PHYS344 - Quantum Physics for non-Physicists
Homework 2

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Notations and conventions

$$0 \in \mathbb{N}$$

A Introduction: classical vs. quantum oscillator models

Exercise 1

For oscillator in state $|\Psi\rangle = |0\rangle$

$$\begin{aligned}\langle 0|\,\hat{x}\,|0\rangle &= x_0 \langle 0| \left(\hat{a} + \hat{a}^\dagger\right) |0\rangle \\ &= x_0 \Big(0 + \langle 0|1\rangle\Big) = 0 \\ \langle 0|\,\hat{p}\,|0\rangle &= -ip_0 \langle 0| \left(\hat{a} - \hat{a}^\dagger\right) |0\rangle \\ &= -ip_0 \Big(0 - \langle 0|1\rangle\Big) = 0\end{aligned}$$

For socillator in states $|\Psi\rangle = |n\rangle$

$$\begin{aligned}\langle n|\,\hat{x}\,|n\rangle &= x_0 \langle n| \left(\hat{a} + \hat{a}^\dagger\right) |n\rangle \\ &= x_0 \Big(\sqrt{n}\,\langle n|n-1\rangle + \sqrt{n+1}\,\langle n|n+1\rangle\Big) = 0 \\ \langle n|\,\hat{p}\,|n\rangle &= -ip_0 \langle n| \left(\hat{a} - \hat{a}^\dagger\right) |n\rangle \\ &= -ip_0 \Big(\sqrt{n}\,\langle n|n-1\rangle - \sqrt{n+1}\,\langle n|n+1\rangle\Big) = 0\end{aligned}$$

In general $\langle n|\,\hat{x}\,|n\rangle = \langle n|\,\hat{p}\,|n\rangle = 0 \quad \forall n \in \mathbb{N}$

Exercise 2

For oscillator in state $|\Psi\rangle = |0\rangle$

$$\begin{aligned}\langle 0|\,\hat{x}^2\,|0\rangle &= x_0^2 \langle 0| \left(\hat{a} + \hat{a}^\dagger\right)^2 |0\rangle \\ &= x_0^2 \langle 0| \Big(\hat{a}^2 + \underbrace{\hat{a}\hat{a}^\dagger}_{=1+\hat{a}^\dagger\hat{a}} + \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\Big) |0\rangle \\ &= x_0^2 \Big(0 + 1 + 0 + 0\Big) = x_0^2 \\ \langle 0|\,\hat{p}^2\,|0\rangle &= -p_0^2 \langle 0| \left(\hat{a} - \hat{a}^\dagger\right)^2 |0\rangle \\ &= -p_0^2 \langle 0| \Big(\hat{a}^2 - \underbrace{\hat{a}\hat{a}^\dagger}_{=1+\hat{a}^\dagger\hat{a}} - \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\Big) |0\rangle \\ &= -p_0^2 \Big(0 - 1 - 0 + 0\Big) = p_0^2\end{aligned}$$

For socillator in states $|\Psi\rangle = |n\rangle$

$$\begin{aligned}\langle n|\,\hat{x}^2\,|n\rangle &= x_0^2 \langle n| \left(\hat{a} + \hat{a}^\dagger\right)^2 |n\rangle \\ &= x_0^2 \langle n| \Big(\hat{a}^2 + \underbrace{\hat{a}\hat{a}^\dagger}_{=1+\hat{a}^\dagger\hat{a}} + \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\Big) |n\rangle \\ &= x_0^2 \Big(0 + \underbrace{1+n} + n + 0\Big) = x_0^2(2n+1) \\ \langle n|\,\hat{p}^2\,|n\rangle &= -p_0^2 \langle n| \left(\hat{a} - \hat{a}^\dagger\right)^2 |n\rangle \\ &= -p_0^2 \langle n| \Big(\hat{a}^2 - \underbrace{\hat{a}\hat{a}^\dagger}_{=1+\hat{a}^\dagger\hat{a}} - \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\Big) |n\rangle \\ &= -p_0^2 \Big(0 - \underbrace{(1+n)} - n + 0\Big) = p_0^2(2n+1)\end{aligned}$$

In general $x_{\mathrm{RMS}} = x_0\sqrt{2n+1}$, $p_{\mathrm{RMS}} = p_0\sqrt{2n+1} \quad \forall n \in \mathbb{N}$

Exercise 3

$$\begin{aligned}x_{\mathrm{RMS}} \cdot p_{\mathrm{RMS}} &= x_0 p_0 (2n+1) \\ &= \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}}(2n+1) \\ &= \frac{\hbar}{2}(2n+1) \geq \frac{\hbar}{2} \quad \forall n \in \mathbb{N}\end{aligned}$$

Exercise 4

$$\begin{aligned}\langle T\rangle &= \langle 0| \frac{\hat{p}^2}{2m} |0\rangle = \frac{1}{2m}\langle \hat{p}^2\rangle = \frac{1}{2m}p_0^2 = \frac{1}{2m}\frac{\hbar m\omega}{2} \\ &= \frac{\hbar\omega}{4} \\ \langle V\rangle &= \langle 0| \frac{m\omega^2\hat{x}^2}{2} |0\rangle = \frac{m\omega^2}{2}\langle x^2\rangle = \frac{m\omega^2}{2}x_0^2 = \frac{m\omega^2}{2}\frac{\hbar}{2m\omega} \\ &= \frac{\hbar\omega}{4}\end{aligned}$$

In general, for $n = 0$, $\langle T\rangle = \langle V\rangle$.

