A quasi-polynomial-time classical algorithm for Lindbladian evolution

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Most quantum systems in nature are open, meaning they interact with an environment and experience noise. A critical question, relevant for quantum advantage in simulation, is to what degree such noise impacts the feasibility of classical simulations. In this work, we present a quasi-polynomial-time classical algorithm for simulating continuous-time quantum Hamiltonian dynamics under a broad range of Lindbladian noise processes. Our algorithm computes the expectation value of any local observable with a small average error over input states drawn from an ensemble (e.g. the computational basis), for any bounded-degree local Hamiltonian and any noise process in the depolarizing class. To do so, we extend existing Pauli-truncation classical algorithms from discrete-time quantum circuits to continuous-time Lindbladian evolutions. Our algorithm evolves the system under an effective truncated Hamiltonian in operator space, which leverages the fact that noise dampens non-local correlations in the system exponentially more than local correlations. A natural avenue for future research is to explore the extent to which these results extend to other physically relevant noise processes, such as thermal noise.

A major goal of quantum computers is to simulate and predict properties of quantum systems in nature. While naive simulation of quantum systems requires classical computational resources that scale exponentially with respect to system size, quantum computers have the potential to simulate these systems efficiently by leveraging the fact that both the computational model and Nature are inherently quantum mechanical [1]. To understand how large of a computational advantage we can expect from quantum computers, we need to understand how complex Nature is to simulate on a classical computer in the first place. To this end, one ubiquitous property of quantum systems in nature is that they are open, meaning they interact with an environment and experience noise. Thus, a key question of interest becomes: What is the classical complexity of simulating open-system quantum dynamics?

This question has been extensively studied in the context of *discrete-time* quantum dynamics, i.e. noisy quantum circuits. In particular, classical algorithms based on Pauli truncation have been successfully utilized to estimate output distributions for random quantum circuits in polynomial time [2, 3]. More recently, Ref. [4] showed that for *any* noisy quantum circuit, expectation values can be estimated in quasi-polynomial time with small average error over the computational basis. However, whether these results extend to continuous-time quantum systems, i.e. Lindbladian dynamics, is currently unknown.. Naive extensions of discrete-time algorithms via Trotterization do not succeed due to large overheads in the translation of continuous-time to discrete-time noise rates. Hence, to date, progress on rigorous, efficient classical algorithms for continuous-time noisy dynamics has been limited to spatially local systems experiencing large noise rates above a threshold [5]. To what extent efficient classical simulation is possible below this threshold remains unclear, aside from strict no-go results for fine-tuned dynamics and input states that enable quantum error correction [6–8].

In this work, we provide a quasi-polynomial-time classical algorithm for estimating local expectation values under a broad range of continuous-time Lindbladian evolutions. Our algorithm succeeds for *any* local, bounded-degree Hamiltonian and any continuous-time noise in the depolarizing class for most input states. The Hamiltonian may be geometrically non-local and can have arbitrary time-dependence. We emphasize that our restriction to "most" input states is both fundamental–since fine-tuned input states and dynamics are known to enable error correction–and also a relatively weak restriction, as a wide range of input state ensembles are allowed (e.g. the computational basis states) [4]. Crucially, in contrast to earlier work for continuous-time systems [5, 9], our algorithm does not exhibit thresholding behavior and instead succeeds for arbitrarily low noise rates and any polynomial evolution time. As for all noisy circuit algorithms, our algorithm runtime grows exponentially in the inverse noise rate, which is fundamental.

Building off of recent Pauli-truncation-based algorithms for noisy quantum circuits [4], our algorithm is based on the key insight that sets of Pauli operators that are hard-to-classically-simulate are also strongly affected by noise, and thereby can be truncated. To extend to these ideas to continuous-time dynamics, we make several technical advances. These include rigorous and general bounds on the flow of operator support under local bounded-degree Hamiltonians, and a simple method to bound continuous-time truncations of this support in the presence of noise. Our result represents the first rigorous application of Pauli truncation algorithms to continuous-time quantum dynamics.

Background.—We consider the task of computing expectation values, $\mathrm{tr}(O\rho(t))$, of a local operator O after continuous time-evolution of a state ρ for a time t. Within the Born-Markov approximation [10], the time-evolution of $\rho(t)$ is governed by Lindbladian evolution,

$$\partial_t \rho(t) = -i[H, \rho(t)] + \sum_\alpha \gamma_\alpha \left(L_\alpha \rho(t) L_\alpha^\dagger - \frac{1}{2}\{\rho(t), L_\alpha^\dagger L_\alpha\} \right), \tag{1}$$

where H is the Hamiltonian describing the unitary dynamics of the system, L_α are jump operators that describe the dissipative dynamics of the system, and γ_α is the relaxation rate associated with each L_α. Equivalently, we can describe the dynamics of the system in the Heisenberg picture by evolving the observable, O, instead of the state ρ. This is governed by,

$$\partial_t O(t) = i[H, O(t)] + \sum_\alpha \gamma_\alpha \left(L_\alpha^\dagger O(t) L_\alpha - \frac{1}{2}\{O(t), L_\alpha^\dagger L_\alpha\} \right), \tag{2}$$

where $\mathrm{tr}(O\rho(t)) = \mathrm{tr}(O(t)\rho)$.

In this work, we will focus on a particularly simple noise process, depolarizing noise. This will allow us to focus on the unique aspects of extending from discrete to continuous time-evolutions, leaving a generalization to other noise models, such as amplitude damping and thermal noise, to future work. The Heisenberg master equation for depolarizing noise simplifies to

$$\partial_t O(t) = i[H, O(t)] - \gamma \mathcal{L}\{O(t)\}, \tag{3}$$

where γ is the depolarizing noise rate, and $\mathcal{L} : P \to w[P]P$ measures the weight of each Pauli operator $P \in \{I, X, Y, Z\}^{\otimes n}$. The weight $w[P]$ is equal to the number of non-identity components of P.

We make only modest and very general assumptions on the Hamiltonian H and the observable O. We assume that H is bounded, k-local, and d-degree, i.e.

$$H = \sum_P h_P P, \tag{4}$$

where $h_P \in [-1, 1]$, $w[P] \le k$ for all non-zero h_P, and there are at most d Pauli operators with non-zero coefficients in H with support on qubit i, for all i. We assume that O is a bounded, few-body observable, i.e. $\|O\|_\infty \le 1$ and O can be written as a sum of Pauli operators with weight at most w_0.

Finally, in common with recent work on discrete-time noisy circuits [4], we assume the initial state $\rho = \rho(0)$ is drawn from a *low-average* ensemble, which we define as any ensemble, $\mathcal{E} = \{\rho\}$, whose mixture is close to the maximally mixed state (see Appendix I). The simplest example is any complete basis of states, such as the computational basis. We will bound the average squared error over states drawn from any low-average ensemble.

We conclude our background discussion with a short remark on notation. For the ease of analysis, we will vectorize the operators, which will allow us to write the master equation in terms of superoperators,

$$\partial_t \|O(t)\rangle\rangle = i\boldsymbol{H}\|O(t)\rangle\rangle - \gamma \boldsymbol{W}\|O(t)\rangle\rangle \equiv \boldsymbol{L}\|O(t)\rangle\rangle, \tag{5}$$

where we use capital bold font to represent the superoperators,

$$\boldsymbol{H}\|O\rangle\rangle = \|[H, O]\rangle\rangle, \tag{6}$$

which commutes an operator with H, and

$$\boldsymbol{W}\|P\rangle\rangle = w[P]\|P\rangle\rangle, \tag{7}$$

which multiplies each Pauli operator by its weight. In the vectorized notation, we have $\|O(t)\rangle\rangle = e^{\boldsymbol{L}t}\|O\rangle\rangle$. We define the inner product $\langle\langle A\|B\rangle\rangle = \frac{1}{2^n}\mathrm{tr}\big(A^\dagger B\big)$. The vector norm therefore corresponds to the operator Frobenius norm, $\langle\langle A\|A\rangle\rangle = \frac{1}{2^n}\mathrm{tr}\big(A^\dagger A\big) = \|A\|_F^2$.

Algorithm.—To efficiently compute the expectation value of O, we evolve the observable under a truncated superoperator $\boldsymbol{H}_c$, whose Hilbert space dimension grows only quasi-polynomially with respect to the system size, n. Our intuition for this approach, inspired by [4], relies on the insight that depolarizing noise damps high-weight Pauli operators exponentially more than low-weight Pauli operators. Depolarizing noise thus preferentially selects for low-weight operators, so when we do *not* evolve high-weight operators, we incur only a small, bounded error (see Fig. 1).

In more detail, our algorithm is as follows. First, we consider the matrix elements of $\boldsymbol{H}$ in the Pauli basis, $\langle\langle Q\|\boldsymbol{H}\|P\rangle\rangle$ for Pauli operators Q, P, which can be interpreted as the transition amplitude from P to Q under the Hamiltonian H. Then, we truncate away all transitions involving Paulis P or Q with weight greater than a fixed threshold, $\ell = \mathcal{O}(\log\left(\frac{t}{\epsilon}\right)\gamma^{-1})$. Here, ϵ is the desired root-mean-square error and t is the evolution time. This truncation leaves us with the effective Hamiltonian, $\boldsymbol{H}_c = \mathcal{P}_{\le \ell}\boldsymbol{H}\mathcal{P}_{\le \ell}$, where $\mathcal{P}_{\le \ell}$ is the projector onto Pauli operators of weight less than or equal to ℓ. Finally, we use classical matrix exponentiation methods to calculate the time-evolution of O under $\boldsymbol{H}_c$,

$$\|O_c(t)\rangle\rangle = e^{(i\boldsymbol{H}_c - \gamma \boldsymbol{W})t}\|O\rangle\rangle \tag{8}$$

and, from this, we compute our classical approximation of the expectation value, $\mathrm{tr}\,(\rho O_c(t))$.

Our main result is that this algorithm succeeds and runs in time growing quasi-polynomially in n, for any $t, 1/\varepsilon = \mathrm{poly}\, n$.

Theorem 1 (Quasi-polynomial algorithm for Lindbladian evolution)**.** *Consider any local operator O, bounded*

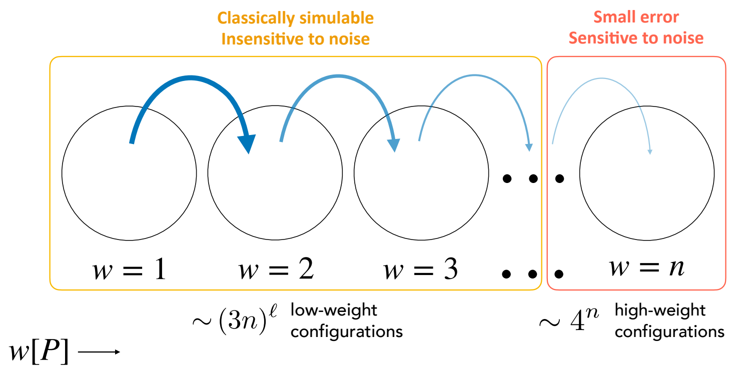

FIG. 1. Intuition for our classical algorithm. We decompose the observable O in the Pauli basis at all times, and group components of the observable by their weight. Each bubble represents all Pauli operators with a given weight. The blue arrows represent how the observable can evolve over time, jumping from one weight to another, under a local Hamiltonian and depolarizing noise. Each arrow corresponds to one application of the Hamiltonian. The opacity of each arrow corresponds to the amount of operator norm flowing from the starting to the ending weight. At each time step, the k-local Hamiltonian allows the observable to evolve support on Pauli operators with weight $k-1$ more than its current support (in this schematic, k being 2). However, at each time step, depolarizing noise exponentially dampens the operator norm based on operator weight. Combining the two effects, we find that the components of the observable above some threshold ℓ has small norm, which can be truncated away in our approximation. The number of low-weight Pauli operators, with weight below ℓ is quasi-polynomial in n, which yields our classical algorithm.

k-local d-degree Hamiltonian H, depolarizing noise of rate γ, and low-average ensemble of states $\mathcal{E} = \{\rho\}$. Assume local Pauli expectation values of ρ can be efficiently computed. Then, there is a classical algorithm that computes the expectation values, $\mathrm{tr}(\rho(t)O)$, in time

$$n^{\mathcal{O}\left(\gamma^{-1}\log\left(\frac{t}{\epsilon}\right)\right)}, \tag{9}$$

with root-mean-square error ϵ over the ensemble $\mathcal{E}$.

