Homework 3

A. Vector space for multiple qubits. Tensor product.

Exercise 1:
Verify by examining all the relevant inner products of four-component column and row vectors, that states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ form an orthonormal set.

We check that the inner products of the vectors are null:

$$\begin{aligned}\langle 00|01\rangle &= (\langle 0_A|0_A\rangle)(\langle 0_B|1_B\rangle) \\ &= 1 \cdot 0 \\ &= 0\end{aligned}$$

$$\langle 00|10\rangle = 0$$

$$\langle 00|11\rangle = 0$$

$$\langle 01|10\rangle = 0$$

and so on

We also check that they're normalized:

$$\begin{aligned}\langle 00|00\rangle &= (\langle 0_A|0_A\rangle)(\langle 0_B|0_B\rangle) \\ &= 1 \cdot 1 \\ &= 1\end{aligned}$$

$$\langle 10|10\rangle = 1$$

$$\langle 01|01\rangle = 1$$

$$\langle 11|11\rangle = 1$$

Answer 1:
The inner products of same vectors are equal to 1 (normalized), and of different vectors are equal to 0 (orthogonal). The vectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ are orthonormal.

Exercise 2:
Use both Dirac and matrix notations to apply the operator $\hat{X} \otimes \hat{I}$ to remaining three basis states $|01\rangle, |10\rangle, |11\rangle$. Which method do you like better?

In Dirac notation:

$$\begin{aligned}&\hat{X}_A|01\rangle \\ &= \hat{X}_A|0_A\rangle \otimes |1_B\rangle \\ &= |1_A\rangle \otimes |1_B\rangle \\ &= |11\rangle\end{aligned}$$

$$\begin{aligned}&\hat{X}_A|10\rangle \\ &= \hat{X}_A|1_A\rangle \otimes |0_B\rangle \\ &= |0_A\rangle \otimes |0_B\rangle \\ &= |00\rangle\end{aligned}$$

$$\begin{aligned}&\hat{X}_A|11\rangle \\ &= \hat{X}_A|1_A\rangle \otimes |1_B\rangle \\ &= |0_A\rangle \otimes |1_B\rangle \\ &= |01\rangle\end{aligned}$$

In matrix notation:

$$\hat{X}_A|01\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle$$

$$\hat{X}_A|10\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle$$

$$\hat{X}_A|11\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$$

Answer 2:
We get $\hat{X}_A|01\rangle = |11\rangle, \hat{X}_A|10\rangle = |00\rangle, \hat{X}_A|11\rangle = |01\rangle$. The Dirac notation is way better.

Exercise 3:
Use both Dirac and matrix notations to apply the operator $\hat{I} \otimes \hat{X}$ to remaining three basis states $|01\rangle, |10\rangle, |11\rangle$.

In Dirac notation:

$$\begin{aligned} &\hat{X}_B|01\rangle \\ &= \hat{X}_B|0_A\rangle \otimes |1_B\rangle \\ &= |0_A\rangle \otimes |0_B\rangle \\ &= |00\rangle \end{aligned} \qquad \begin{aligned} &\hat{X}_B|10\rangle \\ &= \hat{X}_B|1_A\rangle \otimes |0_B\rangle \\ &= |1_A\rangle \otimes |1_B\rangle \\ &= |11\rangle \end{aligned} \qquad \begin{aligned} &\hat{X}_B|11\rangle \\ &= \hat{X}_B|1_A\rangle \otimes |1_B\rangle \\ &= |1_A\rangle \otimes |0_B\rangle \\ &= |10\rangle \end{aligned}$$

In matrix notation:

$$\begin{aligned} \hat{X}_B|01\rangle &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle \end{aligned} \qquad \begin{aligned} \hat{X}_B|10\rangle &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle \end{aligned} \qquad \begin{aligned} \hat{X}_B|11\rangle &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle \end{aligned}$$

Answer 3:
We get $\hat{X}_B|01\rangle = |00\rangle, \hat{X}_B|10\rangle = |11\rangle, \hat{X}_B|11\rangle = |10\rangle$.

