Math 206A Probability: Chapter 4 Martingales: written by Rick Durrett

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Chapter 4 Martingales

Chapter 4.1 Conditional Expectation

Lemma 4.1.1 If Y satisfies (i) and (ii) then it is integrable.

Theorem 4.1.2 If $X_1 = X_2$ on $B \in \mathcal{F} = E(X_2|\mathcal{F})$ a.s. on B.

Chapter 4.1.2 Properties

Theorem 4.1.9 In the first two parts we assume $E|X|, E|Y| < \infty$

(a) Conditional expectation is linear

$$E(aX + Y|\mathcal{F}) = aE(X|\mathcal{F}) + E(Y|\mathcal{F})$$

(b) If $X \leq Y$ then

$$E(X|\mathcal{F}) \leq E(Y|\mathcal{F})$$

(c) If $X_n \geq 0$ and $X_n \uparrow X$ with $EX < \infty$ then

$$E(X_n|\mathcal{F}) \uparrow E(X|\mathcal{F})$$

Theorem 4.1.10 If φ is convex and $E|X|, E|\varphi(X)| < \infty$ then

$$\varphi(E(X|\mathcal{F})) \leq E(\varphi(X)|\mathcal{F})$$

Theorem 4.1.11 Conditional expectation is a contraction in $L^p, p \geq 1$

Theorem 4.1.12 If $\mathcal{F} \subset \mathcal{G}$ and $E(X|\mathcal{G}) \in \mathcal{F}$ then $E(X|\mathcal{F}) = E(X|\mathcal{G})$.

Theorem 4.1.13 If $\mathcal{F}_1 \subset \mathcal{F}_2$ then (i) $E(E(X|\mathcal{F}_1)|\mathcal{F}_2) = E(X|\mathcal{F}_1)$ (ii) $E(E(X|\mathcal{F}_2)|\mathcal{F}_2) = E(X|\mathcal{F}_1)$.

Theorem 4.1.14 If $X \in \mathcal{F}$ and $E|Y|, E|XY| < \infty$ then

$$E(XY|\mathcal{F}) = XE(Y|\mathcal{F})$$

Theorem 4.1.15 Suppose $EX^2 < \infty$. $E(X|\mathcal{F})$ is the variable $Y \in \mathcal{F}$ that minimizes the "mean square error" $E(X - Y)^2$

Chapter 4.1.3 Regular Conditional Probabilities

Theorem 4.1.16 Let $\mu(\omega, A)$ be a r.c.d. for X given $\mathcal{F}$. If $f : (S, \mathcal{S}) \to (R, \mathcal{R})$ has $E|f(X)| < \infty$ then

$$E(f(X)|\mathcal{F}) = \int \mu(\omega, dx) f(x) \qquad \text{a.s.}$$

Theorem 4.1.17 r.c.d.'s exist if $(S, \mathcal{S})$ is nice.

Theorem 4.1.18 Suppose X and Y take values in a nice space $(S, \mathcal{S})$ and $\mathcal{G} = \sigma(Y)$. There is a function $\mu : Sx\mathcal{S} \to [0, 1]$ so that
(i) for each A, $\mu(Y(\omega), A)$ is a version of $P(X \in A|\mathcal{G})$
(ii) for a.e. $\omega, A \to \mu(Y(\omega), A)$ is a probability measure on $(S, \mathcal{S})$

Chapter 4.2 Martingales, Almost Sure Convergence

Theorem 4.2.4 If X_n is a supermartingale then for $n > m, E(X_n|\mathcal{F}_m) \le X_m$.

Theorem 4.2.5 (i) If X_n is a submartingale then for $n > m, E(X_n|\mathcal{F}_m) \ge X_m$
(ii) If X_n is a martingale then for $n > m, E(X_n|\mathcal{F}_m) = X_m$

Theorem 4.2.6 If X_n is a martingale w.r.t. $\mathcal{F}_n$ and φ is a convex function with $E|\varphi(X_n)| < \infty$ for all n then $\varphi(X_n)$ is a submartingale w.r.t. $\mathcal{F}_n$. Consequently if $p \ge 1$ and $E|X_n|^p < \infty$ for all n, then $|X_n|^p$ is a submartingale w.r.t. $\mathcal{F}_n$.

