ISyE 4803 - Homework 6
Charu Thomas
903126680

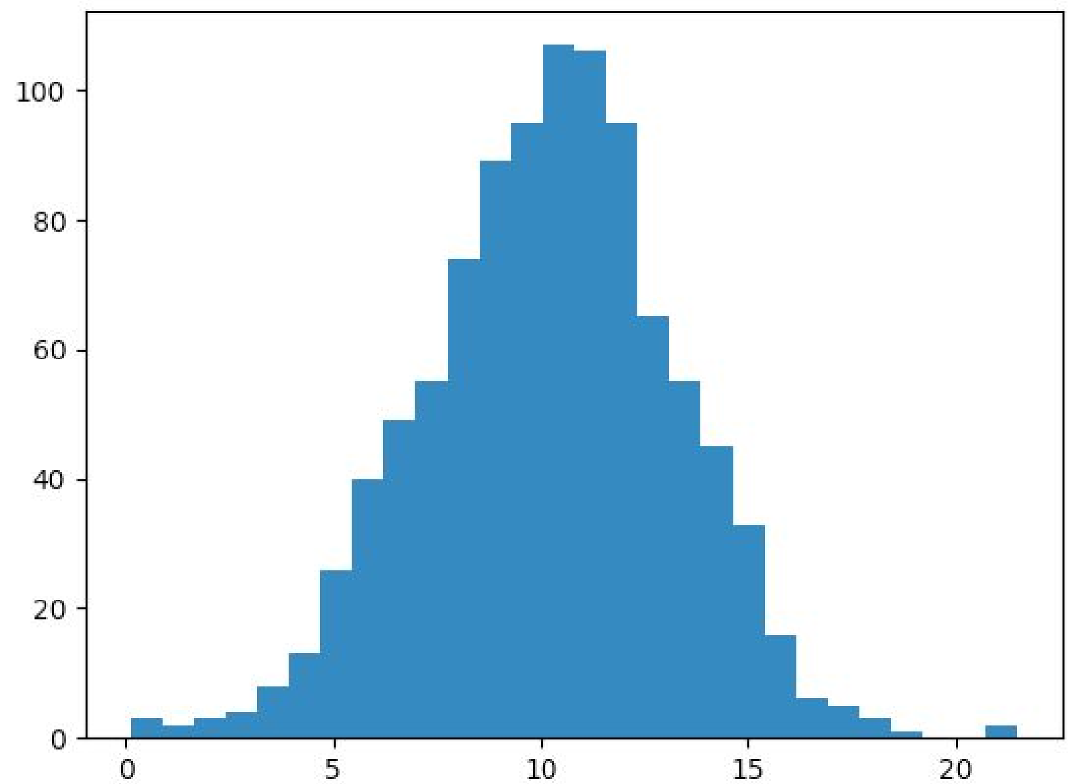
Problem 1.
(a) Checked the site
(b) Uniform random numbers
(c) mt19937ar
(d) period $= 2^{191}$
(e) C code - simul.iro.umontreal.ca/rng/MRG32k3a.c
Java code - github.com/umontreal-simul/ssj/blob/master/src/main/java/umontreal/ssj/rng/MRG32k3a.java
(f) Mersenne Twister
(g) Mersenne Twister
Others: Super-Duper, Wichmann-Hill, etc...
(h) See Script in zip file, run to see histogram
(i) Wichmann-Hill - Recursions on PDF
(j) Yes - passes tests of randomness

Problem 2.
(a) $m=18,\ a=1,\ b=18$
(b) Check script

Problem 3.
$F(x) = \frac{1}{(1+e^{-\frac{(x-\gamma)}{\beta}})}$ $\quad X = F^{-1}(u) = \gamma - \beta \ln\left(\frac{1}{u-1}\right)$
$\hookrightarrow X = \gamma - \beta \ln\left(\frac{1}{u-1}\right)$

