Homework 3

A. Vector space for multiple qubits. Tensor product.

Exercise 1:
Verify by examining all the relevant inner products of four-component column and row vectors, that states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ form an orthonormal set.

We check that the inner products of the vectors are null:

$$\begin{aligned}\langle 00|01\rangle &= (\langle 0_A|0_A\rangle)(\langle 0_B|1_B\rangle) \\ &= 1 \cdot 0 \\ &= 0\end{aligned}$$

$$\langle 00|10\rangle = 0$$

$$\langle 00|11\rangle = 0$$

$$\langle 01|10\rangle = 0$$

and so on

We also check that they're normalized:

$$\begin{aligned}\langle 00|00\rangle &= (\langle 0_A|0_A\rangle)(\langle 0_B|0_B\rangle) \\ &= 1 \cdot 1 \\ &= 1\end{aligned}$$

$$\langle 10|10\rangle = 1$$

$$\langle 01|01\rangle = 1$$

$$\langle 11|11\rangle = 1$$

Answer 1:
The inner products of same vectors are equal to 1 (normalized), and of different vectors are equal to 0 (orthogonal). The vectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ are orthonormal.

Exercise 2:
Use both Dirac and matrix notations to apply the operator $\hat{X} \otimes \hat{I}$ to remaining three basis states $|01\rangle, |10\rangle, |11\rangle$. Which method do you like better?

In Dirac notation:

$$\begin{aligned}&\hat{X}_A|01\rangle \\ &= \hat{X}_A|0_A\rangle \otimes |1_B\rangle \\ &= |1_A\rangle \otimes |1_B\rangle \\ &= |11\rangle\end{aligned}$$

$$\begin{aligned}&\hat{X}_A|10\rangle \\ &= \hat{X}_A|1_A\rangle \otimes |0_B\rangle \\ &= |0_A\rangle \otimes |0_B\rangle \\ &= |00\rangle\end{aligned}$$

$$\begin{aligned}&\hat{X}_A|11\rangle \\ &= \hat{X}_A|1_A\rangle \otimes |1_B\rangle \\ &= |0_A\rangle \otimes |1_B\rangle \\ &= |01\rangle\end{aligned}$$

In matrix notation:

$$\hat{X}_A|01\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle$$

$$\hat{X}_A|10\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle$$

$$\hat{X}_A|11\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$$

Answer 2:
We get $\hat{X}_A|01\rangle = |11\rangle, \hat{X}_A|10\rangle = |00\rangle, \hat{X}_A|11\rangle = |01\rangle$. The Dirac notation is way better.

Exercise 3:
Use both Dirac and matrix notations to apply the operator $\hat{I} \otimes \hat{X}$ to remaining three basis states $|01\rangle, |10\rangle, |11\rangle$.

In Dirac notation:

$$\begin{aligned} &\hat{X}_B|01\rangle \\ &= \hat{X}_B|0_A\rangle \otimes |1_B\rangle \\ &= |0_A\rangle \otimes |0_B\rangle \\ &= |00\rangle \end{aligned} \qquad \begin{aligned} &\hat{X}_B|10\rangle \\ &= \hat{X}_B|1_A\rangle \otimes |0_B\rangle \\ &= |1_A\rangle \otimes |1_B\rangle \\ &= |11\rangle \end{aligned} \qquad \begin{aligned} &\hat{X}_B|11\rangle \\ &= \hat{X}_B|1_A\rangle \otimes |1_B\rangle \\ &= |1_A\rangle \otimes |0_B\rangle \\ &= |10\rangle \end{aligned}$$

In matrix notation:

$$\begin{aligned} \hat{X}_B|01\rangle &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle \end{aligned} \qquad \begin{aligned} \hat{X}_B|10\rangle &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle \end{aligned} \qquad \begin{aligned} \hat{X}_B|11\rangle &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle \end{aligned}$$

Answer 3:
We get $\hat{X}_B|01\rangle = |00\rangle, \hat{X}_B|10\rangle = |11\rangle, \hat{X}_B|11\rangle = |10\rangle$.

Exercise 4:
Show that the regular matrix product of $\hat{X} \otimes \hat{I}$ and $\hat{I} \otimes \hat{X}$ equals to the matrix given by a tensor product $\hat{X} \otimes \hat{X} = \begin{pmatrix} 0\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} & 1\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ 1\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} & 0\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \end{pmatrix} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$

We have:

$$\hat{X} \otimes \hat{I} = \hat{X}_A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \hat{I} \otimes \hat{X} = \hat{X}_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

So we get:

$$\left(\hat{X} \otimes \hat{I}\right)\left(\hat{I} \otimes \hat{X}\right) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
$$= \hat{X} \otimes \hat{X}$$

Answer 4:
See above.

Exercise 5:
Apply operator $\hat{X}_A \hat{X}_B$ to all four computational basis states. Show that it exchanges the states $|01\rangle$ and $|10\rangle$.

We have:

$$\begin{array}{llll} \hat{X}_A \hat{X}_B |00\rangle & \hat{X}_A \hat{X}_B |01\rangle & \hat{X}_A \hat{X}_B |10\rangle & \hat{X}_A \hat{X}_B |11\rangle \\ = \hat{X}_A |0_A\rangle \otimes \hat{X}_B |0_B\rangle & = \hat{X}_A |0_A\rangle \otimes \hat{X}_B |1_B\rangle & = \hat{X}_A |1_A\rangle \otimes \hat{X}_B |0_B\rangle & = \hat{X}_A |1_A\rangle \otimes \hat{X}_B |1_B\rangle \\ = |1_A\rangle \otimes |1_B\rangle & = |1_A\rangle \otimes |0_B\rangle & = |0_A\rangle \otimes |1_B\rangle & = |0_A\rangle \otimes |0_B\rangle \\ = |11\rangle & = |10\rangle & = |01\rangle & = |00\rangle \end{array}$$

Answer 5:
We get $\hat{X}_A \hat{X}_B |00\rangle = |11\rangle, \hat{X}_A \hat{X}_B |01\rangle = |10\rangle, \hat{X}_A \hat{X}_B |10\rangle = |01\rangle, \hat{X}_A \hat{X}_B |11\rangle = |00\rangle$. Specifically, it does exchange states $|01\rangle$ and $|10\rangle$.