Exercise 5

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty}\Big[\psi_n \exp(-in\omega t)\Big]|n\rangle$$

$$\begin{aligned}
\langle\hat{a}\rangle &= \sum_{n=0}^{\infty}\psi_n^*(t)\langle n|\hat{a}|\Psi(t)\rangle \\
&= \sum_{n=0}^{\infty}\psi_n^*(t)\sqrt{n}\psi_{n-1}(t)\exp(-i\omega t)
\end{aligned}$$

$$\begin{cases}\frac{\partial\psi_n(t)}{\partial t} \\ \frac{\partial\psi_n^*(t)}{\partial t}\end{cases} = \begin{cases}-in\omega\psi_n(t) \\ in\omega\psi_{n-1}(t)\end{cases}$$

$$\begin{aligned}
\underline{\frac{\partial\langle a\rangle}{\partial t}} &= -i\omega\sum_{n=0}^{\infty}\sqrt{n}\Big(n\psi_n^*(t)\psi_{n-1}(t) - (n-1)\psi_n^*(t)\psi_{n-1}(t)\Big) \\
&\cdot\exp(-i\omega t) \\
&= \underline{-i\omega\langle a\rangle}
\end{aligned}$$

$$\begin{aligned}
\underline{\frac{\partial\langle\hat{a}^\dagger\rangle}{\partial t}} &= \text{using same techinques} \\
&= \underline{i\omega\langle a^\dagger\rangle}
\end{aligned}$$

Exercise 6

- $|\Psi(t=0)\rangle = |2\rangle$
  By solving the differential equation $\frac{\partial\langle\hat{a}\rangle}{\partial t} = -i\omega\langle a\rangle$, we get that $\langle a\rangle(t) = \sqrt{2}e^{-i\omega t}$, but we also know that $\langle a\rangle(t) = \frac{1}{2}\left(\frac{\langle x\rangle(t)}{x_0} + i\frac{\langle p\rangle(t)}{p_0}\right)$, so we can identify

  $$\underline{\frac{\langle x\rangle(t)}{x_0} = \sqrt{2}\cos(\omega t) \qquad \frac{\langle p\rangle(t)}{p_0} = -\sqrt{2}\sin(\omega t)}$$

  This would describe a circle or radius $\sqrt{2}$.
- $|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
  With the same method, we end up with

  $$\underline{\frac{\langle x\rangle(t)}{x_0} = \frac{1}{\sqrt{2}}\cos(\omega t) \qquad \frac{\langle p\rangle(t)}{p_0} = -\frac{1}{\sqrt{2}}\sin(\omega t)}$$

  This would describe a circle of radius $1/\sqrt{2}$.

TODO Sketches

Exercise 7

Let $n = 1$ for simplicity.

$$\begin{aligned}
\langle 0|\hat{H}|0\rangle &= \frac{1}{2}\hbar\omega \\
\langle n|\hat{H}|n\rangle &= \left(n + \frac{1}{2}\right)\hbar\omega \\
\underline{\langle\hat{H}\rangle} &= \frac{1}{3}\frac{1}{2}\hbar\omega + \frac{2}{3}\left(n + \frac{1}{2}\right)\hbar\omega \\
&= \left(\frac{1}{2} + \frac{2n}{3}\right)\hbar\omega\Big|_{n=1} = \underline{\frac{7}{6}\hbar\omega}
\end{aligned}$$

Energy is not quantized, because it depends on both states of $n = 0$ and $n > 0$, basically a weighted average of their energy states. As it is an average, it can take any value in $\mathbb{R}$ wihtout restriction.

B Coherent states

Exercise 8

$$\begin{aligned}
\underline{\hat{a}|\alpha\rangle} &= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\hat{a}|n\rangle \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{m=-1}^{\infty}\frac{\alpha^{m+1}}{\sqrt{(m+1)!}}\sqrt{m+1}|m\rangle \\
&= \alpha\underbrace{\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}|m\rangle} \\
&= \underline{\alpha|\alpha\rangle}
\end{aligned}$$

Exercise 9

$$\begin{aligned}
\hat{a}^\dagger\sum_{n=0}^{\infty}\psi_n|n\rangle &= \sum_{n=0}^{\infty}\psi_n\hat{a}^\dagger|n\rangle \\
&= \sum_{n=0}^{\infty}\psi_n\sqrt{n+1}|n+1\rangle \\
\lambda \text{ is an eignenvalue of } \hat{a}^\dagger \to &= \lambda\sum_{n=0}^{\infty}\psi_n|n\rangle \\
\langle 0|\hat{a}^\dagger|\Psi\rangle &= \langle 0|\sum_{n=0}^{\infty}\psi_n\sqrt{n+1}|n+1\rangle\Big|_{n=0} \\
&= \psi_0\sqrt{1}\langle 0|1\rangle = 0 \\
\langle 0|\lambda|\Psi\rangle &= \lambda\langle 0|\Psi\rangle = \lambda\psi_0 \\
\to \lambda &= 0 \quad\text{or } \psi_0 = 0
\end{aligned}$$

If $\psi_0 \neq 0$, then we must have $\lambda = 0$, but it would contradict $\hat{a}^\dagger|\Psi\rangle = \lambda|\Psi\rangle$. If $\psi_0 = 0$, then $\hat{a}^\dagger$ would increase the level of energy of $|\Psi\rangle$, and no such eigenstate exists.

