Homework 2

A. Introduction: classical vs. quantum oscillator models

Exercise 1:
Calculate the mean values of $\langle\Psi|\hat{x}|\Psi\rangle$ and $\langle\Psi|\hat{p}|\Psi\rangle$ operators for an oscillator in states $|\Psi\rangle = |0\rangle$ and $|\Psi\rangle = |n\rangle$.

We have:

$$\hat{x} = x_0(\hat{a} + \hat{a}^\dagger)$$

$$\hat{p} = -ip_0(\hat{a} - \hat{a}^\dagger)$$

$$\begin{aligned}\langle\Psi|\hat{x}|\Psi\rangle &= \langle\Psi|x_0(\hat{a} + \hat{a}^\dagger)|\Psi\rangle \\ &= x_0\langle\Psi|\hat{a} + \hat{a}^\dagger|\Psi\rangle \\ &= x_0\langle\Psi|\hat{a}|\Psi\rangle + x_0\langle\Psi|\hat{a}^\dagger|\Psi\rangle\end{aligned}$$

$$\begin{aligned}\langle\Psi|\hat{p}|\Psi\rangle &= \langle\Psi|-ip_0(\hat{a} - \hat{a}^\dagger)|\Psi\rangle \\ &= -ip_0\langle\Psi|\hat{a} - \hat{a}^\dagger|\Psi\rangle \\ &= -ip_0\langle\Psi|\hat{a}|\Psi\rangle + ip_0\langle\Psi|\hat{a}^\dagger|\Psi\rangle\end{aligned}$$

So we get, for $|\Psi\rangle = |0\rangle$:

$$\begin{aligned}&\langle 0|\hat{x}|0\rangle \\ &= x_0\langle 0|\hat{a}|0\rangle + x_0\langle 0|\hat{a}^\dagger|0\rangle \\ &= x_0\langle 0|\hat{a}^\dagger|0\rangle, \text{ since } \hat{a} \text{ annihilates } |0\rangle \\ &= x_0\langle 0|(\hat{a}^\dagger|0\rangle) \\ &= x_0\langle 0|1\rangle \\ &= 0\end{aligned}$$

$$\begin{aligned}&\langle 0|\hat{p}|0\rangle \\ &= -ip_0\langle 0|\hat{a}|0\rangle + ip_0\langle 0|\hat{a}^\dagger|0\rangle \\ &= ip_0\langle 0|\hat{a}^\dagger|0\rangle \\ &= \text{... same as before} \\ &= 0\end{aligned}$$

And for $|\Psi\rangle = |n\rangle$:

$$\begin{aligned}&\langle n|\hat{x}|n\rangle \\ &x_0\langle n|\hat{a}|n\rangle + x_0\langle n|\hat{a}^\dagger|n\rangle \\ &= x_0\langle n|(\hat{a}|n\rangle) + x_0\langle n|(\hat{a}^\dagger|n\rangle) \\ &= x_0\langle n|(\sqrt{n}|n-1\rangle) + x_0\langle n|(\sqrt{n+1}|n+1\rangle) \\ &= x_0\sqrt{n}\langle n|n-1\rangle + x_0\sqrt{n+1}\langle n|n+1\rangle \\ &= 0, \text{ since Fock states are orthogonal}\end{aligned}$$

$$\begin{aligned}&\langle n|\hat{p}|n\rangle \\ &-ip_0\langle n|\hat{a}|n\rangle + ip_0\langle n|\hat{a}^\dagger|n\rangle \\ &= \text{... same as before} \\ &= 0\end{aligned}$$

Answer 1:
Respectively: $0, 0, 0$ and 0

Exercise 2:
Calculate $x_{RMS} = \sqrt{\langle\Psi|\hat{x}^2|\Psi\rangle}$ and $p_{RMS} = \sqrt{\langle\Psi|\hat{p}^2|\Psi\rangle}$ for an oscillator in states $|\Psi\rangle = |0\rangle$ and $|\Psi\rangle = |n\rangle$.

We have:

$$\begin{aligned} x_{RMS} &= \sqrt{\langle\Psi|\hat{x}^2|\Psi\rangle} \\ &= x_0\sqrt{\left\langle\Psi\middle|(\hat{a}+\hat{a}^\dagger)^2\middle|\Psi\right\rangle} \\ &= x_0\sqrt{\left\langle\Psi\middle|\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+(\hat{a}^\dagger)^2\middle|\Psi\right\rangle} \end{aligned}$$

$$\begin{aligned} p_{RMS} &= \sqrt{\langle\Psi|\hat{p}^2|\Psi\rangle} \\ &= p_0\sqrt{\left\langle\Psi\middle|(\hat{a}-\hat{a}^\dagger)^2\middle|\Psi\right\rangle} \\ &= p_0\sqrt{\left\langle\Psi\middle|\hat{a}^2-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}+(\hat{a}^\dagger)^2\middle|\Psi\right\rangle} \end{aligned}$$

And:

$$\begin{aligned} x_{RMS} &= \sqrt{\langle 0|\hat{x}^2|0\rangle} \\ &= x_0\sqrt{\left\langle 0\middle|\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+(\hat{a}^\dagger)^2\middle|0\right\rangle} \\ &= x_0\sqrt{\langle 0|0\rangle} \\ &= x_0\sqrt{1} \\ &= x_0 \end{aligned}$$

$$\begin{aligned} p_{RMS} &= \sqrt{\langle 0|\hat{p}^2|0\rangle} \\ &= p_0\sqrt{\left\langle 0\middle|\hat{a}^2-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}+(\hat{a}^\dagger)^2\middle|0\right\rangle} \\ &= p_0\sqrt{\langle 0|0\rangle} \\ &= p_0 \end{aligned}$$

$$\begin{aligned} x_{RMS} &= \sqrt{\langle n|\hat{x}^2|n\rangle} \\ &= x_0\sqrt{\left\langle n\middle|\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+(\hat{a}^\dagger)^2\middle|n\right\rangle} \\ &= x_0\sqrt{\begin{matrix} n\langle n-1|n-1\rangle+\langle n-1|n+1\rangle \\ +\langle n+1|n-1\rangle+(n+1)\langle n+1|n+1\rangle \end{matrix}} \\ &= x_0\sqrt{2n+1} \end{aligned}$$

$$\begin{aligned} p_{RMS} &= \sqrt{\langle n|\hat{p}^2|n\rangle} \\ &= p_0\sqrt{\left\langle n\middle|\hat{a}^2-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}+(\hat{a}^\dagger)^2\middle|n\right\rangle} \\ &= p_0\sqrt{\begin{matrix} \langle n|n-2\rangle+\langle n-1|n+1\rangle \\ +\langle n+1|n-1\rangle+\langle n|n+2\rangle \end{matrix}} \\ &= p_0\sqrt{2n+1} \end{aligned}$$