Theorem 4.2.7 If X_n is a submartingale w.r.t. $\mathcal{F}_n$ and φ is an increasing convex function with $E|\varphi(X_n)| < \infty$ for all n, then $\varphi(X_n)$ is a submartingale w.r.t. $\mathcal{F}_n$. Consequently (i) If X_n is a submartingale then $(X_n - a)^+$ is a submartingale. (ii) If X_n is a supermartingale then $X_n \wedge a$ is a supermartingale.

Theorem 4.2.8 Let $X_n, n \ge 0$ be a supermartingale. If $H_n \ge 0$ is predictable and each H_n is bounded then $(H \cdot X)_n$ is a supermartingale.

Theorem 4.2.10 Upcrossing inequality If $X_m, m \ge 0$ is a submartingale then

$$(b - a)EU_n \le E(X_n - a)^+ - E(X_0 - a)^+$$

Theorem 4.2.11 Martingale convergence theorem If X_n is a submartingale with $\sup EX_n^+ < \infty$ then as $n \to \infty, X_n$ converges a.s. to a limit X with $E|X| < \infty$

Theorem 4.2.12 If $X_n \ge 0$ is a supermartingale then as $n \to \infty, X_n \to X$ a.s. and $EX \le EX_0$

Chapter 4.3 Examples

Chapter 4.3.1 Bounded Increments

Theorem 4.3.1 Let $X_1, X_2, ...$ be a martingale with $|X_{n+1} - X_n| \le M < \infty$. Let

$$C = \{limX_n \text{ exists and is finite }\}$$
$$D = \{limsupX_n = +\infty \text{ and } \liminf X_n = -\infty\}$$

Then $P(C \cup D) = 1$

Theorem 4.3.2 Doob's decomposition Any submartingale $X_n, n \ge 0$ can be written in a unique way as $X_n = M_n + A_n$ where M_n is a martingale and A_n is a predictable increasing sequence with $A_0 = 0$

Theorem 4.3.4 Second Borel-Cantelli lemma, II Let $\mathcal{F}_n, n \le 0$ be a filtration with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and let $B_n, n \ge 1$ a sequence of events with $B_n \in \mathcal{F}_n$. Then

$$\{B_n i.o.\} = \{\textstyle\sum_{n=1}^{\infty} P(B_n|\mathcal{F}_{n-1} = \infty\}$$

Chapter 4.3.2 Polya's Urn Scheme

Chapter 4.3.3 Radon-Nikodym Derivatives

Theorem 4.3.5 Suppose $\mu_n << v_n$ for all n. Let $X_n = d\mu_n/dv_n$ and let X = lim sup X_n. Then

$$\mu(A) = \int_A X dv + \mu(A \cap \{X = \infty\})$$

Lemma 4.3.6 X_n (defined on $(\Omega, \mathcal{F}, v)$) is a martingale w.r.t. $\mathcal{F}_n$.

THeorem 4.3.8 $\mu << v$ or $\mu \perp v$, according as $\prod_{m=1}^{\infty} \int \sqrt{q_m} dG_m > 0$ or $= 0$.

Chapter 4.3.4 Branching Processes

Lemma 4.3.9 Let $\mathcal{F}_n = \sigma(\xi_i^m : i \geq 1, 1 \leq m \leq n)$ and $\mu = E\xi_i^m \in (0, \infty)$. Then Z_n/μ^n is a martingale w.r.t. $\mathcal{F}_n$

Theorem 4.3.10 If $\mu < 1$ then $Z_n = 0$ for all n sufficiently large, so $Z_n/\mu^n \to 0$

Theorem 4.3.11 If $\mu = 1$ and P($\xi_i^m = 1$) < 1 then $Z_n = 0$ for all n sufficiently large.

Theorem 4.3.12 Suppose $\mu > 1$. If $Z_0 = 1$ then $P(Z_n = 0$ for some n$) =$ p the only solution of $\varphi(p) = pin[0, 1)$.