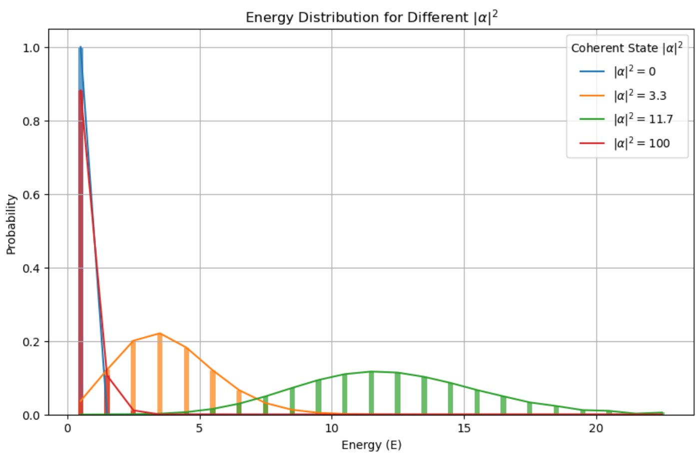
Exercise 10

$$\begin{aligned}
\langle\hat{H}\rangle &= \langle\alpha|\hat{H}|\alpha\rangle \\
&= \langle\alpha|\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|\alpha\rangle \\
&= \hbar\omega\langle\alpha|\hat{a}^\dagger|\alpha\rangle + \frac{\hbar\omega}{2}\langle\alpha|\alpha\rangle \\
\underline{\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle} &= \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = \alpha\langle\alpha|\hat{a}^\dagger|\alpha\rangle \\
&= |\alpha|^2\langle\alpha|\alpha\rangle = \underline{|\alpha|^2} \\
\to \underline{\langle\hat{H}\rangle} &= \underline{\hbar\omega|\alpha|^2 + \frac{\hbar\omega}{2}}
\end{aligned}$$

Exercise 11

$$
\begin{aligned}
\hat{H}^2 &= \hbar^2\omega^2\left(\underbrace{\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}}_{=\hat{A}^2} + \underbrace{\hat{a}^\dagger\hat{a}}_{=\hat{A}} + \frac{1}{4}\right) \\
\langle\alpha|\,\hat{H}^2\,|\alpha\rangle &= \hbar^2\omega^2\left(\langle\alpha|\,A^2\,|\alpha\rangle + \langle\alpha|\,\hat{A}\,|\alpha\rangle + \frac{1}{4}\right) \\
&= \hbar^2\omega^2\left(\underbrace{|\alpha|^4 + |\alpha|^2} + |\alpha|^2 + \frac{1}{4}\right) \\
\left(\langle\alpha|\,\hat{H}\,|\alpha\rangle\right)^2 &= \left(\hbar\omega|\alpha|^2 + \frac{\hbar\omega}{2}\right)^2 \\
&= \hbar^2\omega^2\left(|\alpha|^4 + |\alpha|^2 + \frac{1}{4}\right) \\
E_{\mathrm{RMS}}^2 &= \underbrace{\langle\alpha|\,\hat{H}^2\,|\alpha\rangle}_{\langle\hat{H}^2\rangle} - \underbrace{\left(\langle\alpha|\,\hat{H}\,|\alpha\rangle\right)^2}_{\langle\hat{H}\rangle^2} \\
&= \hbar^2\omega^2|\alpha|^2
\end{aligned}
$$

Exercise 12



Exercise 13

$$
\begin{aligned}
\left(\langle\alpha|\,\hat{x}\,|\alpha\rangle\right)^2 &= \left(\sqrt{\frac{\hbar}{2m\omega}}\,\langle\alpha|\,(\hat{a}+\hat{a}^\dagger)\,|\alpha\rangle\right)^2 \\
&= \frac{-\hbar m\omega}{2}(\alpha-\alpha^*)^2 \\
\langle\alpha|\,\hat{x}^2\,|\alpha\rangle &= \frac{-\hbar m\omega}{2}\,\langle\alpha|\,(\hat{a}+\hat{a}^\dagger)^2\,|\alpha\rangle \\
&= \frac{\hbar}{2m\omega}\left(\alpha^2 + (\alpha^*)^2 - 2|\alpha|^2 - 1\right) \\
\hat{x}_{\mathrm{RMS}} &= \sqrt{\langle\alpha|\,\hat{x}^2\,|\alpha\rangle - \left(\langle\alpha|\,\hat{x}\,|\alpha\rangle\right)^2} \\
&= \sqrt{\frac{\hbar}{2m\omega}}
\end{aligned}
$$

There is nothing in $\sqrt{\hbar/(2m\omega)}$ that depends on the time, so no.

Exercise 14

$$
\begin{aligned}
\left(\langle\alpha|\,\hat{x}\,|\alpha\rangle\right)^2 &= \sqrt{\frac{-\hbar m\omega}{2}}\left(\langle\alpha|\,(\hat{a}+\hat{a}^\dagger)\,|\alpha\rangle\right)^2 \\
&= \frac{1}{2m\omega}(\alpha+\alpha^*)^2 \\
\langle\alpha|\,\hat{x}^2\,|\alpha\rangle &= \frac{-\hbar m\omega}{2}\,\langle\alpha|\,(\hat{a}+\hat{a}^\dagger)^2\,|\alpha\rangle \\
&= \frac{-\hbar m\omega}{2}\left(\alpha^2 + (\alpha^*)^2 - 2|\alpha|^2 - 1\right) \\
\hat{p}_{\mathrm{RMS}} &= \sqrt{\langle\alpha|\,\hat{x}^2\,|\alpha\rangle - \left(\langle\alpha|\,\hat{x}\,|\alpha\rangle\right)^2} \\
&= \sqrt{\frac{\hbar m\omega}{2}}
\end{aligned}
$$

$$
x_{\mathrm{RMS}} \cdot p_{\mathrm{RMS}} = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}} = \frac{\hbar}{2}
$$

There is nothing in $\hbar/2$ that depends on α, so no.

