Math 206A Probability: Chapter 1 continued
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Chapter 1.3 Random Variables

Theorem 1.3.1 If $\{\omega : X(\omega) \in A\} \in \mathcal{F}$ for all A $\in \mathcal{A}$ and $\mathcal{A}$ generates S (i.e. S is the smallest $\sigma-$field that contains $\mathcal{A}$) then X is measurable.

Theorem 1.3.4 If $X : (\Omega, \mathcal{F}) \to (S, S)$ and $f : (S, S) \to (T, \mathcal{T})$ are measurable maps, then f(X) is a measurable map from $\Omega, \mathcal{F}$ to $(T, \mathcal{T})$

Theorem 1.3.5 If $X_1, ..., X_n$ are random variables and $f : (R^n, \mathcal{R}^n) \to (R, \mathcal{R})$ is measurable, then $f(X_1, ..., X_n)$ is a random variable.

Theorem 1.3.6 If $X_1, ..., X_n$ are random variables then $X_1 + ... + X_n$ is a random variable.

Theorem 1.3.7 If $X_1, X_2, ...$ are random variables then so are

$$infX_n \qquad supX_n \qquad limsupX_n \qquad liminfX_n$$

n n n n

The n is supposed to under each object: substack wasnt cooperating so this is what I produced

Chapter 1.4 Integration

Lemma 1.4.1 Let φ and ψ be simple functions
(i) If $\varphi \geq 0$ a.e. then $\int \varphi d\mu \geq 0$
(ii) For any $a \in R$, $\int a\varphi d\mu = a \int \varphi d\mu$
(iii) $\int \varphi + \psi d\mu = \int \varphi d\mu + \int \psi d\mu$

Lemma 1.4.2 If (i) and (iii) hold then we have
(iv) If $\varphi \leq \psi$ a.e. then $\int \varphi d\mu \leq \int \psi d\mu$
(v) If $\varphi = \psi$ a.e. then $\int \varphi d\mu = \int \psi d\mu$
If in addition, (ii) holds when a = -1 we have
(vi) $|\int \varphi d\mu| \leq \int |\varphi| d\mu$

Lemma 1.4.3 Let E be a set with $\mu(E) < \infty$. If f and g are bounded function that vanish on E^c then:
(i) If $f \geq 0$ a.e. then $\int f d\mu \geq 0$
(ii) For any $a \in R$, $\int af d\mu = a \int f d\mu$
(iii) $\int f + g d\mu = \int f d\mu + \int g d\mu$
(iv) If $g \leq f$ a.e. then $\int g d\mu \leq \int f d\mu$

(v) If g = f a.e. then $\int gd\mu = \int fd\mu$
(vi) $|\int fd\mu| \leq \int |f|d\mu$

Lemma 1.4.4 Let $E_n \uparrow \Omega$ have $\mu(E_n) < \infty$ and let $a \wedge b = min(a,b)$. Then

$$\int_{E_n} f \wedge nd\mu \uparrow \int fd\mu \qquad \text{as } n \uparrow \infty$$

Lemma 1.4.5 Suppose f, $g \geq 0$
(i) $\int fd\mu \geq 0$
(ii) If $a > 0$ then $\int afd\mu = a\int fd\mu$
(iii) $\int f + gd\mu = \int fd\mu + \int gd\mu$
(iv) If $0 \leq g \leq f$ a.e. then $\int gd\mu \leq \int fd\mu$
(v) If $0 \leq g = f$ a.e. then $\int gd\mu = \int fd\mu$

Lemma 1.4.6 If $f = f_1 - f_2$ where $f_1, f_2 \geq 0$ and $\int f_i d\mu < \infty$ then

$$\int fd\mu = \int f_1 d\mu - \int f_2 d\mu$$

Theorem 1.4.7 Suppose f and g are integrable
(i) If $f \geq 0$ a.e. then $\int fd\mu \geq 0$
(ii) For all $a \in R$, $\int afd\mu = a\int fd\mu$
(iii) $\int f + gd\mu = \int fd\mu + \int gd\mu$
(iv) If $g \leq f$ a.e. then $\int gd\mu \leq \int gd\mu$
(v) If g = f a.e. then $\int gd\mu = \int fd\mu$ (vi) $|\int fd\mu| \leq \int |f|d\mu$