The detailed proof of Theorem 1 is provided in Appendix I. In the remainder of the main text, we provide a high-level summary of the key proof ideas that allow us to extend from discrete to continuous-time evolutions, and discuss several extensions of Theorem 1 to time-dependent Hamiltonians and any noise model in the depolarizing class.

Proof Overview.—To set up our proof, we first borrow a lemma from [4] to reduce the task of bounding the root-mean-square error in our algorithm to bounding the Frobenius norm of the different between the actual and approximate time-evolved observable,

$$\|O(t) - O_c(t)\|_F, \tag{10}$$

which can be interpreted as the operator norm of all Pauli paths that traverse through the high-weight regime in the evolution of O to time t, where high-weight refers to all Pauli operators with weight more than the threshold ℓ.

Our approach proceeds in three steps. First, we resolve Equation 10 into an integral of the high-weight support amassed in a δt time-step at each given point in time. This allows us to bound Equation 10 in terms of 1) the spectral norm of the superoperator that has been truncated, $\|\boldsymbol{H} - \boldsymbol{H}_c\|_\infty$, and 2) the maximum support of O at weights greater than $\ell - k$,

$$\max_{0 \le t' \le t} \{\|\mathcal{P}_{>\ell-k} e^{\boldsymbol{L}t'} \|O\rangle\rangle\|_F\}, \tag{11}$$

over the time interval $[0, t]$. To facilitate step 1), we also prove that truncating $\boldsymbol{H} - \boldsymbol{H}_c$ and truncating $\boldsymbol{H}_{\mathrm{mid}} = \boldsymbol{H} - \boldsymbol{H}_c - \mathcal{P}_{>\ell+k}\boldsymbol{H}\mathcal{P}_{>\ell+k}$ equivalently produce $O_c(t)$. This equivalence follows from the restriction imposed on the allowed transitions between Pauli operators by the k locality of the Hamiltonian (see Figure 2). The equivalence allows us to bound the spectral norm of the truncation in terms of $\|\boldsymbol{H}_{\mathrm{mid}}\|_\infty$ instead of $\|\boldsymbol{H} - \boldsymbol{H}_c\|_\infty$, which enables a much tighter bound.

Our second step is extremely brief: we prove that $\|\boldsymbol{H}_{\mathrm{mid}}\|_\infty$ can be bound simply in terms of the locality, k, and degree of interaction, d, of the Hamiltonian.

Finally, to bound the maximum high-weight support, we utilize the *operator weight distribution*,

$$W(t) = \{w, P(w,t)\}, \tag{12}$$

a tool originating from quantum gravity and quantum information scrambling used to quantify operator growth[11, 12]. The high-weight support of O can be reformulated in the language of the weight distribution as

$$\|\mathcal{P}_{>\ell-k} e^{\boldsymbol{L}t} \|O\rangle\rangle\|_F^2 = \sum_{w=\ell-k+1}^{n} P(w,t), \tag{13}$$

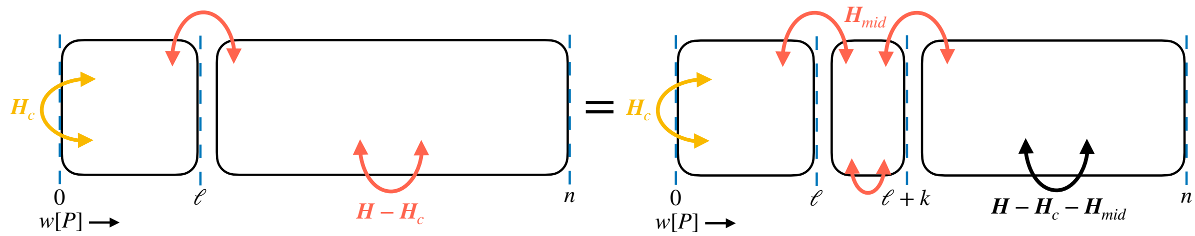

FIG. 2. Depiction of our classical algorithm, where $w[P]$ denotes the weight of a Pauli operator P and the blocks denote the matrix elements of $\boldsymbol{H}$ between Pauli operators of a given range of weights. Our algorithm only simulates the low-weight dynamics embedded in $\boldsymbol{H}$, based on the weight threshold ℓ (left). For a few-body observable composed of Pauli operators of weight less than ℓ evolving under a k-local Hamiltonian, simulating transitions from low-weight Pauli operators to other low-weight Pauli operators is equivalent to simulating *all* transitions except those that allow low-weight Pauli operators to become high-weight Pauli operators. We encode these low-to-high and high-to-low transitions in $\boldsymbol{H}_{\text{mid}}$ (right), and show the equivalence of our algorithm and removing just $\boldsymbol{H}_{\text{mid}}$ in Lemma 2, which we prove in detail in Appendix II.

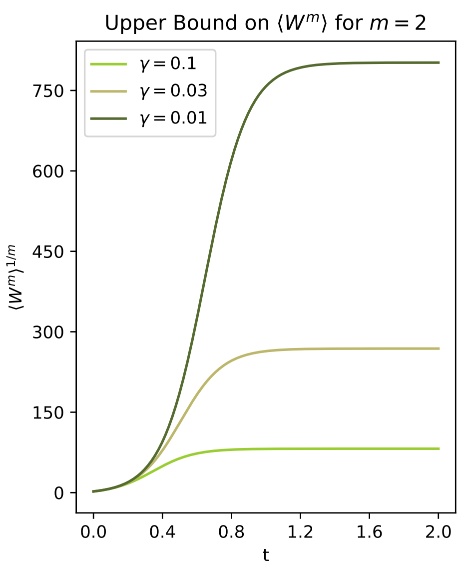

FIG. 3. Plot of our rigorous upper bound on the second moment of the operator weight distribution over time, for $k, d = 2$ and various noise rates γ. The bound quickly saturates to a constant upper bound, whose magnitude scales exponentially in the inverse noise rate.

or the probability of being at weights more than $\ell + k$. Here, the probability of being at weight w is quantified by $P(w,t) = \langle\langle O(t) \| \mathcal{P}_w \| O(t) \rangle\rangle = \| \mathcal{P}_w \| O(t) \rangle\rangle \|_F^2$ where $\mathcal{P}_w$ is the projector onto Pauli operators of weight w. We tightly tail bound the high-weight probability using the moments of the operator weight distribution. We prove that in the presence of noise, the moments of the operator weight distribution never exceed a constant threshold. The threshold depends only on the noise rate, γ, and the properties of the Hamiltonian, k, d (see Figure 3 for an example and Appendix IV for the exact form of the upper bound).

Following these steps, we prove that our classical algorithm can approximate the expectation value of O within small average error ϵ for an ensemble with a weight threshold of $\ell = O(\log\left(\frac{t}{\epsilon}\right)\gamma^{-1})$, resulting in an algorithm that runs in quasi-polynomial time with respect to the number of spins, n. For the full proof, please see Appendix I.

Generality.—We conclude by discussing several immediate extensions of our results. Our proof can be extended to *time-dependent* Hamiltonians with no changes, since none of our techniques rely on the time-independence of H. Our proof can also be extended to any single-qubit noise process in the *depolarizing class*. Any noise model in the depolarizing class can be written as $\mathcal{L}(\rho) = U\mathcal{L}_P(\rho)V^\dagger$, where U, V are single-qubit unitary rotations and

$$\begin{aligned}\mathcal{L}_P(\rho) = (1 - \gamma_X - \gamma_Y - \gamma_Z)\rho \\ + \gamma_X X\rho X + \gamma_Y Y\rho Y + \gamma_Z Z\rho Z.\end{aligned} \tag{14}$$

We set $\gamma = \min(\gamma_X, \gamma_Y, \gamma_Z)$ and our proof extends straightforwardly.

Outlook.—We have shown that the rigorous analysis of Pauli truncation methods can extend to continuous-time quantum dynamics in addition to discrete-time quantum circuits. Our results raise several interesting questions for future work. For one, to what extent do our results carry over to more general noise channels, such as amplitude damping and thermal noise? Several no-go results rule out strict and completely general extensions [4, 5, 13], yet physical arguments suggest that, in many circumstances, such noise channels also preferentially select for operators of low-weight [12]. In the case of thermal noise specifically, can insights from our proof techniques allow us to identify some Lindbladian dynamics and Gibbs states that are more classically intractable than others? In a more separate direction, our algorithm bears a close similarity to many practical classical algorithms used to estimate expectation values of noiseless quantum circuits and many-body dynamics [14–25]. Can our proof techniques be used to analyze the widely observed successes of these methods in noiseless settings?

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Appendix: A quasi-polynomial-time classical algorithm for Lindbladian evolution

CONTENTS

I. PROOF OF THEOREM 1: QUASI-POLYNOMIAL-TIME CLASSICAL ALGORITHM FOR LINDBLADIAN EVOLUTION

In this section, we prove our main result, Theorem 1. For the ease of the reader, we begin by providing a high-level overview of our proof, and summarizing several key technical lemmas. We then provide a short and complete proof of Theorem 1 from these lemmas. We defer the detailed proofs of each technical lemmas to the later sections.

To prove that our classical algorithm is accurate, we would like to bound the root-mean-square error of the expectation values obtained from our classical approximation over the ensemble $\mathcal{E} = \{\rho\}$,

$$\sqrt{\frac{1}{|\mathcal{E}|}\sum_{\rho}\big(\operatorname{tr}(\rho O(t)) - \operatorname{tr}(\rho O_c(t))\big)^2}, \tag{15}$$

in terms of our weight threshold, ℓ, noise rate, γ, evolution time, t, locality, k, and bounded degree of interaction, d, in the Hamiltonian H.

The first step of our proof borrows a simple result from [4], which shows that the root-mean-square error over any *low-average ensemble* of quantum states is upper bounded by the Frobenius distance between the approximate and exact time-evolved observable. A low-average ensemble of states with purity c is defined as follows.

Definition 1. (Low-average ensembles of states) *We say that an ensemble of quantum states $\mathcal{E} = \{\rho\}$ is a low-average ensemble with purity c if $\|\frac{1}{|\mathcal{E}|}\sum_{\rho}\rho\|_\infty \le c/2^n$.*

With this assumption, we can bound the root-mean-square error in expectation value with a Frobenius norm.

