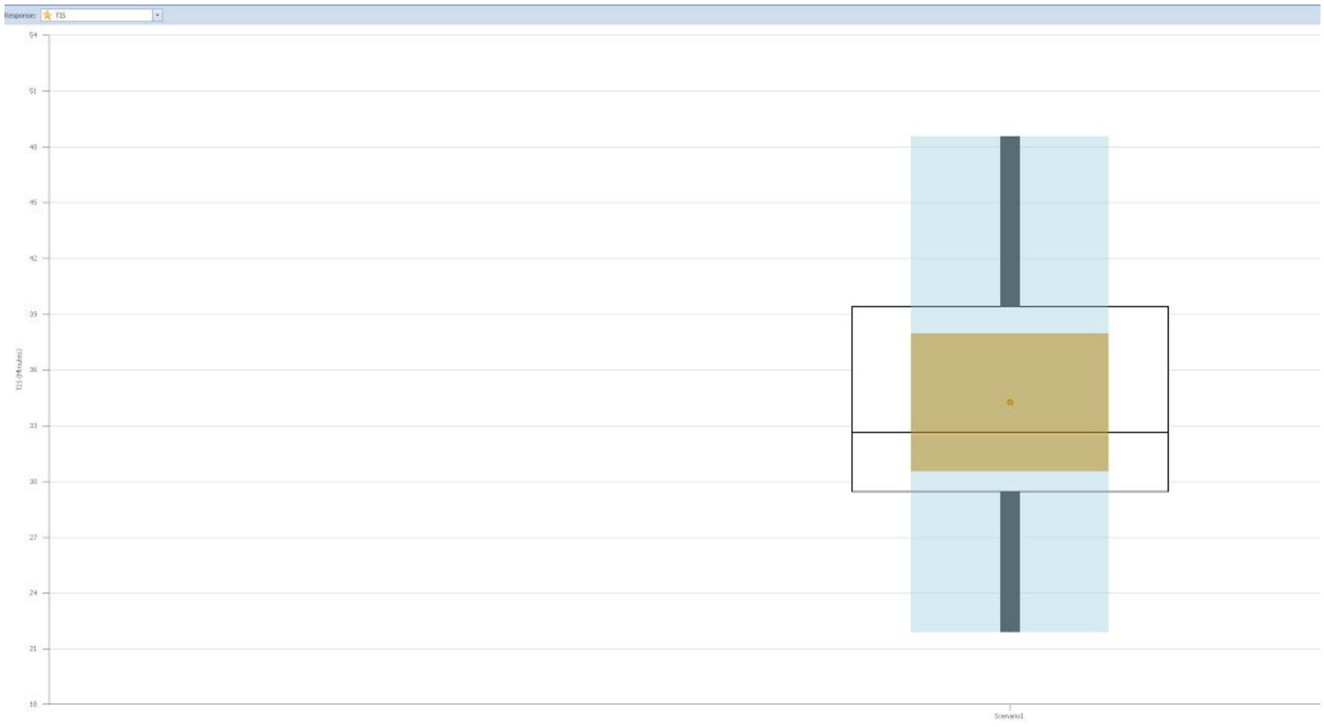
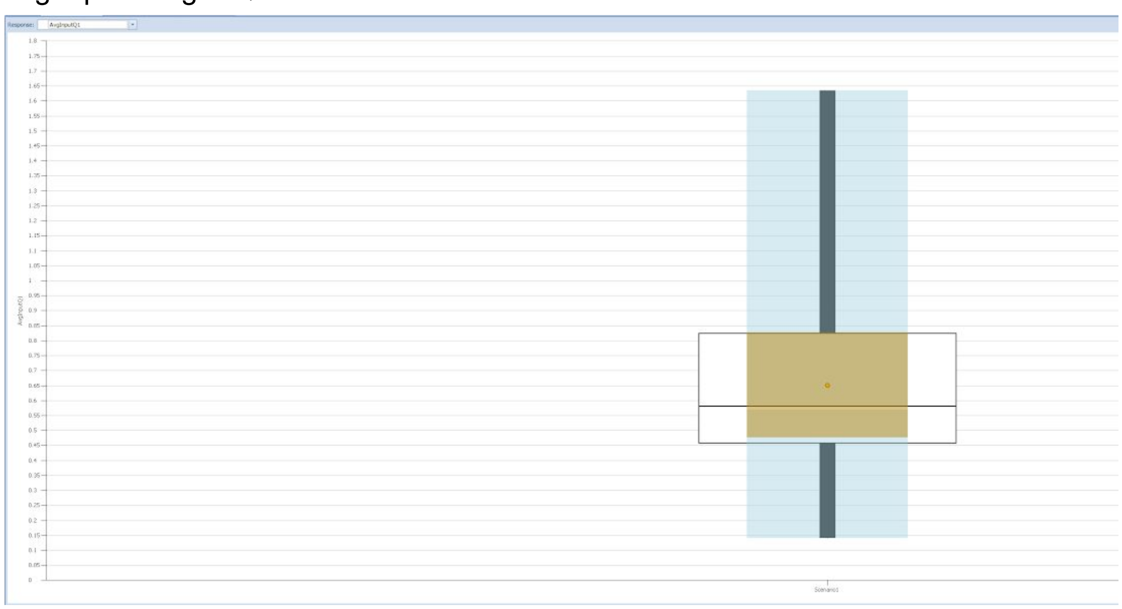
Problem 1.