Exercise 6:
Find the column vectors corresponding to $|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle$.

The vector for $|{+}{+}\rangle$ is already given. for the others:

$$|{+}{-}\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle + |1_A\rangle) \otimes \frac{1}{\sqrt{2}}(|0_B\rangle - |1_b\rangle)$$
$$= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$
$$= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$|{-}{+}\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle - |1_A\rangle) \otimes \frac{1}{\sqrt{2}}(|0_B\rangle + |1_b\rangle)$$
$$= \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$$
$$= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$
\begin{aligned}
|{-}{-}\rangle &= \frac{1}{\sqrt{2}}(|0_A\rangle - |1_A\rangle) \otimes \frac{1}{\sqrt{2}}(|0_B\rangle - |1_b\rangle) \\
&= \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) \\
&= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}
\end{aligned}
$$

Answer 6:
We get $|{+}{-}\rangle = \frac{1}{2}(1,-1,1,-1)^\mathsf{T}$, $|{-}{+}\rangle = \frac{1}{2}(1,1,-1,-1)^\mathsf{T}$, $|{-}{-}\rangle = \frac{1}{2}(1,-1,-1,1)$.

Exercise 7:
Show by an explicit calculation that the four column vectors above are indeed the eigenvectors of the matrix $\hat{X} \otimes \hat{X}$.

We have:

$$
\begin{aligned}
&\hat{X} \otimes \hat{X}|{+}{+}\rangle \\
&= \frac{1}{2}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = |{+}{+}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\hat{X} \otimes \hat{X}|{-}{+}\rangle \\
&= \frac{1}{2}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = -|{-}{+}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\hat{X} \otimes \hat{X}|{+}{-}\rangle \\
&= \frac{1}{2}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = -|{+}{-}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\hat{X} \otimes \hat{X}|{-}{-}\rangle \\
&= \frac{1}{2}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = |{-}{-}\rangle
\end{aligned}
$$

Answer 7:
We find $\hat{X} \otimes \hat{X}|{+}{+}\rangle = +|{+}{+}\rangle$, $\hat{X} \otimes \hat{X}|{+}{-}\rangle = -|{+}{-}\rangle$, $\hat{X} \otimes \hat{X}|{-}{+}\rangle = -|{-}{+}\rangle$, $\hat{X} \otimes \hat{X}|{-}{-}\rangle = +|{-}{-}\rangle$, so these are indeed eigenvectors of $\hat{X} \otimes \hat{X}$.

Exercise 8:
Find the matrix for the operator $\hat{Z}_A + \hat{Z}_B$. Check that the computational states are the eigenstates. Find the corresponding eigenvalues.

We have:

$$\begin{aligned}\hat{Z}_A + \hat{Z}_B &= \hat{Z} \otimes \hat{I} + \hat{I} \otimes \hat{Z} \\ &= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & & & \\ & 0 & & \\ & & 0 & \\ & & & -2 \end{pmatrix}\end{aligned}$$

We check that the computational states are eigenstates:

$$\hat{Z}_A + \hat{Z}_B|00\rangle = 2|00\rangle \qquad \hat{Z}_A + \hat{Z}_B|01\rangle = 0|01\rangle \qquad \hat{Z}_A + \hat{Z}_B|10\rangle = 0|10\rangle \qquad \hat{Z}_A + \hat{Z}_B|11\rangle = -2|11\rangle$$

Answer 8:
We get $\hat{Z}_A + \hat{Z}_B = \begin{pmatrix} 2 & & & \\ & 0 & & \\ & & 0 & \\ & & & -2 \end{pmatrix}$, with the computational states as eigenstates and $2, 0, 0, -2$ as their respective eigenvalues.

Exercise 9:
Verify that for any two single-qubit states satisfying $\langle \Psi_{A,B} | \Psi_{A,B} \rangle = 1$, a two-qubit state $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$ would also satisfy braket(Psi) = 1.

We have:

$$\begin{aligned}\langle \Psi|\Psi\rangle &= (\langle \Psi_A| \otimes \langle \Psi_B|)(|\Psi_A\rangle \otimes |\Psi_B\rangle) \\ &= (\langle \Psi_A|\Psi_A\rangle)(\langle \Psi_B|\Psi_B\rangle) \\ &= 1 \cdot 1 \\ &= 1\end{aligned}$$

Answer 9:
See above.

Exercise 10:
Show that $\left(\hat{Z}_A \hat{X}_B\right)(|\Psi_A\rangle \otimes |\Psi_B\rangle) = \left(\hat{Z}|\Psi_A\rangle\right) \otimes \left(\hat{X}|\Psi_B\rangle\right)$. Use matrix notation.

We have:

$$\left(\hat{Z}_A\hat{X}_B\right)(|\Psi_A\rangle \otimes |\Psi_B\rangle) = \left(\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right)\begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0\beta_1 \\ \alpha_0\beta_0 \\ -\alpha_1\beta_1 \\ -\alpha_1\beta_0 \end{pmatrix}$$

And:

$$\left(\hat{Z}|\Psi_A\rangle\right) \otimes \left(\hat{X}|\Psi_B\rangle\right) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 \\ -\alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \beta_0 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0\beta_1 \\ \alpha_0\beta_0 \\ -\alpha_1\beta_1 \\ -\alpha_1\beta_0 \end{pmatrix}$$

Answer 10:
See above.

Exercise 11:
Consider an N-qubit product state, defined as a tenssor product of N single-qubit states, $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle \otimes |\Psi_C\rangle \otimes \ldots$ What is the length of the resulting column vector? How many real numbers do we need to describe such an N-qubit state?