Lemma 1 (Lemma 3 of [4]). *For a low-average ensemble $\mathcal{E} = \{\rho\}$ with purity c, the root-mean-square difference between the expectation value of any two observables O, O_c is less than $\sqrt{c} \cdot \|O - O_c\|_F$*

Hence, we can prove Theorem 1 by bounding the Frobenius distance, $\|O - O_c\|_F$.

We proceed in three distinct steps.

1. First, we bound the total error of our classical algorithm in terms of the integral of small instantaneous errors accumulated at each time. This allows us to upper bound the desired Frobenius distance via the product of two quantities: 1) the spectral norm of the component of $\boldsymbol{H}$ that transitions Pauli operators across the weight threshold, and 2) the probability of $O(t)$ being above the $\ell - k$ weight threshold at any given time t.

2. Second, we bound the spectral norm of the component of $\boldsymbol{H}$ that transitions across the weight threshold. This step is especially simple using the sparsity of $\boldsymbol{H}$.

3. Finally, we bound the Frobenius norm of the support of $O(t)$ above the weight threshold in terms of the noise strength and weight threshold. To do so, we reformulate the high-weight support norm in terms of the probability of $O(t)$ having support above the threshold and bound the probability with the moments of the probability distribution.

We formalize each step into its own technical lemma, which we prove in detail in the subsequent sections. We now state each lemma and apply them to prove Theorem 1.

The three technical lemmas are as follows.

Lemma 2. (Upper bound on truncation error in terms of high-weight support). *Consider any observable O evolved under a k-local, d-degree Hamiltonian H. The Frobenius norm of $O_c(t) - O(t)$ is bounded by*

$$\|\boldsymbol{H}_{mid}\|_\infty \cdot t \max_{0\le t'\le t}\{\|\mathcal{P}_{>\ell-k}e^{\boldsymbol{L}t'}\|O\rangle\!\rangle\|_F\} \tag{16}$$

where $\boldsymbol{H}_{mid} = \boldsymbol{H} - \mathcal{P}_{>\ell+k}\boldsymbol{H}\mathcal{P}_{>\ell+k} - \mathcal{P}_{\le\ell}\boldsymbol{H}\mathcal{P}_{\le\ell}$.

Lemma 3. *The spectral norm of $\boldsymbol{H}_{mid}$ is bounded by $2d(\ell+2k)$.*

Lemma 4 (Upper bound on the high-weight support of noisy-time-evolved operators)**.** *Consider any observable O with initial weight w_0, time-evolved under a k-local, d-degree Hamiltonian H and Lindbladian depolarizing noise of rate γ. For all times $t \ge 0$, we have*

$$\|\mathcal{P}_{>\ell-k}e^{\boldsymbol{L}t'}\|O\rangle\!\rangle\|_F \le e^{-A_{d,k,\gamma}\gamma(l-k)}, \tag{17}$$

given the (mild) assumption $\ell \ge 8(1+\sqrt{1+2\gamma})w_0 + k$. The constant $A_{d,k,\gamma}$ is defined as,

$$A_{d,k,\gamma} = \frac{1}{\sqrt{1+2\gamma}\,\cdot\,6dk^2} = \frac{1}{6dk^2} + \mathcal{O}(\gamma) \tag{18}$$

We can now apply this approach and set of lemmas to prove Theorem 1.

Proof of Theorem 1. We assume ρ is drawn from a low-average ensemble with purity c; we have $c = 1$ in the statement of the theorem. As described, we first upper bound the root-mean-square error of our algorithm using Lemma 1,

$$\sqrt{\frac{1}{|\mathcal{E}|}\sum_\rho\big(\operatorname{tr}(\rho O(t)) - \operatorname{tr}(\rho O_c(t))\big)^2} \le \sqrt{c}\,\cdot\,\|O_c(t) - O(t)\|_F. \tag{19}$$

Then, we apply Lemma 2 to bound the Frobenius distance,

$$\|O_c(t) - O(t)\|_F \le \|\boldsymbol{H}_{\mathrm{mid}}\|_\infty \cdot t \max_{0\le t'\le t}\{\|\mathcal{P}_{>\ell-k}e^{\boldsymbol{L}t'}\|O\rangle\!\rangle\|_F\}. \tag{20}$$

We bound the spectral norm on the right hand side using Lemma 3,

$$\|\boldsymbol{H}_{\mathrm{mid}}\|_\infty \le 2d(\ell+2k). \tag{21}$$

Meanwhile, we bound the maximum Frobenius norm using Lemma 4,

$$\max_{0\le t'\le t}\{\|\mathcal{P}_{>\ell-k}e^{\boldsymbol{L}t'}\|O\rangle\!\rangle\|_F\} \le e^{-A_{d,k,\gamma}\gamma(l-k)}. \tag{22}$$

In total, this bounds the root-mean-square error of our classical algorithm by

$$2d(\ell+2k)t\sqrt{c}\,\cdot e^{-A_{d,k,\gamma}\gamma(l-k)}. \tag{23}$$

To ensure that the root-mean-square error is less than ϵ, we set

$$\ell \ge \frac{1}{2A_{d,k,\gamma}}\frac{1}{\gamma}\left(\ln c + 4\ln\left(\frac{2dt}{\epsilon}\right)\right) + k \tag{24}$$

where we assume $\epsilon \le \frac{2dt}{\ell+2k}$ for simplicity. We also require $\ell \ge 8(1+\sqrt{1+2\gamma})w_0 + k$ to ensure that Lemma 4 holds. Thus, in total, we set the weight threshold as

$$\ell \ge \max\left[8(1+\sqrt{1+2\gamma})w_0, \frac{1}{2\gamma A_{d,k,\gamma}}\left(\ln c + 4\ln\left(\frac{2dt}{\epsilon}\right)\right)\right] + k. \tag{25}$$

For any $w_0 = \mathcal{O}(\log\left(\frac{t}{\epsilon}\right)/\gamma)$, the maximum selects the second term, which is of order $\ell = \mathcal{O}(\log\left(\frac{t}{\epsilon}\right)\gamma^{-1})$.

It remains only to characterize the runtime of our classical algorithm. Our algorithm performs exact operator time-evolution under the truncated effective Hamiltonian. This can be done in time $\mathcal{O}(N^3)$, where N is the dimension of the accessible operator space. There are $N \equiv \sum_{w=0}^{\ell}\binom{n}{w}3^w$ Pauli operators with weight less than or equal to ℓ, which is less than or equal to $\frac{3}{2}\frac{(3n)^\ell}{\ell!}$ when $\ell \le n/2$ [4]. Hence, the runtime of our algorithm is $\mathcal{O}\left(\left((3n)^\ell/\ell!\right)^3\right)$, which we simplify to $n^{\mathcal{O}(\ell)}$. Setting $\ell = \mathcal{O}(\log\left(\frac{t}{\epsilon}\right)\gamma^{-1})$ as described yields the quasi-polynomial runtime $n^{\mathcal{O}(\log\left(\frac{t}{\epsilon}\right)\gamma^{-1})}$. $\square$

II. PROOF OF LEMMA 2: BOUNDING IN TERMS OF HIGH-WEIGHT SUPPORT

In this section, we use the locality of H, O to bound $\|O_c(t) - O(t)\|_F$ in terms of the probability of $O(t)$ being "high-weight" at any given time and the spectral norm of $\boldsymbol{H}_{\text{mid}}$. We define $\boldsymbol{H}_{\text{mid}}$ as the component of $\boldsymbol{H}$ connecting the low-weight, simulable component, $\boldsymbol{H}_c \equiv \mathcal{P}_{\leq \ell} \boldsymbol{H} \mathcal{P}_{\leq \ell}$, to the high-weight component, $\boldsymbol{H}_{\text{high}} \equiv \mathcal{P}_{>\ell+k} \boldsymbol{H} \mathcal{P}_{>\ell+k}$, or equivalently

$$\boldsymbol{H}_{\text{mid}} \equiv \boldsymbol{H} - \boldsymbol{H}_c - \boldsymbol{H}_{\text{high}}. \tag{26}$$

To do this, we bound the total Frobenius norm by the change in norm at each time step integrated over time. We also prove that evolving $\boldsymbol{H}_c$ is equivalent to only not evolving $\boldsymbol{H}_{\text{mid}}$. Intuitively, the equivalence is true because if O starts at a low weight, it can only become high weight and thus classically unsimulable if it transitions from low to high weight and vice versa via $\boldsymbol{H}_{\text{mid}}$. This allows us to use a tighter spectral norm for bounding.

We can rewrite $\|O_c(t) - O(t)\|_F$ as

$$\begin{aligned} \| \| O_c(t) \rangle\rangle - \| O(t) \rangle\rangle \|_F &= \| e^{(i\boldsymbol{H}_c - \gamma \boldsymbol{W})t} \| O \rangle\rangle - e^{\boldsymbol{L}t} \| O \rangle\rangle \|_F \\ &= \| e^{(\boldsymbol{L} - i\boldsymbol{H}_{\text{mid}})t} \| O \rangle\rangle - e^{\boldsymbol{L}t} \| O \rangle\rangle \|_F \\ &= \| (e^{(\boldsymbol{L} - i\boldsymbol{H}_{\text{mid}})t} - e^{\boldsymbol{L}t}) \| O \rangle\rangle \|_F. \end{aligned} \tag{27}$$

In the first line, we apply Fact 1, as proved below, and in the second line, we factor out $\|O\rangle\rangle$.

Fact 1 (Evolving only low-weight transitions is equivalent to not evolving low-to-high and high-to-low transitions)**.** *For a k-local Hamiltonian H acting on a local observable $O = \sum_P c_P P$, where $c_P = 0$ if $w[P] > \ell$,*

$$e^{i(\boldsymbol{H} - \boldsymbol{H}_{mid})t} \| O \rangle\rangle = e^{i\boldsymbol{H_c}t} \| O \rangle\rangle. \tag{28}$$

Proof. Our proof proceeds as follows:

$$\begin{aligned} e^{i(\boldsymbol{H} - \boldsymbol{H}_{\text{mid}})t} \| O \rangle\rangle &= e^{i(\boldsymbol{H}_{\text{high}} + \boldsymbol{H_c})t} \| O \rangle\rangle \\ &= e^{i\boldsymbol{H_c}t} e^{i\boldsymbol{H}_{\text{high}}t} \| O \rangle\rangle, \end{aligned} \tag{29}$$

where we break the exponential into two, since $\boldsymbol{H_c}$ and $\boldsymbol{H}_{\text{high}}$ act nontrivially on different subspaces, with $\boldsymbol{H_c}$ acting on Pauli operators with weight at most ℓ and $\boldsymbol{H}_{\text{high}}$ acting on operators of weight at least $\ell + k + 1$, and thus commute. We then expand the $\boldsymbol{H}_{\text{high}}$ exponential into a power series,

$$\begin{aligned} e^{i\boldsymbol{H}_{\text{high}}t} \| O \rangle\rangle &= \boldsymbol{I} \| O \rangle\rangle + \sum_{x=1}^{\infty} \frac{(it)^x}{x!} \boldsymbol{H}_{\text{high}}^x \| O \rangle\rangle \\ &= \| O \rangle\rangle + \sum_{x=1}^{\infty} 0 \\ &= \| O \rangle\rangle. \end{aligned} \tag{30}$$

