sqrt{2}} = 0 \end{aligned}$$

And:

$$\begin{aligned} \widehat{M}|-i\rangle = |1\rangle &\Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} \frac{a-ib}{\sqrt{2}} \\ \frac{c-id}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &\Leftrightarrow \frac{a-ib}{\sqrt{2}} = 0 \land \frac{c-id}{\sqrt{2}} = 1 \end{aligned}$$

Therefore, we have:

$$\begin{cases} a + ib = \sqrt{2} \\ a - ib = 0 \\ c + id = 0 \\ c - id = \sqrt{2} \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{\sqrt{2}} \\ b = -\frac{i}{\sqrt{2}} \\ c = \frac{1}{\sqrt{2}} \\ d = \frac{i}{\sqrt{2}} \end{cases}$$

And our conversion matrix $\widehat{M} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$. Let's check that it works the other way around as well:

$$\begin{aligned}\widehat{M}^\dagger|0\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\\ &= \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\widehat{M}^\dagger|1\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\\ &= \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix}\end{aligned}$$

> **Exercise 13**
> Same question as above but this time let's convert between the basis $|+\rangle, |-\rangle$ into $|+i\rangle, |-i\rangle$.

Going from the basis $(|+\rangle, |-\rangle)$ to the basis $(|+i\rangle, |-i\rangle)$ is the same as going from $(|+\rangle, |-\rangle)$ into $(|0\rangle, |1\rangle)$ (using $\hat{H}$) and then into $(|+i\rangle, |-i\rangle)$ (using $M^\dagger$). The final matrix is computed by composing the other two conversion matrices.

Let the final matrix be called $\hat{N}$. Then:

$$\begin{aligned}\hat{N} &= \widehat{M}^\dagger \hat{H}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & 2i\end{pmatrix}\\ &= \boxed{\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}}\end{aligned}$$

And the matrix going the other way around is $\boxed{\hat{N}^\dagger = \begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix}}$

B. Quantum measurement rules

> **Exercise 14**
> Consider Pauli operators $\hat{X} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \hat{Y} = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \hat{Z} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ (with their matrices written in the computational basis). Consider also new operators, obtained from the Pauli operators: $\hat{X} \pm \hat{Z}, \hat{X} \pm \hat{Y}, \hat{X} \pm i\hat{Y}$. Which one(s) cannot represent an observable?

We check that they can represent an observable by looking at whether they're Hermitian operators:

$$\hat{X} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \hat{X}^\dagger \qquad \hat{Y} = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \hat{Y}^\dagger \qquad \hat{Z} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \hat{Z}^\dagger$$

$$\begin{aligned}\hat{X} + \hat{Z} &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\ &= \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\ &= \hat{H}\\ &= \hat{H}^\dagger\end{aligned}$$

$$\begin{aligned}\hat{X} + \hat{Y} &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\\ &= \begin{pmatrix}0 & 1-i\\ 1+i & 0\end{pmatrix}\\ &= \begin{pmatrix}0 & 1-i\\ 1+i & 0\end{pmatrix}^\dagger\end{aligned}$$

$$\begin{aligned}\hat{X} + i\hat{Y} &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\\ &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\\ &= \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}\\ &\neq \begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix}\end{aligned}$$

$$\hat{X} - \hat{Z} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -\hat{H} = -\hat{H}^\dagger$$

$$\hat{X} - \hat{Y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}^\dagger$$

$$\hat{X} - i\hat{Y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

The two matrices that cannot represent an observable are $\hat{X} \pm i\,\hat{Y}$.

> **Exercise 15**
> Cosnider many copies of a qubit prepared in state $|+\rangle$. We measure $\hat{Z}$ for each qubit. What would be the mean value of the outcome? Same question for state $|-\rangle$.

The probability of reading $+1$ is given by:

$$\langle 0|+\rangle^2 = \left[\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right]^2 = \left[\frac{1}{\sqrt{2}}\right]^2 = \frac{1}{2}$$

The probability of reading -1 is given by:

$$\langle 1|+\rangle^2 = \left[\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right]^2 = \left[\frac{1}{\sqrt{2}}\right]^2 = \frac{1}{2}$$

Therefore, the mean value of the outcome for states in $|+\rangle$ is $\frac{1}{2} \cdot (+1) + \frac{1}{2} \cdot (-1) = 0$.

In the same way, we get that the probabilities of reading $+1, -1$ in the case of states $|-\rangle$ are both $\frac{1}{2}$.

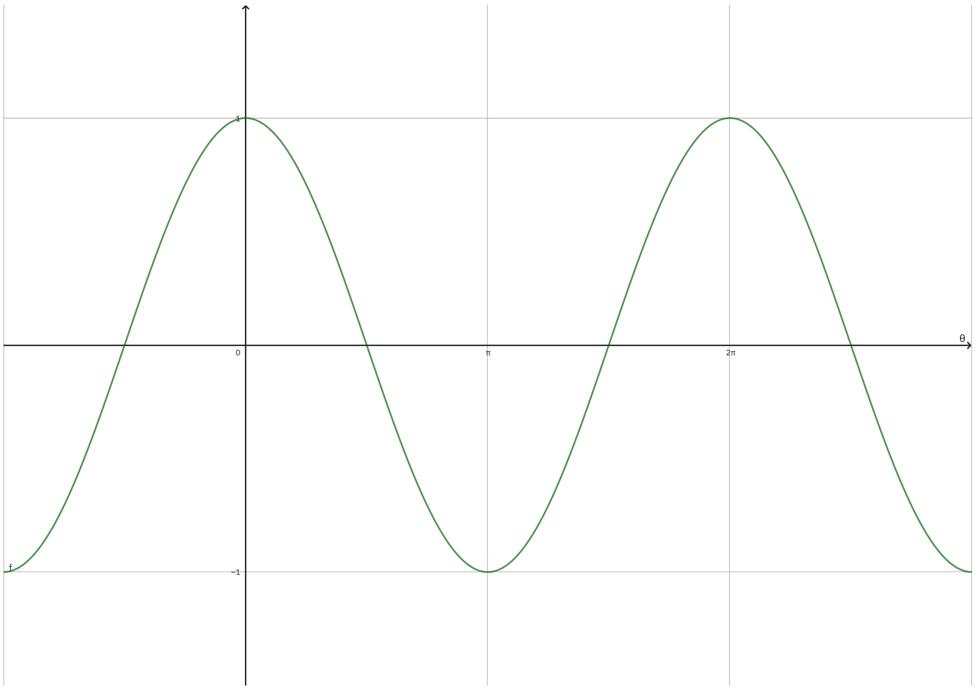
Therefore, the mean value of the outcome for states in $|-\rangle$ is $\frac{1}{2} \cdot (+1) + \frac{1}{2} \cdot (-1) = 0$.

> **Exercise 16**
> Consider the same experiment as in the exercise above but the qubit state is now a general qubit state $|\Psi\rangle$. Plot the mean value of the measurement outcome as a function of θ. Is the answer somewhat consistent with the interpretation of our qubit as a classical arrow oriented at an angle θ with respect to the Z-axis?

Let $|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \exp(i\phi)\sin\left(\frac{\theta}{2}\right)|1\rangle$. The probability of measuring $+1$ is $p(+1) = \cos^2\left(\frac{\theta}{2}\right)$, and the probability of measuring -1 is $p(-1) = \sin^2\left(\frac{\theta}{2}\right)$. The mean value of the outcome is given by:

$$p(+1) \cdot (+1) + p(-1) \cdot (-1) = \boxed{\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}$$

Here is a plot of that function with respect to angle θ:



This is consistent with what we expect of a classical arrow.

> **Exercise 17**
> Show that the average measurement value for the Z-projection in the previous exercise can be compactly written as $\left\langle \Psi \middle| \hat{Z} \middle| \Psi \right\rangle$.