Chapter 1.5 Properties of the Integral

Theorem 1.5.1 Jensen's inequality Suppose φ is convex, that is

$$\lambda\varphi(x) + (1-\lambda)\varphi(y) \geq \varphi(\lambda x + (1-\lambda)y)$$

for all $\lambda \in (0,1)$ and $x, y \in R$ If μ is a probabillity measure and f and $\varphi(f)$ are integrable, then

$$\varphi(\int fd\mu) \leq \int \varphi(f)d\mu$$

Theorem 1.5.2 Holder's inequality If $p, q \in (1,\infty)$ with 1/p+1/q=1 then

$$\int |fg|d\mu \leq ||f||_p ||g||_q$$

Theorem 1.5.3 Bounded convergence theorem Let E be a set with $\mu(E) < \infty$. Suppose f_n vanishes on $E^c, |f_n(x)| \leq M$, and $f_n \to f$ in measure then

$$\int fd\mu = \lim_{n\to\infty} \int f_n d\mu$$

Theorem 1.5.5 Fatou's lemma If $f_n \geq 0$ then

$$\liminf_{n\to\infty} \int f_n d\mu \geq \int (liminf f_{n_{n\to\infty}})d\mu$$

Theorem 1.5.7 Monotone convergence theorem If $f_n \geq 0$ and $f_n \uparrow f$ then

$$\int f_n d\mu \uparrow \int fd\mu$$

Theorem 1.5.8 Dominated converges theorem If $f_n \to f$ a.e. $|f_n| \leq g$ for all n, and g is integrable, then $\int f_n d\mu \to \int f d\mu$

Chapter 1.6 Expected Value

Theorem 1.6.1 Suppose $X, Y \leq 0$ or $E|X|, E|Y| < \infty$
(a) E(X + Y) = EX + EY
(b) E(aX + b) = aE(X) + b for any real numbers a,b
(c) If $X \geq Y$ then $EX \geq EY$

Chapter 1.6.1 Inequalities: Theorem 1.6.2 Jensen's inequality Suppose φ is convex, that is

$$\lambda\varphi(x) + (1 - \lambda)\varphi(y) \geq (\lambda x + (1 - \lambda)y)$$

for all $\lambda \in (0, 1)$ and $x, y \in R$. Then

$$E(\varphi(X)) \geq \varphi(EX)$$

provided both expectations exist, i.e. $E|X|$ and $E|\varphi(X)| < \infty$

Theorem 1.6.3 Holder's inequality If $p, q \in [1, \infty]$ with 1/p+1/q - 1 then

$$E|XY| \leq ||X||_p||Y||_q$$

Here $||X||_r = (E|X|^r)^{1/r}$ for $r \in [1, \infty)$; $||X||_\infty = inf\{M : P(|X| > M) = 0\}$

Theorem 1.6.4 Chebyshev's inequality Suppose $\varphi : R \to R$ has $\varphi \geq 0$, let $A \in \mathcal{R}$ and let $i_A = inf\{\varphi(y) : y \in A\}$

$$i_A P(X \in A) \leq E(\varphi(X); X \in A) \leq E_\varphi(X)$$

1.6.2 Integration to the Limit: Theorem 1.6.5 Fatou's lemma If $X_n \geq 0$ then

$$\liminf_{n\to\infty} EX_n \geq E(\liminf_{n\to\infty} X_n)$$

Theorem 1.6.6 Monotone Convergence theorem If $0 \leq X_n \uparrow X$ then $EX_n \uparrow EX$.

Theorem 1.6.7 Dominated convergence theorem If $X_n \to X$ a.s., $|X_n| \leq Y$ for all n, and $EY < \infty$, then $EX_n \to EX$

Theorem 1.6.8 Suppose $X_n \to X$ a.s. Let g,h be continous functions with
(i) $g \geq 0$ and $g(x) \to \infty$ as $|x| \to \infty$
(ii) $|h(x)|/g(x) \to 0$ as $|x| \to \infty$
and (iii) $EG(X_n) \leq K < \infty$ for all n
Then $Eh(X_n) \to Eh(X)$.

Chapter 1.6.3 Computing Expected Values: Theorem 1.6.9 Change of variables formula Let X be a random element of (S,S) with distribution μ, i.e. $\mu(A) = P(X \in A)$. If f is a measurable function from (S,S) to (R, $\mathcal{R}$) so that $f \geq 0$ or $E|f(X)| < \infty$, then

$$Ef(X) = \int_S f(y)\mu(dy)$$

Chapter 1.7 Product Measures, Fubini's Theorem

Theorem 1.7.1 There is a unique measure μ on $\mathcal{F}$ with

$$\mu(AxB) = \mu_1(A)\mu_2(B)$$

Theorem 1.7.2 Fubini's Theorem If $f \geq 0$ or $\int |f| d\mu < \infty$ then (*)

$$\int_X \int_Y f(x,y)\mu_2(dy)\mu_1(dx) = \int_{XxY} f d\mu = \int_Y \int_X f(x,y)\mu_1(dx)\mu_2(dy)$$

Lemma 1.7.3 If $E \in \mathcal{F}$ then $E_x \in \mathcal{B}$
Lemma 1.7.4 If $E \in \mathcal{F}$ then $g(x) = \mu_2(E_x)$ is $\mathcal{A}$ measurable and

$$\int_X g d\mu_1 = \mu(E)$$