Since all components of $\|O\rangle\rangle$ have weight at most $\ell + k$, $\boldsymbol{H}_{\text{high}}$ acts trivially on $\|O\rangle\rangle$ and sends it to zero. Thus, we are left with just the 0^{th} order term from the power series expansion, $\boldsymbol{I}\|O\rangle\rangle$, and $e^{i\boldsymbol{H}_{\text{high}}t}\|O\rangle\rangle = \|O\rangle\rangle$. Hence, we can rewrite $e^{i(\boldsymbol{H} - \boldsymbol{H}_{\text{mid}})t}\|O\rangle\rangle = e^{i\boldsymbol{H_c}t}\|O\rangle\rangle$. □

Fact 2. *For any $\boldsymbol{L}$, $\boldsymbol{H}_{mid}$, we have the following identity,*

$$e^{(\boldsymbol{L} - i\boldsymbol{H}_{mid})t} - e^{\boldsymbol{L}t} = -\int_0^t e^{(\boldsymbol{L} - i\boldsymbol{H}_{mid})(t-t')} (i\boldsymbol{H}_{mid}) e^{\boldsymbol{L}t'} dt'. \tag{31}$$

Proof. We would like to express the difference between the unitaries as an integral of instantaneous difference accumulated over time. We can write,

$$\begin{aligned} e^{(\boldsymbol{L} - i\boldsymbol{H}_{\text{mid}})t} - e^{\boldsymbol{L}t} &= e^{(\boldsymbol{L} - i\boldsymbol{H}_{\text{mid}})t} (1 - e^{-(\boldsymbol{L} - i\boldsymbol{H}_{\text{mid}})t} e^{\boldsymbol{L}t}) \\ &= e^{(\boldsymbol{L} - i\boldsymbol{H}_{\text{mid}})t} \int_0^t e^{-(\boldsymbol{L} - i\boldsymbol{H}_{\text{mid}})t'} (-i\boldsymbol{H}_{\text{mid}}) e^{\boldsymbol{L}t'} dt' \\ &= -\int_0^t e^{(\boldsymbol{L} - i\boldsymbol{H}_{\text{mid}})(t-t')} (i\boldsymbol{H}_{\text{mid}}) e^{\boldsymbol{L}t'} dt'. \end{aligned} \tag{32}$$

In the first line, we take the derivative of $1 - e^{\boldsymbol{L}t}e^{-(\boldsymbol{L}-i\boldsymbol{H}_{\text{mid}})t}$ and then integrate it with respect to time. In the second line, we pull the leading factor into the integral and combine with the first factor. ☐

Using Fact 2, we know,

$$\begin{aligned} \|\|O_c(t)\rangle\rangle - \|O(t)\rangle\rangle\|_F &= \|-\int_0^t dt'\, e^{(\boldsymbol{L}-i\boldsymbol{H}_{\text{mid}})(t-t')}(i\boldsymbol{H}_{\text{mid}})e^{\boldsymbol{L}t'}\|O\rangle\rangle\|_F \\ &\leq \int_0^t dt' \|(-e^{(\boldsymbol{L}-i\boldsymbol{H}_{\text{mid}})(t-t')}(i\boldsymbol{H}_{\text{mid}})e^{\boldsymbol{L}t'})\|O\rangle\rangle\|_F \\ &= \int_0^t dt' \|e^{(\boldsymbol{L}-i\boldsymbol{H}_{\text{mid}})(t-t')}\boldsymbol{H}_{\text{mid}}e^{\boldsymbol{L}t'}\|O\rangle\rangle\|_F \end{aligned} \tag{33}$$

In the first line, we apply the triangle inequality to pull the integral out of the norm. In the second line, we drop the negative i, since matrix norm is invariant to global phase.

We know $\boldsymbol{L} - i\boldsymbol{H}_{\text{mid}} = i(\boldsymbol{H_c} + \boldsymbol{H}_{\text{high}}) - \gamma\boldsymbol{W}$. Since $i(\boldsymbol{H_c} + \boldsymbol{H}_{\text{high}})$ is Hermitian, $e^{i(\boldsymbol{H_c}+\boldsymbol{H}_{\text{high}})t}$ is unitary and norm-preserving, whereas $e^{-\gamma\boldsymbol{W}t}$ either preserves or decreases the norm of operators for $t \geq 0$. Thus, we know that $e^{(\boldsymbol{L}-i\boldsymbol{H}_{\text{mid}})(t-t')}$ can never increase the norm of any operator when $t' \in (0, t)$:

$$\begin{aligned} \|\|O_c(t)\rangle\rangle - \|O(t)\rangle\rangle\|_F &\leq \int_0^t dt' \|e^{(\boldsymbol{L}-i\boldsymbol{H}_{\text{mid}})(t-t')}\boldsymbol{H}_{\text{mid}}e^{\boldsymbol{L}t'}\|O\rangle\rangle\|_F \\ &\leq \int_0^t dt' \|\boldsymbol{H}_{\text{mid}}e^{\boldsymbol{L}t'}\|O\rangle\rangle\|_F \\ &= \int_0^t dt' \|\boldsymbol{H}_{\text{mid}}\mathcal{P}_{>\ell-k}e^{\boldsymbol{L}t'}\|O\rangle\rangle\|_F \\ &\leq \|\boldsymbol{H}_{\text{mid}}\|_\infty \int_0^t dt' \|\mathcal{P}_{>\ell-k}e^{\boldsymbol{L}t'}\|O\rangle\rangle\|_F \\ &= \|\boldsymbol{H}_{\text{mid}}\|_\infty \cdot t \max_{0\leq t'\leq t}\{\|\mathcal{P}_{>\ell-k}e^{\boldsymbol{L}t'}\|O\rangle\rangle\|_F\}. \end{aligned} \tag{34}$$

In the second line, we apply the identity $\boldsymbol{H}_{\text{mid}} = \boldsymbol{H}_{\text{mid}}\mathcal{P}_{>\ell-k}$, where $\mathcal{P}_{>\ell-k}$ is a projector onto the Pauli operators with weight greater than $\ell - k$. This identity is true since $\langle\langle Q\|\boldsymbol{H}_{\text{mid}}\|P\rangle\rangle$ is only nonzero if at least one of $\|Q\rangle\rangle$ and $\|P\rangle\rangle$ has weight more than ℓ. Since H is k-local, this implies that the other operator has at least weight $\ell - k + 1$, so for all operators $\|P\rangle\rangle$ with weight less than $\ell - k + 1$, $\boldsymbol{H}_{\text{mid}}\|P\rangle\rangle = 0$ (See Fact 7 for a more precise proof). We obtain the third line by applying Holder's inequality and observing that $\boldsymbol{H}_{\text{mid}}$ is independent of time. Finally, in line 4, we upper bound the integral of $\|\mathcal{P}_{>\ell-k}e^{\boldsymbol{L}t'}\|O\rangle\rangle\|_F$ over time by the maximum value it attains. ☐

III. PROOF OF LEMMA 3: BOUNDING THE SPECTRAL NORM OF $\boldsymbol{H}_{\text{mid}}$

In this section, we use the locality, bounded degree of interaction, and the bounded coefficients of our Hamiltonian H, to bound the spectral norm of our truncation, $\boldsymbol{H}_{\text{mid}}$. To do so, we first bound spectral norm in terms of the absolute row and column sum norms for any matrix. We then bound the value of the absolute column sum based on the number of non-zero matrix elements and the allowed range of values in $\boldsymbol{H}$. For intuition, each Pauli operator can transition to only a limited number of operators given that the Hamiltonian is local and has bounded degree of interaction. Since the coefficients of H are bounded, the values the matrix elements can attain are as well. Combining these facts together, we can upper bound the spectral norm. We first introduce the facts we will use in our proof, and then we prove the lemma.

Fact 3. *For any matrix A, $\|A\|_\infty \leq \sqrt{\|A\|_{col}\|A\|_{row}}$*

Proof. We know that $\|A\|_\infty^2 = \lambda_{\max}(A^\dagger A)$, where $\lambda_{\max}(A^\dagger A)$ is the largest eigenvalue of $A^\dagger A$. Since the maximum absolute row sum of a matrix is an induced matrix norm, we know that $\lambda_{\max}(A^\dagger A) \leq \|A^\dagger A\|_{\text{row}}$. By sub-multiplicativity of matrix norms, we know $\|A^\dagger A\|_{\text{row}} \leq \|A^\dagger\|_{\text{row}}\|A\|_{\text{row}}$. We also know that $\|A^\dagger\|_{\text{row}} = \|A\|_{\text{col}}$. Putting this all together, we get that $\|A\|_\infty \leq \sqrt{\|A\|_{\text{col}}\|A\|_{\text{row}}}$. ☐

Fact 4. *Under any Hamiltonian $H = \sum_P h_P P$, the transition amplitude $\langle\langle Q \| \boldsymbol{H} \| A \rangle\rangle$ between two Pauli operators Q, A is*

$$\frac{h_{P'}}{\alpha}(1 - (-1)^{\langle P', A\rangle}), \tag{35}$$

where P' is the unique Pauli operator such that $QA = \alpha P'$, where $\alpha \in \{\pm 1, \pm i\}$.

Proof. We can write $\langle\langle Q \| \boldsymbol{H} \| A \rangle\rangle$ in terms of H as

$$\begin{aligned} \langle\langle Q \| \boldsymbol{H} \| A \rangle\rangle &= \langle\langle Q \| [H, A] \rangle\rangle \\ &= \langle\langle Q \| \sum_P h_P \| [P, A] \rangle\rangle \\ &= \langle\langle Q \| \sum_P h_P (1 - (-1)^{\langle P', A\rangle}) \| PA \rangle\rangle \\ &= \sum_P h_P (1 - (-1)^{\langle P', A\rangle}) \langle\langle Q \| PA \rangle\rangle. \end{aligned} \tag{36}$$

In the first line, we expand out H in the Pauli basis. In the second line, we note that for Pauli operators, $PA = (-1)^{\langle P, A\rangle} AP$ where $\langle P, A\rangle$ is 1 if P and A anti-commute and 0 otherwise. We then rewrite the inner product as a sum of inner products with each Pauli operator in the third line. Since P and A are Pauli operators, PA is also a Pauli operator up to some factor $\alpha \in \{\pm 1, \pm i\}$. Thus, PA is either proportional to Q or orthogonal to it. We also know that for any given Q, A, there is one and exactly one P' such that $\alpha P' = QA$. Since A is a Pauli operator, $A = A^{-1}$ so $\alpha P' A = Q$ where P' is unique. Knowing this, we reduce the summation to

$$\begin{aligned} \langle\langle Q \| \boldsymbol{H} \| A \rangle\rangle &= h_{P'} (1 - (-1)^{\langle P', A\rangle}) \langle\langle Q \| P' A \rangle\rangle \\ &= \frac{h_{P'}}{\alpha} (1 - (-1)^{\langle P', A\rangle}) \langle\langle Q \| Q \rangle\rangle \\ &= \frac{h_{P'}}{\alpha} (1 - (-1)^{\langle P', A\rangle}). \end{aligned} \tag{37}$$

□

Fact 5. *For any Hamiltonian $H = \sum_P h_P P$, where $h_P \in [-1, 1]$, the magnitude of the transition amplitude $\langle\langle Q \| \boldsymbol{H} \| A \rangle\rangle$ for any Pauli operators A, Q is bounded by 2.*

Proof. From Fact 4, we know that

$$\begin{aligned} |\langle\langle Q \| \boldsymbol{H} \| A \rangle\rangle| &= |\frac{h_{P'}}{\alpha} (1 - (-1)^{\langle P', A\rangle})| \\ &\leq |\alpha^{-1}| \cdot |h_{P'}| \cdot |(1 - (-1)^{\langle P', A\rangle})| \\ &\leq 2, \end{aligned} \tag{38}$$

since $\alpha \in \{\pm 1, \pm i\}$, $h_{P'} \in [-1, 1]$, and $(1 - (-1)^{\langle P', A\rangle})$ is at most 2 when $f(P', A)$ is odd. □