Exercise 15

TODO

C Displacement operator

Exercise 16

$$
\begin{aligned}
e^{-\frac{|\alpha|^2}{2}}e^{\alpha a^\dagger}\,|0\rangle &= e^{-\frac{|\alpha|^2}{2}}\left(1 + \alpha\hat{a}^\dagger + \frac{\alpha^2\hat{a}^\dagger\hat{a}^\dagger}{2!} + \dots\right)|0\rangle \\
&= e^{-\frac{|\alpha|^2}{2}}\left(|0\rangle + \alpha\,|1\rangle + \frac{\alpha^2}{2!}\sqrt{2!}\,|2\rangle + \dots\right) \\
&= |\alpha\rangle \\
e^{\alpha^*\hat{a}}\,|0\rangle &= \left(1 + \alpha^* a + \frac{(\alpha^*)^2\hat{a}\hat{a}}{2!} + \dots\right)|0\rangle \\
&= |0\rangle + \alpha^*\cdot 0 + \frac{(\alpha^*)^2}{2!}\sqrt{2!}\cdot 0 + \dots \\
&= |0\rangle
\end{aligned}
$$

Exercise 17

$$
\begin{aligned}
e^{-\frac{|\alpha|^2}{2}}\cdot e^{\alpha\hat{a}^\dagger}\cdot e^{\alpha\hat{a}} &= e^{-\frac{|\alpha|^2}{2}}\cdot e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}\cdot e^{\frac{1}{2}[\alpha\hat{a}^\cdot - \alpha^*\hat{a}]} \\
&= e^{-\frac{|\alpha|^2}{2}}\cdot e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}\cdot e^{\frac{1}{2}|\alpha|^2} \\
&= e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}
\end{aligned}
$$

Exercise 18

$$
\begin{aligned}
\hat{D}(-\alpha) &= e^{-\alpha^*\hat{a}+\alpha\hat{a}^\dagger} \\
&= e^{\left(-\alpha\hat{a}^\dagger+\alpha^*\hat{a}\right)^\dagger} \\
&= \left(e^{-\alpha\hat{a}^\dagger+\alpha^*\hat{a}}\right)^\dagger \\
&= \hat{D}^\dagger(\alpha)
\end{aligned}
$$

$$
\begin{aligned}
\hat{D}(\alpha)\hat{D}(-\alpha) &= e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}}\cdot e^{-\alpha\hat{a}^\dagger+\alpha^*\hat{a}} = \hat{D}(-\alpha)\hat{D}(\alpha) \\
&= \hat{I} \to \text{Unitary}
\end{aligned}
$$

Exercise 19

For equation 43,

$$
\begin{aligned}
\hat{D}^\dagger(\alpha)\hat{a}\left|\beta\right\rangle &= \beta\hat{D}\dagger(\alpha)\left|\beta\right\rangle \\
&= \beta\left|-\alpha+\beta\right\rangle \\
\Big(\hat{a}+\alpha\Big)\hat{D}^\dagger(\alpha)\left|\beta\right\rangle &= (\hat{a}+\alpha)\left|-\alpha+\beta\right\rangle \\
&= \beta\left|-\alpha+\beta\right\rangle \\
\to \hat{D}^\dagger(\alpha)\hat{a}\left|\beta\right\rangle &= \Big(\hat{a}+\alpha\Big)\hat{D}^\dagger(\alpha)\left|\beta\right\rangle
\end{aligned}
$$

For the equation 44,

$$
\begin{aligned}
\hat{D}^\dagger(\alpha)\hat{a}^\dagger\left|\beta\right\rangle &= \beta^*\hat{D}\dagger(\alpha)\left|\beta\right\rangle \\
&= \beta^*\left|-\alpha+\beta\right\rangle \\
\Big(\hat{a}+\alpha^*\Big)\hat{D}^\dagger(\alpha)\left|\beta\right\rangle &= (\hat{a}+\alpha^*)\left|-\alpha+\beta\right\rangle \\
&= \beta^*\left|-\alpha+\beta\right\rangle \\
\to \hat{D}^\dagger(\alpha)\hat{a}^\dagger\left|\beta\right\rangle &= \Big(\hat{a}+\alpha^*\Big)\hat{D}^\dagger(\alpha)\left|\beta\right\rangle
\end{aligned}
$$

Exercise 20

$$
\begin{aligned}
\hat{D}(\alpha) &= e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}} = e^{\alpha(\hat{a}\dagger-\hat{a}} \\
&= \exp\left(\alpha\left[\left(\frac{1}{2}\frac{\hat{x}}{x_0}-\frac{i}{2}\frac{\hat{p}}{p_0}\right)-\left(\frac{1}{2}\frac{\hat{x}}{x_0}+\frac{i}{2}\frac{\hat{p}}{p_0}\right)\right]\right) \\
&= \exp\left(-i\alpha\frac{\hat{p}}{p_0}\right) \\
&= \exp\left(-i\alpha\frac{\hat{p}}{\frac{\hbar}{2x_0}}\right) = \exp\left(-i\frac{\hat{p}\cdot 2\alpha x_0}{\hbar}\right)
\end{aligned}
$$