We have:

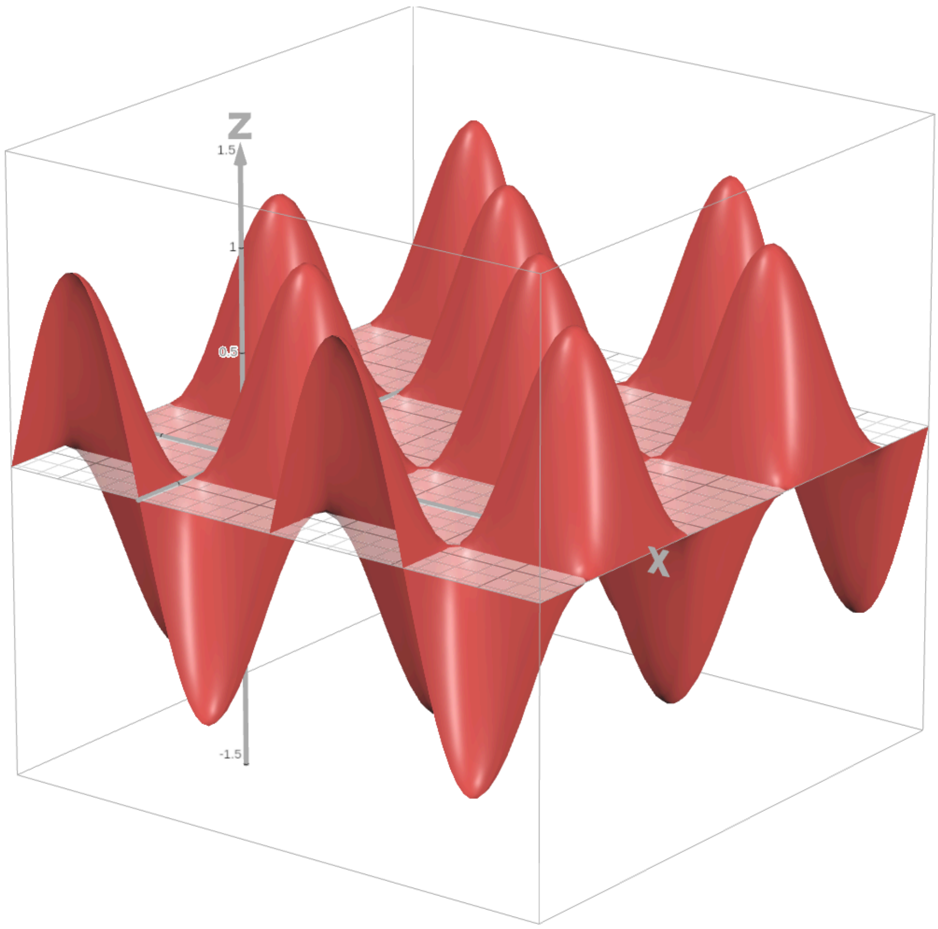
$$\begin{aligned}
\left\langle \Psi \middle| \hat{Z} \middle| \Psi \right\rangle &= \langle \Psi | \left(\hat{Z} | \Psi \rangle \right) \\
&= \langle \Psi | \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \exp(i\phi)\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \right) \\
&= \langle \Psi | \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ -\exp(i\phi)\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & \exp(-i\phi)\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ -\exp(i\phi)\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \\
&= \cos^2\left(\frac{\theta}{2}\right) - \exp(-i\phi)\exp(i\phi)\sin^2\left(\frac{\theta}{2}\right) \\
&= \boxed{\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}
\end{aligned}$$

Exercise 18

Plot $\langle\Psi|\hat{X}|\Psi\rangle$ as a function of the angles θ and ϕ. Compare it to the previously calculated $\langle\Psi|\hat{X}|\Psi\rangle$. Do both quantities behave as X- and Z-projections of a classical arrow with a unit length?

$$\begin{aligned}\langle\Psi|\hat{X}|\Psi\rangle &= \left(\cos\left(\frac{\theta}{2}\right) \;\; \exp(-i\phi)\sin\left(\frac{\theta}{2}\right)\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \exp(i\phi)\sin\left(\frac{\theta}{2}\right)\end{pmatrix}\\ &= \left(\cos\left(\frac{\theta}{2}\right) \;\; \exp(-i\phi)\sin\left(\frac{\theta}{2}\right)\right)\begin{pmatrix}\exp(i\phi)\sin\left(\frac{\theta}{2}\right)\\ \cos\left(\frac{\theta}{2}\right)\end{pmatrix}\\ &= \exp(i\phi)\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) + \exp(-i\phi)\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\\ &= \boxed{\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)(\exp(i\phi)+\exp(-i\phi))}\end{aligned}$$

In a plot, this gives us (with θ on the x-axis and ϕ on the y-axis):



This also is the expected behavior!

Exercise 19

Consider a qubit in state $|0\rangle$ and a measurement of $\hat{Z}$ and $\hat{X}$. We know that if we repeat each measurement many times (each time with a fresh qubit initialized to state $|0\rangle$), the mean value for $\hat{Z}$ would be $+1$ and the mean value for $\hat{X}$ would be 0, that is $\langle 0|\hat{Z}|0\rangle = +1$ and $\langle 0|\hat{X}|0\rangle = 0$. Let's calculate the variance of the measurement outcome, that is $\langle 0|\hat{Z}^2|0\rangle - \langle 0|\hat{Z}|0\rangle^2$ and $\langle 0|\hat{X}^2|0\rangle$.

We have:

$$\begin{aligned}\langle 0|\hat{Z}^2|0\rangle - \langle 0|\hat{Z}|0\rangle^2 &= (1\ \ 0)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^2\begin{pmatrix}1\\0\end{pmatrix} - (+1)^2\\ &= (1\ \ 0)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} - 1\\ &= (1\ \ 0)\begin{pmatrix}1\\0\end{pmatrix} - 1\\ &= 1 - 1\\ &= 0\end{aligned}$$

And:

$$\begin{aligned}\langle 0|\hat{X}^2|0\rangle &= (1\ \ 0)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^2\begin{pmatrix}1\\0\end{pmatrix}\\ &= (1\ \ 0)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\ &= (1\ \ 0)\begin{pmatrix}1\\0\end{pmatrix}\\ &= 1\end{aligned}$$

Exercise 20

Suppose we have qubit in state $|+\rangle$ and measure operator $\hat{Y}$ (measure the spin's Y-projection). What reading would we get after one measurement, and what would be the mean value of the readings after many measurements (each time starting with a fresh qubit in state $|+\rangle$)?

After one measurement, we would get $+1$ (collapsed onto $|+i\rangle$) or -1 (collapsed onto $|-i\rangle$) with probability $\frac{1}{2}$. This would happen with all the fresh qubits. Therefore, after many measurements each on fresh qubits, the mean value would be 0.

Exercise 21

Suppose you have a qubit prepared in state $|0\rangle$. What do we get if we alternate the measurements $\hat{Z}, \hat{X}, \hat{Z}, \hat{X}, \ldots$?

Measuring the qubit along the Z-axis the first time would yield $+1$, since the qubit is exactly in state $|0\rangle$. Measuring it along the X-axis would output ± 1 with probability $\frac{1}{2}$ and make it collapse towards the associated $|+\rangle$ or $|-\rangle$ state. Measuring again along the Z-axis would therefore yield ± 1 at random, and make it collapse to the associated $|0\rangle$ or $|1\rangle$ state.

Alternating the measurements like this would always give us a random sequence of $+1$ or -1 (except for the first measurement since we are measuring $|0\rangle$ along the Z-axis).

Exercise 22

Let's take a fresh qubit in state Ψ every time we measure. We measure the sequence $\hat{Z}, \hat{X}, \hat{Z}, \hat{X}, \ldots$. Compare to averaging the outcome in the experimental protocol of the previous exercise.

Here, the average value of all $\hat{X}$ (or $\hat{Z}$) readings would be $\left\langle \Psi \middle| \hat{X} \middle| \Psi \right\rangle$ (or $\hat{Z}$), which depends on θ and ϕ. However, in the previous exercise, the average value would be 0, since the outcomes of the measurements are randomly chosen between ± 1.

C. Unitary and Hermitian operators

Exercise 23

Check that any unitary operator $\hat{U}$ applied to a state $|\Psi\rangle$ creates a state $|\Psi'\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ where $\alpha_0\alpha_0^* + \alpha_1\alpha_1^* = 1$.

We know that $\langle\psi|\psi\rangle = \alpha_\psi\alpha_\psi^* + \beta_\psi\beta_\psi^*$.