Fact 6. *Under the application of a d-degree Hamiltonian H, any Pauli operator A can transition to at most $d \cdot w[A]$ Pauli operators.*

Proof. We know from Fact 4 that, for any Pauli operator Q,

$$\langle\langle Q \| \boldsymbol{H} \| A \rangle\rangle = \frac{h_{P'}}{\alpha} (1 - (-1)^{\langle P, A\rangle}), \tag{39}$$

where P is the unique Pauli operator such that $\alpha PA = Q$, meaning there is a one-to-one correspondence between P and Q. Thus, to find the number of Pauli operators Q that A can transition to, we can simply analyze how many P exist such that $\frac{h_{P'}}{\alpha}(1 - (-1)^{\langle P, A\rangle})$ is nonzero. This can be expressed as

$$|\{P \mid h_P \neq 0, (1 - (-1)^{\langle P, A\rangle}) \neq 0\}|, \tag{40}$$

since α is always non-zero and finite. For $(1-(-1)^{\langle P,A\rangle}) \neq 0$, P and A must have anti-commuting elements at an odd number of sites. We know that A has non-identity elements at $w[A]$ sites, so P needs to have at least one non-identity element at one of these sites for $\langle P, A\rangle \neq 0$. However, since H has a bounded degree, there are only at most d Pauli operators P that interact with a given site and have non-zero coefficients h_P. As a result, there are at most $d \cdot w[A]$ Pauli operators that could simultaneously satisfy $h_P \neq 0$ and $(1-(-1)^{\langle P,A\rangle}) \neq 0$. □

Fact 7. *For any k-local Hamiltonian H, the transition amplitude $\langle\!\langle Q\|\boldsymbol{H}\|A\rangle\!\rangle$ between two Pauli operators A and Q is non-zero only if $|w[A]-w[Q]| \leq k-1$.*

Proof. Once again, we use Fact 4,

$$\langle\!\langle Q\|\boldsymbol{H}\|A\rangle\!\rangle = \frac{h_{P'}}{\alpha}(1-(-1)^{\langle P,A\rangle}), \tag{41}$$

where P is the unique Pauli operator such that $\alpha PA = Q$. Since H is k-local, $w[P] \leq k$. We know that for $\langle P, A\rangle \neq 0$, P has to have at least one of its non-identity elements at a site where A already has a non-identity element, which leaves at most $k-1$ non-identity elements in P that could overlap with an identity element in A and increase the weight of PA. Thus, $w[Q] = w[PA] \leq w[A]+k-1$. $w[PA]$ is the smallest when every non-identity element of P overlaps with the same non-identity element in A, resulting in PA having an identity element at that site. However, in this case $\langle P, A\rangle = 0$ since P commutes with A at every site. Thus, there has to be at least one site at which P and A anti-commute, resulting in a non-identity element at that site in PA. P has at most k non-identity elements and at least one of them has to produce a non-identity element in PA, so $w[Q] = w[PA] \geq w[A]-k+1$. □

Proof of Lemma 3. We can now apply these facts to bound the spectral norm of $\boldsymbol{H}_{\text{mid}} \equiv \boldsymbol{H} - \mathcal{P}_{>\ell+k}\boldsymbol{H}\mathcal{P}_{>\ell+k} - \mathcal{P}_{\leq\ell}\boldsymbol{H}\mathcal{P}_{\leq\ell}$. First, we can apply Fact 3 to obtain $\|\boldsymbol{H}_{\text{mid}}\|_\infty \leq \sqrt{\|\boldsymbol{H}_{\text{mid}}\|_{\text{col}}\|\boldsymbol{H}_{\text{mid}}\|_{\text{row}}}$. Since $\boldsymbol{H}_{\text{mid}}$ is a sum of Hermitian matrices, it itself is Hermitian, so $\|\boldsymbol{H}_{\text{mid}}\|_{\text{col}} = \|\boldsymbol{H}_{\text{mid}}\|_{\text{row}}$ and

$$\|\boldsymbol{H}_{\text{mid}}\|_\infty \leq \|\boldsymbol{H}_{\text{mid}}\|_{\text{col}}. \tag{42}$$

Since $\langle\!\langle Q\|\boldsymbol{H}_{\text{mid}}\|P\rangle\!\rangle = 0$ if both $w[P], w[Q] > \ell+k$, either P or Q has to have weight less than $\ell+k$. Combining this with Fact 7, we know $w[P] \leq \ell+2k$ for any non-zero transition amplitude in $\boldsymbol{H}_{\text{mid}}$. Applying Fact 6, we know that the number of operators to which P can transition in $\boldsymbol{H}_{\text{mid}}$ is at most $d\cdot w[P] \leq d(\ell+2k)$. Thus, the maximum number of non-zero entries in a column of $\boldsymbol{H}_{\text{mid}}$ is $d(\ell+2k)$. By Fact 5, the maximum absolute value of each of these entries is 2, so

$$\|\boldsymbol{H}_{\text{mid}}\|_\infty \leq \|\boldsymbol{H}_{\text{mid}}\|_{\text{col}} \leq 2d(\ell+2k), \tag{43}$$

bounding the truncation norm. □

IV. PROOF OF LEMMA 4: BOUNDING THE HIGH-WEIGHT PROBABILITY

In this section, we derive an upper bound on

$$\max_{0\leq t'\leq t}\{\|\mathcal{P}_{>\ell-k}e^{\boldsymbol{L}t'}\|O\rangle\!\rangle\|_F\}. \tag{44}$$

Our proof utilizes the notion of an operator weight distribution, first introduced in [11]. The operator weight distribution is a probability distribution, $W(t) = \{w, P(w,t)\}$, where

$$P(w,t) = \langle\!\langle O(t)\|\mathcal{P}_w\|O(t)\rangle\!\rangle = \|\mathcal{P}_w\|O(t)\rangle\!\rangle\|_F^2, \tag{45}$$

for each $w = 0, 1, 2, \ldots, n$. In terms of the operator weight distribution, our quantity of interest, $\|\mathcal{P}_{>\ell-k}e^{\boldsymbol{L}t'}\|O\rangle\!\rangle\|_F$, is equal to the cumulative probability of weights greater than $\ell-k$.

Our strategy will be to apply Markov's inequality to obtain a tight tail bound on this cumulative probability using only the moments of the weight distribution. We show that the differential equations for the moments have steady-state solutions. We prove the steady-state solutions upper bound the moments at all times. We then translate the upper bounds on the moments into a tail bound by extending Markov's inequality to higher moments. Intuitively, these steady-state solutions exist because the "rate" at which the H can increase high-weight support and the "rate" at which the noise exponentially dampens this support depends on how high-weight the operator already is, so there is a critical value at which these rates are equivalent, and the growth of the moments goes to zero. For the ease of the reader, we will defer certain mathematical details of the proof to Section V, which we will reference as we go through the proof.

First, note that if the derivative of the moment is bounded, the integral of the upper bound also bounds the moment itself. Thus, we specifically focus on the derivative of the moments of $W(t)$,

$$\begin{aligned}\partial_t \langle W(t)^m\rangle &= \partial_t \sum_{w=0}^{n} wP(w,t) \\ &= \sum_{w=0}^{n} w\cdot \partial_t P(w,t),\end{aligned} \tag{46}$$

where $\langle W(t)^m\rangle$ is the m^{th} moment of $W(t)$. We know

$$\begin{aligned}\partial_t P(w,t) &= \langle\langle O'(t)\|\mathcal{P}_w\|O(t)\rangle\rangle + \langle\langle O(t)\|\partial_t\mathcal{P}_w\|O(t)\rangle\rangle + \langle\langle O(t)\|\mathcal{P}_w\|O'(t)\rangle\rangle \\ &= \langle\langle O(t)\|(-i\boldsymbol{H}-\gamma\boldsymbol{W}^\dagger)\mathcal{P}_w\|O(t)\rangle\rangle + \langle\langle O(t)\|\mathcal{P}_w(i\boldsymbol{H}-\gamma\boldsymbol{W})\|O(t)\rangle\rangle \\ &= i\langle\langle O(t)\|\mathcal{P}_w\boldsymbol{H}-\boldsymbol{H}\mathcal{P}_w\|O(t)\rangle\rangle - \gamma\langle\langle O(t)\|\mathcal{P}_w\boldsymbol{W}+\boldsymbol{W}\mathcal{P}_w\|O(t)\rangle\rangle \\ &= i\langle\langle O(t)\|\mathcal{P}_w\boldsymbol{H}-\boldsymbol{H}\mathcal{P}_w\|O(t)\rangle\rangle - 2\gamma\langle\langle O(t)\|\boldsymbol{W}\mathcal{P}_w\|O(t)\rangle\rangle \\ &= i\langle\langle O(t)\|\mathcal{P}_w\boldsymbol{H}-\boldsymbol{H}\mathcal{P}_w\|O(t)\rangle\rangle - 2w\gamma P(w,t).\end{aligned} \tag{47}$$

In the second line, we use $\boldsymbol{W}=\boldsymbol{W}^\dagger$ to simplify the expression. In the third line, we note that $\boldsymbol{W}$ sends Pauli operators to themselves, so it can commute with $\mathcal{P}_w$. Finally, we apply the fact that $\boldsymbol{W}\|P\rangle\rangle = w[P]\|P\rangle\rangle$, meaning $\boldsymbol{W}\mathcal{P}_w = w\mathcal{P}_w$.

Based on Fact 7, we can only transition from operator components of weight w to components with weights in the range $[w-(k-1), w+(k-1)]$ under a k-local Hamiltonian. This means that $\boldsymbol{H}\mathcal{P}_w = \sum_{s=-(k-1)}^{k-1}\boldsymbol{H}_{w-s,w}$, where we let,

$$\boldsymbol{H}_{w-s,w} = \mathcal{P}_{w-s}\boldsymbol{H}\mathcal{P}_w = \sum_{w[Q]=w-s, w[P]=w}\langle\langle Q\|\boldsymbol{H}\|P\rangle\rangle, \tag{48}$$

denote the component of $\boldsymbol{H}$ that enacts transitions from weight w to weight $w-s$. Similarly, $\mathcal{P}_w\boldsymbol{H} = \sum_{s=-(k-1)}^{k-1}\boldsymbol{H}_{w,w-s}$. For any unphysical weight transitions (i.e. $w, w-s\notin[0,n]$), we have $\boldsymbol{H}_{w-s,w} = \boldsymbol{H}_{w,w-s} = 0$. Inserting this decomposition into our differential equation, we have

$$\begin{aligned}\partial_t P(w,t) &= i\langle\langle O(t)\|\sum_{s=-(k-1)}^{k-1}(\boldsymbol{H}_{w,w-s}-\boldsymbol{H}_{w-s,w})\|O(t)\rangle\rangle - 2w\gamma P(w,t) \\ &= \sum_{s=1}^{k-1} i\langle\langle O(t)\|(\boldsymbol{H}_{w,w-s}-\boldsymbol{H}_{w-s,w})\|O(t)\rangle\rangle + \sum_{s=1}^{k-1} i\langle\langle O(t)\|(\boldsymbol{H}_{w,w+s}-\boldsymbol{H}_{w+s,w})\|O(t)\rangle\rangle - 2w\gamma P(w,t).\end{aligned} \tag{49}$$