> **Answer 2:**
> Respectively, $x_0, p_0, x_0\sqrt{2n+1}$ and $p_0\sqrt{2n+1}$.

> **Exercise 3:**
> Use the results of the previous exercise and demonstrate that $x_{RMS}p_{RMS} \geq \frac{\hbar}{2}$.

We have:

$$\begin{aligned} x_{RMS}p_{RMS} &= x_0\sqrt{2n+1}p_0\sqrt{2n+1} \\ &= x_0p_0(2n+1) \\ &= \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar\ m\omega}{2}}(2n+1) \\ &= \frac{\hbar}{2}(2n+1) \\ &\geq \frac{\hbar}{2} \text{ since } 2n+1 \text{ is positive} \end{aligned}$$

> **Answer 3:**
> Proof, see above.

> **Exercise 4:**
> Claculate and compare mean kinetic $\left\langle 0\middle|\frac{\hat{p}^2}{2m}\middle|0\right\rangle$ and mean potential $\left\langle 0\middle|\frac{m\omega^2\hat{x}^2}{2}\right\rangle$ energies of the oscillator in its ground state $|0\rangle$.

We have:

$$\left\langle 0 \middle| \frac{\hat{p}^2}{2m} \middle| 0 \right\rangle$$
$$= \frac{1}{2m} \langle 0|\hat{p}^2|0\rangle$$
$$= \frac{p_0^2}{2m}$$
$$= \frac{\hbar\ m\omega}{4m}$$
$$= \frac{\omega\ \hbar}{4}$$

$$\left\langle 0 \middle| m\omega^2 \frac{\hat{x}^2}{2} \middle| 0 \right\rangle$$
$$= \frac{m\omega^2}{2} \langle 0|\hat{x}^2|0\rangle$$
$$= \frac{m\omega^2}{2} x_0^2$$
$$= \frac{m\omega^2}{2} \frac{\hbar}{2m\omega}$$
$$= \frac{\hbar\ \omega}{4}$$

Comparing, we have that kinetic energy equals potential energy when the oscillator is in its ground state.

Answer 4:
$\left\langle 0 \middle| \frac{\hat{p}^2}{2m} \middle| 0 \right\rangle = \left\langle 0 \middle| \frac{m\omega^2 \hat{x}^2}{2} \middle| 0 \right\rangle = \frac{\hbar\ \omega}{4}$

Exercise 5:
Show by a direct calculation that

$$\frac{\partial}{\partial t} \langle \hat{a} \rangle = -i\omega \langle \hat{a} \rangle$$
$$\frac{\partial}{\partial t} \langle \hat{a}^\dagger \rangle = +i\omega \langle \hat{a}^\dagger \rangle$$

First, we caclulate $\langle \Psi(t)|\hat{a}|\Psi(t)\rangle$. We have that:

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} \psi_n \exp(-in\omega t)|n\rangle$$
$$\langle \Psi(t)| = \sum_{k=0}^{\infty} \psi_k^* \exp(+ik\omega t)\langle k|$$

So we get:

$$\begin{aligned}
\langle \Psi(t)|\hat{a}|\Psi(t)\rangle &= \left(\sum_{k=0}^{\infty} \psi_k^* \exp(+ik\omega t)\langle k| \right) \hat{a} \left(\sum_{n=0}^{\infty} \psi_n \exp(-in\omega t)|n\rangle \right) \\
&= \left(\sum_{k=0} \psi_k^* \exp(+ik\omega t)\langle k| \right) \left(\sum_{n=0} \psi_n \exp(-in\omega t)\hat{a}|n\rangle \right) \\
&= \sum_{k=0} \sum_{n=0} \psi_k^* \psi_n \exp(+ik\omega t) \exp(-in\omega t) \langle k|\hat{a}|n\rangle \\
&= \sum_{k=0} \sum_{n=0} \psi_k^* \psi_n \exp(+ik\omega t) \exp(-in\omega t) \langle k|\sqrt{n}|n-1\rangle \\
&= \sum_{k=0} \sum_{n=0} \sqrt{n} \psi_k^* \psi_n \exp(+ik\omega t) \exp(-in\omega t) \langle k|n-1\rangle
\end{aligned}$$

Since $\langle k|n-1\rangle = 0$ if $k \neq n-1$, we get a single sum over n (and we skip the $n = 0$ step since it would make $\sqrt{n} = 0$):

$$
\begin{aligned}
&= \sum_{n=1} \sqrt{n}\psi_{n-1}^* \psi_n \exp(+i(n-1)\omega t) \exp(-in\omega t) \\
&= \sum_{n=1} \sqrt{n}\psi_{n-1}^* \psi_n \exp(-i\omega t) \\
&= \exp(-i\omega t) \sum_{n=1} \sqrt{n}\psi_{n-1}^* \psi_n
\end{aligned}
$$

Then, we find its derivative:

$$
\begin{aligned}
\frac{\partial}{\partial t} \langle \Psi(t) | \hat{a} | \Psi(t) \rangle &= \frac{\partial}{\partial t} \exp(-i\omega t) \sum_{n=1} \sqrt{n}\psi_{n-1}^* \psi_n \\
&= \sum_{n=1} \sqrt{n}\psi_{n-1}^* \psi_n \frac{\partial}{\partial t} \exp(-i\omega t) \\
&= -i\omega \sum_{n=1} \sqrt{n}\psi_{n-1}^* \psi_n \exp(-i\omega t) \\
&= -i\omega \langle \Psi(t) | \hat{a} | \Psi(t) \rangle
\end{aligned}
$$

The computation is the same for $\hat{a}^\dagger$, but taking its conjugate. We get:

$$
\frac{\partial}{\partial t} \langle \Psi(t) | \hat{a}^\dagger | \Psi(t) \rangle = +i\omega \langle \Psi(t) | \hat{a}^\dagger | \Psi(t) \rangle
$$

Answer 5:
Proof, see above.