Exercise 4:
Show that the regular matrix product of $\hat{X} \otimes \hat{I}$ and $\hat{I} \otimes \hat{X}$ equals to the matrix given by a tensor product $\hat{X} \otimes \hat{X} = \begin{pmatrix} 0\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} & 1\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ 1\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} & 0\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \end{pmatrix} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$

We have:

$$\hat{X} \otimes \hat{I} = \hat{X}_A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \hat{I} \otimes \hat{X} = \hat{X}_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

So we get:

$$\left(\hat{X} \otimes \hat{I}\right)\left(\hat{I} \otimes \hat{X}\right) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
$$= \hat{X} \otimes \hat{X}$$

Answer 4:
See above.

Exercise 5:
Apply operator $\hat{X}_A \hat{X}_B$ to all four computational basis states. Show that it exchanges the states $|01\rangle$ and $|10\rangle$.

We have:

$$\begin{aligned} &\hat{X}_A \hat{X}_B |00\rangle \\ &= \hat{X}_A |0_A\rangle \otimes \hat{X}_B |0_B\rangle \\ &= |1_A\rangle \otimes |1_B\rangle \\ &= |11\rangle \end{aligned} \qquad \begin{aligned} &\hat{X}_A \hat{X}_B |01\rangle \\ &= \hat{X}_A |0_A\rangle \otimes \hat{X}_B |1_B\rangle \\ &= |1_A\rangle \otimes |0_B\rangle \\ &= |10\rangle \end{aligned} \qquad \begin{aligned} &\hat{X}_A \hat{X}_B |10\rangle \\ &= \hat{X}_A |1_A\rangle \otimes \hat{X}_B |0_B\rangle \\ &= |0_A\rangle \otimes |1_B\rangle \\ &= |01\rangle \end{aligned} \qquad \begin{aligned} &\hat{X}_A \hat{X}_B |11\rangle \\ &= \hat{X}_A |1_A\rangle \otimes \hat{X}_B |1_B\rangle \\ &= |0_A\rangle \otimes |0_B\rangle \\ &= |00\rangle \end{aligned}$$

Answer 5:
We get $\hat{X}_A \hat{X}_B |00\rangle = |11\rangle, \hat{X}_A \hat{X}_B |01\rangle = |10\rangle, \hat{X}_A \hat{X}_B |10\rangle = |01\rangle, \hat{X}_A \hat{X}_B |11\rangle = |00\rangle$. Specifically, it does exchange states $|01\rangle$ and $|10\rangle$.

Exercise 6:
Find the column vectors corresponding to $|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle$.

The vector for $|{+}{+}\rangle$ is already given. for the others:

$$\begin{aligned} |{+}{-}\rangle &= \frac{1}{\sqrt{2}}(|0_A\rangle + |1_A\rangle) \otimes \frac{1}{\sqrt{2}}(|0_B\rangle - |1_b\rangle) \\ &= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) \\ &= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \end{aligned} \qquad \begin{aligned} |{-}{+}\rangle &= \frac{1}{\sqrt{2}}(|0_A\rangle - |1_A\rangle) \otimes \frac{1}{\sqrt{2}}(|0_B\rangle + |1_b\rangle) \\ &= \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle) \\ &= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}|--\rangle &= \frac{1}{\sqrt{2}}(|0_A\rangle - |1_A\rangle) \otimes \frac{1}{\sqrt{2}}(|0_B\rangle - |1_b\rangle) \\ &= \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) \\ &= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}\end{aligned}$$

Answer 6:
We get $|+-\rangle = \frac{1}{2}(1,-1,1,-1)^{\top}, |-+\rangle = \frac{1}{2}(1,1,-1,-1)^{\top}, |--\rangle = \frac{1}{2}(1,-1,-1,1)$.

Exercise 7:
Show by an explicit calculation that the four column vectors above are indeed the eigenvectors of the matrix $\hat{X} \otimes \hat{X}$.