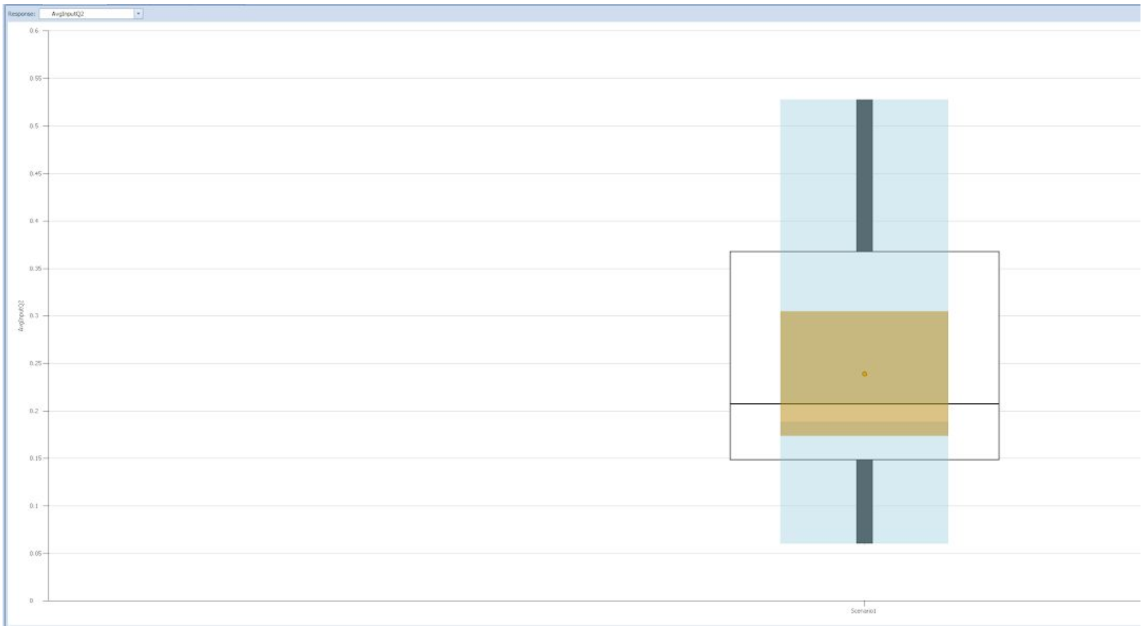
(a) Avg Time In System:



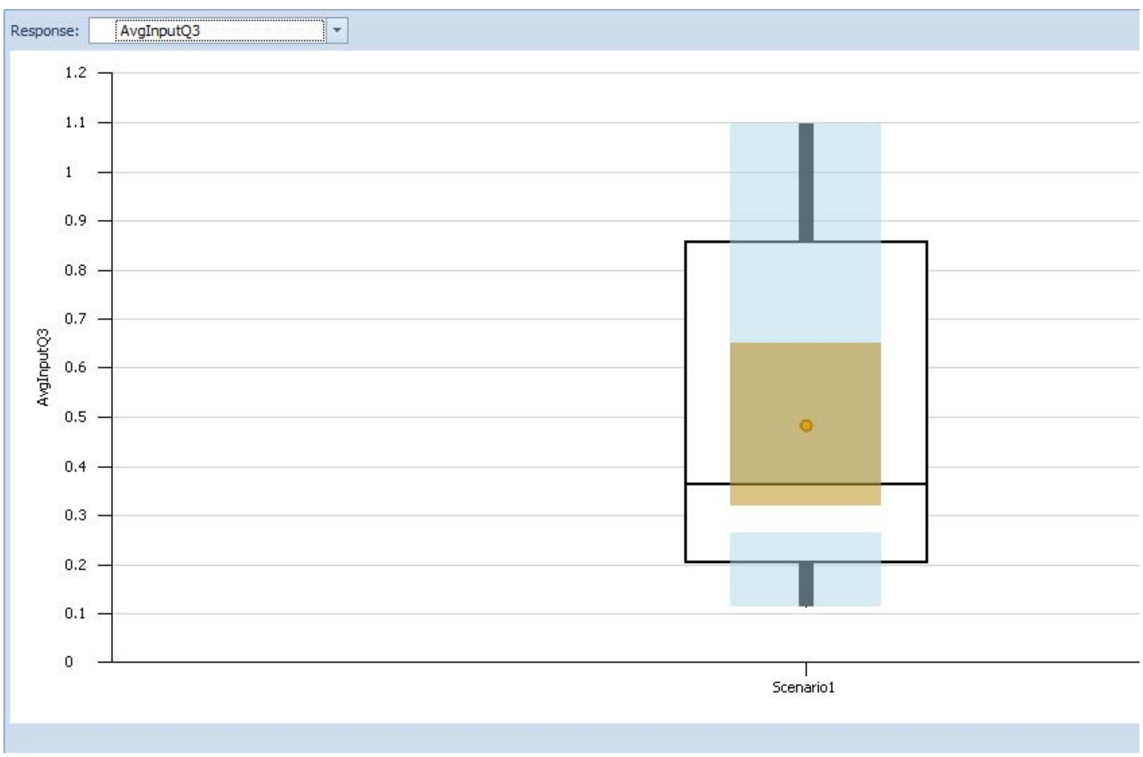
Avg Input Length Queue 1:



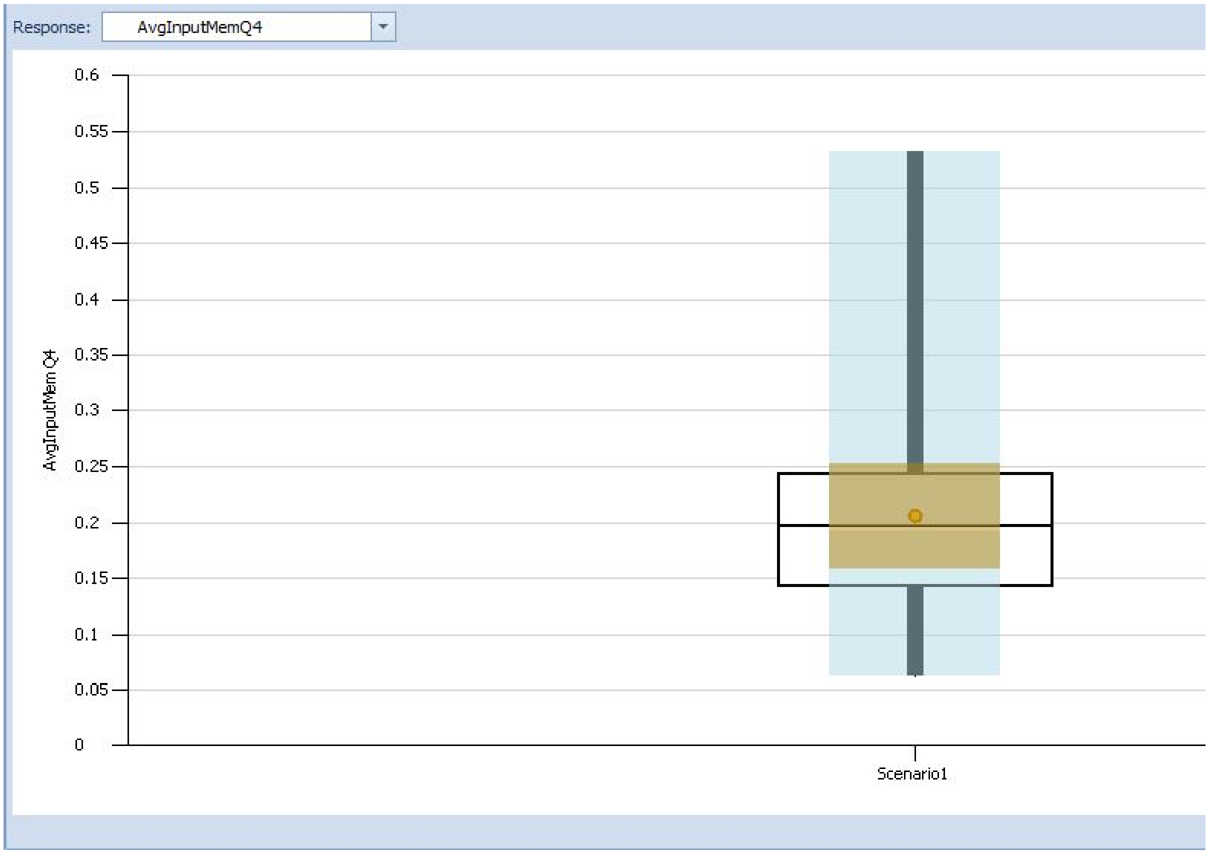
Avg Input Length Queue 2:



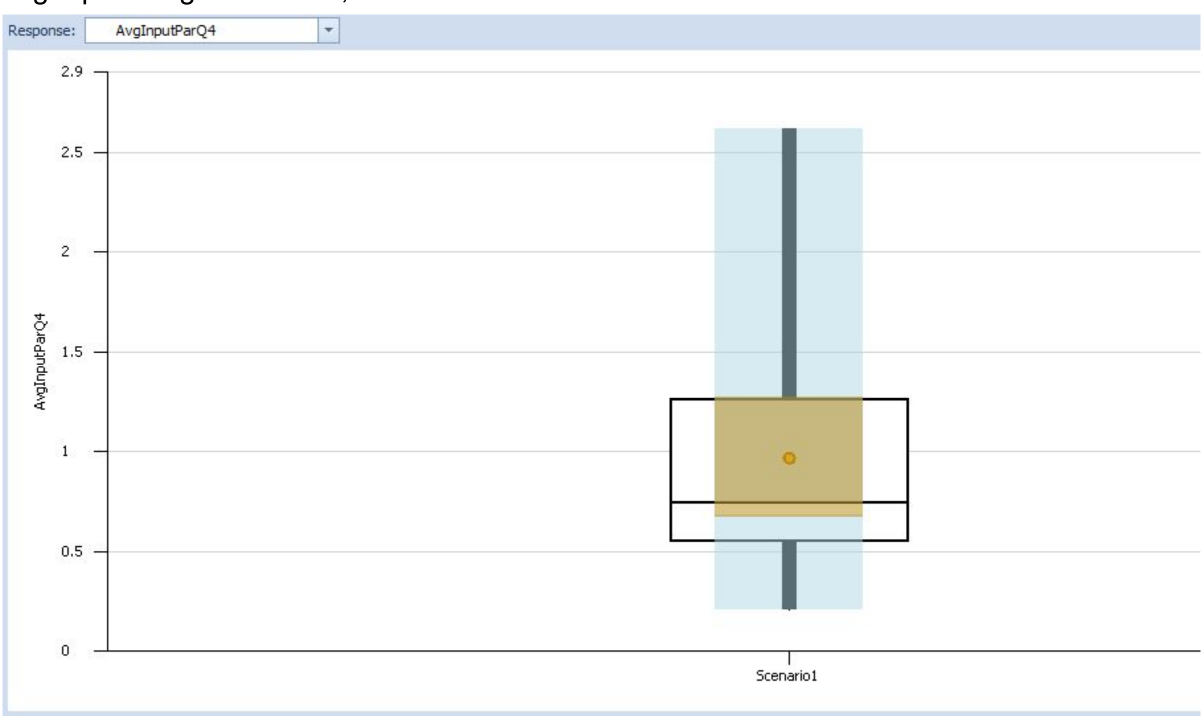
Avg Input Length Queue 3:



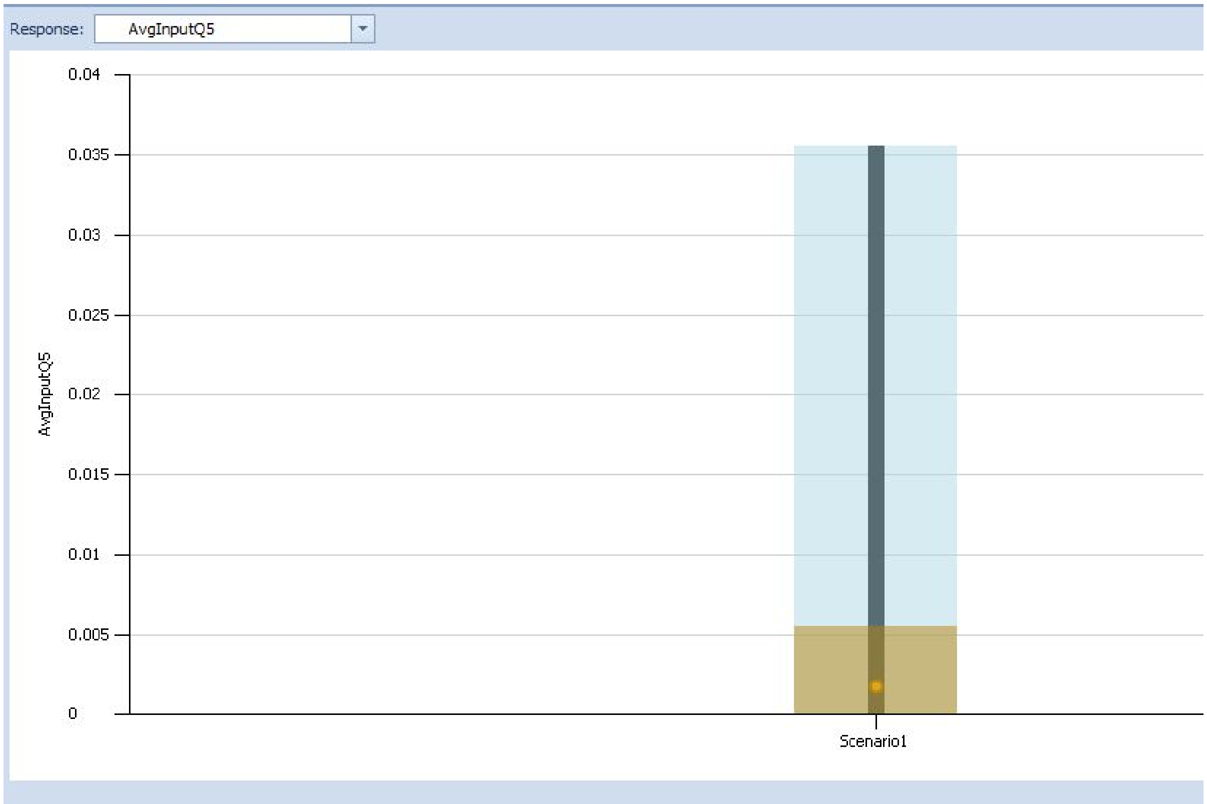
Avg Input Length Queue 4, Member:



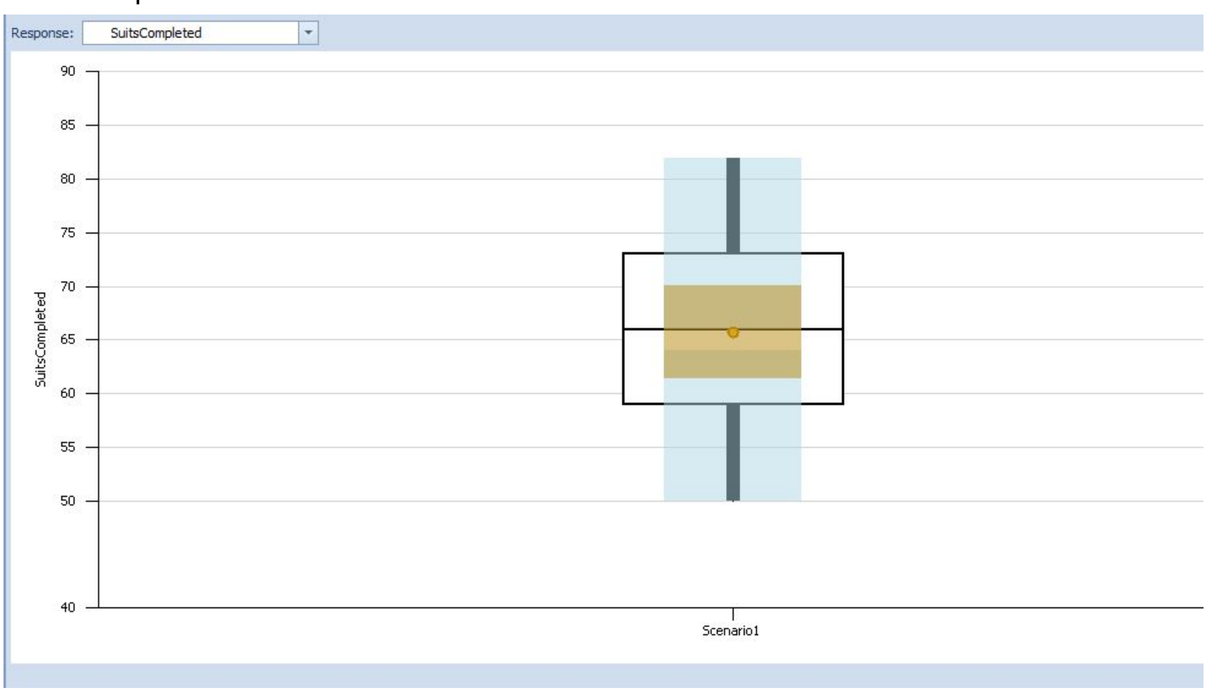
Avg Input Length Queue 4, Parent:



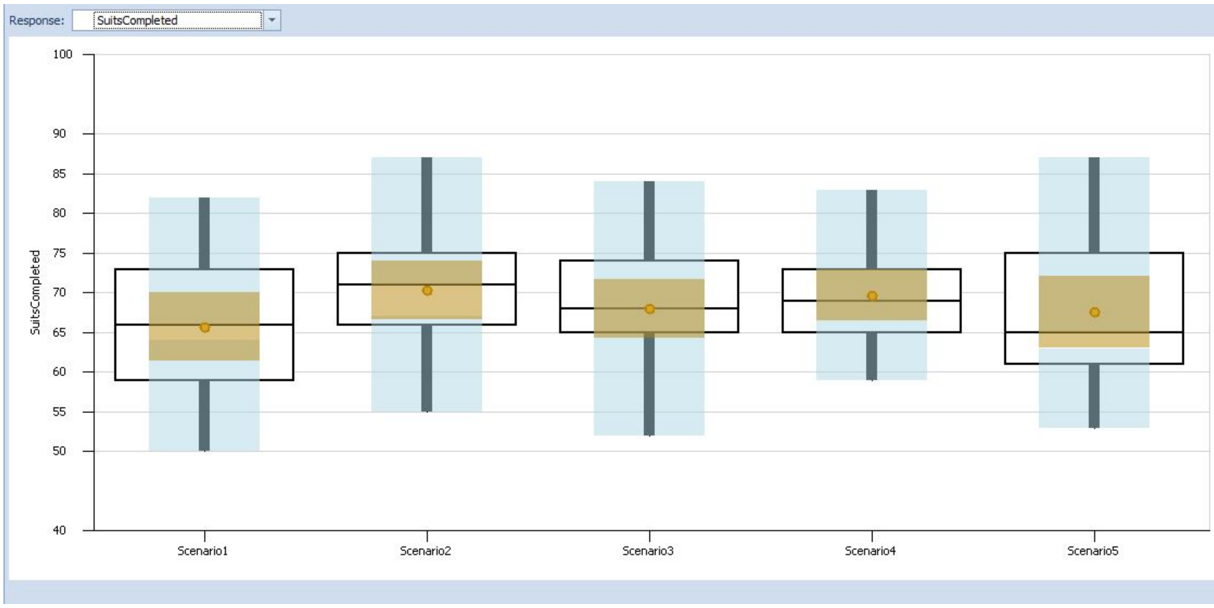
Avg Input Length Queue 5:



Suits Completed:



(b) Choose to add an extra Server to Station 1, less variation than adding to Station 4.



Problem 2.

Problem 2.

(a) $f_X(x)=\int_1^2 (y-x)\,dy = \left[\frac{y^2}{2}-xy\right]_1^2 = 2-2x-(\frac{1}{2}-x)$

$= \frac{3}{2}-x$

$$f_X(x)=\begin{cases}\frac{3}{2}-x, & 0<x<1\\ 0, & ow\end{cases}$$

$f_Y(y)=\int_0^1 (y-x)\,dx = \left[\frac{-x^2}{2}+xy\right]_0^1 = \left[-\frac{1}{2}+y-(0)\right]$

$$f_Y(y)=\begin{cases}-\frac{1}{2}+y, & 1<y<2\\ 0, & ow\end{cases}$$

(b)

$f(x,y)=y-x \overset{?}{=} f_X(x)\,f_Y(y) = (\frac{3}{2}-x)(-\frac{1}{2}+y)$

$= -\frac{3}{4}+\frac{x}{2}+\frac{3y}{2}-xy$

$\therefore$ X, Y are not independent

(c) $F_X(x)=P\{X\le x\}=\int_0^x(\frac{3}{2}-x)\,dx = \left[\frac{3}{2}x-\frac{x^2}{2}\right]_0^x$

$$F_X(x)=\begin{cases}\frac{3x}{2}-\frac{x^2}{2}, & 0<x<1\\ 0, & x\le 0\\ 1, & x\ge 1\end{cases}$$

$F_Y(y)=P\{Y\le y\}=\int_1^y(-\frac{1}{2}+y)\,dy = \left[-\frac{y}{2}+\frac{y^2}{2}\right]_1^y$

$$F_Y(y)=\begin{cases}-\frac{y}{2}+\frac{y^2}{2}, & 1<y<2\\ 0, & y\le 1\\ 1, & y\ge 2\end{cases}$$

(d) $E[X]=\int_0^1 x(\frac{3}{2}-x)\,dx=\int_0^1(\frac{3}{2}x-x^2)\,dx = \left[\frac{3x^2}{4}-\frac{x^3}{3}\right]_0^1$

$= \frac{5}{12}$

$E[X^2]=\int_0^1 x^2(\frac{3}{2}-x)\,dx=\int_0^1(\frac{3}{2}x^2-x^3)\,dx = \left[\frac{x^3}{2}-\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{11}{144}$$

$$E[Y] = \int_1^2 y\left(-\frac{1}{2}+y\right)dy = \int_1^2 \left(-\frac{y}{2}+y^2\right)dy = \left[-\frac{y^2}{4}+\frac{y^3}{3}\right]_1^2$$

$$= \left(-1+\frac{8}{3}\right) - \left(-\frac{1}{4}+\frac{1}{3}\right) = -\frac{3}{4}+\frac{7}{3} = \frac{19}{12}$$

$$E[Y^2] = \int_1^2 y^2\left(-\frac{1}{2}+y\right)dy = \int_1^2\left(-\frac{y^2}{2}+y^3\right)dy = \left[-\frac{y^3}{6}+\frac{y^4}{4}\right]_1^2$$

$$= \left(-\frac{8}{6}+\frac{16}{4}\right) - \left(-\frac{1}{6}+\frac{1}{4}\right) = -\frac{7}{6}+\frac{15}{4} = \frac{31}{12}$$

