2.)

Backward Error Analysis of Bordered algo.

we know def of bordered algorithm that

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & \hat{A}_{TR} \\ \hline \hat{A}_{BL} & \hat{A}_{BR} \end{array}\right)_{A} \quad L_{TL} U_{TL} = \hat{A}_{TL}$$

For backward error analysis, can be computed as:

$$\left(\begin{array}{c|c} [\check{L}\backslash\check{U}]_{TL} = [LU(A_{TL})] & \check{U}_{TR} = 0 \\ \hline \check{L}_{BL} = 0 & [\check{L}\backslash\check{U}_{BR}] = 0 \end{array}\right)$$

computed factors $\check{L}$ and $\check{U}$ satisfy $\check{L}\check{U} = A + \delta A$

we will use a proof by induction.

a.) Base case: $n=1$

If $A^{1\times 1}$, then A is just a real valued scalar.

From 6.2.2, we defined error from storing a real number as

$|\Delta A| \leq \varepsilon_{mach}|A|$,

~~where~~ Additionally, we will define

γ_n as $\gamma_n = \dfrac{n\,\varepsilon_{mach}}{1 - n\,\varepsilon_{mach}}$ (Thm 6.3.2.3) for $\forall n \geq 1$ & $n\varepsilon_{mach} < 1$

For base case $n=1$, $\gamma_n = \dfrac{\varepsilon_{mach}}{1-\varepsilon_{mach}} \approx \varepsilon_{mach}$

Thus: $|\Delta A| \leq \gamma_n |\check{L}||\check{U}|$

$\Rightarrow |\Delta A| \leq \gamma_n$ holds true for $n=1$

$(|LU| \leq |L||U|$ due to Δ inequality$)$.

2) Inductive Step

Assume that for $A_{00} \in \mathbb{R}^{n \times n}$, bordered algo factorization computes $\check{L}$ and $\check{U}$ where $A_{00} + \Delta A_{00} = \check{L}\check{U}$ w/ $|\Delta A| \leq \gamma_n |\check{L}||\check{U}|$ result holds for $n-1$.

Apply bordered algo to A.

Define

$$A = \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \quad L = \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \quad U = \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right)$$

At the top of the loop, after A

$$\left(\begin{array}{c|cc} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & a_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array}\right) \rightarrow \left(\begin{array}{cc|c} L \backslash U_{00} & u_{01} & \hat{A}_{02} \\ \ell_{10}^T & \upsilon_{11} & \hat{a}_{12}^T \\ \hline \hat{A}_{20} & \hat{a}_{21} & \hat{A}_{22} \end{array}\right) \wedge \begin{pmatrix} L_{00} & 0 \\ \ell_{10}^T & 1 \end{pmatrix} \begin{pmatrix} U_{00} & u_{01} \\ 0 & \upsilon_{11} \end{pmatrix} = \begin{pmatrix} \hat{A}_{00} & \hat{a}_{01} \\ \hat{a}_{10} & \hat{a}_{11} \end{pmatrix}$$

~~We want to prove~~

Matrix Mult: $\begin{pmatrix} L_{00} & 0 \\ \ell_{10}^T & 1 \end{pmatrix} \begin{pmatrix} U_{00} & u_{01} \\ 0 & \upsilon_{11} \end{pmatrix} = \begin{pmatrix} L_{00}U_{00} & L_{00}u_{01} \\ \ell_{10}^T U_{00} & \ell_{10}^T u_{01} + \upsilon_{11} \end{pmatrix}$ want to prove each element is bounded.