We have:

$$\begin{aligned}&\hat{X} \otimes \hat{X}|++\rangle \\ &= \frac{1}{2}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = |++\rangle\end{aligned}$$

$$\begin{aligned}&\hat{X} \otimes \hat{X}|-+\rangle \\ &= \frac{1}{2}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = -|-+\rangle\end{aligned}$$

$$\begin{aligned}&\hat{X} \otimes \hat{X}|+-\rangle \\ &= \frac{1}{2}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = -|+-\rangle\end{aligned}$$

$$\begin{aligned}&\hat{X} \otimes \hat{X}|--\rangle \\ &= \frac{1}{2}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = |--\rangle\end{aligned}$$

Answer 7:
We find $\hat{X} \otimes \hat{X}|++\rangle = +|++\rangle, \hat{X} \otimes \hat{X}|+-\rangle = -|+-\rangle, \hat{X} \otimes \hat{X}|-+\rangle = -|-+\rangle, \hat{X} \otimes \hat{X}|--\rangle = +|--\rangle$, so these are indeed eigenvectors of $\hat{X} \otimes \hat{X}$.

Exercise 8:
Find the matrix for the operator $\hat{Z}_A + \hat{Z}_B$. Check that the computational states are the eigenstates. Find the corresponding eigenvalues.

We have:

$$
\begin{aligned}
\hat{Z}_A + \hat{Z}_B &= \hat{Z} \otimes \hat{I} + \hat{I} \otimes \hat{Z} \\
&= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & & & \\ & 0 & & \\ & & 0 & \\ & & & -2 \end{pmatrix}
\end{aligned}
$$

We check that the computational states are eigenstates:

$$
\hat{Z}_A + \hat{Z}_B|00\rangle = 2|00\rangle \qquad \hat{Z}_A + \hat{Z}_B|01\rangle = 0|01\rangle \qquad \hat{Z}_A + \hat{Z}_B|10\rangle = 0|10\rangle \qquad \hat{Z}_A + \hat{Z}_B|11\rangle = -2|11\rangle
$$

Answer 8:
We get $\hat{Z}_A, \hat{Z}_B = \begin{pmatrix} 2 & & & \\ & 0 & & \\ & & 0 & \\ & & & -2 \end{pmatrix}$, with the computational states as eigenstates and $2, 0, 0, -2$ as their respective eigenvalues.

Exercise 9:
Verify that for any two single-qubit states satisfying $\langle \Psi_{A,B} | \Psi_{A,B} \rangle = 1$, a two-qubit state $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$ would also satisfy braket(Psi) = 1.

We have:

$$
\begin{aligned}
\langle \Psi | \Psi \rangle &= (\langle \Psi_A| \otimes \langle \Psi_B|)(|\Psi_A\rangle \otimes |\Psi_B\rangle) \\
&= (\langle \Psi_A | \Psi_A \rangle)(\langle \Psi_B | \Psi_B \rangle) \\
&= 1 \cdot 1 \\
&= 1
\end{aligned}
$$

Answer 9:
See above.

Exercise 10:
Show that $\left(\hat{Z}_A \hat{X}_B\right)(|\Psi_A\rangle \otimes |\Psi_B\rangle) = \left(\hat{Z}|\Psi_A\rangle\right) \otimes \left(\hat{X}|\Psi_B\rangle\right)$. Use matrix notation.

We have:

$$\left(\hat{Z}_A \hat{X}_B\right)(|\Psi_A\rangle \otimes |\Psi_B\rangle) = \left(\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right) \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0\beta_1 \\ \alpha_0\beta_0 \\ -\alpha_1\beta_1 \\ -\alpha_1\beta_0 \end{pmatrix}$$

And:

$$\left(\hat{Z}|\Psi_A\rangle\right) \otimes \left(\hat{X}|\Psi_B\rangle\right) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 \\ -\alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \beta_0 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0\beta_1 \\ \alpha_0\beta_0 \\ -\alpha_1\beta_1 \\ -\alpha_1\beta_0 \end{pmatrix}$$

Answer 10:
See above.

