Homework 0 (self-study warmup assignment)

Version as of Sept 13, 2025, a few typos fixed

Due date for HW0 is **Fri, Sept 19, 20:00.**

Rules

[1] File Format: Submit only one PDF file. The filename must follow the format: Firstname_Lastname_homework0.pdf.

[2] Submission Method: Using LaTeX or a handwriting app on an iPad is strongly recommended. If you want to submit handwritten paperwork, it must be scanned by a printer and saved as a PDF pictures converted to PDF are not acceptable.

[3]Answer Order: Answer the questions in the same order they are listed. Do not change the order or mix them up.

[4] Highlighting Answers: Underline each final answer clearly.

[5] QuTiP Outputs: When using QuTiP, include only a screenshot of the final result. Do not include the entire code.

[6] Legibility of Handwritten Figures: Handwritten figures must be clear and easy to read. Illegible figures will result in a loss of points.

[7] Deadline: Late submissions will not be accepted under any circumstances.

Each exercise is worth 1 point.
Extra point for each error reported on the forum (minus trivial typos).

Classical Harmonic Oscillator

A harmonic oscillator (HO) is probably the most commonly encountered mechanical system. Think of a pendulum, or a mass on a spring, or planetary motion, or even electromagnetic fields confined between two mirrors. It is a good idea to familiarize yourself with the basic mathematics of HO and use such a system as a reference classical object for our Quantum course. We will study the quantum mechanics of harmonic oscillators in detail later in our course.

HO evolution equations in the matrix form

Classical HO (let's imagine a mass m on a spring) is defined by its Hamiltonian function $H(x,p) = p^2/2m + kx^2/2$. Here x, p is the pair of position and momentum variables, respectively, m is the mass and k is the spring stiffness (the larger the k, the harder it is to stretch the spring). Think of the Hamiltonian function as a rigorous way to define a mechanical system. Every time someone describes to you a mechanical system just ask what's the Hamiltonian function and you don't need to know more. Just like every gadget comes with a manual, every mechanical system comes with its Hamiltonian function.

The Hamiltonian equations of motion are:

$$\begin{aligned} \dot{x} &= \partial H/\partial p = p/m \\ \dot{p} &= -\partial H/\partial x = -kx \end{aligned} \tag{1}$$

Solving the differential equations Eqs. (1) and using initial conditions $x(t=0) = x_0$ and $p(t=0) = p_0$ provides us with the values of $x(t), p(t)$ at any given time t. Including $t < 0$! Knowing the Hamiltonian function as well as the values of x and p at any time moment tells you everything about the past and the future of a mechanical system.

A common way to solve Eqs. (1) is to eliminate p and obtain a single 2nd-order equation $\ddot{x} + \omega^2 x = 0$, where $\omega = \sqrt{k/m}$ turns out to be the natural (free) vibration frequency for the oscillator. Here we consider a different route, more appropriate for the rest of the course material. Let us start by rescaling x and p such that they have the same units. This way we would be allowed to add or subtract these variables! After checking the units of t, x, p, m, k, ω one can verify that the new variables $X = x\sqrt{m\omega}$ and $P = p/\sqrt{m\omega}$ would indeed have the same units.

Exercise 1.1 What are the units of X and P?

Now we can consider X and P on equal footing. We can define a column vector $\vec{s} = (X, P)^T$ (here ^{T} means transpose). We can also define a matrix $\hat{\Omega} = \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and rewrite Eqs. (1) as

$$\dot{\vec{s}} = \hat{\Omega}\vec{s} \tag{2}$$

The solution of Eq. (2) is given by

$$\vec{s}(t) = \exp(\hat{\Omega}t)\vec{s}(t=0). \tag{3}$$

Here $\exp(\hat{\Omega}t)$ is also a 2x2 matrix defined by the procedure of matrix exponentiation, using the power series of the exponential function $\exp(x) = 1 + x + x^2/2! + x^3/3! + ...$:

$$\exp(\hat{\Omega}t) = \hat{I} + \hat{\Omega}t + \hat{\Omega}^2 t^2/2! + \hat{\Omega}^3 t^3/3! + \hat{\Omega}^4 t^4/4! + ... \ , \tag{4}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called identity matrix.

Exercise 1.2 Check by explicit multiplication that $\hat{I}^2 = \hat{I}$ and $\hat{\Omega}^2 = -\omega^2 \hat{I}$

Exercise 1.3 Check that expression (3) is indeed the solution of Eq. (2) by taking the time-derivative of $\vec{s}(t)$ and the above definition of the matrix exponentiation.

Oscillator motion in real space is a rotation in the (X, P)-space

Taking into account the results of Exercise 1.2 we obtain a neat expression for the $\exp\hat{\Omega}t$ matrix:

$$\exp(\hat{\Omega}t) = \hat{I}\cos\omega t + \frac{\hat{\Omega}}{\omega}\sin\omega t = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \tag{5}$$

And hence the final solution for the oscillator equations is $\vec{s}(t) = (X(t), P(t))^T$, where $X(t) = X_0 \cos\omega t + P_0 \sin\omega t$, $P(t) = -X_0 \sin\omega t + P_0 \cos\omega t$.

