2.)

Backward Error Analysis of Bordered algo.

we know ~~for~~ def of bordered algorithm that

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L \backslash U_{TL} & \hat{A}_{TR} \\ \hline \hat{A}_{BL} & \hat{A}_{BR} \end{array}\right)_{\Delta} \quad L_{TL} U_{TL} = \hat{A}_{TL}$$

For backward error analysis, can be computed as:

$$\left(\begin{array}{c|c} [\check{L} \backslash \check{U}]_{TL} = [LU(A_{TL})] & \check{U}_{TR} = 0 \\ \hline \check{L}_{BL} = 0 & [\check{L} \backslash \check{U}_{BR}] = 0 \end{array}\right)$$

computed factors $\check{L}$ and $\check{U}$ satisfy

$\check{L}\check{U} = A + \delta A$

we will use a proof by induction.

a.) Base case: $n=1$

If $A^{1\times1}$, then A is just a real valued scalar.

From 6.2.2, we defined error from storing a real number as

$|\Delta A| \leq \varepsilon_{mach}|A|$,

~~where~~ Additionally, we will define

γ_n as $\gamma_n = \dfrac{n\,\varepsilon_{mach}}{1 - n\,\varepsilon_{mach}}$ (Thm 6.3.2.3)

for $\forall n \geq 1$ & $n\varepsilon_{mach} < 1$

For base case $n=1$, $\gamma_n = \dfrac{\varepsilon_{mach}}{1-\varepsilon_{mach}} \approx$ ~~≤~~ ε_{mach}

Thus: $|\Delta A| \leq \gamma_n |\check{L}||\check{U}|$

$\Rightarrow |\Delta A| \leq \gamma_n$ holds true for $n=1$

$(|LU| \leq |L||U|$ due to Δ inequality$)$.

2) Inductive Step

Assume that for $A_{00} \in \mathbb{R}^{n \times n}$, bordered algo factorization computes $\check{L}$ and $\check{U}$ where $A_{00} + \Delta A_{00} = \check{L}\check{U}$ w/ $|\Delta A| \le \gamma_n |\check{L}||\check{U}|$

result holds for $n-1$.

Apply bordered algo to A:

Define

$$A = \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \quad L = \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \quad U = \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right)$$

At the top of the loop, after A

$$\left(\begin{array}{c|cc} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & a_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array}\right) \rightarrow \left(\begin{array}{cc|c} L \backslash U_{00} & u_{01} & \hat{A}_{02} \\ l_{10}^T & v_{11} & \hat{a}_{12}^T \\ \hline \hat{A}_{20} & \hat{a}_{21} & \hat{A}_{22} \end{array}\right) \wedge \begin{pmatrix} L_{00} & 0 \\ l_{10}^T & 1 \end{pmatrix}\begin{pmatrix} U_{00} & u_{01} \\ 0 & v_{11} \end{pmatrix} = \begin{pmatrix} \hat{A}_{00} & \hat{a}_{01} \\ \hat{a}_{10} & \hat{a}_{11} \end{pmatrix}$$

~~We want to prove~~

Matrix Mult: $\begin{pmatrix} L_{00} & 0 \\ l_{10}^T & 1 \end{pmatrix}\begin{pmatrix} U_{00} & u_{01} \\ 0 & v_{11} \end{pmatrix} = \begin{pmatrix} L_{00}U_{00} & L_{00}u_{01} \\ l_{10}^T U_{00} & l_{10}^T u_{01} + v_{11} \end{pmatrix}$ want to prove each element is bounded.

Cholesky factorization

$L_{00}U_{00} = A_{00}$

From assumption above $A_{00} + \Delta A_{00} = \check{L}\check{U}$ w/ $|\Delta A| \le \gamma_n |\check{L}||\check{U}|$

Next, we prove $L_{00}u_{01} = a_{01}$

we know $\check{L}_{00}\check{u}_{01} + \Delta\check{L}_{00}\check{u}_{01} = a_{01}$ (Corollary 6.4.1.4)