Exercise 11:
Consider an N-qubit product state, defined as a tenssor product of N single-qubit states, $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle \otimes |\Psi_C\rangle \otimes \ldots$ What is the length of the resulting column vector? How many real numbers do we need to describe such an N-qubit state?

A qubit has two elements, so the length of the vector double with each tensor product. For an N states product, we get a vector of length 2^N.

Each of the N single-qubits needs 2 real numbers (accounting for the normalization and the global phase), so we need $2N$ real numbers.

Answer 11:
The vector is 2^N elements long and can be represented using $2N$ real numbers.

Exercise 12:
Consider a general N-qubit state of the form

$$|\Psi\rangle = \psi_{00\ldots0}|00\ldots0\rangle + \psi_{00\ldots1}|00\ldots1\rangle + \ldots$$

The length of the N-qubit column vecotr is the same as the product state. But how many real numbers do we need to describe this state?

Each of the column vectros have 2^N elements. Elements are complex numebrs, so we need 2 real numbers to represent them. In total, this is $2 \cdot 2^N = 2^{N+1}$ real numbers.

We remove from that the global phase an normalization parameters, so we get $2^{N+1} - 2$ real numbers.

> **Answer 12:**
> We would need $2^{N+1} - 2$ real numbers.

> **Exercise 13:**
> Consider a register fo $N = 256$ qubits. How many real numbers would we need to store a product state? A general quantum state? If we use one atom to store one real number, do we have enough in the Universe?

We set $N = 256$. For a product state, we get:

$$2N = 2 \cdot 256 = 512 \text{ real numbers}$$

And for a general quantum state, we get:

$$2^{N+1} - 2 = 2^{257} - 2 \approx 2.3 \cdot 10^{77} \text{ real numbers}$$

Both are lower than 10^{80}, so it would be possible to store such states in the Universe.

> **Answer 13:**
> Respectively, we would need 512 and $2.3 \cdot 10^{77}$ real numbers to represent each state. We can you the 1-atom storage solution for both.

> **Exercise 14:**
> Prove that these states cannot be written as product states:
>
> $$|\Psi_{E1}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$
>
> $$|\Psi_{E2}\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

Let $|\Psi_A\rangle = (\alpha_0, \alpha_1)^\top$ and $|\Psi_B\rangle = (\beta_0, \beta_1)^t$ arbitrary single-qubit states.

Assuming $|\Psi_{E1}\rangle$ can be written as a product state, we get:

$$\left|\Psi_{E_1}\right\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$$

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix}$$

This has no solution, so $\left|\Psi_{E_1}\right\rangle$ cannot be written as a product state. The proof for $\left|\Psi_{E_2}\right\rangle$ is similar.

> **Answer 14:**
> See above.

B. Quantum measurement of composite systems

Exercise 15:
Consider $|\Psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$. What is the probability to measure measure $M = -2$?

The state corresponding to eigenvalue -2 is $|11\rangle$. We have:

$$\begin{aligned}\text{probability} &= |\langle 11|\Psi\rangle|^2 \\ &= |\psi_{11}|^2 \\ &= \frac{1}{4}\end{aligned}$$

Answer 15:
The probability is $\frac{1}{4}$.

Exercise 16:
What is the probability to measure neither $M = 2$ not $M = -2$.

This would mean measuring $M = 0$, which corresponds to eigenstates $|01\rangle$ and $|10\rangle$. We have:

$$\begin{aligned}\text{probability} &= |\langle \Psi_0|\Psi\rangle|^2 \\ &= \left|\frac{1}{\sqrt{|\psi_{01}|^2 + |\psi_{10}|^2}}(\psi_{01}|01\rangle + \psi_{10}|10\rangle)\right|^2 \\ &= |\psi_{01}|^2 + |\psi_{10}|^2 \\ &= \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2}\end{aligned}$$

Answer 16:
The probability is $\frac{1}{2}$.

Exercise 17:
What is the probability to measure $M = 0$?

Exact same calculations as above.

Answer 17:
The probability is $\frac{1}{2}$.