A qubit has two elements, so the length of the vector double with each tensor product. For an N states product, we get a vector of length 2^N.

Each of the N single-qubits needs 2 real numbers (accounting for the normalization and the global phase), so we need $2N$ real numbers.

Answer 11:
The vector is 2^N elements long and can be represented using $2N$ real numbers.

Exercise 12:
Consider a general N-qubit state of the form

$$|\Psi\rangle = \psi_{00\ldots0}|00\ldots0\rangle + \psi_{00\ldots1}|00\ldots1\rangle + \ldots$$

The length of the N-qubit column vecotr is the same as the product state. But how many real numbers do we need to describe this state?

Each of the column vectros have 2^N elements. Elements are complex numebrs, so we need 2 real numbers to represent them. In total, this is $2 \cdot 2^N = 2^{N+1}$ real numbers.

We remove from that the global phase an normalization parameters, so we get $2^{N+1} - 2$ real numbers.

> **Answer 12:**
> We would need $2^{N+1} - 2$ real numbers.

> **Exercise 13:**
> Consider a register fo $N = 256$ qubits. How many real numbers would we need to store a product state? A general quantum state? If we use one atom to store one real number, do we have enough in the Universe?

We set $N = 256$. For a product state, we get:

$$2N = 2 \cdot 256 = 512 \text{ real numbers}$$

And for a general quantum state, we get:

$$2^{N+1} - 2 = 2^{257} - 2 \approx 2.3 \cdot 10^{77} \text{ real numbers}$$

Both are lower than 10^{80}, so it would be possible to store such states in the Universe.

> **Answer 13:**
> Respectively, we would need 512 and $2.3 \cdot 10^{77}$ real numbers to represent each state. We can you the 1-atom storage solution for both.

> **Exercise 14:**
> Prove that these states cannot be written as product states:
>
> $$|\Psi_{E1}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$
>
> $$|\Psi_{E2}\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

Let $|\Psi_A\rangle = (\alpha_0, \alpha_1)^{\mathsf{T}}$ and $|\Psi_B\rangle = (\beta_0, \beta_1)^t$ arbitrary single-qubit states.

Assuming $|\Psi_{E1}\rangle$ can be written as a product state, we get:

$$\begin{aligned} \left|\Psi_{E_1}\right\rangle &= |\Psi_A\rangle \otimes |\Psi_B\rangle \\ \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle &= \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix} \end{aligned}$$

This has no solution, so $\left|\Psi_{E_1}\right\rangle$ cannot be written as a product state. The proof for $\left|\Psi_{E_2}\right\rangle$ is similar.

> **Answer 14:**
> See above.

B. Quantum measurement of composite systems

> **Exercise 15:**
> Consider $|\Psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$. What is the probability to measure measure $M = -2$?

The state corresponding to eigenvalue -2 is $|11\rangle$. We have:

$$\begin{aligned} \text{probability} &= |\langle 11|\Psi\rangle|^2 \\ &= |\psi_{11}|^2 \\ &= \frac{1}{4} \end{aligned}$$

> **Answer 15:**
> The probability is $\frac{1}{4}$.

> **Exercise 16:**
> What is the probability to measure neither $M = 2$ not $M = -2$.

This would mean measuring $M = 0$, which corresponds to eigenstates $|01\rangle$ and $|10\rangle$. We have:

$$\begin{aligned} \text{probability} &= |\langle \Psi_0|\Psi\rangle|^2 \\ &= \left| \frac{1}{\sqrt{|\psi_{01}|^2 + |\psi_{10}|^2}} (\psi_{01}|01\rangle + \psi_{10}|10\rangle) \right|^2 \\ &= |\psi_{01}|^2 + |\psi_{10}|^2 \\ &= \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2} \end{aligned}$$

> **Answer 16:**
> The probability is $\frac{1}{2}$.

> **Exercise 17:**
> What is the probability to measure $M = 0$?

Exact same calculations as above.

> **Answer 17:**
> The probability is $\frac{1}{2}$.

> **Exercise 18:**
> How many times do we have to try the measurement on a product state $|+_A\rangle \otimes |+_B\rangle$, on average, in order to obtain an entagled state?

Using the Bernouilli experiment formula with probability $\frac{1}{2}$ of getting $M = 0$:

$$\text{number of tries} = \frac{1}{\frac{1}{2}} = 2$$

Answer 18:
2 times on average.

Exercise 19:
Start with a general state $|\Psi\rangle$ and measure sequentially $\hat{Z}_A$ and then $\hat{Z}_B$. Consider the four possible measurement outcomess for $\left(\hat{Z}_A, \hat{Z}_B\right) \in \{(1,1),(1,-1),(-1,1),(-1,-1)\}$. Describe the states after the first and the second measurement in each of the four cases and calculate the probabilities of each outcome following the described above measurement rule.

We measure $\hat{Z}_A$ first. By cases on the result:

- $Z_A = +1$: probability $|\psi_{00}|^2 + |\psi_{01}|^2$
  The state collapses onto $|0_A\rangle \otimes \frac{\psi_{00}|0_B\rangle + \psi_{01}|1_B\rangle}{\sqrt{|\psi_{00}|^2 + |\psi_{01}|^2}}$
  We get: $|\Psi_A\rangle = |0_A\rangle$ and $|\Psi_B\rangle$ unknown.
  - $Z_B = +1$: probability $\frac{|\psi_{00}|^2}{|\psi_{00}|^2 + |\psi_{01}|^2}$.
    $|\Psi_B\rangle$ collapses onto state $|0_B\rangle$
    We get: final state $|00\rangle$ with probability $|\psi_{00}|^2$.
  - $Z_B = -1$: probability $\frac{|\psi_{01}|^2}{|\psi_{00}|^2 + |\psi_{01}|^2}$.
    $|\Psi_B\rangle$ collapses onto state $|1_B\rangle$
    We get: final state $|01\rangle$ with probability $|\psi_{01}|^2$.
- $Z_A = -1$: probability $|\psi_{10}|^2 + |\psi_{11}|^2$.
  The state collapses onto $|1_A\rangle \otimes \frac{\psi_{10}|0_B\rangle + \psi_{11}|1_B\rangle}{\sqrt{|\psi_{10}|^2 + |\psi_{11}|^2}}$
  We get: $|\Psi_A\rangle = |1_A\rangle$ and $|\Psi_B\rangle$ unknown.
  - $Z_B = +1$: probability $\frac{|\psi_{10}|^2}{|\psi_{10}|^2 + |\psi_{11}|^2}$.
    $|\Psi_B\rangle$ collapses onto state $|0_B\rangle$
    We get: final state $|10\rangle$ with probability $|\psi_{10}|^2$.
  - $Z_B = -1$: probability $\frac{|\psi_{11}|^2}{|\psi_{10}|^2 + |\psi_{11}|^2}$.
    $|\Psi_B\rangle$ collapses onto state $|1_B\rangle$
    We get: final state $|11\rangle$ with probability $|\psi_{11}|^2$.

Answer 19:
The probabilities are respectively $\left|\psi_{ij}\right|^2$.

Exercise 20:
Repeat the previous exercise in reverse order, first measure Z_B and then Z_A. Do you expect anny change in the probability of the four possible outcomes?

The computations are similar to the previous exercise. We get the same results.

Answer 20:
I did not expect any change. Indeed there weren't.

Exercise 21:
Let us consider a measurement operator $\widehat{M} = \hat{Z}_A \hat{Z}_B$. Suppose we start in a state $|\Psi\rangle = |+_A\rangle \otimes |+_B\rangle$. Describe all possible measurement outcomes.

We have:

$$\begin{aligned}|\Psi\rangle &= \frac{1}{\sqrt{2}}(|0_A\rangle + |1_A\rangle) \otimes \frac{1}{\sqrt{2}}(|0_B\rangle + |1_b\rangle) \\ &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\end{aligned}$$

Then:

$$\begin{aligned}|\Psi_{+1}\rangle &= \frac{\frac{1}{2}(|00\rangle + |11\rangle)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} \\ &= \frac{|00\rangle + |11\rangle}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\text{probability} &= |\psi_{00}|^2 + |\psi_{11}|^2 \\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}|\Psi_{-1}\rangle &= \frac{\frac{1}{2}(|01\rangle + |10\rangle)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} \\ &= \frac{|01\rangle + |10\rangle}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\text{probability} &= |\psi_{01}|^2 + |\psi_{10}|^2 \\ &= \frac{1}{2}\end{aligned}$$

Answer 21:
The possible measurement outcomes are ± 1 with states $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$, $\frac{|01\rangle + |10\rangle}{\sqrt{2}}$, and with probability $\frac{1}{2}$ for both.

C. Spooky properties of entagled states

Exercise 22:
Consider a general product state $|\Psi_p\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$. We can choose $|\Psi_A\rangle = \alpha_0 |0_A\rangle + \alpha_1 |1_a\rangle$. Prove that $\left|\langle \Psi_p | \hat{X}_A | \Psi_p \rangle\right|^2 + \left|\langle \Psi_p | \hat{Y}_A | \Psi_p \rangle\right|^2 + \left|\langle \Psi_p | \hat{Z}_A | \Psi_p \rangle\right|^2 = 1$

We have:

$$\begin{aligned}&\left|\langle \Psi_p | \hat{X}_A | \Psi_p \rangle\right|^2 + \left|\langle \Psi_p | \hat{Y}_A | \Psi_p \rangle\right|^2 + \left|\langle \Psi_p | \hat{Z}_A | \Psi_p \rangle\right|^2 \\ &= \left|\langle \Psi_A | \hat{X}_A | \Psi_A \rangle \langle \Psi_B | I | \Psi_B \rangle\right|^2 + \left|\langle \Psi_A | \hat{Y}_A | \Psi_A \rangle \langle \Psi_B | I | \Psi_B \rangle\right|^2 + \left|\langle \Psi_A | \hat{Z}_A | \Psi_A \rangle \langle \Psi_B | I | \Psi_B \rangle\right|^2 \\ &= \left|\langle \Psi_A | \hat{X}_A | \Psi_A \rangle \langle \Psi_B | \Psi_B \rangle\right|^2 + \left|\langle \Psi_A | \hat{Y}_A | \Psi_A \rangle \langle \Psi_B | \Psi_B \rangle\right|^2 + \left|\langle \Psi_A | \hat{Z}_A | \Psi_A \rangle \langle \Psi_B | \Psi_B \rangle\right|^2 \\ &= \left|\langle \Psi_A | \hat{X}_A | \Psi_A \rangle\right|^2 + \left|\langle \Psi_A | \hat{Y}_A | \Psi_A \rangle\right|^2 + \left|\langle \Psi_A | \hat{Z}_A | \Psi_A \rangle\right|^2 \\ &= \sin^2(\theta)\cos^2(\phi) + \sin^2(\theta)\sin^2(\phi) + \cos^2(\theta) \\ &= 1\end{aligned}$$

Answer 22:
See above.

Exercise 23:
Prove that the maximal value of $\langle \Psi_p | \hat{X}_A \hat{X}_B | \Psi_p \rangle + \langle \Psi_p | \hat{Y}_A \hat{Y}_B | \Psi_p \rangle + \langle \Psi_p | \hat{Z}_A \hat{Z}_B | \Psi_p \rangle$ is 1 and the minimal value is -1.