Problem 4.
(a) $F(x) = 1 - P\{X > x\} = 1 - P\{\min(U_1, U_2) > x\} = 1 - P\{U_1 > x, U_2 > x\}$
$= 1 - P\{U_1 > x\}P\{U_2 > x\} = 1-(1-x)^2, \quad -1 \le x \le 1$
$1-(1-x)^2 = U \rightarrow \quad X = 1-\sqrt{1-U}$
(b) $F(x) = P\{X \le x\} = P\{U_1 \le x, U_2 \le x\} = P\{U_1 \le x\}P\{U_2 \le x\}$
$= (x)^2, \ -1 \le x \le 1 \qquad x^2 = U \rightarrow x = \sqrt{U}$

Problem 1H.



Problem 1I.

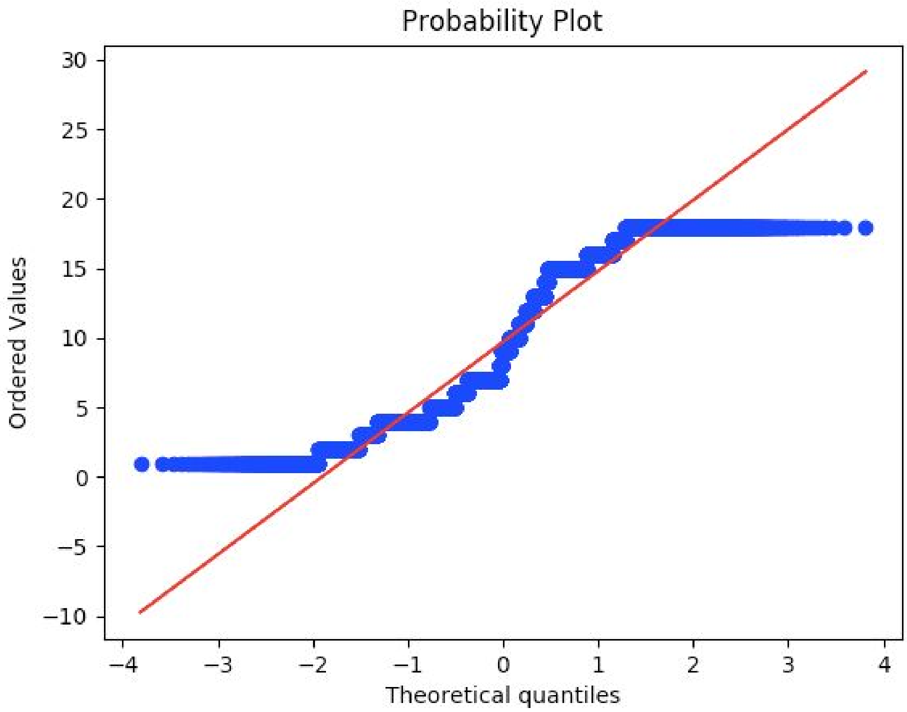
$X_{1,i+1} = 171X_{1,i} \bmod 30269$

$X_{2,i+1} = 172X_{2,i} \bmod 30307$

$X_{3,i+1} = 170X_{3,i} \bmod 30323$

$= (X_{1,i+1}/30269 + X_{2,i+1}/30307 + X_{3,i+1}/30323) - \text{Floor}((X_{1,i+1}/30269 + X_{2,i+1}/30307 + X_{3,i+1}/30323))$

Problem 2B.



Probability Plot
Ordered Values
Theoretical quantiles
30
25
20
15
10
5
0
−5
−10
−4
−3
−2
−1
0
1
2
3
4

