Homework 0

Classical Harmonic Oscillator

HO evolution equations in the matrix form

> **Exercise 1.1**
> What are the units of X and P?

We know the units of every component of the formulae for X and P, so we can deduce their units.
- x is position $[m]$
- m is mass $[kg]$
- k is spring stiffness $\left[\frac{N}{m}\right]$
- p is momentum $[N \cdot s]$

We get:

$$\begin{aligned}\omega = \sqrt{\frac{k}{m}} \text{ so } \left[\sqrt{\frac{N}{m \cdot kg}}\right] &= \left[\sqrt{\frac{\frac{kg \cdot m}{s^2}}{m \cdot kg}}\right] \\ &= \left[\sqrt{\frac{1}{s^2}}\right] \\ &= \left[\frac{1}{s}\right] \text{ (frequency, as expected)}\end{aligned}$$

$$\begin{aligned}X = x\sqrt{m \cdot \omega} \text{ so } [X] = [m]\sqrt{[kg]\left[\frac{1}{s}\right]} &= \left[\frac{m \cdot \sqrt{kg}}{\sqrt{s}}\right] \\ &= \left[\frac{m \cdot kg^{\frac{1}{2}}}{s^{\frac{1}{2}}}\right] = \left[m \cdot kg^{\frac{1}{2}} \cdot s^{-\frac{1}{2}}\right]\end{aligned}$$

$$\begin{aligned}P = \frac{p}{\sqrt{m \cdot \omega}} \text{ so } [P] = \frac{[N \cdot s]}{\sqrt{[kg] \cdot \left[\frac{1}{s}\right]}} &= \left[\frac{N \cdot s}{\sqrt{\frac{kg}{\frac{1}{s}}}}\right] \\ &= \left[\frac{\frac{kg \cdot m \cdot s}{s^2}}{\sqrt{\frac{kg}{\frac{1}{s}}}}\right] \\ &= \left[\frac{kg \cdot m \cdot s}{s^2 \sqrt{\frac{kg}{\frac{1}{s}}}}\right] \\ &= \left[\frac{kg \cdot m}{kg^{\frac{1}{2}} \cdot s^{\frac{1}{2}}}\right] \\ &= \left[\frac{kg^{\frac{1}{2}} \cdot m}{s^{\frac{1}{2}}}\right] = \left[m \cdot kg^{\frac{1}{2}} \cdot s^{-\frac{1}{2}}\right]\end{aligned}$$

> **Exercise 1.2**
> Check by explicit multiplication that $I^2 = I$ and $\hat{\Omega}^2 = -\omega^2 I$.

$$I^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\hat{\Omega}^2 = \omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$= \omega^2\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= -\omega^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\omega^2 I$$

> **Exercise 1.3**
> Check that expression (3) is indeed the solution of equation (2) by taking the time-derivative of $\vec{s}(t)$ and the above definition of the matrix exponentiation.

We have:
$(3) \equiv \vec{s}(t) = \exp\left(\hat{\Omega}t\right)\vec{s}(t = 0)$

And, taking the time-derivative:

$$\dot{\vec{s}}(t) = \hat{\Omega}\exp\left(\hat{\Omega}t\right)\vec{s}(t = 0)$$
$$= \hat{\Omega}\vec{s}(t) \equiv (2)$$

Oscillator motion in real space is a rotation in the (X, P)-space

> **Exercise 1.4**
> Prove the above formula for $\exp\left(\hat{\Omega}t\right)$.

We have:
$(5) \equiv \exp\left(\hat{\Omega}t\right) = I\cos(\omega t) + \frac{\hat{\Omega}}{\omega}\sin(\omega t)$

$$= \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

Computing the powers of $\hat{\Omega}$, we find that:

$\hat{\Omega}^{4k} = \omega^{4k}I \quad ; \quad \hat{\Omega}^{4k+1} = \omega^{4k}\hat{\Omega} \quad ; \quad \hat{\Omega}^{4k+2} = -\omega^{4k+2}I \quad ; \quad \hat{\Omega}^{4k+3} = -\omega^{4k+2}\hat{\Omega}$