We transform the equation into a dot product:

$$\begin{aligned}
&\langle \Psi_p | \hat{X}_A \hat{X}_B | \Psi_p \rangle + \langle \Psi_p | \hat{Y}_A \hat{Y}_B | \Psi_p \rangle + \langle \Psi_p | \hat{Z}_A \hat{Z}_B | \Psi_p \rangle \\
&= \langle \Psi_A | \hat{X}_A | \Psi_A \rangle \langle \Psi_B | \hat{X}_B | \Psi_B \rangle + \langle \Psi_A | \hat{Y}_A | \Psi_A \rangle \langle \Psi_B | \hat{Y}_B | \Psi_B \rangle + \langle \Psi_A | \hat{Z}_A | \Psi_A \rangle \langle \Psi_B | \hat{Z}_B | \Psi_B \rangle \\
&= \begin{pmatrix} \langle \Psi_A | \hat{X}_A | \Psi_A \rangle \\ \langle \Psi_A | \hat{Y}_A | \Psi_A \rangle \\ \langle \Psi_A | \hat{Z}_A | \Psi_A \rangle \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \langle \Psi_B | \hat{X}_B | \Psi_B \rangle \\ \langle \Psi_B | \hat{Y}_B | \Psi_B \rangle \\ \langle \Psi_B | \hat{Z}_B | \Psi_B \rangle \end{pmatrix}
\end{aligned}$$

Then we use Cauchy-Schwartz:

$$\left| \begin{pmatrix} \langle \Psi_A | \hat{X}_A | \Psi_A \rangle \\ \langle \Psi_A | \hat{Y}_A | \Psi_A \rangle \\ \langle \Psi_A | \hat{Z}_A | \Psi_A \rangle \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \langle \Psi_B | \hat{X}_B | \Psi_B \rangle \\ \langle \Psi_B | \hat{Y}_B | \Psi_B \rangle \\ \langle \Psi_B | \hat{Z}_B | \Psi_B \rangle \end{pmatrix} \right| \leq \left\| \begin{pmatrix} \langle \Psi_A | \hat{X}_A | \Psi_A \rangle \\ \langle \Psi_A | \hat{Y}_A | \Psi_A \rangle \\ \langle \Psi_A | \hat{Z}_A | \Psi_A \rangle \end{pmatrix} \right\| \left\| \begin{pmatrix} \langle \Psi_B | \hat{X}_B | \Psi_B \rangle \\ \langle \Psi_B | \hat{Y}_B | \Psi_B \rangle \\ \langle \Psi_B | \hat{Z}_B | \Psi_B \rangle \end{pmatrix} \right\|$$
$$\leq 1 \text{ since they are unit vectors}$$

So we get:

$$-1 \leq \begin{pmatrix} \langle \Psi_A | \hat{X}_A | \Psi_A \rangle \\ \langle \Psi_A | \hat{Y}_A | \Psi_A \rangle \\ \langle \Psi_A | \hat{Z}_A | \Psi_A \rangle \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \langle \Psi_B | \hat{X}_B | \Psi_B \rangle \\ \langle \Psi_B | \hat{Y}_B | \Psi_B \rangle \\ \langle \Psi_B | \hat{Z}_B | \Psi_B \rangle \end{pmatrix} \leq 1$$

Answer 23:
See above.

Exercise 24:
Show that the four Bell states are indeed orthogonal to each other and each has length 1.

We have:

$$\langle B_0 | B_0 \rangle = \frac{1}{2}(0, 1, -1, 0) \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 1 \qquad \langle B_2 | B_2 \rangle = \frac{1}{2}(1, 0, 0, -1) \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = 1$$

$$\langle B_1 | B_1 \rangle = \frac{1}{2}(0, 1, 1, 0) \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 1 \qquad \langle B_3 | B_3 \rangle = \frac{1}{2}(1, 0, 0, 1) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 1$$

And:

$$\langle B_0|B_1\rangle = \frac{1}{2}(0,1,-1,0)\begin{pmatrix}0\\1\\1\\0\end{pmatrix} = 0 \qquad \langle B_1|B_3\rangle = \frac{1}{2}(0,1,1,0)\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = 0$$

$$\langle B_0|B_2\rangle = \frac{1}{2}(0,1,-1,0)\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} = 0 \qquad \langle B_2|B_3\rangle = \frac{1}{2}(1,0,0,-1)\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = 0$$

... etc ... etc

Answer 24:
We get that $\langle \Psi_i | \Psi_j \rangle = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$, so the Bell states are orthonormal.

Exercise 25:
Suppose we used a measurement of $\hat{Z}_A + \hat{Z}_B$ to create an entangled state $|B_1\rangle$. Verify that:

$$|B_0\rangle = \hat{Z}_A|B_1\rangle$$
$$|B_2\rangle = \hat{X}_A\hat{Z}_A|B_1\rangle$$
$$|B_3\rangle = \hat{X}_A|B_1\rangle$$

We have::

$$\begin{aligned}\hat{Z}_a|B_1\rangle &= \frac{1}{\sqrt{2}}\left(\hat{Z}_A|01\rangle + \hat{Z}_A|10\rangle\right)\\ &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\\ &= |B_0\rangle\end{aligned}$$

$$\begin{aligned}\hat{X}_A|B_1\rangle &= \frac{1}{\sqrt{2}}\left(\hat{X}_A|01\rangle + \hat{X}_A|10\rangle\right)\\ &= \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)\\ &= |B_3\rangle\end{aligned}$$

$$\begin{aligned}\hat{X}_A\hat{Z}_A|B_1\rangle &= \frac{1}{\sqrt{2}}\left(\hat{X}_A\hat{Z}_A|01\rangle + \hat{X}_A\hat{Z}_A|10\rangle\right)\\ &= \frac{1}{\sqrt{2}}\left(\hat{X}_A|01\rangle - \hat{X}_A|10\rangle\right)\\ &= \frac{1}{\sqrt{2}}(|11\rangle - |00\rangle)\\ &= -\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\\ &= -|B_2\rangle \text{ undistinguishable from } |B_2\rangle\end{aligned}$$

Answer 25:
See above.