In the second line, we reorganize the summations and, noting that $\boldsymbol{H}_{w,w}-\boldsymbol{H}_{w,w}=0$, reindex s to start at 1 instead of 0. Here, it is useful to use the notion of currents,

$$J_{w,s}(t) = i\langle\langle O(t)\|\boldsymbol{H}_{w,w-s}-\boldsymbol{H}_{w-s,w}\|O(t)\rangle\rangle \tag{50}$$

as described in [4]. We rewrite $\partial_t P(w,t)$ in terms of currents:

$$\partial_t P(w,t) = \left(\sum_{s=1}^{k-1} J_{w,s}(t) - J_{w+s,s}(t)\right) - 2w\gamma P(w,t). \tag{51}$$

Plugging this into our differential equation for the moment, we find

$$\begin{aligned}\partial_t\langle W(t)^m\rangle &= \sum_{w=0}^{n} w^m\partial_t P(w,t) \\ &= \left(\sum_{w=0}^{n}\sum_{s=1}^{k-1} w^m J_{w,s}(t) - w^m J_{w+s,s}(t)\right) - \sum_{w=0}^{n} 2w^{m+1}\gamma P(w,t) \\ &= -2\gamma\langle W(t)^{m+1}\rangle + \sum_{s=1}^{k-1}\sum_{w=0}^{n}(w^m-(w-s)^m)J_{w,s}(t).\end{aligned} \tag{52}$$

In the second line, we rewrite the noise term in terms of the $m+1$ moment and we reindex the second term inside the summation from $w \to w+s$. To simplify this expression further, we prove the following proposition:

Proposition 1. *$J_{w,s}(t)$ is upper-bounded by $4d\sqrt{wP(w,t)}\sqrt{(w-s)P(w-s,t)}$.*

Proof. We know $J_{w,s}(t) = \langle\langle O(t)\|\boldsymbol{H}_{w,w-s} - \boldsymbol{H}_{w-s,w}\|O(t)\rangle\rangle$, so

$$|J_{w,s}(t)| \le |\langle\langle O(t)\|\boldsymbol{H}_{w,w-s}\|O(t)\rangle\rangle| + |\langle\langle O(t)\|\boldsymbol{H}_{w-s,w}\|O(t)\rangle\rangle|. \tag{53}$$

Since $\boldsymbol{H}_{w,w-s} = \mathcal{P}_w \boldsymbol{H}_{w,w-s}\mathcal{P}_{w-s}$,

$$\begin{aligned}\langle\langle O(t)\|\boldsymbol{H}_{w,w-s}\|O(t)\rangle\rangle^2 &= \langle\langle O(t)\|\mathcal{P}_w\boldsymbol{H}_{w,w-s}\mathcal{P}_{w-s}\|O(t)\rangle\rangle^2 \\ &\le \langle\langle O(t)\|\mathcal{P}_w\mathcal{P}_w^\dagger\|O(t)\rangle\rangle \cdot \langle\langle O(t)\|\mathcal{P}_{w-s}^\dagger\boldsymbol{H}_{w,w-s}^\dagger\boldsymbol{H}_{w,w-s}\mathcal{P}_{w-s}\|O(t)\rangle\rangle \\ &= \langle\langle O(t)\|\mathcal{P}_w\|O(t)\rangle\rangle \cdot \|\boldsymbol{H}_{w,w-s}\mathcal{P}_{w-s}\|O(t)\rangle\rangle\|_F^2.\end{aligned} \tag{54}$$

In the first line, we apply the Cauchy–Schwarz inequality. In the second line, we use the fact that $\mathcal{P}_w^\dagger = \mathcal{P}_w$ to simplify the first term, and we rewrite the second term as a Frobenius norm. Using Holder's inequality, we can break up the second term into two,

$$\begin{aligned}\langle\langle O(t)\|\boldsymbol{H}_{w,w-s}\|O(t)\rangle\rangle^2 &\le P(w,t)\|\boldsymbol{H}_{w,w-s}\|_\infty^2\|\mathcal{P}_{w-s}\|O(t)\rangle\rangle\|_F^2 \\ &= P(w,t)P(w-s,t)\|\boldsymbol{H}_{w,w-s}\|_\infty^2 \\ &= P(w,t)P(w-s,t)\|\boldsymbol{H}_{w,w-s}\|_{\text{col}}\|\boldsymbol{H}_{w,w-s}\|_{\text{row}}.\end{aligned} \tag{55}$$

In the fourth line, we rewrite $\|\mathcal{P}_{w-s}\|O(t)\rangle\rangle\|_F^2$ as $P(w-s,t)$ by definition, and in the last line, we apply Fact 3. Since $\boldsymbol{H}$ is Hermitian, we know $\langle\langle Q\|\boldsymbol{H}\|P\rangle\rangle = \overline{\langle\langle P\|\boldsymbol{H}\|Q\rangle\rangle}$, which means that $\boldsymbol{H}_{w,w-s} = \boldsymbol{H}_{w-s,w}^\dagger$ and $\|\boldsymbol{H}_{w,w-s}\|_{\text{row}} = \|\boldsymbol{H}_{w-s,w}\|_{\text{col}}$. Thus,

$$\begin{aligned}\langle\langle O(t)\|\boldsymbol{H}_{w,w-s}\|O(t)\rangle\rangle^2 &\le P(w,t)P(w-s,t)\|\boldsymbol{H}_{w,w-s}\|_{\text{col}}\|\boldsymbol{H}_{w-s,w}\|_{\text{col}} \\ &\le P(w,t)P(w-s,t)(2d(w-s))(2dw),\end{aligned} \tag{56}$$

where we apply Lemmas 5 and 6. Using the same technique, we find the same upper bound for $\langle\langle O(t)\|\boldsymbol{H}_{w-s,w}\|O(t)\rangle\rangle^2$. From this, we know

$$\begin{aligned}J_{w,s}(t) &\le |J_{w,s}(t)| \\ &\le 2\sqrt{P(w,t)P(w-s,t)(2d(w-s))(2dw)} \\ &= 4d\sqrt{wP(w,t)}\sqrt{(w-s)P(w-s,t)},\end{aligned} \tag{57}$$

which completes the proof of the proposition. ☐

Using the upper bound on currents from Proposition 1, we find that the growth of the m^{th} moment is bound by

$$\partial_t\langle W(t)^m\rangle \le -2\gamma\langle W(t)^{m+1}\rangle + \sum_{s=1}^{k-1}\sum_{w=0}^{n}(w^m - (w-s)^m)4d\sqrt{wP(w,t)}\sqrt{(w-s)P(w-s,t)}. \tag{58}$$

We bound the geometric mean $\sqrt{wP(w,t)}\sqrt{(w-s)P(w-s,t)}$ with the arithmetic mean $\frac{1}{2}(wP(w,t) + (w-s)P(w-s,t))$, and through various algebraic manipulations which allow us to reformulate the bound in terms of the m^{th} moment itself, we find

$$\partial_t\langle W(t)^m\rangle \le -2\gamma\langle W(t)^m\rangle^{\frac{m+1}{m}} + 4dm(k-1)^2(\langle W(t)^m\rangle^{\frac{1}{m}} + k - 1)^m. \tag{59}$$

We defer the details of this derivation to Appendix V. We abbreviate $u = \langle W(t)^m\rangle^{\frac{1}{m}}$, which allows us to express the bound in a simplified form:

$$\partial_t u^m \le -2\gamma u^{m+1} + 4dm(k-1)^2(u+k-1)^m. \tag{60}$$

The right hand side is a first-order differential equation with one strictly decreasing term and one strictly increasing term. The derivative is equal to zero when the two terms cancel, at a value $u = u^*$. Deferring algebraic details to Section V, we find that u^* is upper bounded, $u^* \leq u_{\mathrm{ub}}$, by,

$$u_{\mathrm{ub}} = \left(1 + \sqrt{1+2\gamma}\right) \frac{dk^2}{\gamma} m. \tag{61}$$

Hence, if $u(0) \leq u^* \leq u_{\mathrm{ub}}$ at time zero, then $u(t) \leq u^* \leq u_{\mathrm{ub}}$ at all future times t.

We now return to our original expression of interest, $\max_{0\leq t'\leq t}\{\|\mathcal{P}_{>\ell-k} e^{\boldsymbol{L}t'}\|O\rangle\!\rangle\|_F\}$. We can rewrite this in terms of the operator weight distribution $P(w,t)$ as follows,

$$\begin{aligned}
\|\mathcal{P}_{>\ell-k} e^{\boldsymbol{L}t'}\|O\rangle\!\rangle\|_F^2 &= \frac{1}{2^n} \operatorname{tr}\left((\mathcal{P}_{>\ell-k} O(t'))^\dagger (\mathcal{P}_{>\ell-k} O(t'))\right) \\
&= \frac{1}{2^n} \operatorname{tr}\left(O(t')(\sum_{w=\ell-k+1}^{n} \mathcal{P}_w)(\sum_{w=\ell-k+1}^{n} \mathcal{P}_w) O(t')\right) \\
&= \sum_{w,w'=\ell-k+1}^{n} \frac{1}{2^n} \operatorname{tr}\left(O(t')\mathcal{P}_{w'}\mathcal{P}_w O(t')\right) \\
&= \sum_{w,w'=\ell-k+1}^{n} \frac{1}{2^n} \operatorname{tr}\left(O(t')\delta_{w'=w}\mathcal{P}_w O(t')\right) \\
&= \sum_{w=\ell-k+1}^{n} \|\mathcal{P}_w \|O(t')\rangle\!\rangle\|_F^2 \\
&= \sum_{w=\ell-k+1}^{n} P(w,t').
\end{aligned} \tag{62}$$

In the first line, we decompose $\mathcal{P}_{>\ell-k}$ into $\sum_{w=\ell-k+1}^{n} \mathcal{P}_w$, each of which projects onto an orthogonal subspace. In the third line, we use the linearity of trace to pull the summations out of the trace. In the fourth line, we apply the fact that $\mathcal{P}_{w'}\mathcal{P}_w = 0$ when $w' \neq w$ since they are projectors for orthogonal subspaces, and $\mathcal{P}_{w'}\mathcal{P}_w = \mathcal{P}_w$ when $w' = w$. In line 5, we rewrite the trace in terms of the Frobenius norm, and then in line 6, we re-express this in terms of $P(w,t')$. Thus, we find that $\|\mathcal{P}_{>\ell-k} e^{\boldsymbol{L}t'}\|O\rangle\!\rangle\|_F = \Pr(W(t') \geq \ell - k + 1)^{\frac{1}{2}}$.