Exercise 21

$$
\begin{aligned}
\hat{D}^\dagger(\alpha)\hat{x}\hat{D}(\alpha) &= \exp\left(i\frac{\hat{p}\cdot 2\alpha x_0}{\hbar}\right)\cdot\hat{x}\exp\left(-i\frac{\hat{p}\cdot 2\alpha x_0}{\hbar}\right) \\
&= x_0\hat{D}^\dagger(\alpha)(\hat{a}+\hat{a}^\dagger)\hat{D}(\alpha) \\
&= x_0\Big(\underbrace{\hat{D}^\dagger\hat{a}\hat{D}(\alpha)}_{=\hat{a}+\alpha}+\underbrace{\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha)}_{=\hat{a}^\dagger+\alpha^*}\Big) \\
&= \hat{x}+2\alpha x_0
\end{aligned}
$$

D Matrix representation of quantum oscillators

Exercise 22

$$
\begin{aligned}
\sum_n\left|n\right\rangle\left\langle n\right| &= \begin{pmatrix}1 & 0 & 0 & \cdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}+\begin{pmatrix}0 & 1 & 0 & \cdots\end{pmatrix}\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix} \\
&= \begin{pmatrix}1 & 0 & 0 & \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \\ & \vdots & & \ddots\end{pmatrix}+\begin{pmatrix}0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ & \vdots & & \ddots\end{pmatrix} = \hat{I}
\end{aligned}
$$

Exercise 23

$$
\begin{aligned}
\hat{x} &= x_0(\hat{a}^\dagger+\hat{a}) \\
&= x_0\left[\begin{pmatrix}0&0&0&0&0\\ \sqrt{1}&0&0&0&0\\ 0&\sqrt{2}&0&0&0\\ 0&0&\sqrt{3}&0&0\\ 0&0&0&\sqrt{4}&0\end{pmatrix}+\begin{pmatrix}0&\sqrt{1}&0&0&0\\ 0&0&\sqrt{2}&0&0\\ 0&0&0&\sqrt{3}&0\\ 0&0&0&0&\sqrt{4}\\ 0&0&0&0&0\end{pmatrix}\right] \\
&= x_0\begin{pmatrix}0&\sqrt{1}&0&0&0\\ \sqrt{1}&0&\sqrt{2}&0&0\\ 0&\sqrt{2}&0&\sqrt{3}&0\\ 0&0&\sqrt{3}&0&\sqrt{4}\\ 0&0&0&\sqrt{4}&0\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\hat{p} &= -ip_0(\hat{a}^\dagger-\hat{a}) \\
&= x_0\left[\begin{pmatrix}0&0&0&0&0\\ \sqrt{1}&0&0&0&0\\ 0&\sqrt{2}&0&0&0\\ 0&0&\sqrt{3}&0&0\\ 0&0&0&\sqrt{4}&0\end{pmatrix}-\begin{pmatrix}0&\sqrt{1}&0&0&0\\ 0&0&\sqrt{2}&0&0\\ 0&0&0&\sqrt{3}&0\\ 0&0&0&0&\sqrt{4}\\ 0&0&0&0&0\end{pmatrix}\right] \\
&= -ip_0\begin{pmatrix}0&-\sqrt{1}&0&0&0\\ \sqrt{1}&0&-\sqrt{2}&0&0\\ 0&\sqrt{2}&0&-\sqrt{3}&0\\ 0&0&\sqrt{3}&0&-\sqrt{4}\\ 0&0&0&\sqrt{4}&0\end{pmatrix}
\end{aligned}
$$

Exercise 24

$$
\begin{aligned}
\hat{a}^\dagger\hat{a} &= \begin{pmatrix}0&0&0&0&0\\ \sqrt{1}&0&0&0&0\\ 0&\sqrt{2}&0&0&0\\ 0&0&\sqrt{3}&0&0\\ 0&0&0&\sqrt{4}&0\end{pmatrix}\begin{pmatrix}0&\sqrt{1}&0&0&0\\ 0&0&\sqrt{2}&0&0\\ 0&0&0&\sqrt{3}&0\\ 0&0&0&0&\sqrt{4}\\ 0&0&0&0&0\end{pmatrix} \\
&= \begin{pmatrix}0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&2&0&0\\ 0&0&0&3&0\\ 0&0&0&0&4\end{pmatrix} \\
\hat{a}\hat{a}^\dagger &= \begin{pmatrix}0&\sqrt{1}&0&0&0\\ 0&0&\sqrt{2}&0&0\\ 0&0&0&\sqrt{3}&0\\ 0&0&0&0&\sqrt{4}\\ 0&0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0&0\\ \sqrt{1}&0&0&0&0\\ 0&\sqrt{2}&0&0&0\\ 0&0&\sqrt{3}&0&0\\ 0&0&0&\sqrt{4}&0\end{pmatrix} \\
&= \begin{pmatrix}1&0&0&0&0\\ 0&2&0&0&0\\ 0&0&3&0&0\\ 0&0&0&4&0\\ 0&0&0&0&0\end{pmatrix}
\end{aligned}
$$

$$
\hat{a}^\dagger \neq \hat{a}\hat{a}^\dagger
$$

Exercise 25

$$
\begin{aligned}
[\hat{a}, \hat{a}^\dagger] &= \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} \\
&= \begin{pmatrix} 1&0&0&0&0 \\ 0&2&0&0&0 \\ 0&0&3&0&0 \\ 0&0&0&4&0 \\ 0&0&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&2&0&0 \\ 0&0&0&3&0 \\ 0&0&0&0&4 \end{pmatrix} \\
&= \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&-4 \end{pmatrix}
\end{aligned}
$$