Exercise 26:
Work out a unitary transformation $\hat{U}$ which changes the basis from states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ to states $|B_0\rangle, |B_1\rangle, |B_2\rangle, |B_3\rangle$. Write down the matrix for $\hat{U}$ in the computational basis.

We want to find $\hat{U}$ such that:

$$\hat{U}(|00\rangle\ \ |01\rangle\ \ |10\rangle\ \ |11\rangle) = (|B_0\rangle\ \ |B_1\rangle\ \ |B_2\rangle\ \ |B_3\rangle)$$

$$\hat{U}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Answer 26:
We find $\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$.

Exercise 27:
Prove that $\left\langle B_0\middle|\hat{X}_A\middle|B_0\right\rangle = \left\langle B_0\middle|\hat{Y}_A\middle|B_0\right\rangle = \left\langle B_0\middle|\hat{Z}_A\middle|B_0\right\rangle = 0$. Can you imagine a random 3D vector with such a property?

We have:

$$\begin{aligned} &\left\langle B_0\middle|\hat{X}_A\middle|B_0\right\rangle \\ &= \frac{1}{2}(\langle 01| - \langle 10|)\hat{X}_A(|01\rangle - |10\rangle) \\ &= \frac{1}{2}(\langle 01| - \langle 10|)(|11\rangle - |00\rangle) \\ &= \frac{1}{2}(\langle 01|11\rangle - \langle 01|00\rangle - \langle 10|11\rangle + \langle 10|00\rangle) \\ &= 0 \end{aligned}$$

$$\begin{aligned} &\left\langle B_0\middle|\hat{Y}_A\middle|B_0\right\rangle \\ &= \frac{1}{2}(\langle 01| - \langle 10|)\hat{Y}_A(|01\rangle - |10\rangle) \\ &= \frac{1}{2}(\langle 01| - \langle 10|)(i|11\rangle + i|00\rangle) \\ &= \frac{i}{2}(\langle 01|11\rangle - \langle 01|00\rangle - \langle 10|11\rangle - \langle 10|00\rangle) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \left\langle B_0\middle|\hat{Z}_A\middle|B_0\right\rangle &= \frac{1}{2}(\langle 01| - \langle 10|)\hat{Z}_A(|01\rangle - |10\rangle) \\ &= \frac{1}{2}(\langle 01| - \langle 10|)(|01\rangle + |10\rangle) \\ &= \frac{1}{2}(\langle 01|01\rangle - \langle 01|10\rangle - \langle 10|01\rangle - \langle 10|10\rangle) \\ &= 0 \end{aligned}$$

Answer 27:
See above. The only 3D vector that has this property is $\vec{0}$.

Exercise 28:
Prove the above but for the operators of qubit B.

$$\begin{aligned}
&\left\langle B_0 \middle| \hat{X}_B \middle| B_0 \right\rangle \\
&= \frac{1}{2}(\langle 01| - \langle 10|)\hat{X}_B(|01\rangle - |10\rangle) \\
&= \frac{1}{2}(\langle 01| - \langle 10|)(|00\rangle - |11\rangle) \\
&= \frac{1}{2}(\langle 01|00\rangle - \langle 01|11\rangle - \langle 10|00\rangle + \langle 10|11\rangle) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
&\left\langle B_0 \middle| \hat{Y}_B \middle| B_0 \right\rangle \\
&= \frac{1}{2}(\langle 01| - \langle 10|)\hat{Y}_B(|01\rangle - |10\rangle) \\
&= \frac{1}{2}(\langle 01| - \langle 10|)(-i|00\rangle - i|11\rangle) \\
&= \frac{i}{2}(-\langle 01|00\rangle - \langle 01|11\rangle + \langle 10|00\rangle + \langle 10|11\rangle) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\left\langle B_0 \middle| \hat{Z}_B \middle| B_0 \right\rangle &= \frac{1}{2}(\langle 01| - \langle 10|)\hat{Z}_B(|01\rangle - |10\rangle) \\
&= \frac{1}{2}(\langle 01| - \langle 10|)(-|01\rangle - |10\rangle) \\
&= \frac{1}{2}(-\langle 01|01\rangle - \langle 01|10\rangle + \langle 10|01\rangle + \langle 10|10\rangle) \\
&= 0
\end{aligned}$$

Answer 28:
See above.

Exercise 29:
Prove that $\left\langle B_0 \middle| \hat{X}_A \hat{X}_B \middle| B_0 \right\rangle = \left\langle B_0 \middle| \hat{Y}_A \hat{Y}_B \middle| B_0 \right\rangle = \left\langle B_0 \middle| \hat{Z}_a \hat{Z}_B \middle| B_0 \right\rangle = -1$.

We have:

$$\begin{aligned}
&\left\langle B_0 \middle| \hat{X}_A \hat{X}_B \middle| B_0 \right\rangle \\
&= \frac{1}{2}(\langle 01| - \langle 10|)\hat{X}_A \hat{X}_B(|01\rangle - |10\rangle) \\
&= \frac{1}{2}(\langle 01| - \langle 10|)(|10\rangle - |01\rangle) \\
&= \frac{1}{2}(\langle 01|10\rangle - \langle 01|01\rangle - \langle 10|10\rangle + \langle 10|01\rangle) \\
&= \frac{1}{2}(-1 - 1) = -1
\end{aligned}$$