Exercise 18:
How many times do we have to try the measurement on a product state $|+_A\rangle \otimes |+_B\rangle$, on average, in order to obtain an entagled state?

Using the Bernouilli experiment formula with probability $\frac{1}{2}$ of getting $M = 0$:

$$\text{number of tries} = \frac{1}{\frac{1}{2}} = 2$$

Answer 18:
2 times on average.

Exercise 19:
Start with a general state $|\Psi\rangle$ and measure sequentially $\hat{Z}_A$ and then $\hat{Z}_B$. Consider the four possible measurement outcomess for $\left(\hat{Z}_A, \hat{Z}_B\right) \in \{(1,1),(1,-1),(-1,1),(-1,-1)\}$. Describe the states after the first and the second measurement in each of the four cases and calculate the probabilities of each outcome following the described above measurement rule.

We measure $\hat{Z}_A$ first. By cases on the result:

- $Z_A = +1$: probability $|\psi_{00}|^2 + |\psi_{01}|^2$
  The state collapses onto $|0_A\rangle \otimes \frac{\psi_{00}|0_B\rangle + \psi_{01}|1_B\rangle}{\sqrt{|\psi_{00}|^2 + |\psi_{01}|^2}}$
  We get: $|\Psi_A\rangle = |0_A\rangle$ and $|\Psi_B\rangle$ unknown.
  - $Z_B = +1$: probability $\frac{|\psi_{00}|^2}{|\psi_{00}|^2 + |\psi_{01}|^2}$.
    $|\Psi_B\rangle$ collapses onto state $|0_B\rangle$
    We get: final state $|00\rangle$ with probability $|\psi_{00}|^2$.
  - $Z_B = -1$: probability $\frac{|\psi_{01}|^2}{|\psi_{00}|^2 + |\psi_{01}|^2}$.
    $|\Psi_B\rangle$ collapses onto state $|1_B\rangle$
    We get: final state $|01\rangle$ with probability $|\psi_{01}|^2$.
- $Z_A = -1$: probability $|\psi_{10}|^2 + |\psi_{11}|^2$.
  The state collapses onto $|1_A\rangle \otimes \frac{\psi_{10}|0_B\rangle + \psi_{11}|1_B\rangle}{\sqrt{|\psi_{10}|^2 + |\psi_{11}|^2}}$
  We get: $|\Psi_A\rangle = |1_A\rangle$ and $|\Psi_B\rangle$ unknown.
  - $Z_B = +1$: probability $\frac{|\psi_{10}|^2}{|\psi_{10}|^2 + |\psi_{11}|^2}$.
    $|\Psi_B\rangle$ collapses onto state $|0_B\rangle$
    We get: final state $|10\rangle$ with probability $|\psi_{10}|^2$.
  - $Z_B = -1$: probability $\frac{|\psi_{11}|^2}{|\psi_{10}|^2 + |\psi_{11}|^2}$.
    $|\Psi_B\rangle$ collapses onto state $|1_B\rangle$
    We get: final state $|11\rangle$ with probability $|\psi_{11}|^2$.

Answer 19:
The probabilities are respectively $\left|\psi_{ij}\right|^2$.

Exercise 20:
Repeat the previous exercise in reverse order, first measure Z_B and then Z_A. Do you expect anny change in the probability of the four possible outcomes?

The computations are similar to the previous exercise. We get the same results.

Answer 20:
I did not expect any change. Indeed there weren't.

> **Exercise 21:**
> Let us consider a measurement operator $\widehat{M} = \hat{Z}_A \hat{Z}_B$. Suppose we start in a state $|\Psi\rangle = |+_A\rangle \otimes |+_B\rangle$. Describe all possible measurement outcomes.

We have:

$$\begin{aligned}|\Psi\rangle &= \frac{1}{\sqrt{2}}(|0_A\rangle + |1_A\rangle) \otimes \frac{1}{\sqrt{2}}(|0_B\rangle + |1_b\rangle) \\ &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\end{aligned}$$

Then:

- $M = 1$: probabilty
  $|\Psi\rangle$ collapses onto