$$Var(Y) = E[Y^2] - (E[Y])^2 = \frac{31}{12} - \left(\frac{19}{12}\right)^2 = \frac{11}{144}$$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{2}{3} - \left(\frac{5}{12}\right)\left(\frac{19}{12}\right) = \frac{1}{144}$$

$$E[XY] = \int_0^1\int_1^2 xy(y-x)\,dy\,dx = \int_0^1\int_1^2 (xy^2 - x^2y)\,dy\,dx$$

$$= \int_0^1 \left[\frac{xy^3}{3} - \frac{x^2y^2}{2}\right]_1^2 dx = \int_0^1\left(\frac{7x}{3} - \frac{3x^2}{2}\right)dx$$

$$= \left[\frac{7x^2}{6} - \frac{x^3}{2}\right]_0^1 = \frac{7}{6} - \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$$

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{(1/144)}{\sqrt{11/144}\sqrt{11/144}} = \frac{1}{11}$$

Problem 3.

Problem 3.

$$(4.6)\quad E[S_n^2] = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x}_n)^2\right]$$

$$= \frac{1}{n-1}E\left[\sum_{i=1}^{n}(x_i-\bar{x}_n)^2\right]$$

$$= \frac{1}{n-1}E\left[\sum^{n}(x_i-\mu)^2 - n(\bar{x}_n-\mu)^2\right]$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\underbrace{E[(x_i-\mu)^2]}_{var(x_i)} - \frac{1}{n-1}\,n\,var(\bar{x}_n)$$

$$= \frac{1}{n-1}\left[n\,var(x_i) - n\,var(\bar{x}_n)\right]$$

$$= \frac{1}{n-1}\left[n\sigma^2 - n\left(\frac{\sigma^2}{n}\left[1+2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k\right)\right]\right.$$

$$= \frac{1}{n-1}\left[n\sigma^2 - \sigma^2 - 2\sigma^2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k\right]$$

$$= \frac{1}{n-1}\left[\sigma^2(n-1) - 2\sigma^2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k\right]$$

$$= \sigma^2 - \frac{2\sigma^2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k}{n-1}$$

$$= \sigma^2\left[1 - 2\frac{\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k}{n-1}\right]$$

$$(4.7)\quad var(\bar{x}_n) = \frac{\sigma^2\left[1+2\sum_{j=1}^{n-1}(1-j/n)\rho_j\right]}{n} = \frac{\sigma^2 a}{n}$$

$$(4.8)\quad E\left[\frac{S_n^2}{n}\right] = \frac{1}{n}E[S_n^2] = \frac{1}{n}\left(\sigma^2\left[1-2\frac{\sum_{k=1}^{n-1}(1-k/n)\rho_k}{n-1}\right]\right)$$

$$= \frac{\sigma^2}{n}\left[1-2\frac{\sum_{k=1}^{n-1}(1-k/n)\rho_k}{n-1}\right]$$

$$= \frac{\sigma^2}{n}\left[1-\frac{2}{n-1}\sum_{k=1}^{n-1}(1-k/n)\rho_k\right]$$

$$= \frac{\sigma^2}{n(n-1)}\left[n-1-2\sum_{k=1}^{n}(1-k/n)\rho_k\right]$$

$$= \frac{\sigma^2}{n-1} - \frac{\sigma^2}{n(n-1)}\left[\left(1+2\sum_{k=1}^{n}(1-k/n)\rho_k\right)\right]$$

$$= \frac{\sigma^2}{n-1} - \frac{\sigma^2}{n(n-1)} a$$

$$= \frac{n\sigma^2 - \sigma^2 a}{n(n-1)} = \frac{\sigma^2}{n-1} - \frac{1}{n-1} \operatorname{Var}(\bar{X}_n)$$

$$= \frac{\frac{\operatorname{Var}(\bar{X}_n) n}{a}}{n-1} - \frac{1}{n-1} \operatorname{Var}(\bar{X}_n)$$

$$= \frac{\operatorname{Var}(\bar{X}_n)}{n-1} \left(\frac{n}{a} - 1 \right)$$

Problem 4 & 5.

Problem 4.

(a) marginal mean = 5

$q = 2$

$var(x_i) = 3^2 \sum_{i=0}^{q} B_i^2 = 9\left(1 + \frac{4}{9} + \frac{1}{9}\right) = 14$

(b) $Cov(X_i, X_{i+j}) = \sigma_\varepsilon^2 \sum_{i=0}^{2-j} B_i B_{i+j} \quad , \forall j \in \{0, \ldots, 2\}$

$= \sigma_\varepsilon^2 \left(B_0 B_{0+j} + B_1 B_{1+j} + \ldots + B_{2-j} B_{2-j+j} \right) \quad \forall j \in \{0, \ldots, 2\}$

For $j=1$

$cov(X_i, X_{i+1}) = \sigma_\varepsilon^2 \sum_{i=0}^{1} B_i B_{i+1} = \sigma_\varepsilon^2 \left(1\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) \right)$

