Question 1 (2018 A1).

- Define the terms *ring, integral domain* and *field.*
- Is it true that every integral domain is a field? Justify your answer.

Question 2 (2018 A2). Suppose that R is a ring and that $R[X]$ is a polynomial ring in one indeterminate over R. Suppose that $g \in R[X]$ is a monic element and that f is any element of $R[X]$. Explain the process of *long division* of f by g. Illustrate your answer with the case where $R = \mathbb{Z}$, $g = X^2 - 5$ and $f = 2X^3 + 1$.

Question 3 (2018 A3).

- If R, S are rings, define the notion of a *homomorphism* from R to S.
- Define the notion of the *kernel* of a homomorphism.
- Define the notion of an *ideal* of a ring R.
- Is the set of all odd integers an ideal in the ring $\mathbb{Z}$? Justify your answer.

Question 4 (2016 A1).

- Show that if
$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$
is an ascending chain of ideals, then $\bigcup_{n\geq 1} I_n$ is also an ideal of A.
- Consider the ideals (2) and (3) in the ring $\mathbb{Z}$. Is $(2) \cup (3)$ also an ideal in $\mathbb{Z}$?

Question 5 (2018 A4). Construct a homomorphism $\phi : \mathbb{Z}[X] \to \mathbb{C}$ whose kernel equals $(X^2 + 5)$, the principal ideal generated by $X^2 + 5$. Justify your answer.

Question 6 (2017 A1). An element of a ring R is said to be *nilpotent* if there is an integer $m > 0$ such that $x^m = 0$. Show that if x and y are nilpotent elements of A, then $x + y$ is also nilpotent.

Question 7 (2017 A2). Is the polynomial ring $A[X]$ finitely generated as an A-module? Justify your answer.

Question 8 (2017 A3). State Eisenstien's criterion for the irreducibility of a polynomial over $\mathbb{Q}$. Use it to show that $X^3 + 3X + 3$ is irreducible in $\mathbb{Q}[X]$. (You may use any other general result that you wish, provided you state it clearly.)

Question 9 (2017 A4). Define the notion of a *prime ideal* in a ring A, and show that an ideal I in A is prime if and only if A/I is an integral domain.

Question 10 (2016 A4, 2015 A4 (Aug)). Define the notion of a *maximal ideal* in a ring A, and show that an ideal I in A is maximal if and only if A/I is a field.

Question 11 (2016 A2). Show that for any ring A and any element $a \in A$, the ring $A[X]/(X - a)$ is isomorphic to A.

Question 12 (2015 A1 (both)). Show that the rings $\mathbb{Z}[\sqrt{-7}]$ (May) and $\mathbb{Z}[\sqrt{-3}]$ (Aug) are not unique factorisation domains.

Question 13 (2015 A3 (May)).

- Suppose that I is an ideal in the polynomial ring $A[X]$, that f is a monic polynomial of I of degree n and that every nonzero element of I has degree at least n. Show that $I = (f)$, the principal ideal generated by f.
- Prove that $\mathbb{Z}[7^{1/3}]$ is isomorpic to $\mathbb{Z}[X]/(X^3 - 7)$, where X is an indeterminate.

Question 14 (combined 2018 B5, 2017 B5, 2015 A2 (May/Aug)).

- Define the notion of a *Euclidean domain.*
- Prove that $\mathbb{Z}$ and $\mathbb{Z}[i]$ are Euclidean.
- Define the notion of a *principal ideal domain* and show that every Euclidean domain is a principal ideal domain.

Question 15 (2018 B6, 2015 A3 (Aug), 2016 B6).

- Define the notion of a *Noetherian ring.* Prove that, if R is a Noetherian ring, then so is the polynomial ring $R[X]$.
- Consider the homomorphism $\phi : \mathbb{Z}[X] \to \mathbb{R}$ defined by $\phi(X) = \sqrt{5}$. Show that the kernel of ϕ is a principal ideal in $\mathbb{Z}[X]$ and find a generator of this ideal.