Theorem 4.3.13 W = lim Z_n/μ^n is not = 0 if and only if $\sum p_k k log k < \infty$

Chapter 4.4 Doob's inequality, convergence in $L^p, p > 1$

Theorem 4.4.1 If X_n is a submartingale and N is a stopping time with $P(N \leq k) = 1$ then

$$EX_0 \leq EX_n \leq EX_k$$

Theorem 4.4.2 Doob's inequality Let X_m be a submartingale

$$\bar{X}_n = max_{0 \leq m \leq n} X_m^+$$

$\lambda > 0$ and $A = \{\bar{X}_n \geq \lambda\}$. Then

$$\lambda P(A) \leq EX_n 1_A \leq EX_n^+$$

Theorem 4.4.4 L^p maximum inequality. If X_n is a submartingale then for $1 < p < \infty$

$$E(\bar{X}_n^p) \leq (\tfrac{p}{p-1})^p E(X_n^+)^p$$

Consequently if Y_n is a martingale and $Y_n^* = max_{0 \leq m \leq n} |Y_m|$.

$$E|Y_n^*|^p \leq (\tfrac{p}{p-1})^p E(|Y_n|^p)$$

Theorem 4.4.6 L^p convergence theorem If X_n is a martingale with sup $E|X_n|^p < \infty$ where $p > 1$ then $X_n \to X$ a.s. and in L^p

Theorem 4.4.7 Orthogonality of martingale increments Let X_n be a martingale with $EX_n^2 < \infty$ for all n. If $m \leq n$ and $Y \in \mathcal{F}_m$ has $EY^2 < \infty$ then

$$E((X_n - X_m)Y) = 0$$

and hence if $\ell < m < n$

$$E((X_n - X_m)(X_m - X_\ell) = 0$$

Theorem 4.4.8 Conditional variance formula If X_n is amartingale with $EX_n^2 < \infty$ for all n

$$E((X_n - X_m)^2|\mathcal{F}_m) = E(X_n^2|\mathcal{F}_m) - X_m^2$$

Chapter 4.5 Square Integrable Martingales*

Theorem 4.5.1 $E(sup_m|X_m|^2) \leq 4EA_\infty$

Theorem 4.5.2 $lim_{n\to\infty}X_n$ exists and is finite a.s. on $\{A_\infty < \infty\}$.

Theorem 4.5.3 Let $f \geq 1$ be increasing with $\int_0^\infty f(t)^{-2}dt < \infty$. Then $X_n/f(A_n) \to 0$ a.s. on $\{A_\infty = \infty\}$

Theorem 4.5.5 Second Borel Cantelli Lemma III Suppose B_n is adapted to $\mathcal{F}_n$ and let $p_n = P(B_n|\mathcal{F}_{n-1})$. Then

$$\sum_{m=1}^n 1_{B(m)} / \sum_{m=1}^n p_m \to 1 \qquad \text{a.s. on } \{\sum_{m=1}^\infty p_m = \infty\}$$

Theorem 4.5.7 $E(sup_n|X_n|) \leq 3EA_\infty^{1/2}$

Chapter 4.6 Uniform Integrability, Convergence in L^1

Theorem 4.6.1 Given a probability space $(\Omega, \mathcal{F}_0, P)$ and an $X \in L^1$ then $\{E(X|\mathcal{F}) : \mathcal{F} \text{ is a } \sigma - field \subset \mathcal{F}_0\}$ is uniformly integrable.

Theorem 4.6.2 Let $\varphi \geq 0$ be any function with $\varphi(x)/x \to \infty$ as $x \to \infty$, e.g., $\varphi(x) = x^p$ with $p > 1$ or $\varphi(x) = xlog^+x$. If $E\varphi(|X_i|) \leq C$ for all $i \in I$, then $\{X_i : i \in I\}$ is uniformly integrable.

Theorem 4.6.3 Suppose that $E|X_n| < \infty$ for all n. If $X_n \to X$ in probability then the following are equivalent:
(i) $\{X_n : n \geq 0\}$ is unifromly integrable
(ii) $X_n \to X$ in L^1
(iii) $E|X_n| \to E|X| < \infty$

Theorem 4.6.4 For a submartingale, the following are equivalent
(i) It is uniformly integrable
(ii) It converges a.s. and in L^1
(iii) It converges in L^1.