There exists a matrix notation st $(L + \Delta L)\hat{x} = y$ where $|\Delta L| \le \gamma_n |L|$

applying to eqn above

$$\check{L}_{00}\check{u}_{01} = a_{01} - \Delta\check{L}_{00}\check{u}_{01}$$

Thus $\epsilon_{01} = -\Delta\check{L}_{00}\check{u}_{01} - \delta a_{01}$

Hence $|\epsilon_{01}| \le \overset{\gamma_{n-1}}{\cancel{\gamma_n}} |\check{L}_{00}\check{u}_{01}| \le \gamma_n |\check{L}_{00}||\check{u}_{01}|$

Next, we prove

$l^T_{10} U_{00} = a^T_{10}$

Similar to above, we show

$\hat{l}^T_{10} \hat{U}_{00} = a^T_{10} - \hat{l}^T_{10} \Delta U_{00}$

$\varepsilon^T_{10} = -l^T_{10} \Delta \hat{U}_{00} = \delta a^T_{10}$

So $|\varepsilon^T_{10}| \le \gamma_n |\hat{l}^T_{10}||\check{U}_{00}|$ (corollary 6.3.3.2)

Finally, we must bound $l^T_{10} u_{01} + v_{11} = \alpha_{11}$

Rewrite $\alpha_{11} - \check{l}^T_{10} \check{u}_{01} + \delta a_{11} = \check{v}_{11}$

Lemma: Let α and λ be scalar, x & y be vectors in $\mathbb{R}^{k-1}$, consider assignment $k := \frac{\alpha - x^T y}{\lambda}$ and $\hat{k} = (\alpha - x^T y)$

Thus, $|\alpha - x^T y - \hat{k}\lambda| \le \gamma_k (|x|^T |y| + |\hat{k}||\lambda|) = \gamma_k |v|^T |z|$

where $v = \left(\frac{x}{\lambda}\right)$ and $z = \left(\frac{y}{k}\right)$

Thus, if $\lambda = 1$

$\varepsilon_{11} - \delta a_{11} = \hat{v}_{11} - a_{11} + \hat{l}^T_{10} \hat{u}_{01}$

apply lemma from above:

$|\varepsilon_{11}| \le \gamma_n \left|\left(\frac{l_{10}}{1}\right)^T\right| \left|\left(\frac{\hat{u}_{01}}{\hat{v}_{11}}\right)\right|$

All four have been bounded, apply backward error to matrix $\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a^T_{10} & \alpha_{11} \end{array}\right)$

$A + \Delta A = \check{L}\check{U}$

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a^T_{10} & a_{11} \end{array}\right) + \left(\begin{array}{c|c} \Delta A_{00} & \delta a_{01} \\ \hline \delta a^T_{10} & \delta a_{11} \end{array}\right) = \left(\begin{array}{c|c} \check{L}_{00} & 0 \\ \hline \check{l}_{10} & 1 \end{array}\right) \left(\begin{array}{c|c} \check{U}_{00} & \check{u}_{01} \\ \hline 0 & \check{v}_{00} \end{array}\right)$$

$$\left|\left(\begin{array}{c|c} \Delta A_{00} & \delta a_{01} \\ \hline \delta a^T_{10} & \delta a_{11} \end{array}\right)\right| \le \underbrace{\left(\begin{array}{c|c} \gamma_n |\check{L}_{00}||\check{U}_{00}| & \gamma_n |\check{L}_{00}||\check{u}_{01}| \\ \hline \gamma_n |\check{l}^T_{10}||\check{U}_{00}| & \gamma_n \left|\left(\frac{\hat{l}_{10}}{1}\right)\right|^T \left|\left(\frac{\hat{u}_{01}}{\hat{v}_{11}}\right)\right| \end{array}\right)}_{\text{factor out } \gamma_n \text{ and separate } L \text{ \& } U}$$

$$= \gamma_n \left|\left(\begin{array}{c|c} \check{L}_{00} & 0 \\ \hline \check{l}^T_{10} & 1 \end{array}\right)\right| \left|\left(\begin{array}{c|c} \hat{U}_{00} & \hat{u}_{01} \\ \hline 0 & \hat{v}_{11} \end{array}\right)\right| = \text{Right hand side}$$

Citation: 6.4.2. Paolo Bientinesi, Science of Deriving Stable Algo