We have $|\Psi'\rangle = \hat{U}|\Psi\rangle$ and $\langle\Psi'| = \langle\Psi|\hat{U}^\dagger$. Then:

$$\begin{aligned}
\alpha_0\alpha_0^* + \alpha_1\alpha_1^* &= \langle\Psi'|\Psi'\rangle \\
&= \left\langle \Psi \middle| \hat{U}^\dagger\hat{U} \middle| \Psi \right\rangle \\
&= \left\langle \Psi \middle| \hat{I} \middle| \Psi \right\rangle \\
&= \langle\Psi|\Psi\rangle \\
&= \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) \\
&= 1
\end{aligned}$$

Exercise 24

Check that the Pauli operators $\hat{X}, \hat{Y}, \hat{Z}$ are both hermitian and unitary. Illustrate both properties of $\hat{X}, \hat{Y}$, and $\hat{Z}$ using vectors $|0\rangle, |1\rangle, |+\rangle, |-\rangle, |+i\rangle, |-i\rangle$.

We have:

Hermitian:

$$\hat{X}^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hat{X}$$

$$\hat{Y}^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \hat{Y}$$

$$\hat{Z}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \hat{Z}$$

Unitary:

$$\hat{X}^\dagger\hat{X} = \hat{X}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hat{I}$$

$$\hat{Y}^\dagger\hat{Y} = \hat{Y}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \hat{I}$$

$$\hat{Z}^\dagger\hat{Z} = \hat{Z}^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \hat{I}$$

What's more, we check the hermitian and unitary properties (resp. real eigenvalues and orthogonal eigenvectors, and inner product preservation):

$\hat{X}$: eigenvvectors: $|+\rangle, |-\rangle$

eigenvalues: `markhl[`

+1, −1 in RR

]

orthogonal: $\langle +|-\rangle = 0$

inner product preservation:

$\left\langle \hat{X}+ \middle| \hat{X}+\right\rangle = 1 = \langle +|+\rangle$

$\left\langle \hat{X}- \middle| \hat{X}-\right\rangle = -1 = \langle -|-\rangle$

$\hat{Y}$: eigenvvectors: $|+i\rangle, |-i\rangle$

eigenvalues: `markhl[`

$+1, -1$ in RR
]

orthogonal: $\langle +i|-i\rangle = 0$

inner product preservation:

$\left\langle \hat{Y}+i \middle| \hat{Y}+i\right\rangle = 1 = \langle +i|+i\rangle$

$\left\langle \hat{Y}-i \middle| \hat{Y}-i\right\rangle = -1 = \langle -i|-i\rangle$

$\hat{Z}$: eigenvvectors: $|0\rangle, |1\rangle$

eigenvalues: `markhl[` $+1$,
-1 in RR
]

orthogonal: $\langle 0|1\rangle = 0$

inner product preservation:

$\left\langle \hat{Z}0 \middle| \hat{Z}0\right\rangle = 1 = \langle 0|0\rangle$

$\left\langle \hat{Z}1 \middle| \hat{Z}1\right\rangle = -1 = \langle 1|1\rangle$

Exercise 25
Repeat the steps above for finding the matrix for $\hat{X}$-operator using its eigenvectors $|\pm\rangle$ and eigenvalues ± 1.

The eigenvectors for $\hat{X}$ are $h_0 = |+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $h_1 = |-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$, with eigenvalues $h_0 = +1, h_1 = -1$ respectively. Then:

$$\begin{aligned} \hat{X} &= \sum_{i=0}^{1} h_i |h_i\rangle\langle h_i| \\ &= h_0|h_0\rangle\langle h_0| + h_1|h_1\rangle\langle h_1| \\ &= |+\rangle\langle +| - |-\rangle\langle -| \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Exercise 26
Do the same as above but for $\hat{Y}$, using its eigenvectors $|\pm i\rangle$ and eigenvalues ± 1.

The eigenvectors for $\hat{Y}$ are $h_0 = |+i\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$ and $h_1 = |-i\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$, with eigenvalues $h_0 = +1, h_1 = -1$ respectively. Then:

$$\begin{aligned} \hat{Y} &= \sum_{i=0}^{1} h_i |h_i\rangle\langle h_i| \\ &= h_0|h_0\rangle\langle h_0| + h_1|h_1\rangle\langle h_1| \\ &= |+i\rangle\langle +i| - |-i\rangle\langle -i| \\ &= \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{aligned}$$

Exercise 27
Use the representation of a Hermitian operator above to prove that $\hat{H}^n = \sum_{\text{all eigenstates}} h_i^n |h_i\rangle\langle h_i|$.

The Hermitian is defined as: $\begin{cases} \hat{H}|h_i\rangle = h_i|h_i\rangle \\ \hat{H} = \sum_{\text{all eigenstates}} h_i|h_i\rangle\langle h_i| \end{cases}$, with $f\left(\hat{H}\right) = \sum_{\text{all eigenstates}} f(h_i)|h_i\rangle\langle h_i|$. A Hermitian matrix of size $N \times N$ has N eigenvectors and associated eigenvalues.

$$\hat{H}^n = \left(\sum_{i=0}^{N-1} h_i|h_i\rangle\langle h_i|\right)^n$$

We note that since eigenvectors are orthogonal, we have that $\forall i \neq j, |h_i\rangle\langle h_j| = \langle h_i|h_j\rangle = 0$. Therefore, when developping the product, every term mixing $i \neq j$ is equal to zero. We only keep the terms where $i = j$:

$$\hat{H}^n = h_0^n(|h_0\rangle\langle h_0|)^n + h_1^n(|h_1\rangle\langle h_1|)^n + \dots + h_{N-1}^n(|h_{N-1}\rangle\langle h_{N-1}|)^n$$

We also note that:

$$\begin{aligned} (|a\rangle\langle a|)^n &= (|a\rangle\langle a|)(|a\rangle\langle a|)(|a\rangle\langle a|)\dots(|a\rangle\langle a|) \\ &= |a\rangle\langle a|a\rangle\langle a|a\rangle\langle a|\dots|a\rangle\langle a| \\ &= |a\rangle\langle a| \end{aligned}$$

Therefore:

$$\begin{aligned} \hat{H}^n &= h_0^n|h_0\rangle\langle h_0| + h_1^n|h_1\rangle\langle h_1| + \dots + h_{N-1}^n|h_{N-1}\rangle\langle h_{N-1}| \\ &= \sum_{i=0}^{N-1} h_i^n|h_i\rangle\langle h_i| \end{aligned}$$

> **Exercise 28**
> Show that any unitary operator $\hat{U}$ (represented by an $N \times N$ matrix) can be written as $\hat{U} = \exp\left(i\alpha\hat{H}\right)$, where α is a real number and $\hat{H}$ is some hermitian operator matrix.

Let $\lambda_0, \dots\lambda_{N-1}$ the eigenvalues of $\hat{U}$. We can write $\hat{U} = \begin{pmatrix} \lambda_0 & & \\ & \dots & \\ & & \lambda_{N-1} \end{pmatrix}$ since $\hat{U}$ diagonalizable. We have:

$$\begin{aligned} \hat{U}\hat{U}^\dagger = \hat{I} &\Leftrightarrow \begin{pmatrix} \lambda_0 & & \\ & \dots & \\ & & \lambda_{N-1} \end{pmatrix} \begin{pmatrix} \lambda_0^* & & \\ & \dots & \\ & & \lambda_{N-1}^* \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \dots & \\ & & 1 \end{pmatrix} \\ &\Leftrightarrow \lambda_i\lambda_i^* = 1 \\ &\Rightarrow \|\lambda_i\| = 1 \\ &\Rightarrow \text{we can write} \lambda_i \text{in their exponential forms:} \lambda_i = \exp\left(i\theta_i\right), \theta_i \in \mathbb{R} \end{aligned}$$

Since $|u_i\rangle$ are orthonormal (eigenvectors of $\hat{U}$), and $\theta_i \in \mathbb{R}$, we can write:

$$\hat{H} = \sum_{i=0}^{N-1} \theta_i|u_i\rangle\langle u_i|$$

Now we use $f\left(\hat{H}\right) = \sum_i f(h_i)|h_i\rangle\langle h_i|$ to find $\hat{U}$ in the basis of eigenvectors of $\hat{H}$. Let $f(x) = \exp(i\alpha x)$. Then:

$$f\left(\hat{H}\right) = \sum_{i=0}^{N-1} f(\theta_i)|U_i\rangle\langle U_i|$$
$$= \sum_{i=0}^{N-1} \exp\left(i\alpha\theta_i\right)|u_i\rangle\langle u_i|$$

For $\alpha = 1$, we get:

$$f\left(\hat{H}\right) = \exp\left(i\hat{H}\right) = \sum_{i=0}^{N-1} \exp(i\theta_i)|u_i\rangle\langle u_i|$$
$$= \sum_{i=0}^{N-1} \lambda_i|u_i\rangle\langle u_i|$$
$$= \hat{U}$$

If we want a general α ($\alpha \neq 1$), we can define $\hat{H}' = \frac{1}{\alpha}\hat{H}$ and use that instead.