Lemma 4.6.5 If integrable random variables $X_n \to X$ in L^1 then

$$E(X_n : A) \to E(X : A)$$

Lemma 4.6.6 if a martingale $X_n \to X$ in L^1 then $X_n = E(X|\mathcal{F}_n)$.

Theorem 4.6.7 For a martingale, the following are equivalent
(i) It is uniformly integrable
(ii) It converges a.s. and in L^1
(iii) It converges in L^1
(iv) There is an integrable random variable X so that $X_n = E(X|\mathcal{F}_n)$.

Theorem 4.6.8 Suppose $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ i.e. $\mathcal{F}_n$ is an increasing sequence of $\sigma - fields$ and $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$. As $n \to \infty$

$$E(X|\mathcal{F}_n) \to E(X|\mathcal{F}_\infty) \qquad \text{a.s. and in } L^1$$

Theorem 4.6.9 Levy's 0-1 law If $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and $A \in \mathcal{F}_\infty$ then $E(1_A|\mathcal{F}_n) \to 1_A$ a.s.

Theorem 4.6.10 Dominated convergence theorem for conditinoal expectations Suppose $Y_n \to Y$ a.s. and $|Y_n| \leq Z$ for all n where $EZ < \infty$. If $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ then

$$E(Y_n|\mathcal{F}_n) \to E(Y|\mathcal{F}_\infty) \qquad \text{a.s.}$$

Chapter 4.7 Backiwards Martingales

Theorem 4.7.1 $X_{-\infty} = lim_{n\to-\infty}X_n$ exists a.s. and in L^1

Theorem 4.7.2 If $X_{-\infty} = lim_{n\to-\infty}X_n$ and $\mathcal{F}_{-\infty} = \cap_n\mathcal{F}_n$, then $X_{-\infty} = E(X_0|\mathcal{F}_{-\infty}$

Theorem 4.7.3 If $\mathcal{F}_n \downarrow \mathcal{F}_{-\infty}$ as $n \downarrow -\infty$ (i.e. $\mathcal{F}_{-\infty} = \cap_n\mathcal{F}_n$), then

$$E(Y|\mathcal{F}_n) \to E(Y|\mathcal{F}_{-\infty}) \qquad \text{a.s. and in } L^1$$

Lemma 4.7.7 Suppose $X_1, X_2, ...$ are i.i.d. and let

$$A_n(\varphi) = \tfrac{1}{(n)_k}\sum_i \varphi(X_{i,1}, ..., X_{ik})$$

where the sum is over all sequences of distinct integers $1 \leq i_1, ..., i_k \leq n$ and

$$(n)_k = n(n-1)\dots(n-k+1)$$

is the number of such sequences. If φ is bounded, $A_n(\varphi) \to E\varphi(X_1, ..., X_k)$ a.s.

Theorem 4.7.9 de Finetti's Theorem if $X_1, X_2, ...$ are exchangable then conditional on $\mathcal{E}, X_1, X_2, ...$ are independent and identically distributed.

Chapter 4.8 Optional Stopping Theorems

Theorem 4.8.1 If X_n is uniformly integrable submartingale then for any stopping time N, $X_{N\wedge n}$ is uniformly integrable

Theorem 4.8.2 If $E|X_N| < \infty$ and $X_n^1 N > n$ is uniformly integrable then $X_{N\wedge n}$ is uniformly integrable and hence $EX_0 \leq EX_N$.

Theorem 4.8.3 If X_n is a uniformly integrable submartingale then for any stopping time $N \leq \infty$, we have $EX_0 \leq EX_N \leq EX_\infty$, where $X_\infty = limX_n$

Theorem 4.8.4 If X_n is a nonnegative supermartingale and $N \leq \infty$ is a stopping time, then $EX_0 \geq EX_N$ where $X_\infty = limX_n$ which exists by Theorem 4,2.12.