Cholesky factorization

$L_{00}U_{00} = A_{00}$

From assumption above $A_{00} + \Delta A_{00} = \check{L}\check{U}$ w/ $|\Delta A| \leq \gamma_n |\check{L}||\check{U}|$

Next, we prove $L_{00}u_{01} = a_{01}$

we know $\check{L}_{00}\check{u}_{01} + \Delta\check{L}_{00}\check{u}_{01} = a_{01}$ (Corollary 6.4.1.4)

There exists a matrix notation st $(L + \Delta L)\hat{x} = y$ where $|\Delta L| \leq \gamma_n |L|$

applying to eqn above

$$\check{L}_{00}\check{u}_{01} = a_{01} - \Delta\check{L}_{00}\check{u}_{01}$$

Thus $\varepsilon_{01} = -\Delta\check{L}_{00}\check{u}_{01} - \delta a_{01}$

Hence $|\varepsilon_{01}| \leq \overset{\gamma_{n-1}}{\cancel{\gamma_n}} |\check{L}_{00}\check{u}_{01}| \leq \gamma_n |\check{L}_{00}||\check{u}_{01}|$

Next, we prove

$l_{10}^T U_{00} = a_{10}^T$

Similar to above, we show

$\hat{l}_{10}^T U_{00} = a_{10}^T - \hat{l}_{10}^T \hat{U}_{00}$ (Corollary 6.3.3.2.

$\varepsilon_{10}^T = -l_{10}^T \Delta U_{00} = \delta a_{10}^T$ so $|\varepsilon_{10}^T| \leq \gamma_n |\hat{l}_{10}^T||U_{00}|$

Finally, we must bound $l_{10}^T u_{01} + v_{11} = \alpha_{11}$

Rewrite $\alpha_{11} - \hat{l}_{10}^T \hat{u}_{01} + \delta a_{11} = \hat{v}_{11}$

We will use an external source to help. In Science of Deriving Stable Analyzes by Paolo Bientineli and Robert Van Geerin, thm 6.4 states

Consider assignments $\mu := \alpha - (x^T y)$ & $v := (\alpha - (x^T y))/\lambda$.

Then, $|\alpha - x^T y - v\lambda| \leq \gamma_n (|x|^T|y| + |v||\lambda|) = \gamma_n |v|^T |z|$

where $v = \begin{pmatrix} x \\ \lambda \end{pmatrix}$ and $z = \begin{pmatrix} y \\ \mu \end{pmatrix}$

backward analysis for dot product

Thus, if $\lambda = 1$ $\varepsilon_{11} = \delta a_{11} = \hat{v}_{11} - a_{11} + \hat{l}_{10}^T \hat{u}_{01}$

apply lemma: $|\varepsilon_{11}| \leq \gamma_n \left|\begin{pmatrix} \hat{l}_{10} \\ 1 \end{pmatrix}\right|^T \left|\begin{pmatrix} \hat{u}_{01} \\ \hat{v}_{11} \end{pmatrix}\right|$

Therefore, ~~Thus~~ if all four elements are bounded, now add backward error

$A + \Delta A = \check{L}\check{U}$

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) + \left(\begin{array}{c|c} \Delta A_{00} & \delta a_{01} \\ \hline \delta a_{10}^T & \delta\alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} \check{L}_{00} & 0 \\ \hline \check{l}_{10}^T & 1 \end{array}\right)\left(\begin{array}{c|c} \check{U}_{00} & \check{u}_{01} \\ \hline 0 & \check{v}_{00} \end{array}\right)$$

$$\left|\left(\begin{array}{c|c} \Delta A_{00} & \delta a_{01} \\ \hline \delta a_{10}^T & \delta\alpha_{11} \end{array}\right)\right| \leq \left(\begin{array}{c|c} \gamma_n |\check{L}_{00}||\check{U}_{00}| & \gamma_n |\check{L}_{00}||\check{u}_{01}| \\ \hline \gamma_n |\check{l}_{10}^T||\check{U}_{00}| & \gamma_n \left|\begin{matrix} \hat{l}_{10} \\ 1 \end{matrix}\right|^T \left|\begin{matrix} u_{01} \\ v_{11} \end{matrix}\right| \end{array}\right)$$

factor & separate

$$= \gamma_n \left(\left|\begin{array}{c|c} \check{L}_{00} & 0 \\ \hline \check{l}_{10}^T & 1 \end{array}\right|\right)\left|\begin{array}{c|c} \check{U}_{00} & \check{u}_{01} \\ \hline 0 & \check{v}_{11} \end{array}\right| = \text{Right hand side}$$