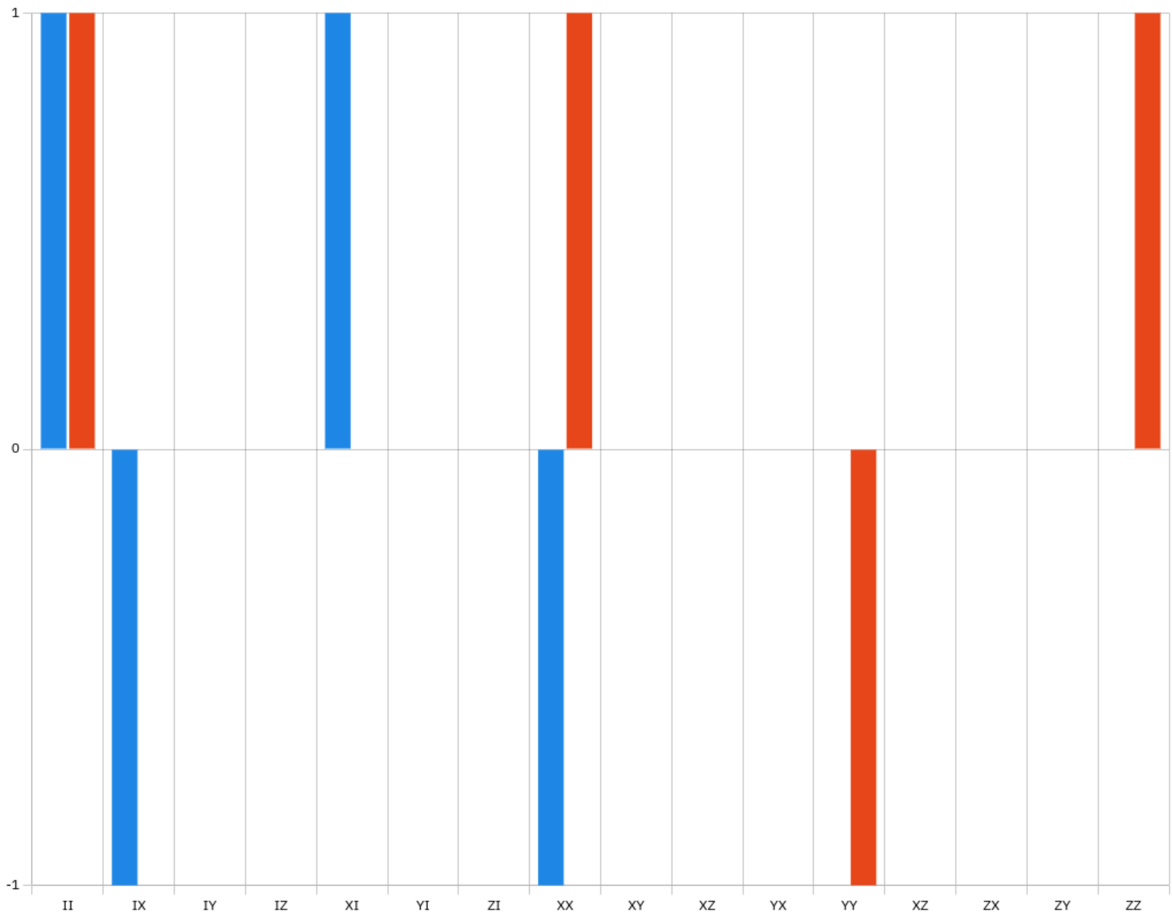
$$\begin{aligned}
&\left\langle B_0 \middle| \hat{Y}_A \hat{Y}_b \middle| B_0 \right\rangle \\
&= \frac{1}{2}(\langle 01| - \langle 10|)\hat{Y}_A \hat{Y}_B(|01\rangle - |10\rangle) \\
&= \frac{1}{2}(\langle 01| - \langle 10|)(|10\rangle - |01\rangle) \\
&= \frac{1}{2}(\langle 01|10\rangle - \langle 01|01\rangle - \langle 10|10\rangle + \langle 10|01\rangle \\
&= \frac{1}{2}(-1 - 1) = -1
\end{aligned}$$

$$\begin{aligned}
\left\langle B_0 \middle| \hat{Z}_A \hat{Z}_B \middle| B_0 \right\rangle &= \frac{1}{2}(\langle 01| - \langle 10|)\hat{Z}_A \hat{Z}_B(|01\rangle - |10\rangle) \\
&= \frac{1}{2}(\langle 01| - \langle 10|)(-|01\rangle + |10\rangle) \\
&= \frac{1}{2}(-\langle 01|01\rangle + \langle 01|10\rangle + \langle 10|01\rangle - \langle 10|10\rangle) \\
&= \frac{1}{2}(-1 - 1) = -1
\end{aligned}$$

Answer 29:
See above.

Exercise 30:
Create the Paule plot for two states, $|{+}{-}\rangle$ and $|B_3\rangle$.

Answer 30:
We have, with $|+-\rangle$ in blue and $|B_3\rangle$ orange:



Exercise 31:
Show that person A measuring $\hat{Z}_A$ would find $Z_A = +1$ with a probability $\frac{1}{2}$ and $Z_Z = -1$ with a probability $\frac{1}{2}$. However, every time person A measures $Z_A = +1$, person B afterwards measures $\hat{Z}_B = +1$ as well.

We have:

$$\begin{aligned}|B_3\rangle &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \\ &= \psi_{00}|00\rangle + \psi_{11}|11\rangle\end{aligned}$$

$$\Rightarrow \text{probability} = |\psi_{00}|^2 = |\psi_{11}|^2 = \frac{1}{2}$$

Therefore, measuring $\hat{Z}_A$ on $|B_3\rangle$, we get ± 1 with probability $\frac{1}{2}$ ($\pm$ 1 are the eigenvalues for $|00\rangle, |11\rangle$ and $|01\rangle, |10\rangle$ respectively).

When measuring $+1$, we know for sure that $|B_3\rangle$ collapses onto $|00\rangle$, so qubit B is in $|0_B\rangle$, so measuring $\hat{Z}_b$ would output $+1$ with probability 1.

Answer 31:
See above.

Exercise 32:
Person A measures $\hat{X}_A$ and right aftewards person B measures $\hat{Z}_B$. Show that now every time person A measures $\hat{X}_A = +1$, person B measures $Z_A = +1$ with a probability $\frac{1}{2}$ and $Z_A = -1$ also with a probability $\frac{1}{2}$.