D. Rotating the qubit state on the Bloch sphere

> **Exercise 29**
> Show that $\exp\left(-i\frac{\alpha}{2}\hat{X}\right)$ is a rotation of the Bloch vector by an angle α around X-axis.

We know that:

$$\exp\left(-i\frac{\alpha}{2}\hat{X}\right) = \hat{I}\cos\left(\frac{\alpha}{2}\right) - i\hat{X}\sin\left(\frac{\alpha}{2}\right)$$
$$= \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) & -i\sin\left(\frac{\alpha}{2}\right) \\ -i\sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) \end{pmatrix}$$

We apply $\exp\left(-i\frac{\alpha}{2}\hat{X}\right)$ to a pair of basis states $|+\rangle$ and $|-\rangle$:

$$\exp\left(-i\frac{\alpha}{2}\hat{X}\right)|+\rangle$$
$$= \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) & -i\sin\left(\frac{\alpha}{2}\right) \\ -i\sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\alpha}{2}\right) - i\sin\left(-\frac{\alpha}{2}\right)\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \exp\left(-i\frac{\alpha}{2}\right)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$
$$= \exp\left(-i\frac{\alpha}{2}\right)|+\rangle$$

$$\exp\left(-i\frac{\alpha}{2}\hat{X}\right)|-\rangle$$
$$= \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) & -i\sin\left(\frac{\alpha}{2}\right) \\ -i\sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\alpha}{2}\right) + i\sin\left(\frac{\alpha}{2}\right)\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$= \exp\left(i\frac{\alpha}{2}\right)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$
$$= \exp\left(i\frac{\alpha}{2}\right)|-\rangle$$

Therefore, this operator rotates the state in the $Y\ Z$-plane.

> **Exercise 30**
> Show that $\exp\left(-i\frac{\alpha}{2}\hat{Y}\right)$ is a rotation of the Bloch vector by an angle α around Y-axis.

In the same way, we find:

$$\begin{cases} \exp\left(-i\frac{\alpha}{2}\hat{Y}\right)|{+i}\rangle = \exp\left(-i\frac{\alpha}{2}\right)|{+i}\rangle \\ \exp\left(-i\frac{\alpha}{2}\hat{Y}\right)|{-i}\rangle = \exp\left(i\frac{\alpha}{2}\right)|{-i}\rangle \end{cases}$$

Therefore, this operator rotates the state in the X Z-plane.

> **Exercise 31**
> Show that a general qubit state $|\Psi\rangle$ can be obtained by first rotating $|0\rangle$ by an angle θ around Y-axis and then rotating by angle ϕ around Z-axis:
>
> $$|\Phi\rangle = \exp\left(-i\frac{\phi}{2}\hat{Z}\right)\exp\left(-i\frac{\theta}{2}\hat{Y}\right)|0\rangle$$

Let $|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \exp(i\phi)\sin\left(\frac{\theta}{2}\right)|1\rangle$.

We first apply the rotation around the Y-axis. Let $\alpha = \theta$:

$$-i\frac{\alpha}{2}\hat{Y} = -i\frac{\theta}{2}\hat{Y} = \begin{pmatrix} 0 & -\frac{\theta}{2} \\ \frac{\theta}{2} & 0 \end{pmatrix}$$

And, using matrix exponentiation in the same way as earlier::

$$\begin{aligned} \exp\left(-i\frac{\alpha}{2}\hat{Y}\right) &= \exp\left(-i\frac{\theta}{2}\hat{Y}\right) \\ &= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \end{aligned}$$

So we find:

$$\begin{aligned} \left|\Psi_{\text{intermediary}}\right\rangle &= \exp\left(-i\frac{\alpha}{2}\hat{Y}\right)|0\rangle \\ &= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin(\theta)|1\rangle \end{aligned}$$

In the same way, we rotate the obtained $\left|\Psi_{\text{intermediary}}\right\rangle$ by ϕ around Z-axis. Let $\alpha = \phi$:

$$-i\frac{\alpha}{2}\hat{Z} = -i\frac{\phi}{2}\hat{Z} = \begin{pmatrix} -\frac{\phi}{2} & 0 \\ 0 & \frac{\phi}{2} \end{pmatrix}$$

Since this matrix is diagonal, its exponentiation is just the exponentiation of its diagonal coefficents, we have:

$$\begin{aligned} \exp\left(-i\frac{\alpha}{2}\hat{Z}\right) &= \exp\left(-i\frac{\theta}{2}\hat{Z}\right) \\ &= \begin{pmatrix} \exp\left(-i\frac{\phi}{2}\right) & \\ & \exp\left(i\frac{\phi}{2}\right) \end{pmatrix} \end{aligned}$$

So we get:

$$\begin{aligned}
\exp\left(-i\frac{\phi}{2}\hat{Z}\right)|\Psi_{\text{intermediate}}\rangle &= \exp\left(-i\frac{\phi}{2}\hat{Z}\right)\left(\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin(\theta)|1\rangle\right) \\
&= \exp\left(-i\frac{\phi}{2}\right)\cos\left(\frac{\theta}{2}\right)|0\rangle + \exp\left(i\frac{\phi}{2}\right)\sin\left(\frac{\theta}{2}\right)|1\rangle \\
&= \exp\left(-i\frac{\phi}{2}\right)\left(\cos\left(\frac{\theta}{2}\right)|0\rangle + \exp(i\phi)\sin\left(\frac{\theta}{2}\right)|1\rangle\right) \\
&= |\Psi\rangle \text{ up to the global phase}
\end{aligned}$$

Exercise 32

Is the order of rotations important in the previous exercise?

The order of rotations is important since rotations about different axes do not commute.

Exercise 33

Let $\widehat{M} = \frac{\hat{X}+\hat{Z}}{\sqrt{2}}$ a hermitian operator and define a rotation $\exp\left(-i\frac{\alpha}{2}\widehat{M}\right)$. Figure out what it does.

$\widehat{M} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the Hadamard operator (this is the matrix that converts from basis $(|0\rangle, |1\rangle)$ to $(|+\rangle, |-\rangle)$ and back). We saw that $\exp\left(-i\frac{\alpha}{2}\widehat{M}\right)$ is the rotation matrix about the axis defined by the eigenvectors of $\widehat{M}$. Let's find its eigenvectors $|m_0\rangle, |m_1\rangle$:

$$\begin{aligned}
&\widehat{M}|m_0\rangle = |m_0\rangle \\
&\Leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \\
&\Leftrightarrow \begin{cases} \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} = a \\ \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}} = b \end{cases} \\
&\Leftrightarrow a = 1 + \sqrt{2} \land b = 1
\end{aligned}$$

$$\begin{aligned}
&\widehat{M}|m_1\rangle = |m_1\rangle \\
&\Leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -c \\ -d \end{pmatrix} \\
&\Leftrightarrow \begin{cases} \frac{c}{\sqrt{2}} + \frac{d}{\sqrt{2}} = -c \\ \frac{c}{\sqrt{2}} - \frac{d}{\sqrt{2}} = -d \end{cases} \\
&\Leftrightarrow c = 1 - \sqrt{2} \land d = 1
\end{aligned}$$

We get the eigenvectors $|m_0\rangle = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}$ and $|m_1\rangle = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$. Therefore, $\exp\left(-i\frac{\alpha}{2}\widehat{M}\right)$ is the rotation around the axis defined by $|m_0\rangle = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}$ and $|m_1\rangle = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$.

Exercise 34

Based on the previous two exercises, is it true that $\exp\left(-i\alpha\left(\hat{X}+\hat{Z}\right)\right) = \exp\left(-i\alpha\hat{X}\right)\exp\left(-i\alpha\hat{Z}\right)$?

No:

$$\exp\left(-i\alpha\left(\hat{X}+\hat{Z}\right)\right) = \exp\left(-i\alpha\left(\hat{Z}+\hat{X}\right)\right)$$

But we saw that:

$$\exp\left(-i\alpha\hat{X}\right)\exp\left(-i\alpha\hat{Z}\right) \neq \exp\left(-i\alpha\hat{Z}\right)\exp\left(-i\alpha\hat{X}\right)$$

Exercise 35

Consider another unitary operator $\exp\left(-i\frac{\alpha}{2}\widehat{M}\right)$, where $\widehat{M} = \frac{\hat{X}+\hat{Y}}{\sqrt{2}}$. What kind of rotation on the Bloch sphere is it?