Applying Markov's inequality to the m-th moment, we know that

$$\Pr(W(t') \geq \ell - k + 1) \leq \min_m \frac{\langle W(t')^m \rangle}{(\ell - k + 1)^m} \tag{63}$$

Given $\langle W(0)^m \rangle = u(0) \leq u^{*m}$, we can upper bound the m-th moment to re-write,

$$\Pr(W(t') \geq \ell - k + 1) \leq \min_m \left(\frac{u_{\mathrm{ub}}}{\ell - k + 1}\right)^m \leq \min_m \left(\frac{dk^2\left(1+\sqrt{1+2\gamma}\right)}{\gamma(\ell-k+1)} m\right)^m. \tag{64}$$

To minimize the expression, we note the following fact,

Fact 8. *The minimum value of $(cx)^x$, for $x, c > 0$, is attained when $x = (ce)^{-1}$, and is equal to $e^{-\frac{1}{ce}}$.*

Proof. We know $(cx)^x = e^{x\ln(cx)}$, so minimizing $(cx)^x$ is equivalent to minimizing $x\ln(cx)$. The first derivative of $x\ln(cx)$ is $\ln(cx)+1$, found using the product rule, and the second derivative is $\frac{1}{cx}+1$. When $x, c > 0$, the second derivative is positive for all x, so $(cx)^x$ is strictly convex and only has a global minimum. The only value at which first derivative is zero is $x = (ce)^{-1}$, so the global minimum occurs when $x = (ce)^{-1}$. □

From Fact 8, we can achieve the tightest possible bound on $\Pr(W(t') \geq \ell - k + 1)$ by setting $m = \frac{\gamma(\ell-k+1)}{dk^2\left(1+\sqrt{1+2\gamma}\right)e}$, which yields the upper bound

$$\Pr(W(t') \geq \ell - k + 1) \leq \exp\left[-\frac{\gamma(\ell-k+1)}{dk^2\left(1+\sqrt{1+2\gamma}\right)e}\right] \leq \exp\left[-2A_{d,k,\gamma}\gamma(\ell-k)\right]. \tag{65}$$

where we abbreviate,

$$A_{d,k,\gamma} = \frac{1}{6dk^2\left(1+\sqrt{1+2\gamma}\right)}, \tag{66}$$

which nearly completes our proof.

It remains only to ensure that $\langle W(0)^m\rangle \leq u^{*m}$ for our selected m, so that our upper bound on the m-th moment applies. Given that O is local (meaning it only has support on Paulis with weight less than w_0), $u(0) = \langle W(0)^m\rangle^{\frac{1}{m}} \leq w_0$. Thus, we can ensure $\langle W(0)^m\rangle \leq u^{*m}$ holds true by ensuring $w_0^m \leq u^{*m}$. Note that since the upper bound on $\langle W(0)^m\rangle$ is governed by a first-order differential equation with a single steady-state solution, all upper bounds monotonically converge to the steady-state solution. Thus, we can define an equivalent condition that the initial condition has a non-negative derivative, or

$$-2\gamma {w_0}^{m+1} + 4dm(k-1)^2(w_0+k-1)^m \geq 0, \tag{67}$$

which holds true if

$$-2\gamma {w_0}^{m+1} + 4dm(k-1)^2 {w_0}^m \geq 0, \tag{68}$$

holds true. Given $m = \frac{\gamma(\ell-k+1)}{dk^2\left(1+\sqrt{1+2\gamma}\right)e}$, we can rewrite this as

$$\begin{aligned} 4dm(k-1)^2{w_0}^m &\geq 2\gamma{w_0}^{m+1} \\ m &\geq \frac{\gamma w_0}{2d(k-1)^2} \\ \frac{\gamma(\ell-k+1)}{dk^2\left(1+\sqrt{1+2\gamma}\right)e} &\geq \frac{\gamma w_0}{2d(k-1)^2} \\ \ell &\geq \frac{k^2\left(1+\sqrt{1+2\gamma}\right)e}{2(k-1)^2}w_0 + k - 1. \end{aligned} \tag{69}$$

We can replace this condition on ℓ with a slightly more strict, but easier to manipulate, condition,

$$\ell \geq 8\left(1+\sqrt{1+2\gamma}\right)w_0 + k \tag{70}$$

since $k \geq 2$. □

V. MATHEMATICAL DETAILS FOR WEIGHT DISTRIBUTION MOMENTS

In this section, we present the mathematical details of the bounds placed on the moments of the operator weight distribution in Appendix IV. In the first subsection, we bound the growth of the m^{th} moment by an autonomous, first-order differential equation. We do so by regrouping terms in the differential equation obtained in Appendix IV and upper bounding the dependencies on other moments by expressions solely dependent on the m^{th} moment. In the second subsection, we analyze the behavior of the solutions to the differential equation to derive a simplified upper bound on the solutions, and thereby the moments.

A. Autonomous ODE upper bound

In this subsection, we demonstrate that

$$\partial_t\langle W(t)^m\rangle \leq -2\gamma\langle W(t)^{m+1}\rangle + \sum_{s=1}^{k-1}\sum_{w=0}^{n}(w^m-(w-s)^m)4d\sqrt{wP(w,t)}\sqrt{(w-s)P(w-s,t)}, \tag{71}$$

can be simplified to

$$\partial_t\langle W(t)^m\rangle \leq -2\gamma u^{m+1} + 4dm(k-1)^2(u+k-1)^m, \tag{72}$$

where $u = \langle W(t)^m \rangle^{\frac{1}{m}}$.

We begin by upper-bounding the double summation,

$$\sum_{s=1}^{k-1} \sum_{w=0}^{n} (w^m - (w-s)^m) 4d \sqrt{wP(w,t)} \sqrt{(w-s)P(w-s,t)}. \tag{73}$$

Since the arithmetic mean always upper bounds the geometric mean, the summation can be upper bounded by

$$2d \sum_{s=1}^{k-1} \sum_{w=0}^{n} (w^m - (w-s)^m)(wP(w,t) + (w-s)P(w-s,t)). \tag{74}$$

Applying the mean value theorem to the function x^m on the closed interval $[w, w-s]$, we find that $w^m - (w-s)^m \le s(mw^{m-1})$, so the summation can once again be upper bounded by,

$$2d \sum_{s=1}^{k-1} \sum_{w=0}^{n} smw^{m-1}(wP(w,t) + (w-s)P(w-s,t)). \tag{75}$$

We simplify the form of the last term of the second summation by upper bounding $w-s$ with w and then break apart the inner summation into two, resulting in

$$2dm \sum_{s=1}^{k-1} s \left(\sum_{w=0}^{n} w^m P(w,t) + \sum_{w=0}^{n} w^m P(w-s,t) \right). \tag{76}$$

By upper bounding w with $w+s$ in the first inner summation and reindex the second inner summation, we find

$$\begin{aligned} 2dm \sum_{s=1}^{k-1} s \left(\sum_{w=0}^{n} w^m P(w,t) + \sum_{w=0}^{n} w^m P(w-s,t) \right) &\le 2dm \sum_{s=1}^{k-1} s \left(\sum_{w=0}^{n} (w+s)^m P(w,t) + \sum_{w=-s}^{n-s} (w+s)^m P(w,t) \right) \\ &\le 4dm \sum_{s=1}^{k-1} s \langle (W(t)+s)^m \rangle \\ &\le 4dm(k-1)^2 \langle (W(t)+k-1)^m \rangle. \end{aligned} \tag{77}$$

In the first line, we upper bound both inner summations in terms of the m^{th} moment of the distribution $W(t)+s$. In the final line, we upper-bound the $k-1$ terms in summation with the largest term $(k-1)\langle (W(t)+k-1)^m \rangle$. Thus,

$$\partial_t \langle W(t)^m \rangle \le -2\gamma \langle W(t)^{m+1} \rangle + 4dm(k-1)^2 \langle (W(t)+k-1)^m \rangle. \tag{78}$$

We can manipulate this upper bound to be in terms of solely the m^{th} moment. Let's define $g(x) = x^{\frac{m+1}{m}}$. We know $g''(x) = \frac{m+1}{m^2} x^{\frac{1-m}{m}} \ge 0$ for all m when $x > 0$. Using Jensen's inequality, we know that $\langle g(W(t)^m) \rangle = \langle W(t)^{m+1} \rangle \ge \langle W(t)^m \rangle^{\frac{m+1}{m}} = g(\langle W(t)^{m+1} \rangle)$. Similarly, we know $h(x) = x^{a/m}$ is concave or linear when $a \le m$ and $x > 0$ since $h''(x) = \frac{a}{m}(\frac{a}{m} - 1)x^{\frac{a-2m}{m}} \le 0$ under those conditions. Thus, by Jensen's inequality, $\langle h(W(t)^m) \rangle = \langle W(t)^a \rangle \le \langle W(t)^m \rangle^{\frac{a}{m}} = h(\langle W(t)^{m+1} \rangle)$, and

$$\begin{aligned} \langle (W(t)+k-1)^m \rangle &= \sum_{w=0}^{n} (w+k-1)^m P(w,t) \\ &= \sum_{w=0}^{n} \sum_{a=0}^{m} \binom{m}{a} (k-1)^{m-a} w^a P(w,t) \\ &= \sum_{a=0}^{m} \binom{m}{a} (k-1)^{m-a} \langle W(t)^a \rangle \\ &\le \sum_{a=0}^{m} \binom{m}{a} (k-1)^{m-a} \langle W(t)^m \rangle^{a/m} \\ &= (\langle W(t)^m \rangle^{\frac{1}{m}} + k - 1)^m. \end{aligned} \tag{79}$$

Knowing this and $\langle W(t)^{m+1}\rangle \geq \langle W(t)^m\rangle^{\frac{m+1}{m}}$, we rewrite our upper bound on $\partial_t\langle W(t)^m\rangle$ as

$$\partial_t\langle W(t)^m\rangle \leq -2\gamma\langle W(t)^m\rangle^{\frac{m+1}{m}} + 4dm(k-1)^2(\langle W(t)^m\rangle^{\frac{1}{m}} + k - 1)^m. \tag{80}$$

To simplify this bound for readability, we can use $u = \langle W(t)^m\rangle^{\frac{1}{m}}$, rewriting the bound as

$$\partial_t\langle W(t)^m\rangle \leq -2\gamma u^{m+1} + 4dm(k-1)^2(u+k-1)^m. \tag{81}$$

□

B. Upper bound on stable steady-state solution to ODE

In this subsection, we prove the existence of a stable steady-state solution for

$$\begin{aligned}\partial_t\langle W(t)^m\rangle &\leq -2\gamma u^{m+1} + 4dm(k-1)^2(u+k-1)^m \\ &= -h(u) + g(u)\end{aligned} \tag{82}$$

where $u = \langle W(t)^m\rangle^{\frac{1}{m}}, h(u) = 2\gamma u^{m+1}$, $g(u) = 4dm(k-1)^2(u+k-1)^m$. Then, we upper bound this solution and demonstrate that if $\partial_t\langle W(0)^m\rangle$ is less than the upper bound, $\partial_t\langle W(t)^m\rangle$ will be less than the upper bound for all $t \geq 0$.

Let $f(u) = -h(u) + g(u)$. We first introduce the various facts and propositions that we use to build up our proof later on.