Exercise 6:
Apply the result of the previous exercise to two cases: $|\Psi(t=0)\rangle = |2\rangle$ and $|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$. Plot or sketch the time-evolution of $\langle \hat{a} \rangle$ in the 2D plane defined by axis $\frac{x}{x_0}$ and $i\frac{p}{p_0}$.

We found:

$$
\frac{\partial}{\partial t} \langle \Psi(t) | \hat{a} | \Psi(t) \rangle = -i\omega \langle \Psi(t) | \hat{a} | \Psi(t) \rangle
$$

So the solution is:

$$
\langle \Psi(t) | \hat{a} | \Psi(t) \rangle = \langle \Psi(0) | \hat{a} | \Psi(0) \rangle \exp(-i\omega t)
$$

For $|\Psi(t=0)\rangle = |2\rangle$:

$$
\begin{aligned}
\langle 2 | \hat{a} | 2 \rangle &= \left\langle 2 \middle| \sqrt{2} \middle| 1 \right\rangle \\
&= 0 \\
&\Rightarrow \langle \Psi(t) | \hat{a} | \Psi(t) \rangle = 0
\end{aligned}
$$

For $|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$:

$$\begin{aligned}
&\left(\frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|\right)\hat{a}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \\
&= \left(\frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|\right)\left(\frac{1}{\sqrt{2}}|0\rangle\right) \\
&= \frac{1}{2}\langle 0|0\rangle + \frac{1}{2}\langle 1|0\rangle \\
&= \frac{1}{2} \\
&\Rightarrow \langle \Psi(t)|\hat{a}|\Psi(t)\rangle = \frac{1}{2}\exp(-i\omega t)
\end{aligned}$$

The results are obtained similarly for the $\hat{a}^\dagger$ equation.

Answer 6:
For $|\Psi(t=0)\rangle = |2\rangle$, $\langle \Psi(t)|\hat{a}|\Psi(t)\rangle = 0$, and the plot has $\langle \hat{a}\rangle$ remaining at the origin. For $|\Psi(t = 0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$, $\langle \Psi(t)|\hat{a}|\Psi(t)\rangle = \frac{1}{2}\exp(-i\omega t)$ and the plot shows $\langle \hat{a}\rangle$ rotating clockwise in a cirlce of radius $\frac{1}{2}$ centered at the origin and with angular frequency ω:

#figure(image("./res/hw2_question6.jpg", width: 60%))

Exercise 7:
Plot or sketch the mean energy of the oscillator $\left\langle \Psi(t)\middle|\hat{H}\middle|\Psi(t)\right\rangle$ as a function of time for $|\Psi(t = 0)\rangle = \sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|n\rangle$. Is energy really quantized in a quantum oscillator this time?

We choose $n = 1$ for simpicity. The hamiltionina is given by $\hbar\ \omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\ \hbar\ \omega$.

We get:

$$\begin{aligned}
|\Psi(t)\rangle &= \sqrt{\frac{1}{3}}\exp(-i\omega t\cdot 0)|0\rangle + \sqrt{\frac{2}{3}}\exp(-i\omega t\cdot 1)|1\rangle \\
&= \sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}\exp(-i\omega t)|1\rangle
\end{aligned}$$

Now, the mean energy is:

$$\begin{aligned}
\left\langle \Psi(t)\middle|\hat{H}\middle|\Psi(t)\right\rangle &= \left(\sqrt{\frac{1}{3}}\langle 0| + \sqrt{\frac{2}{3}}\exp(+i\omega t)\langle 1|\right)\hat{H}\left(\sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}\exp(-i\omega t)|1\rangle\right) \\
&= \left(\sqrt{\frac{1}{3}}\langle 0| + \sqrt{\frac{2}{3}}\exp(+i\omega t)\langle 1|\right)\left(\sqrt{\frac{1}{3}}\frac{1}{2}\ \hbar\ \omega|0\rangle + \sqrt{\frac{2}{3}}\frac{3}{2}\ \hbar\ \omega\exp(-i\omega t)|1\rangle\right) \\
&= \frac{1}{6}\ \hbar\ \omega + \hbar\ \omega \\
&= \frac{7}{6}\ \hbar\ \omega
\end{aligned}$$

Answer 7:
I don't think I understand what the question means, but I can say that $\frac{7}{6}$ ħ ω is definitely not one of the "allowed" energy measurements. However that would be expected since it's a mean, so I don't think this is what the question was getting at, sorry. Here is the plot: (shelf at $\frac{7}{6}$)

#figure(image("./res/hw2_question7.jpg", width: 60%))

B. Coherent states

Exercise 8:
Show that a coherent state $|\alpha\rangle$ is an eigenstate of the lowering operator $\hat{a}$ with eigenvalue α.

We have:

$$\begin{aligned}
\hat{a}|\alpha\rangle &= \hat{a}\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\hat{a}|n\rangle \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sqrt{n}|n-1\rangle \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{(n-1)!}}|n-1\rangle \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^{n+1}}{\sqrt{n!}}|n\rangle \\
&= \alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^{n+1}}{\sqrt{n!}}|n\rangle \\
&= \alpha|\alpha\rangle
\end{aligned}$$

Answer 8:
Proof, see above.