This is a rotation around the axis defined by the eigenvectors of $\frac{\hat{X}+\hat{Y}}{\sqrt{2}} = \begin{pmatrix} 0 & \frac{1-i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & 0 \end{pmatrix}$ (by analogy with the Hadamard matrix we saw earlier).

E. Qubit state tomography

> **Exercise 36**
> Write down the values of θ and ϕ in terms of the mean values $\langle \Psi | \hat{X} | \Psi \rangle$, $\langle \Psi | \hat{Y} | \Psi \rangle$ and $\langle \Psi | \hat{Z} | \Psi \rangle$. Do we really need to know both X- and Y-projections?

We have:

$$\langle \Psi | \hat{Z} | \Psi \rangle = \cos(\theta)$$
$$\Leftrightarrow \theta = \arccos\left(\langle \Psi | \hat{Z} | \Psi \rangle\right)$$

And:

$$\begin{aligned} \langle \Psi | \hat{X} | \Psi \rangle &= \sin(\theta)\cos(\phi) \\ &= \sin\left(\arccos\left(\langle \Psi | \hat{Z} | \Psi \rangle\right)\right)\cos(\phi) \end{aligned}$$

$$\Leftrightarrow \phi = \arccos\left(\frac{\langle \Psi | \hat{X} | \Psi \rangle}{\sin\left(\arccos\left(\langle \Psi | \hat{Z} | \Psi \rangle\right)\right)}\right)$$

We need both projection to differenciate between $\pm\,\phi$.

> **Exercise 37**
> Come up with a specific protocol for measuring $\langle \Psi | \hat{X} | \Psi \rangle$ using an instrument that can only measures $\hat{Z}$.

We can use the Hadamard matrix since it converts from a measurement of $\hat{Z}$ to a measurement of $\hat{X}$ and back. Therefore, we apply $\hat{H}$ to our qubit and measure $\hat{Z}$. We get:

$$\begin{cases} \text{outcome} = +1 \Leftrightarrow \text{measurement of } \hat{X} = +1 \\ \text{outcome} = -1 \Leftrightarrow \text{measurement of } \hat{X} = -1 \end{cases}$$

And therefore, we have:

$$\langle \Psi | \hat{X} | \Psi \rangle = \langle \hat{H}\Psi | \hat{Z} | \hat{H}\Psi \rangle$$

Part 2

A. Time-evolution of the qubit state

> **Exercise 1**
> Verify the following equation for the time evolution of the *mean value* of a measurement outcome of some hermitian operator $\hat{L}$:
>
> $$\frac{\partial}{\partial t}\langle \Psi(t) | \hat{L} | \Psi(t) \rangle = \left\langle \Psi(t) \middle| \left(-\frac{i}{\hbar}\left[\hat{L}, \hat{H}\right]\right) \middle| \Psi(t) \right\rangle,$$
>
> where $\left[\hat{L}, \hat{H}\right] = \hat{L}\hat{H} - \hat{H}\hat{L}$.

We have:

$$\frac{\partial}{\partial t}|\Psi(t)\rangle = -\frac{i}{\hbar}\hat{H}|\Psi(t)\rangle$$

$$\frac{\partial}{\partial t}\langle\Psi(t)| = \frac{i}{\hbar}\langle\Psi(t)|\hat{H}$$

In the mean value, $\hat{L}$ is not time-dependant, so it acts as a scalar, and we can use the product rule on the $\Psi(t)$ braket. We get:

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle\Psi(t)\middle|\hat{L}\middle|\Psi(t)\right\rangle &= \left\langle\frac{\partial}{\partial t}\Psi(t)\middle|\hat{L}\middle|\Psi(t)\right\rangle + \left\langle\Psi(t)\middle|\hat{L}\middle|\frac{\partial}{\partial t}\Psi(t)\right\rangle \\ &= \left\langle\frac{i}{\hbar}\Psi(t)\middle|\hat{H}\hat{L}\middle|\Psi(t)\right\rangle + \left\langle\Psi(t)\middle|\hat{L}\hat{H}\middle|-\frac{i}{\hbar}|\Psi(t)\rangle\right\rangle \\ &= \frac{i}{\hbar}\left\langle\Psi(t)\middle|\hat{H}\hat{L}\middle|\Psi(t)\right\rangle - \frac{i}{\hbar}\left\langle\Psi(t)\middle|\hat{L}\hat{H}\middle||\Psi(t)\rangle\right\rangle \\ &= \frac{i}{\hbar}\left(\left\langle\Psi(t)\middle|\hat{H}\hat{L}\middle|\Psi(t)\right\rangle - \left\langle\Psi(t)\middle|\hat{L}\hat{H}\middle||\Psi(t)\rangle\right\rangle\right) \\ &= \frac{i}{\hbar}\left\langle\Psi(t)\middle|\hat{H}\hat{L} - \hat{L}\hat{H}\middle|\Psi(t)\right\rangle \\ &= \frac{i}{\hbar}\left\langle\Psi(t)\middle|\left[\hat{L},\hat{H}\right]\middle|\Psi(t)\right\rangle\end{aligned}$$

Exercise 2

Plug the above wave funtion into Eq. 1 and obtain the following solution of the Schrodinger's equation:

$$|\Psi(t)\rangle = \exp\left(-i\frac{E_0}{\hbar}t\right) \times \alpha_0(t=0)|E_0\rangle + \exp\left(-i\frac{E_1}{\hbar}t\right) \times \alpha_1(t=0)|E_1\rangle$$

Pluging $|\Psi(t)\rangle = \alpha_0(t)|E_0\rangle + \alpha_1(t)|E_1\rangle$ into both sides of Eq. 1 separately, we get:

$$\begin{aligned}\frac{\partial}{\partial t}|\Psi\rangle &= \frac{\partial}{\partial t}(\alpha_0(t)|E_0\rangle + \alpha_1(t)|E_1\rangle) \\ &= \frac{\partial}{\partial t}\alpha_0(t)|E_0\rangle + \frac{\partial}{\partial t}\alpha_1(t)|E_1\rangle\end{aligned}$$

And:

$$\begin{aligned}-\frac{i}{\hbar}\hat{H}|\Psi\rangle &= -\frac{i}{\hbar}\hat{H}(\alpha_0(t)|E_0\rangle + \alpha_1(t)|E_1\rangle) \\ &= -\frac{i}{\hbar}\alpha_0(t)\hat{H}|E_0\rangle - \frac{i}{\hbar}\alpha_1(t)\hat{H}|E_1\rangle \\ &= -\frac{i}{\hbar}\alpha_0(t)E_0|E_0\rangle - \frac{i}{\hbar}\alpha_1(t)E_1|E_1\rangle\end{aligned}$$

Therefore, we get:

$$\begin{cases}\frac{\partial}{\partial t}\alpha_0(t) = -\frac{i}{\hbar}\alpha_0(t)E_0 \\ \frac{\partial}{\partial t}\alpha_1(t) = -\frac{i}{\hbar}\alpha_1(t)E_1\end{cases} \Rightarrow \begin{cases}\alpha_0(t) = K_0\exp\left(-\frac{i}{\hbar}E_0 t\right) \\ \alpha_1(t) = K_1\exp\left(-\frac{i}{\hbar}E_1 t\right)\end{cases}$$

Exercise 3
Show that if a qubit starts in an energy eigenstate $|E_0\rangle$ or $|E_1\rangle$, it stays in that state, no time-evolution takes place.

