1.) Bordered algorithm for computing Cholesky fact. of SPD matrix A.

$$A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \quad \text{and} \quad L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)$$

substitute partitioned matrices into $A = LL^T$ (def of Cholesky)

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)\left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)^T = \left(\begin{array}{c|c} L_{00}L_{00}^T & L_{00}l_{10} \\ \hline l_{10}^T L_{00}^T & l_{10}^T l_{10} + \lambda_{11}^2 \end{array}\right)$$

from here, we solve for ~~A~~ L's terms

$$A = LL^T \longrightarrow \left.\begin{array}{c|c} \cancel{A}\; L_{00} = \text{Chol}(A)_{00} & \\ \hline l_{10}^T = a_{10}^T L_{00}^{-T} & \lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}} \end{array}\right.$$

So, ~~a~~ to do the bordered algorithm, we must partition A;

make the assumption that $A_{00} := L_{00} = \text{Chol}(A)_{00}$ (Bordered assumption.) has been computed for previous iterations (assume looping for Cholesky), then overwrite two matrix indices:

$$a_{10}^T := l_{10}^T = a_{10}^T L_{00}^{-T}$$

$$\alpha_{11} := \sqrt{\alpha_{11} - l_{10}^T l_{10}}$$

1b) Prove Cholesky factorization using Bordered Algorithm is well-defined for a matrix that is SPD.

We will employ a proof of induction.

Base Case: $n=1$

If $n=1$, then matrix A is 1×1 which is a scalar.
Let $A = [a] = LL^T$ using ~~[illegible]~~ def of Cholesky.

uniquely defined:

Because A is spd, we know that the scalar has SPD properties. Thus, because A can be rewritten as a product of two lower triangular matrices, we can conclude that the L is uniquelly defined.

~~[illegible]~~ well-defined: Because A is SPD, we know eigenvals are positive, thus scalar must be positive when taking square root, which means well-defined

Inductive Step:

We first make the assumption that matrix $A^{n-1 \times n-1}$ is spd, and uniquely and well-defined.

Partion $A^{n \times n}$ into $\left[\begin{array}{c|c} \overset{L_{00}}{\cancel{A_{00}}} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right]$ and $L = \left[\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right]$

for our proof by induction, we must prove that each partition in L is real and uniquely defined.

define $A = LL^T$. Thus

$$\left[\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right] \left[\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right]^T = \left[\begin{array}{c|c} L_{00}L_{00}^T & L_{00}l_{10} \\ \hline l_{10}L_{00}^T & l_{10}^T l_{10} + \lambda_{11}^2 \end{array}\right]$$

we know $L_{00}L_{00}^T$ is equal to A_{00}, which is already defined from the base case to be well-and uniquelly defined.

Next, we must prove that l_{10}^T is ~~real~~ well and uniquely defined.

Right now, $l_{10}^T L_{00}^T = a_{10}^T$

Solve for l_{10}: $L_{00}^{-T}(l_{10}^T L_{00}^T) = (a_{10}^T) L_{00}^{-T}$

Because L_{00} is well-defined, we know L_{00}^{-T} exists. $l_{10}^T = a_{10}^T \underbrace{L_{00}^{-T}}_{\text{well-defined}}$

~~uniquely~~ well defined: If A_{00} is spd, then we know that a_{10}^T is well-defined.

uniquely defined: we know that a_{10}^T is a fixed vector.

In addition, we know there exists only 1 L_{00}^T and it is unique from earlier.

Thus, because both computation elements are unique, ~~then~~ their computation must also be unique. Thus, we have proved ~~inductively~~ inductively for l_{10}^T, and its transpose.

Finally, we want to prove ~~λ_{11}~~ α_{11} is well-defined & unique.

currently ~~λ_{11}~~ $\alpha_{11} = l_{10}^T l_{10} + \lambda^2$.

Solving for $\lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}}$

Thus, for ~~α_{11}~~ λ_{11} to be well-defined, α_{11} must be greater than or equal to $l_{10}^T l_{10}$. We know that of spd matrices, $\alpha_{11} > l_{10}^T l_{10}$

uniquely defined: Because all elements are restricted to be positive, then its elements are unique. Because A is spd, Thm 5.4.4.1. is also α_{11} is real and positive.

It is unique because we only take the positive square root.

thus λ_{11} is uniquely defined.

Thus, each element of L is well & uniquely defined, concluding our ~~inductive~~ proof by induction for matrix $A^{n \times n}$.