It's almost $\hat{I}$, but with the last element on the diagonal being $N_{\max}+1$ too small. We can fix

$$[\hat{a}, \hat{a}^\dagger] + (N_{\max}+1)\,|N_{\max}\rangle\langle N_{\max}| = \hat{I}$$

E Wavefunctions

Exercise 26

$$
\underline{\Psi_1(x)} = \underline{\frac{x}{x_0}\Psi_0(x)} - 0\cdot\Psi_{-1}(x)
$$

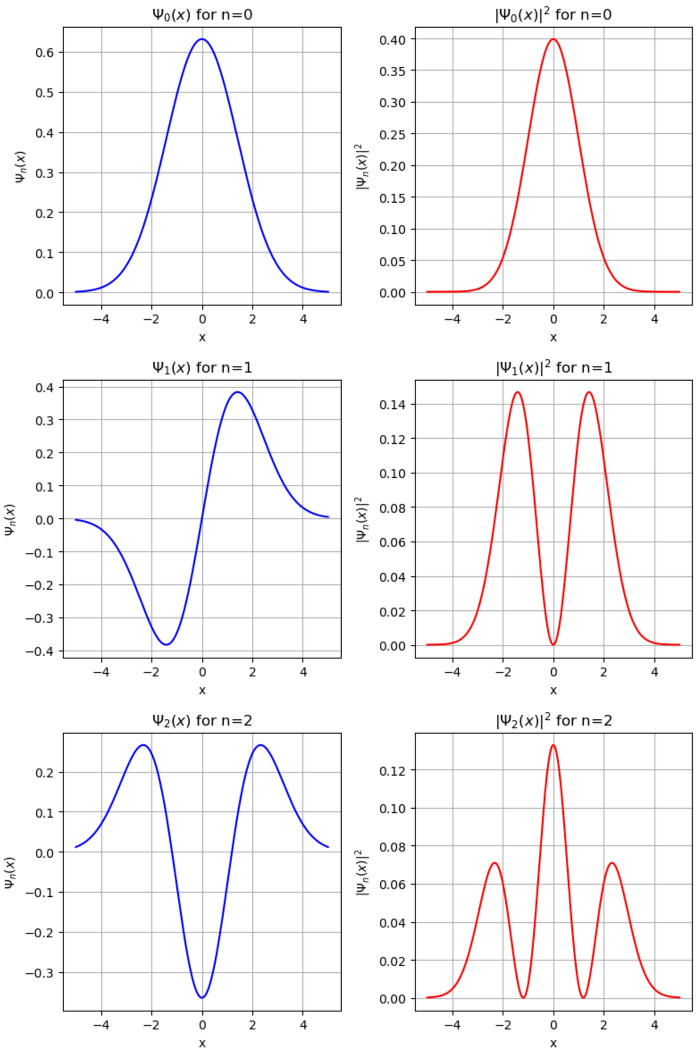
$$
\begin{aligned}
\underline{\Psi_2(x)} &= \frac{1}{\sqrt{2}}\left(\frac{x}{x_0}\Psi_1 - \Psi_0(x)\right) \\
&= \frac{1}{\sqrt{2}}\left(\frac{x}{x_0}\left[\frac{x}{x_0}\Psi_0(x)\right] - \Psi_0(x)\right) \\
&= \underline{\frac{1}{\sqrt{2}}\left[\left(\frac{x}{x_0}\right)^2 - 1\right]\Psi_0(x)}
\end{aligned}
$$

$$
\begin{aligned}
\underline{\Psi_3(x)} &= \frac{1}{\sqrt{3}}\left[\frac{x}{x_0}\Psi_2(x) - \sqrt{2}\Psi_1(x)\right] \\
&= \frac{1}{\sqrt{3}}\left[\frac{x}{x_0}\frac{1}{\sqrt{2}}\left[\left(\frac{x}{x_0}\right)^2 - 1\right]\Psi_0(x) - \sqrt{2}\frac{x}{x_0}\Psi_0(x)\right] \\
&= \frac{1}{\sqrt{3}}\frac{x}{x_0}\left[\frac{1}{\sqrt{2}}\left[\left(\frac{x}{x_0}\right)^2 - 1\right] - \sqrt{2}\right]\Psi_0(x) \\
&= \underline{\frac{1}{\sqrt{6}}\frac{x}{x_0}\left[\left(\frac{x}{x_0}\right)^2 - 1 - 2\right]\Psi_0(x)}
\end{aligned}
$$

Exercise 27

$$
\begin{aligned}
\underline{x\Psi_0(x) + 2x_0^2\frac{\partial\Psi_0(x)}{\partial x}} &= x\Psi_0(x) + 2x\frac{-x}{2x_0^2}\underbrace{\frac{1}{\sqrt[4]{2\pi x_0^2}}e^{-\left(\frac{x}{2x_0^2}\right)^2}}_{=\Psi_0(x)} \\
&= x\Psi_0(x)\left(1 + \underbrace{2x_0^2\frac{-x}{2x_0^2}}_{-1}\right) \\
&= \underline{0}
\end{aligned}
$$

Exercise 28



Node count :

n	# of 0s
0	0
1	1
2	2

I'm not suprised, because given the differential equation 62 and the recursion nature of the waves, I would have expect the extremum of a function to be the 0s of the following function.