Let $|\Psi(0)\rangle = |E_0\rangle$. Then:

$$|\Psi(0)\rangle = |E_0\rangle = 1 \cdot |E_0\rangle + 0 \cdot |E_1\rangle$$
$$\Rightarrow \alpha_0(t=0) = 1 \wedge \alpha_1(t=0) = 0$$

So:

$$\begin{aligned} |\Psi(t)\rangle &= \alpha_0(t=0)|E_0\rangle + \alpha_1(t=0)\exp\left(-i\frac{E_1 - E_0}{\hbar} \cdot t\right)|E_1\rangle \\ &= 1 \cdot |E_0\rangle + 0 \cdot \exp\left(-i\frac{E_1 - E_0}{\hbar} \cdot t\right)|E_1\rangle \\ &= |E_0\rangle \end{aligned}$$

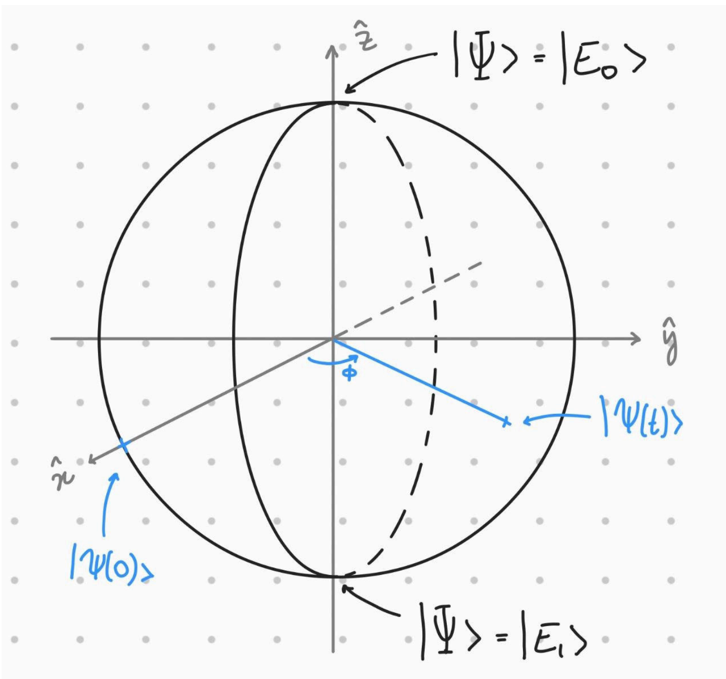
This can be mirrored for $|E_1\rangle$.

Exercise 4
Set up a Bloch sphere in the basis $|E_0\rangle, |E_1\rangle$ and consider $\alpha_0(t=0) = \alpha_1(t=0) = \frac{1}{\sqrt{2}}$. Show the qubit state at $t = 0$ and mark its time-evolution.

For $\alpha_{0,1}(t=0) = \frac{1}{\sqrt{2}}$, we have:

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}|E_0\rangle + \frac{1}{\sqrt{2}}|E_1\rangle$$

On the Bloch sphere, it's at the intersection between the sphere and the X-axis, and its time-evolution $|\Psi(t)\rangle$ is its rotation around the Z-axis with an angle ϕ:



Exercise 5
What would be the perios of oscillations of the qubit in this case? Does it make sense why Big Mac is classical now? How small should the qubit energy difference be for the phase to oscillate in time at a more experimentally accessible frequency?

We have the total energy of a Big Mac:

$$E_1 - E_0 = E_{\text{Big Mac}} = 590 \text{ cal} \cdot 4200 = 2478000\text{J}$$

The period would be:

$$\begin{aligned} T_{\text{Big Mac}} &= h \cdot E_{\text{Big Mac}} \\ &= \frac{6 \cdot 10^{-34}\text{J} \cdot \text{s}}{2478000\text{J}} \\ &= 2.42 \cdot 10^{-40}\text{s} \end{aligned}$$

Since a Big Mac has such a big amount of energy, it's oscillation is classical.

To be observed experimentally, it should oscillate at an accessible frequency. I do not know what such a frequency would be, but you would need to compute $E_{\text{accessible}} = \frac{h}{T_{\text{accessible}}}$ to answer the question with an actual value.

Exercise 6
Show that a modified Hamiltonian $\hat{H} - \hat{I}E_0$ has the same eigenvectors as $\hat{H}$ and the same time-evolution of the qubit state.

Let E_i an eigenvector of $\hat{H}$. We have:

$$\begin{aligned} \left(\hat{H} - \hat{I}E_0\right)|E_i\rangle &= \hat{H}|E_i\rangle - \hat{I}E_0|E_i\rangle \\ &= E_i|E_i\rangle - \hat{I}E_0|E_i\rangle \\ &= (E_i - E_0)|E_i\rangle \end{aligned}$$

And:

$$\begin{aligned} \frac{\partial}{\partial t}|\Psi\rangle &= -\frac{i}{\hbar}\left(\hat{H} - \hat{I}E_0\right)|\Psi\rangle \\ &= -\frac{i}{\hbar}\hat{H}|\Psi\rangle + \frac{i}{\hbar}\hat{I}E_0|\Psi\rangle \end{aligned}$$

Exercise 7
Check that the proposed evolution operator indeed solves the Schrodinger's equation.

We have:

$$\begin{aligned} \frac{\partial}{\partial t}|\Psi\rangle &= \frac{\partial}{\partial t}\exp\left(-\frac{i}{\hbar}\hat{H}t\right)|\Psi(t=0)\rangle \\ &= -\frac{i}{\hbar}\hat{H}\exp\left(-\frac{i}{\hbar}\hat{H}t\right)|\Psi(t=0)\rangle \\ &= -\frac{i}{\hbar}\hat{H}|\Psi\rangle \end{aligned}$$

B. Hamiltonian operator for qubits

Exercise 8

show by an explicit demonstartion that any Hermitian 2×2 matrix can be written as a superposition of

$$\frac{1}{\hbar}\hat{H} = -\omega_I \hat{I} - \frac{\omega_X}{2}\hat{X} - \frac{\omega_Y}{2}\hat{Y} - \frac{\omega_Z}{2}\hat{Z}$$

Any Hermitian matrix can be written as $\frac{1}{\hbar}\hat{H} = \begin{pmatrix} a & x-iy \\ x+iy & b \end{pmatrix}$, so we can check:

$$\begin{pmatrix} a & x-iy \\ x+iy & b \end{pmatrix} = -\omega_I \hat{I} - \frac{\omega_X}{2}\hat{X} - \frac{\omega_Y}{2}\hat{Y} - \frac{\omega_Z}{2}\hat{Z}$$

$$\Leftrightarrow \boxed{\begin{cases} a = \omega_I + \frac{\omega_Z}{2} \\ b = \omega_I - \frac{\omega_Z}{2} \\ x = \frac{\omega_X}{2} \\ y = \frac{\omega_Y}{2} \end{cases}}$$

Exercise 9

Show that eigenvalues of $\frac{\hat{H}}{\hbar}$ in Eq. 7 are given by $\pm\frac{1}{2}\sqrt{\omega_X^2 + \omega_Y^2 + \omega_Z^2}$

We have:

$$\frac{\hat{H}}{\hbar} = \frac{1}{2}\begin{pmatrix} \omega_Z & \omega_X - i\omega_Y \\ \omega_X + i\omega_Y & -\omega_Z \end{pmatrix}$$

Let h_i be an eigenvalue of $\hat{H}$. Then we have:

$$\det\left(-\frac{1}{2}\begin{pmatrix} \omega_Z & \omega_X - i\omega_Y \\ \omega_X + i\omega_Y & -\omega_Z \end{pmatrix} - h_i\hat{I}\right) = 0$$

$$\Leftrightarrow \left(-\frac{\omega_Z}{2} - h_i\right)\left(\frac{\omega_Z}{2} - h_i\right) - \left(-\frac{1}{2}(\omega_X - i\omega_Y)\right)\left(-\frac{1}{2}(\omega_X + i\omega_Y)\right) = 0$$

Solving that, we find:

$$\boxed{h_i = \pm\frac{1}{2}\sqrt{\omega_X^2 + \omega_Y^2 + \omega_Z^2}}$$

Exercise 10

Show that eigenstates of $\frac{\hat{H}}{\hbar}$ in Eq. 7 in the Bloch sphere are aligned along (or against) the axis defined by $(\omega_X, \omega_Y, \omega_Z)$.

TODO

Exercise 11

Create the rotation operators in terms of $\omega_X, \omega_Y, \omega_Z$, create the suspect eigenstates by applying the rotation operators to $|0\rangle$ and $|1\rangle$ and check if the rotated basis states are indeed the eigenstates.

The rotations about the Z- and Y-axes are, respectively:

$$R_Z = \exp\left(-i\frac{\phi}{2}\hat{Z}\right)$$

$$R_Y = \exp\left(-i\frac{\theta}{2}\hat{Y}\right)$$

So the final rotation is given by:

$$\begin{aligned} R &= R_Z R_Y \\ &= \boxed{\exp\left(-i\frac{\phi}{2}\hat{Z}\right)\exp\left(-i\frac{\theta}{2}\hat{Y}\right)} \end{aligned}$$

I suspect solving exercise 10 would yield results that I could use here to express θ and ϕ in terms of $\omega_X, \omega_Y, \omega_Z$.