Exercise 9:
Show that the raising operator $\hat{a}^\dagger$ does not have any eigenstates.

We assume that $\hat{a}^\dagger$ has an eigenstate $|\Psi\rangle$, with eigenvalue λ:

$$\begin{aligned}
\hat{a}^\dagger|\Psi\rangle &= \lambda|\Psi\rangle \\
\hat{a}^\dagger\sum_{n-0}\psi_n|n\rangle &= \lambda\sum_{n-0}\psi_n|n\rangle \\
\sum_{n=0}\psi_n\hat{a}^\dagger|n\rangle &= \lambda\sum_{n=0}\psi_n|n\rangle \\
\sum_{n=0}\psi_n\sqrt{n+1}|n+1\rangle &= \lambda\sum_{n=0}\psi_n|n\rangle
\end{aligned}$$

We can match the coefficents ψ_n:

$$\begin{cases} 0 = \lambda\psi_0, \text{ since there's no coefficient for } |0\rangle \text{ on the left} \\ \qquad\qquad (\text{starts at } |1\rangle) \\ \psi_n\sqrt{n+1} = \lambda\psi_{n+1} \text{ otherwise} \end{cases}$$

Now, we look at $\lambda\psi_0 = 0$ by cases:

- $\lambda = 0$:

  $\Rightarrow \lambda\psi_{n+1} = 0$

  $\Rightarrow \psi_n\sqrt{n+1} = 0$

  $\Rightarrow \forall n, \psi_n = 0$

  $\Rightarrow |\Psi\rangle = 0, \text{ not a state}$

  $\Rightarrow \lambda \text{ can't be } 0$

- $\psi_0 = 0$:

  $\psi_{n+1} = \dfrac{\sqrt{n+1}}{\lambda}\psi_n$

  $\Rightarrow \forall n, \psi_n = 0$

  $\Rightarrow \psi_0 \text{ can't be } 0 \text{ by the same logic}$

Answer 9:
Proof, see above.

Exercise 10:
Show that the mean value of energy is a coherent state $|\alpha\rangle$ is given by $\hbar\,\omega|\alpha|^2 + \frac{1}{2}\,\hbar\,\omega$. Equivalently, $\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2$.

$$\begin{aligned} \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle &= \alpha\langle\alpha|\hat{a}^\dagger|\alpha\rangle \\ &= \alpha\alpha^* \\ &= |\alpha|^2 \end{aligned}$$

Answer 10:
Proof, see above.

Exercise 11:
For an oscillator in a coherent state $|\alpha\rangle$, calculate the variance of the energy, defined by: $E_{RMS}^2 = \left\langle\alpha\middle|\left(\hat{H} - \left\langle\alpha\middle|\hat{H}\middle|\alpha\right\rangle^2\right)\middle|\alpha\right\rangle = \left\langle\alpha\middle|\hat{H}^2\middle|\alpha\right\rangle - \left\langle\alpha\middle|\hat{H}\middle|\alpha\right\rangle^2$.

We know that:

$$\left\langle\alpha\middle|\hat{H}\middle|\alpha\right\rangle = \hbar\,\omega|\alpha|^2 + \frac{1}{2}\,\hbar\,\omega$$

So:

$$\begin{aligned} \left\langle\alpha\middle|\hat{H}\middle|\alpha\right\rangle^2 &= \left(\hbar\,\omega|\alpha|^2 + \frac{1}{2}\,\hbar\,\omega\right)^2 \\ &= \hbar^2\omega^2\left(|\alpha|^4 + |\alpha|^2 + \frac{1}{4}\right) \end{aligned}$$

For $\left\langle\alpha\middle|\hat{H}^2\middle|\alpha\right\rangle$:

$$\begin{aligned}\hat{H}^2 &= \left(\hbar\ \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\right)^2 \\ &= \hbar^2\omega^2\left((\hat{a}^\dagger\hat{a})2 + \hat{a}^\dagger\hat{a} + \frac{1}{4}\right) \\ &= \hbar^2\omega^2\left(\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \hat{a}^\dagger\hat{a} + \frac{1}{4}\right)\end{aligned}$$

So:

$$\begin{aligned}\left\langle\alpha\middle|\hat{H}^2\middle|\alpha\right\rangle &= \hbar^2\omega^2\left\langle\alpha\middle|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \hat{a}^\dagger\hat{a} + \frac{1}{4}\middle|\alpha\right\rangle \\ &= \hbar^2\omega^2\left\langle\alpha\middle|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\middle|\alpha\right\rangle + \hbar^2\omega^2\left\langle\alpha\middle|\hat{a}^\dagger\hat{a}\middle|\alpha\right\rangle + \hbar^2\omega^2\frac{1}{4}\langle\alpha|\alpha\rangle \\ &= \hbar^2\omega^2\left\langle\alpha\middle|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\middle|\alpha\right\rangle + \hbar^2\omega^2|\alpha|^2 + \hbar^2\omega^2\frac{1}{4} \\ &= \hbar^2\omega^2(|\alpha|^4 + |\alpha|^2) + \hbar^2\omega^2|\alpha|^2 + \hbar^2\omega^2\frac{1}{4} \\ &= \hbar^2\omega^2\left(|\alpha|^4 + 2|\alpha|^2 + \frac{1}{4}\right)\end{aligned}$$

Therefore, we get:

$$\begin{aligned}\left\langle\alpha\middle|\hat{H}^2\middle|\alpha\right\rangle - \left\langle\alpha\middle|\hat{H}\middle|\alpha\right\rangle^2 &= \hbar^2\omega^2\left(|\alpha|^4 + 2|\alpha|^2 + \frac{1}{4}\right) - \hbar^2\omega^2\left(|\alpha|^4 + |\alpha|^2 + \frac{1}{4}\right) \\ &= \hbar^2\omega^2|\alpha|^2\end{aligned}$$

Answer 11:
$E_{RMS}^2 = \hbar^2\omega^2|\alpha|^2$.