Exercise 29

After numercial integration, the results were

$$\int_{-\infty}^{+\infty} |\Psi_0|^2 \, \mathrm{d}x = 1$$

$$\int_{-\infty}^{+\infty} |\Psi_1|^2 \, \mathrm{d}x = 1$$

$$\int_{-\infty}^{+\infty} |\Psi_2|^2 \, \mathrm{d}x = 1$$

Indeed, there are nomalized

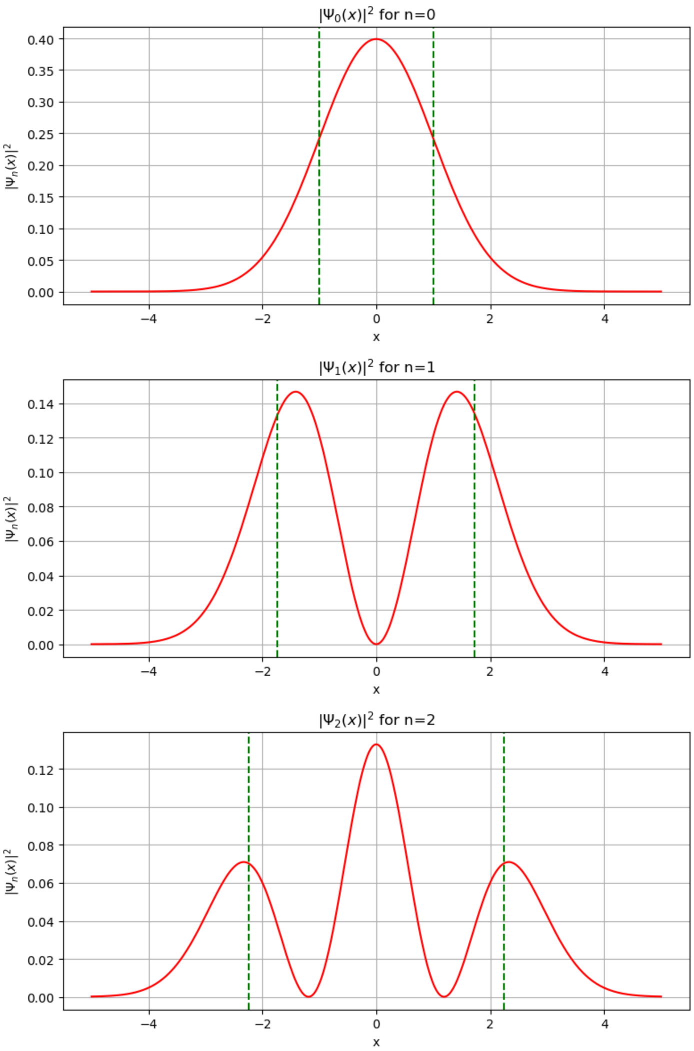
Exercise 30

After numercial integration, the results were

$$\int_{-\infty}^{+\infty} \Psi_0 \Psi_1 \, \mathrm{d}x = 0$$
$$\int_{-\infty}^{+\infty} \Psi_1 \Psi_2 \, \mathrm{d}x = 0$$
$$\int_{-\infty}^{+\infty} \Psi_2 \Psi_0 \, \mathrm{d}x = 0$$