Question 16 (combined 2018 B7, 2015 B5 (Aug)).

- Suppose that R is a Noetherian ring and that M is a finitely generated R-module. Explain how M can be described in terms of matrices with entries in R. (You may assume that every submodule of a finitely generated R-module is also finitely generated.)
- State the structure theorem for finitely generated modules over a Euclidean domain R and explain how it can be derived from a theorem about matrices with entries in R.
- Apply the structure theorem to the $\mathbb{Z}$-module M generated by m_1, m_2, m_3 subject to the relations $m_1 + 2m_2 + 3m_3 = 0$ and $4m_1 + 5m_2 + 6m_3 = 0$.

Question 17 (2016 A3). Prove that $X^3 + X + 1$ is irreducible in $\mathbb{Q}[X]$. (You may use any general result that you wish, provided you state it clearly.)

Question 18 (2015 B6 (Aug/May), 2018 B8). Suppose that R is a unique factorisation domain, with fraction field K.

- Define the notion of *content* $c(f)$ of an element $f \in R[X]$ and prove that $c(fg) = c(f)c(g)$ for $g \in R[X]$.
- Show that if f is an irreducible element of $R[X]$ that is irreducible in $R[X]$, then it is irreducible in $K[X]$.
- Show that $X^3 - 5X + 1$ and $X^3 - 3X + 1$ are irreducible in $\mathbb{Q}[X]$.

Question 19 (2017 B6). Give an example of a principal ideal domain A such that $A[X]$ is also a principal ideal domain, and of a principal ideal domain B such that $B[X]$ is not a principal ideal domain. Justify your answers.

Question 20 (2017 B7)**.** Suppose that A is a unique factorisation domain. Say what it means for an element $f \in A[X]$ to be *primitive*, and that the product of two primitive polynomials is primitive.

Question 21 (2017 B8)**.** Consider the homomorphism $\phi : \mathbb{Q}[X, Y] \to \mathbb{Q}[t]$ given by $\phi(X) = t^3$, $\phi(Y) = t^4$.

- Find an element f of $\ker(\phi)$ such that $\deg(f) = 4$.
- Show that f generates $\ker(\phi)$ as an ideal of $\mathbb{Q}[X, Y]$.

Question 22 (2016 B7)**.** Let A denote the subring $\mathbb{Q}[t^2, t^5]$ of the polynomial ring $\mathbb{Q}[t]$. Determine the kernel of the homomorphism $\phi : \mathbb{Q}[X, Y] \to A$ given by $\phi(X) = t^2$, $\phi(Y) = t^5$.

Question 23 (2015 B7 (Aug/May), 2016 B8)**.** Suppose that A is a ring, I an ideal of A, M an A-module with n genrators and $\phi : M \to M$ an A-homomorphism such that $\phi(M) \subseteq IM$. Show that ϕ satisfies an equation of the form

$$\phi^n + a_{n-1}\phi^{n-1} + \cdots + a_0 = 0$$

where $a_{n-1}, \ldots, a_0 \in I$.

Question 24 (2015 B8 (Aug))**.** If R, S are rings, then the direct sum $R \oplus S$ is the ring of ordered pairs (r, s) with $r \in R$ and $s \in S$ with the ring operations $(r, s)(r', s') = (rr', ss')$ and $(r, s) + (r', s') = (r + r', s + s')$. Show that if I, J are ideals in a ring A such that $I + J = A$, then $A/(I \cap J) = (A/I) \oplus (A/J)$.

Question 25 (2015 B5 (May))**.** Suppose that k is a field, that the characteristic of k is not 2 and that X, Y, Z, S, T are independent indeterminates. Consider the homomorphism

$$\phi : k[X, Y, Z] \to k[S, T]$$

defined by $X \mapsto S^2$, $Y \mapsto ST$, $Z \mapsto T^2$. Describe $\ker(\phi)$ in terms of a generator or generators.