Exercise 12:
Sketch the histogram for $|\alpha|^2 = 0, 3.3, 11.7, 100$. What is the ration of the mean energy to E_{RMS} for $|\alpha|^2 = 100$?

We have:

$$P_n = \exp(-|\alpha|^2) \cdot \frac{|\alpha|^{2n}}{n!}$$

And the sketch for $|\alpha|^2 = 0, 3.3, 11.7, 100$, respectively in red, green, dark blue and light blue (the one for $|\alpha|^2 = 0$ is really not visible since it's directly on the x-axis):

#figure(image(“./res/hw2_question12.jpg”, width: 60%))

This grpah is very similar to the one in figure 3: gaussians that get “more spread out” the bigger n is, and with the highest point being at $|\alpha|^2$.

We find the ration of mean energy and E_{RMS}:

$$\begin{aligned}\langle\hat{H}\rangle &= \hbar\ \omega\left(|\alpha|^2 + \frac{1}{2}\right) \\ &= 100.5\ \hbar\ \omega\end{aligned}$$

$$\begin{aligned}E_{RMS} &= \hbar\ \omega|\alpha| \\ &= 10\ \hbar\ \omega\end{aligned}$$

$$\frac{100.5\ \hbar\ \omega}{10\ \hbar\ \omega}$$
$$= 10.05 \approx 10$$

Answer 12:
See figure above. The ratio is $\frac{\langle \hat{H} \rangle}{E_{RMS}} \approx 10$.

Exercise 13:
Calculate the mean value of $\langle \alpha | \hat{x} | \alpha \rangle$ in a cocherent state $|\alpha\rangle$ as well as $x_{RMS}^2 = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2$. Does this value change in time?

On a:

$$\begin{aligned} \langle \alpha | \hat{x} | \alpha \rangle &= x_0 \langle \alpha | \hat{a} + \hat{a}^\dagger | \alpha \rangle \\ &= x_0 (\alpha + \alpha^*) \end{aligned}$$

And:

$$\begin{aligned} x_{RMS}^2 &= \langle \alpha | \hat{x}^2 | \alpha \rangle - \langle \alpha | \hat{x} | \alpha \rangle^2 \\ &= x_0 * 2 \langle \alpha | \hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a} + \hat{a}^\dagger \hat{a}^\dagger | \alpha \rangle - x_0^2 (\alpha + \alpha^*)^2 \\ &= x_0^2 \left(\alpha^2 + (\alpha^*)^2 + |\alpha|^2 + |\alpha|^2 + 1 \right) - x_0^2 \left(\alpha^2 + (\alpha^*)^2 + 2|\alpha|^2 \right) \\ &= x_0^2 \end{aligned}$$

Answer 13:
$\langle \hat{x} \rangle = x_0(\alpha + \alpha^*)$, and $x_{RMS}^2 = x_0^2$. This value is not dependent on time.

Exercise 14:
Calculate the mean value of $\langle \alpha | \hat{p} | \alpha \rangle$ in a cocherent state $|\alpha\rangle$ as well as $p_{RMS}^2 = \langle p^2 \rangle - \langle \hat{p} \rangle^2$. Check the product $x_{RMS} p_{RMS}$. Does it depend on the value of α?

The computation of $\langle \alpha | \hat{p} | \alpha \rangle$ and p_{RMS}^2 are similar, and we get:

$$\langle \alpha | \hat{p} | \alpha \rangle = -i p_0 (\alpha - \alpha^*) \qquad p_{RMS}^2 = p_0^2$$

We check:

$$\begin{aligned} x_{RMS} p_{RMS} &= x_0 p_0 \\ &= \sqrt{\frac{\hbar}{2m\omega}} \sqrt{\frac{\hbar\ m\omega}{2}} \\ &= \frac{\hbar}{2} \end{aligned}$$

Answer 14:
$\langle \hat{p} \rangle = -i p_0 (\alpha - \alpha^*)$ and $p_{RMS}^2 = p_0^2$. The product is $\frac{\hbar}{2}$. It does not depend on α.

Exercise 15:
Chooste $\alpha(t = 0) = 10$ and plot $\langle \hat{x} \rangle$ as a function of time on a computer. Make the thickness of your line equal to $x_R MS$.

We have $\alpha(t) = 10 \exp(-i\omega t)$ and constant thickness $x_{RMS} = x_0$ (musch smaller), so this is just a cosine osicllation:

#figure(image("./res/hw2_question15.jpg", width: 60%))

Answer 15:
See plot above.

C. Displacement operator

Exercise 16:
Check the following method of creating coherent state:

$$\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(\alpha\hat{a}^\dagger)|0\rangle = |\alpha\rangle$$
$$\exp(\alpha^*\hat{a})|0\rangle = |0\rangle$$

WE have:

$$\begin{aligned}
\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(\alpha\hat{a}^\dagger)|0\rangle &= \exp\left(-\frac{|\alpha|^2}{2}\right)\left(1 + \alpha\hat{a}^\dagger + \frac{\alpha^2\hat{a}^\dagger\hat{a}^\dagger}{2} + \dots\right)|0\rangle \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\left(|0\rangle + \alpha\hat{a}^\dagger|0\rangle + \frac{\alpha^2}{2}\hat{a}^\dagger\hat{a}^\dagger|0\rangle + \dots\right) \\
&= \frac{\alpha^2\hat{a}^\dagger\hat{a}^\dagger}{2}\left(|0\rangle + \alpha|1\rangle + \frac{\alpha^2}{2!}\sqrt{2!}|2\rangle + \dots\right) \\
&= |\alpha\rangle
\end{aligned}$$

And:

$$\begin{aligned}
\exp(\alpha^*\hat{a})|0\rangle &= \left(1 + \alpha^*\hat{a} + \frac{(\alpha^*)^2\hat{a}\hat{a}}{2!} + \dots\right)|0\rangle \\
&= |0\rangle + \alpha^*\hat{a}|0\rangle + (\frac{(\alpha^*)^2}{2!}\hat{a}\hat{a}|0\rangle + \dots \\
&= |0\rangle + \alpha^* \cdot 0 + \frac{(\alpha^*)^2}{2!}\sqrt{2!} \cdot 0 + \dots \\
&= |0\rangle
\end{aligned}$$

Answer 16:
Proof, see above.