Indeed, there are orthogonal

Exercise 31

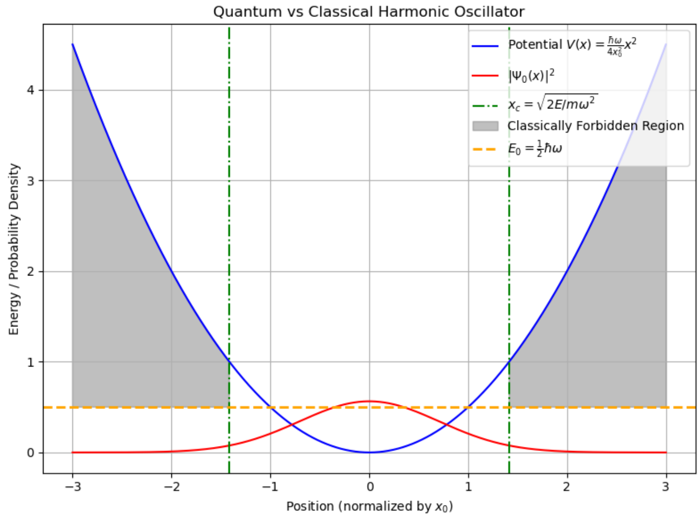


With the probabilities

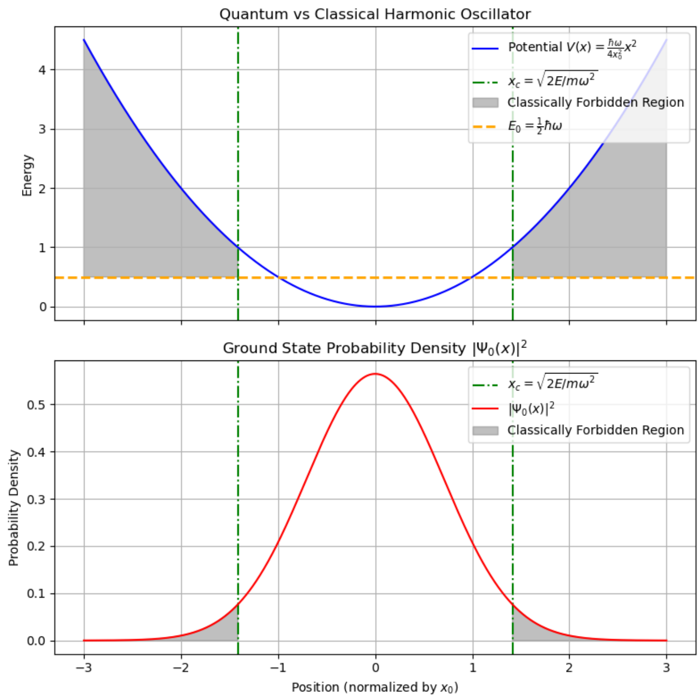
n	$P\left(\lvert x \rvert < x_{\mathrm{RMS}}\right)$
0	0.6807
1	0.3028
2	0.2253

F Discovering quantum mechanics with oscillator wavefunctions

Exercise 32

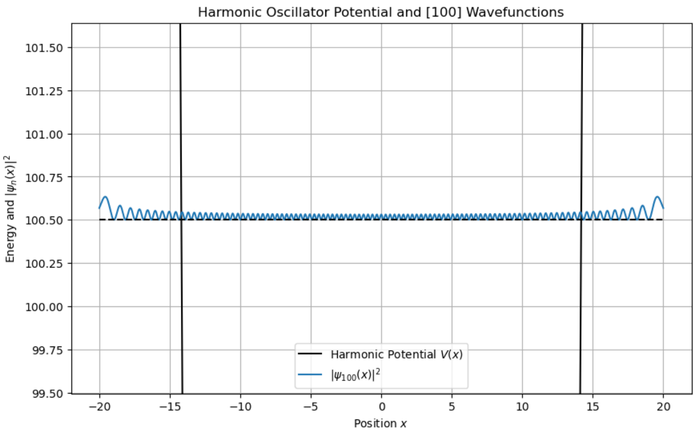


Exercise 33



The region where $|x| > x_c$ is the gray region on the plots.

Exercise 34



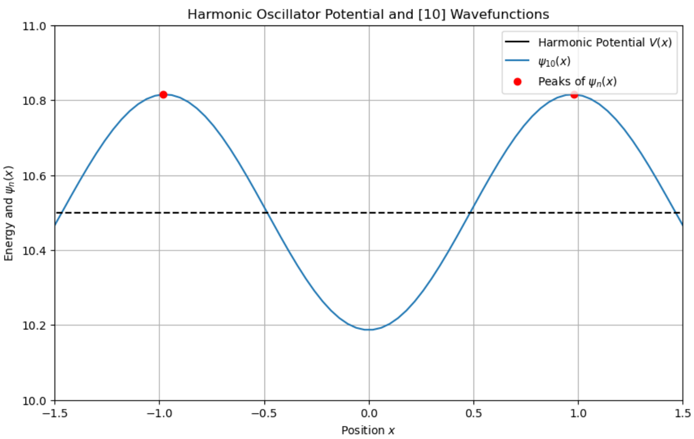
No, because at its limit, the oscillaor is slower, hence more time has passed in this region

Exercise 34'

The cas mass is 4kg, given catster.com, and it's speed is $15\frac{\mathrm{m}}{\mathrm{s}}$, rounded from cats.com. Then,

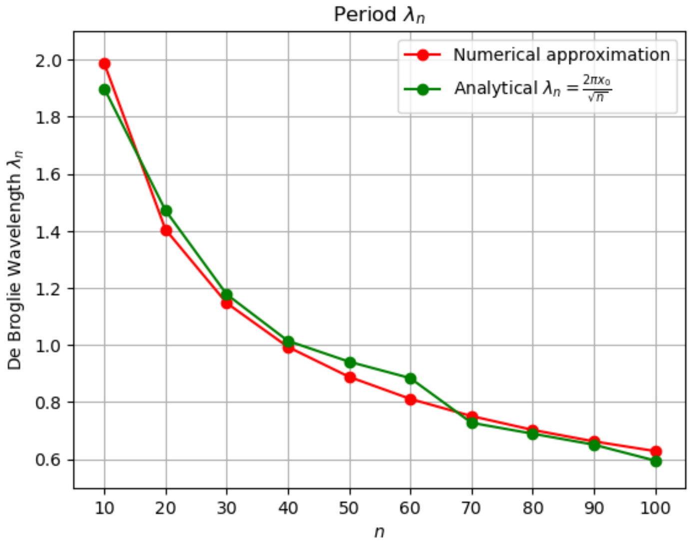
$$\lambda_{\text{cat}} = \frac{\hbar}{mv} = \frac{6.626 \cdot 10^{-34}}{4 \cdot 15} = 1.104 \cdot 10^{-35}$$

Exercise 35



Period is 2, which compared to $\frac{2\pi x_0}{\sqrt{10+\frac{1}{2}}} = 19.3$ is not a bad approximation.

Exercise 36



Exercise 37

TODO

Exercise 38

TODO

Exercise 39

TODO