We rewrite $|00\rangle, |11\rangle$ to have qubit A in the $\hat{X}$ basis:

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \qquad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

So:

$$\begin{aligned} |00\rangle &= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \otimes |0\rangle \\ &= \frac{1}{\sqrt{2}}(|+0\rangle + |-0\rangle) \end{aligned} \qquad \begin{aligned} |11\rangle &= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \otimes |1\rangle \\ &= \frac{1}{\sqrt{2}}(|+1\rangle - |-1\rangle) \end{aligned}$$

And we get:

$$|B_3\rangle = \frac{1}{2}(|+0\rangle + |-0\rangle + |+1\rangle - |-1\rangle)$$

Person A measures $X_A = +1$, $|B_3\rangle$ collapses onto $|++\rangle$. Person B measures $\hat{Z}_B$ on that collapsed state ($|+_B\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$) and can get ± 1 with probability $\frac{1}{2}$.

Answer 32:
See above.

D. Deteministic creation of entanglement

Exercise 33:
Consider the two-qubit dynamics according to the Hamiltonian Eq 3. Consider an initial state $|\Psi(t = 0)\rangle = |01\rangle, |10\rangle$. Show that for both cases nothing happens.

Since $g \ll \omega_{1,2}$, we have:

$$\begin{aligned} \hat{H}|00\rangle &= -\tfrac{1}{2}\omega_1 \hat{Z}_A|00\rangle - \tfrac{1}{2}\omega_2 \hat{Z}_B|00\rangle + g\hat{X}|00\rangle \\ &= -\tfrac{1}{2}\omega_1 \hat{Z}_A|00\rangle - \tfrac{1}{2}\omega_2 \hat{Z}_B|00\rangle + g|11\rangle \\ &\approx -\tfrac{1}{2}\omega_1 \hat{Z}_A|00\rangle - \tfrac{1}{2}\omega_2 \hat{Z}_B|00\rangle \\ &= -2\pi|00\rangle \end{aligned} \qquad \begin{aligned} \hat{H}|00\rangle &= -\tfrac{1}{2}\omega_1 \hat{Z}_A|11\rangle - \tfrac{1}{2}\omega_2 \hat{Z}_B|11\rangle + g\hat{X}|11\rangle \\ &= \tfrac{1}{2}\omega_1 \hat{Z}_A|11\rangle + \tfrac{1}{2}\omega_2 \hat{Z}_B|11\rangle + g|00\rangle \\ &\approx \tfrac{1}{2}\omega_1 \hat{Z}_A|11\rangle + \tfrac{1}{2}\omega_2 \hat{Z}_B|11\rangle \\ &= 2\pi|11\rangle \end{aligned}$$

So both stay in the same state (up to a factor).

Answer 33:
See above.

Exercise 34:
Consider an initial state $|\Psi(t = 0)\rangle = |01\rangle, |10\rangle$. Show that these two states oscillate into one another with a period given by $\frac{2\pi}{g}$.

We have:

$$\begin{aligned}\hat{H}|01\rangle &= -\frac{1}{2}\omega_1|01\rangle + \frac{1}{2}\omega_2|01\rangle + g|10\rangle \\ &= -\pi|01\rangle + \pi|01\rangle + g|10\rangle \\ &= g|10\rangle\end{aligned}$$

$$\begin{aligned}\hat{H}|10\rangle &= -\frac{1}{2}\omega_1|10\rangle + \frac{1}{2}\omega_2|10\rangle + g|10\rangle \\ &= +\pi|10\rangle - \pi|10\rangle + g|01\rangle \\ &= g|01\rangle\end{aligned}$$

So the period is given by $\frac{2\pi}{g}$.

Answer 34:
See above.

Exercise 35:
Consider the initial state $|01\rangle$ and show that at time $t = \frac{1}{2} \cdot \frac{2\pi}{g}$, the state is $|01\rangle + i|10\rangle$ up to a normalization factor. Likewise, start with $|10\rangle$ and show that that one evolves into $|01\rangle - i|10\rangle$.

We have:

$$\begin{aligned}\hat{H}|01\rangle &= g|10\rangle \\ \hat{H}|10\rangle &= g|01\rangle\end{aligned}$$

TODO

Answer 35:
See above.

Exercise 36:
Propose a single-qubit operator which would convert the state $|01\rangle + i|10\rangle$ into one of the Bell states.

We use:

$$\widehat{M} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \\ & i \end{pmatrix}$$

So:

$$\widehat{M}|0\rangle = \frac{1}{\sqrt{2}}|0\rangle$$

$$\widehat{M}|1\rangle = \frac{i}{\sqrt{2}}|1\rangle$$

We apply it to the given state:

$$\begin{aligned}\widehat{M}(|01\rangle + i|10\rangle) &= \widehat{M}|01\rangle + i\widehat{M}|10\rangle \\ &= \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle\end{aligned}$$

Answer 36:
$\widehat{M} = \begin{pmatrix} 1 & \\ & i \end{pmatrix}$

Exercise 37:
Express the 2×2 matrix for $\hat{H}$ in the basis of states $|01\rangle$ and $|10\rangle$. What combination of Pauli operators is it?

We have;

$$\begin{aligned}
\hat{H} &= -\frac{1}{2}\omega_1 \hat{Z}_1 - \frac{1}{2}\omega_2 \hat{Z}_2 + g\hat{X}_1\hat{X}_2 \\
&= \begin{pmatrix} -\frac{1}{2}\omega_1 - \frac{1}{2}\omega_2 & \\ & \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2 \end{pmatrix} + \begin{pmatrix} & g \\ g & \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{2}\omega_1 - \frac{1}{2}\omega_2 & g \\ g & \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2 \end{pmatrix} \\
&= \left(-\frac{1}{2}\omega_1 - \frac{1}{2}\omega_2\right)\hat{Z} + g\hat{X}
\end{aligned}$$

Answer 37:
$\hat{H} = \left(-\frac{1}{2}\omega_1 - \frac{1}{2}\omega_2\right)\hat{Z} + g\hat{X}$