$= \sigma_\varepsilon^2 \left(\frac{8}{9}\right) = 8$

For $j=2$:

$cov(x_i, x_{i+2}) = \sigma_\varepsilon^2 \sum_{i=0}^{0} B_i B_{i+2} = \sigma_\varepsilon^2 \left((1)\left(\frac{1}{3}\right) \right) = \sigma_\varepsilon^2 \left(\frac{1}{3}\right) = 3$

For $j=0$:

$Cov(x_i, x_i) = \sigma_\varepsilon^2 \sum_{i=0}^{2} B_i B_i = \sigma_\varepsilon^2 \left(1 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 \right)$

$= \sigma_\varepsilon^2 \left(\frac{14}{9}\right) = 9\left(\frac{14}{9}\right) = 14$

B/c cov for all j doesn't depend on i, we see that $\{X_i\}$ is weakly stationary.

(c) $\rho_j \equiv Corr(x_t, x_{t+j}) = \frac{c_j}{\sigma_x^2} = \frac{cov(x_t, x_{t+j})}{\sigma_x^2} \quad \forall j \in \{0,$

For $j=0$:

$\rho_0 \equiv Corr(x_t, x_{t+0}) = \frac{cov(x_t, x_{t+0})}{\sigma_x^2} = \frac{\left(\frac{14}{9}\right)\sigma_\varepsilon^2}{\sigma_x^2} = \frac{\left(\frac{14}{9}\right)(9)}{14}$

$= 1$

For $j=1$:

$\rho_1 \equiv Corr(x_t, x_{t+1}) = \frac{cov(x_t, x_{t+1})}{\sigma_x^2} = \frac{8}{14} = \frac{4}{7}$

For $j=2$:

$\rho_2 \equiv Corr(x_t, x_{t+2}) = \frac{cov(x_t, x_{t+2})}{\sigma_x^2} = \frac{3}{14}$

Problem 4.

(d) $Var(\bar{X}_n) = Var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} x_i\right)$

b/c IID

$$= \frac{1}{n^2}\left(Var(x_1) + Var(x_2) + \ldots + Var(x_n) + 2\sum_{i<j} Cov(x_i, x_j)\right)$$

$$= \frac{1}{n^2}\left(n\sigma_x^2 + 2\sum_{i<j} cov(x_i, x_j)\right)$$

$$= \frac{1}{n^2}\Big(n\sigma_x^2 + 2\{c_1 + c_2 + \ldots + c_{n-2} + c_{n-1}$$
$$c_1 + c_2 + \ldots + c_{n-2}$$
$$\vdots$$
$$c_1\}$$

$$= \frac{1}{n^2}\left(n\sigma_x^2 + 2\{(n-1)c_1 + (n-2)c_2 + \ldots + c_{n-1}\}\right)$$

$$= \frac{\sigma_x^2}{n}\left[1 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k\right] = \frac{14}{n}\left[1 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k\right]$$

$$= \frac{14}{n}\left[1 + 2\left\{1 + \left(1-\frac{1}{n}\right)\left(\frac{4}{7}\right) + \left(1-\frac{2}{n}\right)\left(\frac{3}{14}\right)\right\}\right]$$

$$= \frac{14}{n}\left[1 + 2\left\{\frac{25}{14} - \frac{1}{n}\right\}\right] = \frac{14}{n}\left[1 + \frac{25}{7} - \frac{2}{n}\right] = \frac{14}{n}\left[\frac{32}{7} - \frac{2}{n}\right]$$

$$= \frac{64}{n} - \frac{28}{n^2}$$

(e) $E\left[\frac{s_n^2}{n}\right] = \frac{Var(\bar{X}_n)}{n-1}\left(\frac{n}{a} - 1\right)$ where

$$a = \left[1 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k\right] = \frac{32}{7} - \frac{2}{n}$$

$$\frac{(n/a - 1)}{n-1} Var(\bar{X}_n) < Var(\bar{X}_n) \rightarrow \frac{(n/a-1)}{n-1} < 1$$

$$\hookrightarrow \frac{n}{a} - 1 < n - 1 \rightarrow \frac{n}{a} < n \rightarrow a > 1$$

$$1 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k > 1 \rightarrow \sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k > 0 \checkmark$$

(f) $\sigma^2 = \lim_{n\to\infty} n\,Var(\bar{X}_n) = \sum_{k=-\infty}^{\infty} c_k = \sigma_x^2 + 2\sum_{k=1}^{\infty} c_k$

$$= \sigma_x^2 + 2\{c_1 + c_2\} = 14 + 2(3+8) = 36$$

Problem 5.

For CI (1): 70.25%

For CI(2): 91.5%

Yes, this is expected.

For CI (1): 70.25% contain mean
For CI (2): 91.5% contain mean
This is expected. The half width of CI (2) is larger than that of CI (1).