Substituting that in the matrix exponentiation:

$$\begin{aligned}
\exp\left(\hat{\Omega}t\right) &= I + \hat{\Omega}t + \frac{\hat{\Omega}^2t^2}{2!} + \frac{\hat{\Omega}^3t^3}{3!} + \frac{\hat{\Omega}^4t^4}{4!} + \frac{\hat{\Omega}^5t^5}{5!} + \ldots \\
&= I + \hat{\Omega}t + \frac{-\omega^2It^2}{2!} + \frac{-\omega^2\hat{\Omega}t^3}{3!} + \frac{\omega^4It^4}{4!} + \frac{\omega^4\hat{\Omega}t^5}{5!} + \ldots \\
&= I\left(1 - \frac{(\omega t)^2}{2!} + \frac{(\omega t)^4}{4!} - \ldots\right) + \hat{\Omega}\left(t - \frac{\omega^2t^3}{3!} + \frac{\omega^4t^5}{5!} - \ldots\right) \\
&= I\left(1 - \frac{(\omega t)^2}{2!} + \frac{(\omega t)^4}{4!} - \ldots\right) + \frac{\hat{\Omega}}{\omega}\left(\omega t - \frac{(\omega t)^3}{3!} + \frac{(\omega t)^5}{5!} - \ldots\right) \\
&= I\cos(\omega t) + \frac{\hat{\Omega}}{\omega}\sin(\omega t)
\end{aligned}$$

> **Exercise 1.5**
> Show that the matrix $\exp\left(\hat{\Omega}t\right)$ corresponds to a rotation of a vector in the (X, P)-plane by an angle ωt, clockwise.

We have:
$\exp\left(\hat{\Omega}t\right) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}$

This is the standard form for a rotation matrix with angle $\omega\, t$. Using the two proposed vectors, we can check that they are indeed rotated clockwise with a ωt angle and still have the same norm:

$$\begin{aligned} \exp\left(\hat{\Omega}t\right)\begin{pmatrix} X_0 \\ 0 \end{pmatrix} &= \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}\begin{pmatrix} X_0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} X_0\cos(\omega t) \\ -X_0\sin(\omega t) \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \exp\left(\hat{\Omega}t\right)\begin{pmatrix} 0 \\ P_0 \end{pmatrix} &= \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}\begin{pmatrix} 0 \\ P_0 \end{pmatrix} \\ &= \begin{pmatrix} P_0\sin(\omega t) \\ P_0\cos(\omega t) \end{pmatrix} \end{aligned}$$

> **Exercise 1.6**
> Speculate what might be a useful application of the fact that HO's period does not depend on initial conditions. Think watchmaking industry.

In the watchmaking industry, this can be used to keep time accurately since regardless of how much the HO inside the clock is wound, the period stays the same. This is used in metronomes as well (also to keep time).

> **Exercise 1.7**
> Find a matrix $\exp\left(\hat{\Omega}t\right)^{-1}$ which is defined as the inverse of $\exp\left(\hat{\Omega}t\right)$.

The inverse matrix would be a rotation matrix of the same angle but in the opposite direction:

$$\boxed{\exp\left(\hat{\Omega}t\right)^{-1} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}}$$

We can check:

$$\begin{aligned} \exp\left(\hat{\Omega}t\right)\exp\left(\hat{\Omega}t\right)^{-1} &= \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}\begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\omega t)\cos(\omega t) + \sin(\omega t)\sin(\omega t) & -\cos(\omega t)\sin(\omega t) + \sin(\omega t)\cos(\omega t) \\ -\sin(\omega t)\cos(\omega t) + \cos(\omega t)\sin(\omega t) & \sin(\omega t)\sin(\omega t) + \cos(\omega t)\cos(\omega t) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \end{aligned}$$

> **Exercise 1.8**
> Show by explicit calculation using equation (5) that the quantity $X(t)^2 + P(t)^2 = X_0^2 + P_0^2$, that is, the value of the Hamiltonian function — basically the total energy of the system — does not change in time.

We have:
$(5) \equiv \exp\left(\hat{\Omega}t\right) = I\cos(\omega t) + \frac{\hat{\Omega}}{\omega}\sin(\omega t)$
$= \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}$

And:
$X(t) = X_0\cos(\omega t) + P_0\sin(\omega t) \quad ; \quad P(t) = -X_0\sin(\omega t) + P_0\cos(\omega t)$

We can compute:

$$\begin{aligned} X(t)^2 &= (X_0\cos(\omega t) + P_0\sin(\omega t))^2 \\ &= X_0^2\cos^2(\omega t) + P_0^2\sin^2(\omega t) + 2X_0P_0\cos(\omega t)\sin(\omega t) \end{aligned}$$

$$\begin{aligned} P(t)^2 &= (-X_0\sin(\omega t) + P_0\cos(\omega t))^2 \\ &= X_0^2\sin^2(\omega t) + P_0^2\cos^2(\omega t) - 2X_0P_0\cos(\omega t)\sin(\omega t) \end{aligned}$$

So we get:

$$\begin{aligned} X(t)^2 + P(t)^2 &= X_0^2\cos^2(\omega t) + P_0^2\sin^2(\omega t) + X_0^2\sin^2(\omega t) + P_0^2\cos^2(\omega t) \\ &= X_0^2\left(\cos^2(\omega t) + \sin^2(\omega t)\right) + P_0^2\left(\cos^2(\omega t) + \sin^2(\omega t)\right) \\ &= X_0^2 + P_0^2 \end{aligned}$$

Exercise 1.9
Verify that $X(t)$ and $P(t)$ obtained using the complex number formulation agree with the matrix formulation solution.