Exercise 17:
Prove that

$$\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(\alpha\hat{a}^\dagger)\exp(-\alpha^*\hat{a}^\dagger) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$$

We have:

$$\begin{aligned}
&\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(\alpha\hat{a}^\dagger)\exp(-\alpha^*\hat{a}^\dagger) \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})\exp\left(\frac{1}{2}[\alpha\hat{a} - \alpha^*\hat{a}]\right) \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})\exp\left(\frac{|\alpha|^2}{2}\right) \\
&= \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})
\end{aligned}$$

Answer 17:
Proof, see above.

Exercise 18:
Prove that displacement operator redefined is unitary and that $\hat{D}^{-1}(\alpha) = \hat{D}(-\alpha) = \hat{D}^\dagger(\alpha)$.

We have:

$$\begin{aligned}
\hat{D}(-\alpha) &= \exp(-\alpha^*\hat{a} + \alpha\hat{a}^\dagger) \\
&= \exp(-\alpha\hat{a}^\dagger + \alpha * \hat{a})^\dagger \\
&= \hat{D}^\dagger(\alpha)
\end{aligned}$$

And:

$$\begin{aligned}
\hat{D}(\alpha)\hat{D}(-\alpha) &= \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})\exp(-\alpha\hat{a}^\dagger + \alpha^*\hat{a}) \\
&= \hat{I} \\
&\Rightarrow \hat{D}(-\alpha) = \hat{D}^{-1}(\alpha)
\end{aligned}$$

Answer 18:
Proff, see above.

Exercise 19:
Prove the following communation relations:

$$\hat{D}^\dagger(\alpha)\hat{a} = (\hat{a} + \alpha)\hat{D}^\dagger(\alpha)$$

$$\hat{D}^\dagger(\alpha)\hat{a}^\dagger = (\hat{a}^\dagger + \alpha^*)\hat{D}^\dagger(\alpha)$$

We have:

$$\begin{aligned}
\hat{D}(\alpha)\hat{D}(\beta) &= \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})\exp(\beta\hat{a}^\dagger - \alpha^*\hat{a}) \\
&= \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a} + \beta\hat{a}^\dagger - \alpha^*\hat{a})\exp\left(\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)\right) \\
&= \exp((\alpha + \beta)\hat{a}^\dagger - (\alpha^* + \beta^*)\hat{a})\exp\left(\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)\right) \\
&= \hat{D}(\alpha + \beta)\exp\left(\frac{\alpha\beta^* - \beta\alpha^*}{2}\right)
\end{aligned}$$

We apply both sides of the first equation to an abitrary coherent state $|\beta\rangle$ with eigenvale β and see they are equal:

$$\begin{aligned}
&D^\dagger(\alpha)\hat{a}|\beta\rangle \\
&= \beta\hat{D}^\dagger(-\alpha)|\beta\rangle \\
&= \beta\hat{D}(-\alpha)\hat{D}(\beta)|0\rangle \\
&= \beta\hat{D}(\beta-\alpha)\exp\left(\frac{1}{2}(-\alpha\beta^* + \alpha^*\beta)\right)|0\rangle
\end{aligned}$$

$$\begin{aligned}
&(\hat{a}+\alpha)\hat{D}^\dagger(\alpha)|\beta\rangle \\
&= (\hat{a}+\alpha)\hat{D}(-\alpha)|\beta\rangle \\
&= (\hat{a}+\alpha)\hat{D}(\beta-\alpha)\exp\left(\frac{1}{2}(-\alpha\beta^* + \alpha^*\beta)\right)|0\rangle \\
&= \beta\hat{D}(\beta-\alpha)\exp\left(\frac{1}{2}(\beta\alpha^* - \alpha\beta^*)\right)|0\rangle
\end{aligned}$$

The second equation is done in the same way.

Answer 19:
Proof, see above.

Exercise 20:
Show that for a real α (that is $\alpha^* = \alpha$), the displaicement operator becomes:

$$\hat{D}(\alpha) = \exp\left(-i\frac{\hat{p}\cdot 2\alpha x_0}{\hbar}\right)$$

We have:

$$\begin{aligned}
\hat{D}(\alpha) &= \exp(\alpha\hat{a}^\dagger - \alpha\hat{a}) \\
&= \exp(\alpha(\hat{a}^\dagger - \hat{a})) \\
&= \exp\left(\alpha\left(\left(\frac{1}{2}\frac{\hat{x}}{x_0} - \frac{1}{2}i\frac{\hat{p}}{p_0}\right) - \left(\frac{1}{2}\frac{\hat{x}}{x_0} + \frac{1}{2}i\frac{\hat{p}}{p_0}\right)\right)\right) \\
&= \exp\left(-i\alpha\frac{\hat{p}}{p_0}\right) \\
&= \exp\left(-i\frac{\hat{p}\cdot e\alpha x_0}{\hbar}\right)
\end{aligned}$$

Answer 20:
Proof, see above.

Exercise 21:
Prove further that for $\alpha^* = \alpha$:

$$\exp\left(+i\frac{\hat{p}\cdot 2\alpha x_0}{\hbar}\right)\hat{x}\exp\left(-i\frac{\hat{p}\cdot 2\alpha x_0}{\hbar}\right) = \hat{x} + 2\alpha x_0$$

We have:

$$\begin{aligned}\exp\left(+i\frac{\hat{p}\cdot 2\alpha x_0}{\hbar}\right)\hat{x}\exp\left(-i\frac{\hat{p}\cdot 2\alpha x_0}{\hbar}\right) &= \hat{D}^\dagger(\alpha)\hat{x}\hat{D}(\alpha)\\ &= x_0\hat{D}^\dagger(\alpha)\left(\hat{a}+\hat{a}^\dagger\right)\hat{D}(\alpha)\\ &= x_0\left(\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha)+\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha)\right)\\ &= x_0\left(\hat{a}+\alpha+\hat{a}^\dagger+\alpha\right)\\ &= \hat{x}+2\alpha x_0\end{aligned}$$

Answer 21:
Proof, see above.