The suspected eigenstates are:

$$\begin{aligned} |h_0\rangle &= R|0\rangle \\ &= R_Z R_Y|0\rangle \\ &= \boxed{\cos\left(\frac{\theta}{2}\right)|0\rangle + \exp(i\phi)\sin\left(\frac{\theta}{2}\right)|1\rangle} \end{aligned}$$

And:

$$\begin{aligned} |h_1\rangle &= R|1\rangle \\ &= R_Z R_Y|1\rangle \\ &= \boxed{-\sin\left(\frac{\theta}{2}\right)\exp\left(-i\frac{\phi}{2}\right)|0\rangle + \exp(i\phi)\cos\left(\frac{\theta}{2}\right)|1\rangle} \end{aligned}$$

To check that they are indeed eigenstates, we would have to compute $\frac{\hat{H}}{\hbar}|h_i\rangle = \pm|h_i\rangle$. However, not having the answers form exercise 10, I can't verify it.

C. Commutation relations of $\hat{X}, \hat{Y}, \hat{Z}$

Exercise 12

Verify all the other commutation relations involving Pauli matrices by directly multiplying the corresponding matrices.

We find:

$$\begin{aligned} &\left[\hat{X}, \hat{Y}\right] \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= 2i\hat{Z} \end{aligned}$$

$$\begin{aligned} &\left[\hat{Y}, \hat{Z}\right] \\ &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= 2i\hat{X} \end{aligned}$$

$$\begin{aligned} &\left[\hat{Z}, \hat{X}\right] \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= 2i\hat{Y} \end{aligned}$$

For the others, we use the identtities:

$$[M, M] = 0 \qquad\qquad [M, N] = -[N, M]$$

Exercise 13

Let's consider a simplified case of $\frac{\hat{H}}{\hbar} = -\frac{\omega}{2}\hat{Z}$. Describe the evolution of $z(t)$ which follows from Eq. 8.

We have $\frac{\hat{H}}{\hbar} = -\frac{\omega}{2}\hat{Z}$, so:

$$\langle\dot{X}\rangle = \omega\langle\hat{Y}\rangle$$
$$\langle\dot{Y}\rangle = \omega\langle\hat{X}\rangle$$
$$\langle\dot{Z}\rangle = 0$$

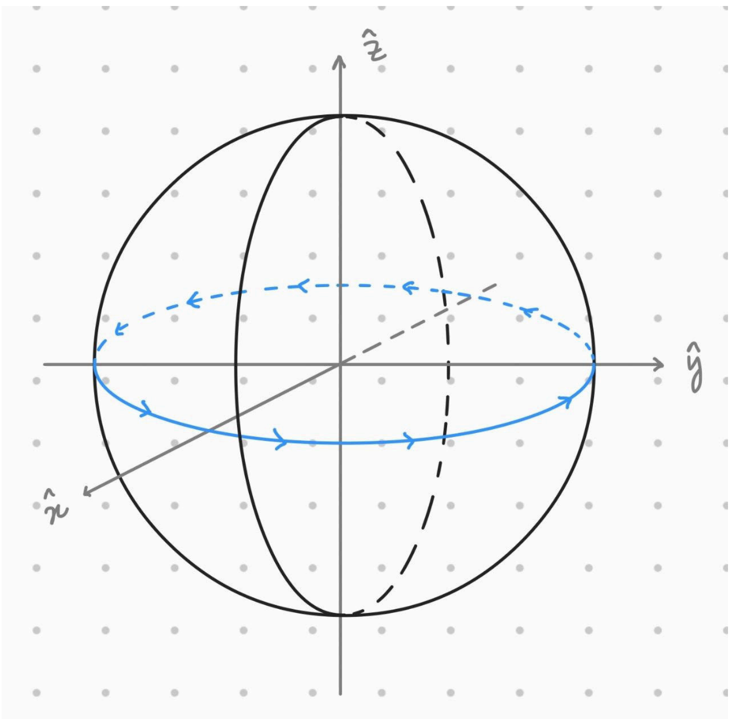
And:

$$\langle\dot{Z}\rangle = 0 \Rightarrow z(t) = \langle Z\rangle = \text{cste}$$

Exercise 14

Show the time-evolution in the XY-plane.

There is only a rotation in the $X\ Y$-plane since both $x(t)$ and $y(t)$ are harmonic oscillations:



Exercise 15

Set $\omega_X = \omega_Y = \omega_Z = 1$ and solve the equation of motion for x, y, z numerically.

With $\omega_X = \omega_Y = \omega_Z = 1$, we get:

$$\widehat{M} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

And the solution to $\dot{\vec{r}} = \widehat{M}\vec{r}$ is:

$$\vec{r}(t) = \exp\left(\widehat{M}t\right)\vec{r}(0)$$

This is a rotation around the axis defined by the eigenvectors of $\widehat{M}$, which are along $(1, 1, 1)$.

D. Changing reference frames: "rotating" frame vs. "lab" frame

Exercise 16

Conider a qubit with a Hamiltonian $\frac{\hat{H}}{\hbar} = -\frac{\omega}{2}\hat{Z}$. Consider that at time $t = 0$ the qubit was initialised in state $|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Show that the time evolution of the qubit state would be

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\exp(-i\omega t)|1\rangle$$

We have the time-evolution:

$$\begin{aligned} |\Psi(t)\rangle &= \exp\left(-\frac{i}{\hbar}\hat{H}t\right)|\Psi_0\rangle \\ &= \exp\left(-i\frac{\omega}{2}\hat{Z}\right)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ &= \frac{1}{\sqrt{2}}\exp\left(-i\frac{\omega}{2}\hat{Z}\right)|0\rangle + \frac{1}{\sqrt{2}}\exp\left(-i\frac{\omega}{2}\hat{Z}\right)|1\rangle \\ &= \frac{1}{\sqrt{2}}\exp\left(-i\frac{\omega}{2}\right)|0\rangle + \frac{1}{\sqrt{2}}\exp\left(-i\frac{\omega}{2}\right)|1\rangle \\ &= \boxed{\exp\left(-i\frac{\omega}{2}\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\exp(-i\omega t)|1\rangle\right)} \end{aligned}$$

Exercise 17

Apply a unitary transformation $\hat{U} = \exp\left(-i\frac{\omega}{2}\hat{Z}t\right)$ to the time-dependant state of the qubit in the previous exercise. What new Hamiltonian describess the time-evolution of this new state?

We have (from the previous exercise):

$$\hat{U}|\Psi(t)\rangle = \exp\left(-i\frac{\omega}{2}\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\exp(-i\omega t)|1\rangle\right)$$

And:

$$\frac{\partial}{\partial t}\hat{U}|\Psi(t)\rangle = \hat{H}'|\Psi\rangle$$

With $\hat{H}'$ the new Hamiltonian. Solving that equation, we get:

$$\hat{H}' = \boxed{-\frac{\hbar}{2}\omega\hat{Z}}$$

Exercise 18

Consider a qubit with a Hamiltonian $\frac{\hat{H}}{\hbar} = -\frac{\omega_0}{2}\hat{Z}$. Considera frame change defined by the unitary $\exp\left(-i\frac{\omega}{2}\hat{Z}t\right)$. What is the Hamiltonian in this new frame? How would you choose ω to make the evolution as simple as possible?

Whe have:

$$\hat{H}' = \hat{U}\hat{H}\hat{U}^\dagger + i\hbar\dot{\hat{U}}\hat{U}^\dagger$$

And we have (since $\hat{H}, \hat{U}$ are diagonal):

$$\begin{aligned}\hat{U}\hat{H}\hat{U}^\dagger &= \hat{U}\hat{U}^\dagger\hat{H} \\ &= \hat{H} \\ &= -\hbar\frac{\omega_0}{2}\hat{Z}\end{aligned}$$

$$\begin{aligned}i\hbar\dot{\hat{U}}\hat{U}^\dagger &= i\hbar\frac{\partial}{\partial t}\left(\hat{U}\right)\hat{U}^\dagger \\ &= i\hbar\frac{\partial}{\partial t}\left(\exp\left(-i\frac{\omega}{2}\hat{Z}t\right)\right)\hat{U}^\dagger \\ &= i\hbar\left(-i\frac{\omega}{2}\hat{Z}\hat{U}\right)\hat{U}^\dagger \\ &= \hbar\frac{\omega}{2}\hat{Z}\end{aligned}$$

So we get:

$$\begin{aligned}\hat{H}' &= -\hbar\frac{\omega_0}{2}\hat{Z} + \hbar\frac{\omega}{2}\hat{Z} \\ &= \hbar\frac{\omega - \omega_0}{2}\hat{Z}\end{aligned}$$

To make it as simple as possible, we can set $\omega = \omega_0$ to get a null matrix and have a constant state.