Theorem 4.8.5 Suppose X_n is a submartingale and $E(|X_{n+1}-X_n||\mathcal{F}_n) \leq B$ a.s. if N is a stopping time with $EN < \infty$ then $X_{N\wedge n}$ is uniformly integrable and hence $EX_n \geq EX_0$

Chapter 4.8.1 Applications to random walks

Theorem 4.8.6 Wald's equation If $\xi_1, \xi_2, ...$ are i.i.d. with $E\xi_i = \mu, S_n = \xi_1 + \cdots + \xi_n$ and N is a stopping time with $EN < \infty$ then $ES_n = \mu EN$

Theorem 4.8.7 Symmetric simple random walk Refers to the special case in which $P(\xi_i = 1) = P(\xi_i = -1) = 1/2$. Suppose $S_0 = x$ and let $N = min\{n : S_n \notin (a, b)\}$. Writing a subscript x to remind us of the starting point

$$\text{(a)} \qquad P_x(S_N = a) = \tfrac{b-x}{b-a} \qquad P_x(S_N = b) = \tfrac{x-a}{b-a}$$

(b) $E_0 N = -ab$ and hence $E_x N = (b-x)(x-a)$

Theorem 4.8.8 Let S_n be symmetric random walk with $S_0 = 0$ and let $T_1 = min\{n : S_n = 1\}$

$$Es^{T1} = \frac{1-\sqrt{1-s^2}}{s}$$

Inverting the generating function we find

$$P(T_1 = 2n-1) = \frac{1}{2n-1} \cdot \frac{(2n)!}{n!n!} 2^{-2n}$$

Theorem 4.8.9 Asymmetric simple random walk refers to the special case in which $P(\xi_i = 1) = p$ and $P(\xi_i = -1) = q = 1-p$ with $p \neq q$
(a) If $\varphi(y) = \{(1-p)/p\}^y$ then $\varphi(S_n)$ is a martingale
(b) If we let $T_z = inf\{n : S_n = z\}$ then for $a < x < b$

$$P_x(T_a < T_b) = \frac{\varphi(b)-\varphi(x)}{\varphi(b)-\varphi(a)} \qquad P_x(T_b < T_a) = \frac{\varphi(x)-\varphi(a)}{\varphi(b)-\varphi(a)}$$

For the last two parts suppose $1/2 < p < 1$
(c) If $a < 0$ then $P(min_n S_n \leq a) = P(T_a < \infty) = \{(1-p)/p\}^{-a}$
(d) If $b > 0$ then $P(T_b < \infty) = 1$ and $ET_b = b/(2p-1)$

Chapter 4.9 Combinatorics of simple random walk

Theorem 4.9.1 Reflection principle If $x, y > 0$ then the number of paths from $(0, x)$ to (n,y) that are 0 at some time is equal to the number of paths from (0, -x) to (n,y).

Theorem 4.9.2 The Ballot Theorem Suppose that in an election candidate A gets α votes and candidate B gets β votes where $\beta < \alpha$. The probability that throughout the counting A always leads B is $(\alpha - \beta)/(\alpha + \beta)$

Lemma 4.9.3 $P(S_1 \neq 0, ..., S_{2n} \neq 0) = P(S_{2n} = 0)$

Lemma 4.9.4 Let $u_{2m} = P(S_{2m} = 0)$. Then $P(L_{2n} = 2k) = u_{2k} u_{2n-2k}$.

Theorem 4.9.5 Arcsine law for the last visit to 0 For $0 < a < b < 1$

$$P(a \leq L_{2n}/2n \leq b) \to \int_a^b \pi^{-1}(x(1-x))^{-1/2} dx$$

Theorem 4.9.6 Arcsine law for time above 0 Let π_{2n} be the number of segments $(k-1, S_{k-1}) \to (k, S_k)$ that lie above the axis (i.e. in $\{(x, y) : y \geq 0\}$), and let $u_m = P(S_m = 0)$

$$P(\pi_{2n} = 2k) = u_{2k} u_{2n-2k}$$

and consequently, if $0 < a < b < 1$

$$P(a \leq \pi_{2n}/2n \leq b) \to \int_a^b \pi^{-1}(x(1-x))^{1/2} dx$$