Fact 9. *If u^* is a real non-negative root of $f(u)$, provided $k > 1$ and $m, \gamma, d \geq 0$, then $f'(u^*) < 0$.*

Proof. We know $f'(u) = -2\gamma(m+1)u^m + 4dm^2(k-1)^2(u+k-1)^{m-1}$ by the power rule. By definition $f(u^*) = 0$, which we can rewrite as

$$\begin{aligned}4dm(k-1)^2(u^*+k-1)^m &= 2\gamma {u^*}^{m+1} \\ 4dm^2(k-1)^2(u^*+k-1)^{m-1} &= 2\gamma {u^*}^{m+1}\frac{m}{u^*+k-1}.\end{aligned} \tag{83}$$

Thus, we know that

$$\begin{aligned}f'(u^*) &= -2\gamma(m+1){u^*}^m + 4dm^2(k-1)^2(u^*+k-1)^{m-1} \\ &= -2\gamma(m+1){u^*}^m + 2\gamma\frac{m}{u^*+k-1}{u^*}^{m+1} \\ &= \frac{2\gamma {u^*}^m}{u^*+k-1}(mu_1 - (m+1)(u^*+k-1)) \\ &= -\frac{2\gamma {u^*}^m}{u^*+k-1}(u^* + (m+1)(k-1)).\end{aligned} \tag{84}$$

Since u^* is non-negative, $f'(u^*) < 0$. □

Fact 10. *$f(u)$ has at most one real non-negative root for $k > 1$ and $m, \gamma, d \geq 0$.*

Proof. We will prove this by contradiction. Suppose that $f(u)$ has multiple non-negative real roots. Let u_1, u_2 be two consecutive non-negative real roots, where $u_1 < u_2$. We know by Fact 9, $f'(u_1) < 0$ and $f'(u_2) < 0$, which means that $f(u_1 + a) < 0$ for some small $a > 0$ and $f(u_2 - b) > 0$ for some small $b > 0$. Since f is a polynomial of u, it is continuous in u. Then, by the intermediate value theorem, there exists at least one value $u_3 \in [u_1 + a, u_2 - b]$ such that $f(u_3) = 0$. However, we began by assuming that u_1, u_2 are consecutive roots, which is a contradiction. Thus, $f(u)$ has at most one real, non-negative root. □

Fact 11. *Let u^* be a real, nonnegative root of $f(u)$ for $k > 1, m > 0$ and $\gamma, d \geq 0$. If $\langle W(0)^m\rangle < {u^*}^m$, then $\langle W(t)^m\rangle \leq {u^*}^m$ for any $t \geq 0$.*

Proof. From Fact 10, we know that there is at most one real, nonnegative root of $f(u)$, so u^* is unique. An steady-state solution for $\partial_t\langle W(t)^m\rangle = f(\langle W(t)^m\rangle^{\frac{1}{m}})$ must satisfy $f(\langle W(t)^m\rangle^{\frac{1}{m}}) = 0$. Since there is one and only one root of $f(u)$, $\langle W(t)^m\rangle = u^{*m}$ is the steady-state solution for $\partial_t\langle W(t)^m\rangle = f(\langle W(t)^m\rangle^{\frac{1}{m}})$.

Also, from Fact 9, we know that $f'(u^*) < 0$. For $m > 0$,

$$
\begin{aligned}
\Big(\frac{df}{d\langle W(t)^m\rangle}\Big)_{u=u^*} &= f'(u^*)\Big(\frac{du}{d\langle W(t)^m\rangle}\Big)_{\langle W(t)^m\rangle = u^{*m}} \\
&= f'(u^*)\Big(\frac{1}{m}\langle W(t)^m\rangle^{\frac{1}{m}-1}\Big)_{u^{*m}} \\
&= f'(u^*)\Big(\frac{1}{m}(u^{*m})^{\frac{1}{m}-1}\Big) \\
&= f'(u^*)\Big(\frac{1}{mu^{*m-1}}\Big) < 0.
\end{aligned} \tag{85}
$$

Thus, $\langle W(t)^m\rangle = u^{*m}$ is a stable steady-state solution for $\partial_t\langle W(t)^m\rangle = f(\langle W(t)^m\rangle^{\frac{1}{m}})$, and the only real nonnegative steady-state solution. We know that $\langle W(t)^m\rangle$ is always real since it is the moment of a weight distribution, and if $\langle W(0)^m\rangle \le u^{*m}$, $\langle W(t)^m\rangle$ cannot grow larger than the stable steady-state solution. Thus, $\langle W(t)^m\rangle \le u^{*m}$ for all $t \ge 0$ when $\partial_t\langle W(t)^m\rangle = f(\langle W(t)^m\rangle^{\frac{1}{m}})$. If $\partial_t\langle W(t)^m\rangle \le f(\langle W(t)^m\rangle^{\frac{1}{m}})$, $\langle W(t)^m\rangle$ is upper-bounded by the solutions to $\partial_t\langle W(t)^m\rangle = f(\langle W(t)^m\rangle^{\frac{1}{m}})$, so $\langle W(t)^m\rangle \le u^{*m}$ for all $t \ge 0$ still holds true. ☐

Proposition 2. *Given $\frac{u}{k-1} > m$, then we have $(u+k-1)^m \le u^m + 2(k-1)mu^{m-1}$.*

Proof. We can expand $(u+k-1)^m$ using binomial expansion:

$$
\begin{aligned}
(u+k-1)^m &= \sum_{p=0}^{m}\binom{m}{p}u^{m-p}(k-1)^p \\
&= u^m + (k-1)u^{m-1}\sum_{p=1}^{m}\binom{m}{p}u^{m-p-(m-1)}(k-1)^{p-1} \\
&= u^m + (k-1)u^{m-1}\sum_{p=1}^{m}\binom{m}{p}(u^{-1}(k-1))^{p-1}.
\end{aligned} \tag{86}
$$

In the first line, we separate the first term of the summation from the rest of the terms and pull out $(k-1)u^{m-1}$ from the rest of them. In the second line, we simplify our expression inside the summation. We can further simplify and upper bound the summation by rewriting the binomial coefficient in terms of a product,

$$
\begin{aligned}
\sum_{p=1}^{m}\binom{m}{p}(u^{-1}(k-1))^{p-1} &= \sum_{p=1}^{m}\Big(\prod_{r=1}^{p}\frac{m-r+1}{r}\Big)(u^{-1}(k-1))^{p-1} \\
&\le m\Big(1 + \sum_{p=2}^{m}\Big(\prod_{r=2}^{p}\frac{m-r+1}{r}\Big)(m^{-1})^{p-1}\Big) \\
&= m\Big(1 + \sum_{p=2}^{m}\prod_{r=2}^{p}\frac{m-r+1}{mr}\Big).
\end{aligned} \tag{87}
$$

In the first line, we separate the $p = 1$ from the rest of the summation terms and upper bound $u^{-1}(k-1) \le m^{-1}$. In the second line, we consolidate the two product terms inside the summation together. Noting that $\frac{m-r+1}{r} \le \frac{m-2+1}{2}$ for $r \ge 2$, we know

$$
\begin{aligned}
\sum_{p=1}^{m}\binom{m}{p}(u^{-1}(k-1))^{p-1} &\le m\Big(1 + \sum_{p=2}^{m}\prod_{r=2}^{p}\frac{m-2+1}{2m}\Big) \\
&\le m\Big(1 + \sum_{p=2}^{m}2^{-(p-1)}\Big) \\
&= 2m
\end{aligned} \tag{88}
$$

Since $\frac{m-1}{2m} \le \frac{1}{2}$, we upper-bound the product with $2^{-(p-1)}$ in the first line. Finally, in the second line, we note that the resulting summation plus the term in front of the summation is upper-bounded by an infinite geometric series, which converges to 2. Thus,

$$(u+k-1)^m \le u^m + 2m(k-1)u^{m-1} \tag{89}$$

□

From Fact 11, we know that if we find u^*, a real, nonnegative root of $f(u)$ for $k, d > 1; m > 0$; and $\gamma \ge 0$, then we know that $\langle W(t)^m \rangle \le u^{*m}$ for all $t \ge 0$ given $\langle W(0)^m \rangle \le u^{*m}$, so we shift our focus to upper-bounding the root of $f(u) = g(u) - h(u)$ instead. Instead of using $g(u) = 4dm(k-1)^2(u+k-1)^m$, we use $g_2(u) = 4dm(k-1)^2(u^m + 2(k-1)mu^{m-1})$, which by Proposition 2, upper bounds $g(u)$ when $u \ge m(k-1)$. Let $\tilde{u}$ be the root of $g_2(u) - h(u)$. Then,

$$\begin{aligned} g_2(\tilde{u}) - h(\tilde{u}) &= 4dm(k-1)^2(\tilde{u}^m + 2(k-1)m\tilde{u}^{m-1}) - 2\gamma\tilde{u}^{m+1} \\ &= 2\tilde{u}^{m-1}(-\gamma\tilde{u}^2 + 2dm(k-1)^2\tilde{u} + 4dm^2(k-1)^3) = 0 \end{aligned} \tag{90}$$

Thus, there are $m-1$ roots where $\tilde{u} = 0$ and 2 roots such that

$$\begin{aligned} \tilde{u} &= \frac{2dm(k-1)^2 \pm \sqrt{(2dm(k-1)^2)^2 + 16\gamma dm^2(k-1)^3}}{2\gamma} \\ &= \frac{1 \pm \sqrt{1 + \frac{4\gamma}{d(k-1)}}}{\gamma} d(k-1)^2 m \end{aligned} \tag{91}$$

We choose $\tilde{u}$ to be the largest root, $\left(1 + \sqrt{1 + \frac{4\gamma}{d(k-1)}}\right)\gamma^{-1} d(k-1)^2 m$. By Fact 12, we know that when $\tilde{u} \ge m(k-1)$, $u^{*m} \le \tilde{u}^m$:

Fact 12. *If $\tilde{u}$ is a root of $g_2(u) - h(u)$ such that $\tilde{u} \ge m(k-1)$, then $u^* \le \tilde{u}$ for $k > 1; m, d > 0$; and $\gamma \ge 0$.*

Proof. We know $g_2(u), g(u), h(u)$ are polynomials of u, so any linear combination of them is continuous in u. We also know that $f(0) = g(0) - h(0) = 4dm(k-1)^{m+2} > 0$ since $d, m, k-1 > 0$. If $\tilde{u}$ is a root of $g_2(u) - h(u)$, $h(\tilde{u}) = g_2(\tilde{u})$. By Proposition 2, $g(\tilde{u}) \le g_2(\tilde{u}) = h(\tilde{u})$ since $\tilde{u} \ge m(k-1)$, which means that $f(\tilde{u}) = g(\tilde{u}) - h(\tilde{u}) \le 0$. Since $f(u) = g(u) - h(u)$ is a continuous function, and $0 \in [f(0), f(\tilde{u})]$, there must exist some value c in $[0, \tilde{u}]$ that is the root of $f(u)$ by the Intermediate Value Theorem. From Fact 10, we know that $f(u)$ has at most one real non-negative root, so $c = u^*$. Thus, $u^* \le \tilde{u}$. □

Thus, $\langle W(t)^m \rangle \le u^{*m} \le \tilde{u}^m$ for all $t \ge 0$ if $\langle W(0)^m \rangle \le u^{*m}$ and $\tilde{u} \ge m(k-1)$, or equivalently,

$$\frac{1 + \sqrt{1 + \frac{4\gamma}{d(k-1)}}}{\gamma} d(k-1) \ge 1, \tag{92}$$

which we ensure for all d, k when $\gamma \le 4$. We can extend our results to $\gamma > 4$ by simply picking the $\tilde{u}$ that corresponds to $\gamma = 4$ for our algorithm instead.

For simplicity, we further upper bound $\tilde{u}$ by

$$\tilde{u} = \left(1 + \sqrt{1 + \frac{4\gamma}{d(k-1)}}\right) \frac{d(k-1)^2}{\gamma} m \le \left(1 + \sqrt{1 + 2\gamma}\right) \frac{dk^2}{\gamma} m \tag{93}$$

by plugging in $d, k = 2$ as the smallest values of d and k. We refer to this final value as $u_{\text{ub}} = \left(1 + \sqrt{1 + 2\gamma}\right) \frac{dk^2}{\gamma} m$. □