E. Implementing quantum gates: evolution after a fast switch

> **Exercise 19**
> Show the evolution on the Bloch sphere. At what time should the experimentalist set $\omega = 0$ to arrive at states $|1\rangle, |+\rangle, |-\rangle$?

For $\frac{\hat{H}}{\hbar} = -\frac{\omega}{2}\hat{Y}$, we have a rotation around the Y-axis:

$$\begin{aligned}\hat{U} &= \exp\left(i\frac{\omega}{2}\hat{Y}t\right) \\ &= \cos\left(\frac{\omega}{2}t\right)|0\rangle - \sin\left(\frac{\omega}{2}t\right)|1\rangle \\ &\Rightarrow \theta = \omega t \\ &\Rightarrow t = \frac{\theta}{\omega}\end{aligned}$$

So we get:

For $|+\rangle$:

$$\begin{aligned}\theta &= \frac{\pi}{2} \\ \Rightarrow t &= \frac{\pi}{2\omega}\end{aligned}$$

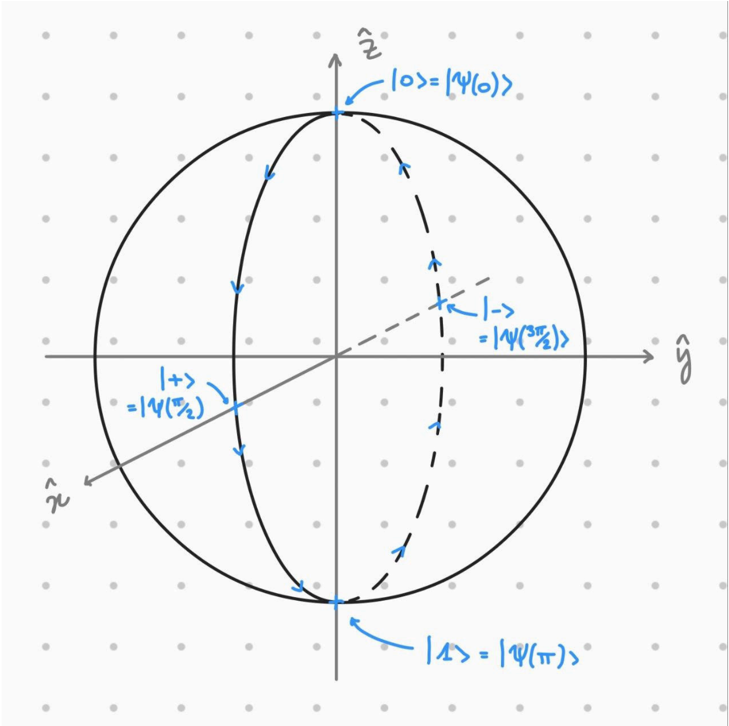
For $|1\rangle$:

$$\begin{aligned}\theta &= \pi \\ \Rightarrow t &= \frac{\pi}{\omega}\end{aligned}$$

For $|-\rangle$:

$$\begin{aligned}\theta &= \frac{3\pi}{2} \\ \Rightarrow t &= \frac{3\pi}{2\omega}\end{aligned}$$

On the Bloch sphere:



> **Exercise 20**
> Show the evolution on the Bloch sphere. At what time should the experimentalist set $\omega = 0$ to arrive at states $|1\rangle, |+i\rangle, |-i\rangle$?

In the same way, we have a rotation around the X-axis, and we get:

For $|-i\rangle$:

$$t = \frac{\pi}{2\omega}$$

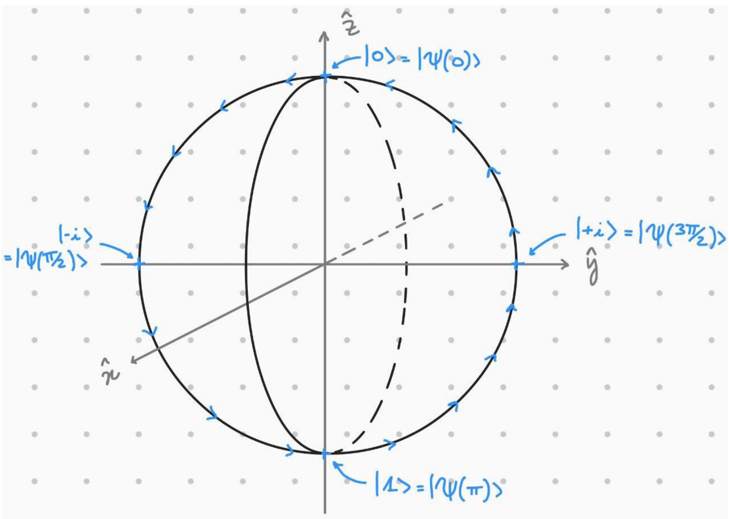
For $|1\rangle$:

$$t = \frac{\pi}{\omega}$$

For $|+i\rangle$:

$$t = \frac{3\pi}{2\omega}$$

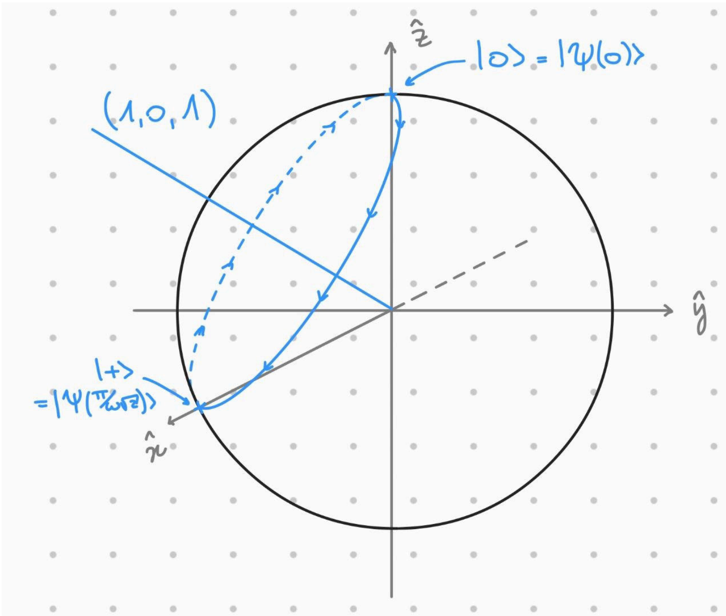
On the Bloch sphere:



> **Exercise 21**
> Solve for the evolution of the state for $t > 0$ and show it on the Bloch sphere. At what time $|0\rangle$ turns into $|+\rangle$?

This is a rotation around the exis $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\end{pmatrix}$ with frequency $\omega' = \frac{\omega}{\sqrt{2}}$. Doing the same as before, we get $t = \frac{\pi}{\omega\sqrt{2}}$.

On the Bloch sphere:



Exercise 22

In the previous exercise, repeat the calculation but start from state $|+\rangle$ at $t = 0$. At what time $|+\rangle$ turns into $|0\rangle$?

The rotation is symmetric, so we get $t = \frac{\pi}{\omega\sqrt{2}}$ as well.

Exercise 23

Consider the evolutin of the qubit starting from state $|0\rangle$ during the time τ_1 when $\omega_X = g$. Show that the evolution is a rotation of the qubit state vector in the Bloch sphere, around the axis defined by ocmponents $\begin{pmatrix}g & 0 & \omega_Z\end{pmatrix}$, and with constant angular velocity $\sqrt{\omega_Z^2 + g^2}$. What's the period of such rotation?

The Hamiltonian is given by:

$$\hat{H} = -\frac{\hbar}{2}\left(g\hat{X} + \omega_Z\hat{Z}\right)$$

As seen in the previous exercises, this means that the rotation axis is given by:

$$\begin{pmatrix}g & 0 & \omega_Z\end{pmatrix}$$

And we have (also as seen previously):

$$\omega = \sqrt{\omega_Z^2 + g^2}$$

And the rotation angle:

$$\theta = \omega t$$

This gives us:

$$
\begin{aligned}
T &= \frac{2\pi}{\omega} \\
&= \boxed{\frac{2\pi}{\sqrt{\omega_Z^2 + g^2}}}
\end{aligned}
$$