We have:
$Z(t) = Z_0\exp(-i\omega t)$ the complex number formulation.

We can expand via the definition of the complex exponential form:

$$\begin{aligned} Z(t) &= Z_0\exp(-i\omega t) \\ &= (X_0 + iP_0)(\cos(\omega t) - i\sin(\omega t)) \\ &= X_0\cos(\omega t) - iX_0\sin(\omega t) + iP_0\cos(\omega t) + P_0\sin(\omega t) \\ &= X_0\cos(\omega t) + P_0\sin(\omega t) + i(P_0\cos(\omega t) - X_0\sin(\omega t)) \end{aligned}$$

And we get:

$$X(t) = \mathrm{Re}(Z(t)) = X_0\cos(\omega t) + P_0\sin(\omega t)$$

$$P(t) = \mathrm{Im}(Z(t)) = P_0\cos(\omega t) - X_0\sin(\omega t)$$

Which are exactly the solutions from the matrix formulation.

Periodically-driven oscillator: resonance

Exercise 1.10
Show that the Hamiltonian equations of motion in the presence of drive, in the matrix form, are

$$\dot{\vec{s}} = \hat{\Omega}\vec{s} + \vec{F}(t),$$

where $\vec{F}(t) = (0, f(t))$, where $f(t) = \frac{F(t)}{\sqrt{m\omega}}$.

The Hamiltonian equation with drive is:
$H_D(x,p) = \frac{p^2}{2m} + \frac{kx^2}{2} - x\vec{F}(t)$

We get:

$$\dot{x} = \frac{\partial H_D}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H_D}{\partial x} = -kx + F(t)$$

Following the same procedure as described earlier in the handout, we can redefine $X = x\sqrt{m\omega}$ and $P = \frac{p}{\sqrt{m\omega}}$, and a column vector $\vec{s} = (X,P)^T$. We also reuse the definition $\hat{\Omega} = \omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We want to find the relation between $\vec{s}$ and $\dot{\vec{s}} = \left(\dot{X}, \dot{P}\right)^T$:

$$\begin{aligned} \dot{X} &= \dot{x}\sqrt{m\omega} \\ &= \frac{p\sqrt{m\omega}}{m} \\ &= \frac{p}{\sqrt{m\omega}}\frac{\sqrt{m\omega}\sqrt{m\omega}}{m} \\ &= P\frac{\sqrt{m\omega}\sqrt{m\omega}}{m} \\ &= \omega P \end{aligned}$$

$$\begin{aligned} \dot{P} &= \frac{\dot{p}}{\sqrt{m\omega}} \\ &= \frac{-kx + F(t)}{\sqrt{m\omega}} \\ &= \frac{\frac{-kx\sqrt{m\omega}}{\sqrt{m\omega}} + F(t)}{\sqrt{m\omega}} \\ &= \frac{\frac{-kX}{\sqrt{m\omega}} + F(t)}{\sqrt{m\omega}} \\ &= \frac{-kX}{m\omega} + \frac{F(t)}{\sqrt{m\omega}} \\ &= \frac{-\omega^2 X}{\omega} + \frac{F(t)}{\sqrt{m\omega}} \\ &= -\omega X + \frac{F(t)}{\sqrt{m\omega}} \end{aligned}$$

As given in the handout, we define $f(t) = \frac{F(t)}{\sqrt{m\omega}}$. We get:

$$\begin{aligned} \dot{\vec{s}} &= \begin{pmatrix} \dot{X} \\ \dot{P} \end{pmatrix} \\ &= \begin{pmatrix} \omega P \\ -\omega X + f(t) \end{pmatrix} \\ &= \begin{pmatrix} \omega P \\ -\omega X \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \end{aligned}$$

Again, defining as given in the handout $\vec{F}(t) = (0, f(t))^T$, we get:

$$\dot{\vec{s}} = \begin{pmatrix} \omega P \\ -\omega X \end{pmatrix} + \vec{F}(t)$$

Finally, using the $\hat{\Omega}$ described above, we have:

$$\dot{\vec{s}} = \hat{\Omega}\vec{s} + \vec{F}(t)$$

Exercise 1.11
Show by a direct substitution that (the case of the scalar equation) $\dot{s_0} = \exp(-\Omega t)F(t)$, and hence $s_0(t) = \int^t \exp(-\Omega t')F(t')dt' + C$ and $s(t) = \exp(\Omega t)\left(C + \int^t \exp(-\Omega t')F(t')dt'\right)$. The constant C defines the initial conditions.