Exercise 1.4 Prove the above formula for $\exp\hat{\Omega}t$. Hint: use the expansions $\cos\alpha = 1 - \alpha^2/2! + \alpha^4/4! - \alpha^6/6! + ...$ and $\sin\alpha = \alpha - \alpha^3/3! + \alpha^5/5! - ...$ and directly multiply the matrices in each term of the series.

Exercise 1.5 Show that the matrix $\exp\hat{\Omega}t$ corresponds to a rotation of a vector in the (X, P)-plane by an angle ωt, clockwise. Hint: set the value of ωt to something convenient, like $\pi/4$ and apply the matrix $\exp\hat{\Omega}t$ to simple vectors such as $(X_0, 0)^T$ or $(0, P_0)^T$.

So (assuming you did the above exercises), we observe that the time-evolution of a HO after a time t is a rotation of the initial state vector (X_0, P_0) by an angle ωt. One can say this is a rotation at a constant rate, given by $\omega/2\pi$, that is the period of rotation is $2\pi/\omega$. No matter at which point in the (X, P)-plane we start, we will come back to this point in time $T = 2\pi/\omega$ along a circular trajectory. For example, for the simple initial condition $P_0 = 0$, we get the familiar oscillatory in time solution for the evolution of the position $X(t) = X_0 \cos\omega t$. The property that the rate of rotation (the oscillator's period) does not depend on initial conditions is a special one, it's called linearity.

Exercise 1.6 Speculate what might be a useful application of the fact that HO's period does not depend on initial conditions. Think watchmaking industry.

Exercise 1.7 Find a matrix $(\exp\hat{\Omega}t)^{-1}$ which is defined as the inverse of $\exp\hat{\Omega}t$. Hint: if the original matrix is a rotation by ωt, what operation would undo that rotation? Verify by performing an explicit matrix multiplication $\exp\hat{\Omega}t(\exp\hat{\Omega}t)^{-1}$.

Exercise 1.8 Show by explicit calculation using Eq. 5 that the quantity $X(t)^2 + P(t)^2 = X_0^2 + P_0^2$, that is, the value of the Hamiltonian function – basically the total energy of the system – does not change in time.

Let us consider one more trick with rotations, this time using complex numbers. We define a complex number $Z = X + iP$. It's real part is X and imaginary part is P, so we can visualize Z as a vector in the (X, P)-plane, just like we did with the state vector $\vec{s}$. How can we write down an operation that rotates this vector by an angle ωt? Note that any complex number can be written as $Z = |Z| \exp i\alpha$, where α is the angle of the vector with the X-axis. Therefore, to rotate this vector by an angle ωt clockwise, we just need to multiply by $\exp(-i\omega t)$. So let us now similarly introduce the initial state of the oscillator $Z_0 = X_0 + iP_0$ and then the solution for the motion of HO is most compactly written as

$$Z(t) = Z_0 \exp(-i\omega t) \tag{6}$$

Again, what we mean by this equation is that $X(t) + iP(t) = (X_0 + iP_0) \times \exp(-i\omega t)$. Now $X(t) = \mathrm{Re}[(X_0 + iP_0) \times \exp(-i\omega t)]$ and $P(t) = \mathrm{Im}[(X_0 + iP_0) \times \exp(-i\omega t)]$.

Exercise 1.9 Verify that $X(t)$ and $P(t)$ obtained using the complex number formulation agree with the matrix formulation solution.

Periodically-driven oscillator: resonance

Next we consider the effect of an external periodic force. We usually say let's "drive" the oscillator. The drive is modeled by modifying the Hamiltonian function as $H \to H - xF(t)$.

Now it's the total energy of the oscillator plus work produced by the external force.

Exercise 1.10 show that the Hamiltonian equations of motion in the presence of drive, in the matrix form, are

$$\dot{\vec{s}} = \hat{\Omega}\vec{s} + \vec{F}(t), \tag{7}$$

where $\vec{F} = (0, f(t))$, where $f(t) = F(t)/\sqrt{m\omega}$

How do we solve the new equations of motion? The trick is the following. Suppose we have a scalar equation, $\dot{s} = \Omega s + F(t)$. In this case, we first take a solution for $F = 0$, which is $s(t) = s_0 \exp(\Omega t)$. Next we take the "initial value" s_0 to be also a function of time, that is $s(t) = s_0(t)\exp(\Omega t)$ and try it as a solution in the case $F \neq 0$.