D. Matric representation of quantum oscillators

Exercise 22:
Show by explicit matrix multiplication that the identity matrix $\hat{I}$ would be given by

$$\hat{I} = \sum_n |n\rangle\langle n|$$

We have:

$$\begin{aligned}\sum_n |n\rangle\langle n| &= \begin{pmatrix}1 & 0 & 0 & \ldots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix} + \begin{pmatrix}0 & 1 & 0 & \ldots\end{pmatrix}\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix} + \ldots\\ &= \begin{pmatrix}1 & 0 & 0 & \ldots\\ 0 & 0 & 0 & \ldots\\ 0 & 0 & 0 & \ldots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix} + \begin{pmatrix}0 & 0 & 0 & \ldots\\ 0 & 1 & 0 & \ldots\\ 0 & 0 & 0 & \ldots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix} + \ldots\\ &= \hat{I}\end{aligned}$$

Answer 22:
Profo, see above.

Exercise 23:
Write down matrices for $\hat{x}$ and $\hat{p}$ operators for $N_{\max} = 4$. Do they come out hermitian?

We have:

$$\begin{aligned}\hat{x} &= x_0\left(\hat{a}^\dagger+\hat{a}\right)\\ &= x_0\left(\begin{pmatrix}0 & \sqrt{1} & 0 & 0 & 0\\ 0 & 0 & \sqrt{2} & 0 & 0\\ 0 & 0 & 0 & \sqrt{3} & 0\\ 0 & 0 & 0 & 0 & \sqrt{4}\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & \sqrt{1} & 0 & 0 & 0\\ 0 & 0 & \sqrt{2} & 0 & 0\\ 0 & 0 & 0 & \sqrt{3} & 0\\ 0 & 0 & 0 & 0 & \sqrt{4}\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}^T\right)\end{aligned}$$

$$= x_0 \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix}$$

And:

$$\begin{aligned} \hat{p} &= -ip_0(\hat{a}^\dagger - \hat{a}) \\ &= -ip_0 \left(\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \right) \\ &= -ip_0 \begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix} \end{aligned}$$

Answer 23:

Respectively, $x_0 \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix}$ and $-ip_0 \begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix}$. Both are trivially hermitian.

Exercise 24:
Write down matrices for $\hat{a}^\dagger \hat{a}$ and $\hat{a}\hat{a}^\dagger$ operators for $N_{\text{max}} = 4$. Are these matrices identical.?

We have:

$$\begin{aligned} \hat{a}^\dagger \hat{a} &= \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \end{aligned}$$

And:

$$\hat{a}\hat{a}^\dagger = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Answer 24:

Respectively, $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. They are not the same.

Exercise 25:

Check the commutation $\left[\hat{a}, \hat{a}^\dagger\right] = \hat{I}$. If it does not exactly match, how do you think we can fix it?

We check:

$$\begin{aligned} \left[\hat{a}, \hat{a}^\dagger\right] &= \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix} \end{aligned}$$

Answer 25:

Prrof, see above. We can fix it by adding $N_{\max} + 1$ at the element in $(N_{\max}, N_{\max})$, which can be done by doing $\left[\hat{a}, \hat{a}^\dagger\right] + (N_{\max} + 1)|N_{\max}\rangle\langle N_{\max}|$ isntead of normal $\left[\hat{a}, \hat{a}^\dagger\right]$.

E. Wavefunctions

Exercise 26:
Use the recurssion relation to derive the following wave-functions of the oscillator's excited state:

$$\Psi_1(x) = \frac{x}{x_0}\Psi_0(x)$$

$$\Psi_2(x) = \frac{1}{\sqrt{2}}\left(\left(\frac{x}{x_0}\right)^2 - 1\right)\Psi_0(x)$$

$$\Psi_3(x) = \frac{1}{\sqrt{6}}\frac{x}{x_0}\left(\left(\frac{x}{x_0}\right)^2 - 3\right)\Psi_0(x)$$

Ψ_1 is found trivially. For the rest, we have:

$$\begin{aligned}\Psi_2(x) &= \frac{1}{\sqrt{2}}(\frac{x}{x_0}\Psi_1(x) - \Psi_0(x) \\ &= \frac{1}{\sqrt{2}}\left(\frac{x}{x_0}\left(\frac{x}{x_0}\Psi_0(x)\right) - \Psi_0(x)\right) \\ &= \frac{1}{\sqrt{2}}\left(\left(\frac{x}{x_0}\right)^2 - 1\right)\Psi_0(x)\end{aligned}$$

And:

$$\begin{aligned}\Psi_3(x) &= \frac{1}{\sqrt{3}}\left(\frac{x}{x_0}\Psi_2(x) - \sqrt{2}\Psi_1(x)\right) \\ &= \frac{1}{\sqrt{3}}\frac{x}{x_0}\left(\frac{1}{\sqrt{2}}\left(\left(\frac{x}{x_0}\right)^2 - 1\right) - \sqrt{2}\right)\Psi_0(x) \\ &= \frac{1}{\sqrt{6}}\frac{x}{x_0}\left(\left(\frac{x}{x_0}\right)^2 - 3\right)\Psi_0(x)\end{aligned}$$

Answer 26:
Proof, see above.