Problem 5. INVERSE METHOD:

$$F(x) = \int_{-1}^{x} \frac{3x^2}{2}\,dx = \left[\frac{x^3}{2}\right]_{-1}^{x} = \frac{x^3}{2} + \frac{1}{2} \quad , -1 \le x \le 1$$

$$F(x) = U = \frac{x^3}{2} + \frac{1}{2} \rightarrow \frac{x^3}{2} = U - \frac{1}{2} \rightarrow x^3 = 2U - 1$$

$$X = \sqrt[3]{2U-1}$$

COMPOSITION METHOD:

$$f(x) = \tfrac{1}{2} f_1(x) + \tfrac{1}{2} f_2(x)$$

$$f_1(x) = 3x^2, \; -1 \le x < 0 \rightarrow F_1(x) = x^3 + 1 \quad\quad x^3 + 1 = V$$

$$f_2(x) = 3x^2, \; 0 \le x \le 1 \rightarrow F_2(x) = x^3 \quad\quad x^3 = V$$

$U, V \sim \text{Unif}(0,1)$

If $U \le \frac{1}{2}$, $X = \sqrt[3]{V-1}$ otherwise $X = \sqrt[3]{V}$

Problem 6.

$f(x) = 20x(1-x)^3, \; 0 \le x \le 1$

(a) $\max f(x) \equiv \max \ln(f(x))$ W.L.O.G.

$$\ln f(x) = \ln(20x(1-x)^3) = \ln(20x) + \ln((1-x)^3)$$

$$= \ln(20) + \ln(x) + 3\ln(1-x)$$

$$\frac{d \ln f(x)}{dx} = \frac{1}{x} - \frac{3}{1-x} = 0 \rightarrow \frac{1}{x} = \frac{3}{1-x} \rightarrow \frac{(1-x)}{x} = 3$$

$$1 - x = 3x \rightarrow 4x = 1 \rightarrow x = \tfrac{1}{4}$$

$$f(x) = f(1/4) = 20(1/4)(3/4)^3 = \frac{540}{256} = 2.109$$

(b) $c = \int_0^1 2.109\,dx = [2.109x]_0^1 = \frac{540}{256}$

$$h(x) = \frac{t(x)}{540/256} = 1 \quad , \quad g(x) = \frac{f(x)}{540/256} = \frac{265}{27} x(1-x)^3$$

do

$U_1, U_2 \sim U(0,1)$

$Y = U_2$

until $U_1 \le \frac{265}{27}(Y(1-Y)^3)$

return $X = Y$

(c) c iterations until realization on avg

↳ In this case, $\frac{540}{256}$ iterations

Problem 7. $f(x)=20x(1-x)^3,\ 0\le x\le 1 \rightarrow p=\frac{1}{20}$ to make function "nicer"

Repeat

generate $U\sim U(0,\frac{3\sqrt{3}}{16})$ and $V\sim U(0,1/8)$

$Z = V/U$

Until $U^2 \le Z(1-Z)^3$

return $X=Z$

mean # of times to generate a single obs $= \dfrac{2}{|S|/|T|} = \dfrac{2(u^*(v^*-0))}{p/2} = \dfrac{2\left(\frac{3\sqrt{3}}{16}\left(\frac{1}{8}\right)\right)}{\frac{1}{40}}$

$= \underline{\underline{3.25}}$

$u(z)=\sqrt{pf(z)} = \sqrt{z(1-z)^3}$ $\quad$ $v(z)=z\sqrt{pf(z)} = z\sqrt{z(1-z)^3}$

$u^* = \frac{3\sqrt{3}}{16}$ $\quad$ $v^* = \frac{1}{8}$

Problem 8.

(a) $E[X] = \frac{1}{2}(-1) + \frac{1}{2}(1) = 0$

$E[X^2] = \frac{1}{2}(-1)^2 + \frac{1}{2}(1)^2 = 1$

$Var(x) = E[X^2] - (E[X])^2 = 1-0 \quad 1$

Problem 9.

The naive algorithm is as follows...

$$T_{i+1} = T_i - \frac{1}{\lambda(T_i)}\ln(U_{i+1})$$

You can end up skipping intervals. This paper posits that the algorithm for generating an NHPP - use Acceptance/Rejection. Find maximal λ^* and generate according to it. Accept w/ probability $\frac{\lambda(T_i)}{\lambda^*}$