Exercise 1.11 Show by a direct substituion that (the case of the scalar equation) $\dot{s_0} = \exp(-\Omega t)F(t)$, and hence $s_0(t) = \int^t \exp(-\Omega t')F(t')dt' + C$ and $s(t) = \exp(\Omega t)\big(C + \int^t \exp(-\Omega t')F(t')dt\big)$. The constant C defines the initial conditions.

Applying the above reasoning to the matrix form equations, we can generalize the above expression to the matrix form equation for the driven HO:

$$\vec{s}(t) = \exp(\hat{\Omega}t)\big(\vec{C} + \int^t \exp(-\hat{\Omega}t')\vec{F}(t')dt'\big) \tag{8}$$

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The most common driving case is when $f(t) = f_0 \cos\omega_d t$, that is the drive signal contains only one frequency, ω_d. Let us resort again to complex numbers to simplify the calculations by rewriting the expression above by analogy with Eq. (6):

$$Z = X + iP = \exp(-i\omega t)\big(C + \int_0^t \exp(i\omega t')(0 + if_0\cos\omega_d t')dt'\big) \tag{9}$$

Now let us consider separately two cases: $\omega_d \neq \omega$ and $\omega_d = \omega$. In the first case, $\omega_d \neq \omega$, we proceed dully with the integration: $Z = \exp(-i\omega t)\big(C + (if_0/2)\int_0^t \exp(i\omega t')(\exp(i\omega_d t') + \exp(-i\omega_d t'))dt'\big)$. Remember, C is now the complex number specifying the initial conditions $C = X_0 + iP_0$. Keep working this out, we get

$$Z = C\exp(-i\omega t) + \exp(i\omega_d t)(f_0/2)/(\omega_d + \omega) + \exp(-i\omega_d t)(f_0/2)/(\omega - \omega_d) \tag{10}$$

The first term corresponds to free oscillations (rotations in the X-P plane) at the oscillator's natural frequency ω. This is a result of it's initial displacement at some point in time, set by constant C. The other terms are more interesting as they are caused entirely by the external periodic force. Let us ignore the effect of the free oscillations by setting $C = 0$ (the oscillator is initially at rest and the spring is not charged):

$$Z = X + iP = \exp(i\omega_d t)(f_0/2)/(\omega_d + \omega) + \exp(-i\omega_d t)(f_0/2)/(\omega - \omega_d) \tag{11}$$

We can see that now the solution for the state vector of the driven oscillator consists of two components. The first one is a vector of length $f_0/(2(\omega_d + \omega))$ rotating counterclockwise at a rate ω_d and a second vector of larger length $f_0/(2|\omega - \omega_d|)$, rotating clockwise also at a rate ω_d. Near the resonance condition, $|\omega - \omega_d| \ll \omega$ we can neglect the counterclockwise rotating vector as it's length is much smaller than that of the clockwise rotating vector. Therefore to a good approximation, the solution near the resonance condition is

$$X + iP = \frac{f_0}{2(\omega - \omega_d)} \exp(-i\omega_d t) \tag{12}$$

The difference with the free oscillations case (Eq. 6) is that now HO oscillates at the drive frequency but with an amplitude which gets the larger the closer the drive frequency is to ω. This is the phenomenon of resonance: when an oscillator is driven at a frequency close to its free oscillations frequency, it becomes very efficient at taking the energy from the drive and oscillate at a larger amplitude.

Now, what happens on resonance, when $\omega_d = \omega$? In this case we get a division by zero in Eq. (12), which means we should have been a bit more careful in working out the integral. Let's go back to it, set $\omega_d = \omega$ and it looks that now we get $Z = \exp(-i\omega t)\big(C + (if_0/2)\int^t (\exp(2i\omega t') + 1)dt'\big)$. Plugging in $\omega_d = \omega$ we get

$$Z(t) = X + iP = i(f_0/2)t\exp(-i\omega t) + i(f_0/2\omega)\sin(\omega t), \tag{13}$$

from which we further obtain:

$$\begin{aligned} X(t) &= (f_0/2)t\sin(\omega t) \\ P(t) &= (f_0/2)t\cos(\omega t) + (f_0/2\omega)\sin(\omega t) \end{aligned} \tag{14}$$

Exercise 1.12 Plot $X(t)$ as well as a t-parametric plot in the $X - P$ plane, starting from $t = 0$ to $t = 17 \times 2\pi/\omega$. Assume that at $t = 0$, $X = 0$, and $P = 0$. Describe in words what happens with a an oscillator driven exactly on resonance. After you are done, look up online and compare with your result.

The solutions for the equations of a driven harmonic oscillator at $\omega_d = \omega$ no longer describe a rotation of the state vector at a rate ω, because the value of $X^2 + P^2$ now increases in time as t^2. A resonant drive causes the oscillator to accept the energy from the drive without limits. By contrast, when we are slightly off-resonant, $\omega_d - \omega \neq 0$, there is a limit to this process, and the oscillation amplitude for X and P is given by Eq. (12).