Exercise 27:
Verify that the following functin satisfies equation 62:

$$\Psi_0(x)\frac{1}{(2\pi x_0^2)^{\frac{1}{4}}}\exp\left(-\left(\frac{x}{2x_0}\right)^2\right)$$

We have:

$$x\Psi_0(x) + 2x_0^2 \frac{\partial}{\partial x}\Psi_0(x) = x\Psi_0(x) + 2x_0^2 \frac{-x}{2x_0^2} \frac{1}{(2\pi x_0^2)^{\frac{1}{4}}} \exp\left(-\left(\frac{x}{2x_0^2}\right)^2\right)$$
$$= x\Psi_0(x) + 2x\frac{-x}{2x_0^2}\Psi_0(x)$$
$$= \Psi_0(x)\left(x + 2x_0^2 \frac{-x}{2x_0^2}\right) \qquad = 0$$

Answer 27:
Proof, see above.

Exercise 28:
Plot the 3 lowest energy eigenstates wavefunctions. Count the number of nodes. Make a similar plot with $|\Psi|^2$. Are you suprised with where the oscillator is more or less likely to be?

Answer 28:
The graphs are on the slides seen in class, so I won't redraw them. There are $0, 1,$ and 2 nodes for Ψ_0, Ψ_1 and Ψ_2 respectively, and same for $|\Psi_i|^2$. Since we saw that in class, I'm not really surprised by anything, but it's worth mentionig that this doesn't follow classical expectations.

Exercise 29:
Verify by numerical integration that $\Psi_0(x), \Psi_1(x)$ and $\Psi_2(x)$ magically come out normalized in the sense of equation 51.

Answer 29:
I did not trudge through those three integrals myself but online calculators did find that those were all equal to 1:

$$\int_{-\infty}^{+\infty} |\Psi_0(x)|^2 \, \mathrm{d}x = 1$$

$$\int_{-\infty}^{+\infty} |\Psi_1(x)|^2 \, \mathrm{d}x = 1$$

$$\int_{-\infty}^{+\infty} |\Psi_2(x)|^2 \, \mathrm{d}x = 1$$

So we can say that they are indeed normalized in the sense of equation 51.

Exercise 30:
Verify by numerical integration that $\Psi_0(x), \Psi_1(x)$ and $\Psi_2(x)$ magically come out orthogonal.

Answer 30:
Again, I did not go throught them myself, but online calculators did find that:

$$\int_{-\infty}^{+\infty} \Psi_0(x)\Psi_1(x)\,\mathrm{d}x = 0$$

$$\int_{-\infty}^{+\infty} \Psi_1(x)\Psi_2(x)\,\mathrm{d}x = 0$$

$$\int_{-\infty}^{+\infty} \Psi_2(x)\Psi_0(x)\,\mathrm{d}x = 0$$

So we can conclude that they are indeed orthognoal.

Exercise 31:
Plot $\left|\Psi_{n(x)}\right|^2$ for $n = 0, 1, 2$. For each n, calculate the probability that $|x| < x_{RMS}$.

Answer 31:
Again, we've seen those plots in class. For the probabilities, we get $0.681, 0.303$ and 0.225 for $n = 0, 1$ and 2 respectively.

F. Discovering quantum mechanics with oscillator wavefunctions

Exercise 32:
Plot the oscillator's potential energy $V(x) = \frac{m\omega^2 x^2}{2} = \hbar\ \omega \frac{x^2}{4x_0^2}$ and identify the classically forbidden reegion geometrically for the ground state $\Psi_0(x)$.

We have:

#figure(image(“./res/hw2_question32.jpg”, width: 60%))

Answer 32:
See plot above. The classically forbidden regions are the areas under the $V(x)$ curve and outside of the zone delimited by $\pm x_c$ (green vertical lines).

Exercise 33:
Stack the plot from the previous exercise on top of a plot for $|\Psi_0(x)|^2$, using exactly the same range of x-axis. Indicate the probability to find teh oscillator at $|x| > x_c$ geometrically.

We have:

#figure(image(“./res/hw2_question33.jpg”, width: 60%))

Answer 33:
Dee plot above. The areas are those under the curve, outside of the $\pm x_c$ lines.

Exercise 34:
Make the “$V(x) - \Psi(x)$” plot for $|\Psi_{100}|^2$ and observe that one is more likely to find a particle near the boudaries of the classically forbidden region. Does this make sense with your classical intuitiion?

Answer 34:
See in class as well. The curve indeed goes higher towards the two ends (outside $\pm V(x)$), so it's more likely to find a particule there. This does make sense with classical intuition, since a classical oscillator would get slower when it reaches its turning points (and therefore spend more time in that area).

Exercise 35:
Calculate De Broglie wavelength of a cat chasing a mouse. Use any realistic assumptions on the mass and the speed of the cat.

We have:

$$\begin{aligned}\lambda_{\text{cat}} &= \frac{\hbar}{mv} \\ &= \frac{6.63 \cdot 10^{-34}}{3 \cdot 10} \\ &= 2.21 \cdot 10^{-35}\end{aligned}$$

Answer 35:
Assuming the cat weighs around $3kg$, and that it runs at about $10m.s^{-1}$, we get that its wavelength is $2.21 \cdot 10^{-35}m$

Exercise 36:
Plot $\Psi_{10}(x)$, zoom in to its period near the center $x = 0$, and extract the period, Verify that this period indeed approximately equals $2\pi\frac{x_0}{\sqrt{10+\frac{1}{2}}}$.

We have:

$$\frac{2\pi}{\sqrt{10 + \frac{1}{2}}}x_0 \approx 1.94x_0$$

Now, on the plot:

#figure(image("./res/hw2_question36.jpg", width: 60%))

We can approximate that the period is around $2x_0$, which matches with the formula.

Answer 36:
See plot above. The period is indeed approximately equal to the one we get using the analytical predition.

Exercise 37:
Reapeat the previous exercise and identify the De Broglie wavelength λ_n for state $|n\rangle, n = 10, 20, 30, ..., 100$. Summarize your answers on a λ_n vs n plot and compare them to the analytical prediction $\lambda_n = 2\pi\frac{x_0}{\sqrt{n}}$.

We have:

#figure(image("./res/hw2_question37.jpg", width: 60%))

Answer 37:
See plot above. The points match up pretty nicely, which was expected.