We have:
$\dot{s} = \Omega s + F(t)$

The ansatz for this equation is of the form:

$$s(t) = s_0(t)\exp(\Omega t)$$

And:

$$\dot{s} = \dot{s}_0 \exp(\Omega t) + s_0 \Omega \exp(\Omega t)$$

We substite into the differential equation to get:

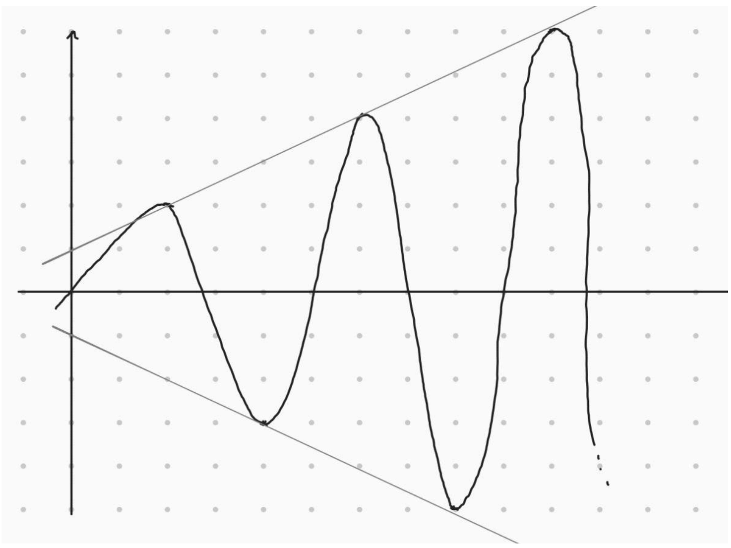
$$\begin{aligned}\dot{s}_0 \exp(\Omega t) + s_0 \Omega \exp(\Omega t) &= s_0 \Omega \exp(\Omega t) + F(t)\\ \dot{s}_0 \exp(\Omega t) &= F(t)\\ \dot{s}_0 &= \exp(-\Omega t)F(t)\\ s_0(t) &= \int^t \exp\left(-\Omega t'\right)F(t')dt' + C\end{aligned}$$

Now, substitue this back into the ansatz:

$$s(t) = \exp\left(\Omega t\right)\left(\int^t \exp(-\Omega t')F(t')dt' + C\right)$$

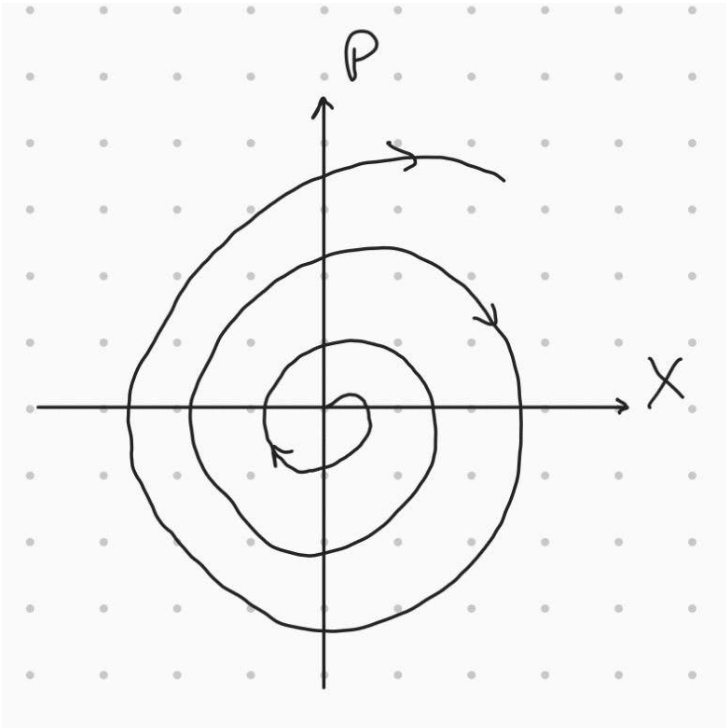
Exercise 1.12
Plot $X(t)$ as well as a t-parametric plot in the $X - P$ plane, starting from $t = 0$ to $t = 17 \times \frac{2\pi}{\omega}$. Assume that at $t = 0$, $X = 0$ and $P = 0$. Describe in words what happens with an oscillator driven exactly in resonance.

Plot of $X(t)$ with respect to time:



(period stays the same but amplitude grows lineraly)

t-parametric plot in the $X - P$ plane:



(clockwise spiral towards the outside. I'm sorry I did not have time to measure precisely)

When an oscillator is driven exactly in resonance, the oscillator continously gets energy from the drive, so its amplitude grows linearly in time: it oscillates always at the same frequency but grows in amplitude.