Math 206A Probability: Chapter 5 Markov Chains: written by Rick Durrett

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Chapter 5.1 Examples

Cahpter 5.2 Construction, Markov Properties

THeorem 5.2.1 X_n is a Markov chain (with respect to $\mathcal{F}_n = \sigma(X_0, X_1, ..., X_n)$) with transition probability p. That is,

$$P_\mu(X_{n+1} \in B|\mathcal{F}_n) = p(X_n, B)$$

Theorem 5.2.2 Monotone class theorem Let $\mathcal{A}$ be a $\pi - system$ that contains Ω and let $\mathcal{H}$ be a collection of real valued functions that satisfies:
(i) If $A \in \mathcal{A}$ then $1_A \in \mathcal{H}$
(ii) If f,g $\in \mathcal{H}$ then f + g, and $cf \in \mathcal{H}$ for any real number c.
(iii) If $f_n \in \mathcal{H}$ are nonnegative and increase to a bounded function f then $f \in \mathcal{H}$
Then $\mathcal{H}$ contains all bounded functions measurable with respect to $\sigma(\mathcal{A})$.

Theorem 5.2.3 The Markov property Let $Y : \Omega_o \to R$ be bounded and measurable

$$E_\mu(Y \circ \theta_m|\mathcal{F}_m) = Ex_mY$$

Theorem 5.2.4 Chapman-Kolmogorov equation

$$P_x(X_{m+n} = z) = \textstyle\sum_y P_x(X_m = y)P_y(X_n = z)$$

Theorem 5.2.5 Strong Markov property Suppose that for each n, $Y_n : \Omega_0 \to R$ is measurable and $|Y_n| \leq M$ for all n. Then

$$E_\mu(Y_n \circ \theta_N|\mathcal{F}_N) = Ex_NY_N \text{ on } \{N < \infty\}$$

where the right hand side is $\varphi(x, n) = E_xY_n$ evaluated at $x = X_N, n = N$

Theorem 5.2.6 $P_x(T_y^k < \infty) = p_{xy}p_{yy}^{k-1}$ Intuitively in order to make k visits to y, we first have to go from x to y and then return k - 1 times to y.

Theorem 5.2.7 Reflection principle $\xi_1, \xi_2, ...$ be independent and identically distributed with a distribution that is symmetric about 0. Let $S_n = \xi_1 + \cdots + \xi_n$ If $a > 0$ then

$$P(sup_{m\leq n}S_m \geq a) \leq 2P(S_n \geq a)$$

Chapter 5.3 Recurrence and Transience

Theorem 5.3.1 y is recurrent if and only if $E_y N(y) = \infty$

Theorem 5.3.2 If x is recurrent and $p_{xy} > 0$ then y is recurrent and $p_{yx} = 1$

Theorem 5.3.3 Let C be a finite closed set. Then C contains a recurrent state. If C is irreducible then all states in C are recurrent.

Theorem 5.3.5 Decomposition Theorem Let $R = \{x : p_{xx} = 1\}$ be the recurrent states of a Markov chain. R can be written as $\cup_i R_i$, where each R_i is closed and irreducible.

Theorem 5.3.8 Suppose S is irreducible and $\varphi \geq 0$ with $E_x \varphi(X_i) \leq \varphi(x)$ for $x \notin F$, a finite set, and $\varphi(x) \to \infty$ as $x \to \infty$ i.e. $\{x : \varphi(x) \leq M\}$ is finite for any $M < \infty$, then the chain is recurrent.

Theorem 5.3.10 If $a < x < b$ then

$$P_x(T_a < T_b) = \tfrac{\varphi(b)-\varphi(x)}{\varphi(b)-\varphi(a)} \qquad P_X(T_b < T_a) = \tfrac{\varphi(x)-\varphi(a)}{\varphi(b)-\varphi(a)}$$

Theorem 5.3.11 0 is recurrent if and only if $\varphi(M) \to \infty$ as $M \to \infty$ i.e.,

$$\varphi(\infty) = \textstyle\sum_{m=0}^{\infty} \prod_{j=1}^{m} \frac{q_j}{p_j} = \infty$$

If $\varphi(\infty) < \infty$ then $P_x(T_0 = \infty) = \varphi(x)/\varphi(\infty)$

Chapter 5.4 Recurrence of Random Walks

Theorem 5.4.1 The set V of recurrent values is either $\emptyset$ or a closed subgroup of R^d. In the second case V = U, the set of possible values.

Theorem 5.4.3 For any random walk, the following are equivalent: (i) $P(\mathcal{T}_1 < \infty) = 1$, (ii) $P(S_m = 0 \text{ i.o.}) = 1$ and (iii) $\sum_{m=0}^{\infty} P(S_m = 0) = \infty$

Theorem 5.4.4 Simple random walk is recurrent in $d \leq 2$ and transient in $d \geq 3$

Lemma 5.4.5 If $\sum_{n=1}^{\infty} P(||S_n|| < \epsilon) < \infty$ then $P(||S_n|| < \epsilon \text{ i.o.}) = 0$. If $\sum_{n=1}^{\infty} P(||S_n|| < \epsilon) = \infty$ then $P(||S_n|| < 2\epsilon \text{ i.o.}) = 1$.

Lemma 5.4.6 Let m be an integer ≥ 2

$$\textstyle\sum_{n=0}^{\infty} P(||S_n|| < m\epsilon) \leq (2m)^d \sum_{n=0}^{\infty} P(||S_n|| < \epsilon)$$

Theorem 5.4.7 The convergence (resp. divergence) of $\sum_n P(||S_n|| < \epsilon)$ for a single value of $\epsilon > 0$ is sufficient for transience (resp. recurrence).

Theorem 5.4.8 Chung Fuchs theorem Suppose d=1. If the weak law of large numbers holds in the form $S_n/n \to 0$ in probability, then S_n is recurrent.

Theorem 5.4.9 If S_n is a random walk in R^2 and $S_n/n^{1/2} \Rightarrow$ a nondegenerate normal distribution then S_n is recurrent.

Theorem 5.4.10 Let $\delta > 0.S_n$ is recurrent if and only if

$$\textstyle\int_{(-\delta,\delta)^d} Re \frac{1}{1-\varphi(y)} dy = \infty$$

Theorem 5.4.11 Let $\delta > 0 S_n$ is recurrent if and only if

$$sup_{r<1} \textstyle\int_{(-\delta,\delta)^d} Re \frac{1}{1-\varphi(y)} dy = \infty$$

Lemma 5.4.12 Parseval relation Let μ and v be probability measures on R^d with ch.f.'s φ and ψ

$$\int \psi(t)\mu(dt) = \int \varphi(x)v(dx)$$

lemma 5.4.13 If $|x| \leq \pi/3$ then $1 - cosx \geq x^2/4$

Theorem 5.4.14 No truly three dimensional random walk is recurrent.

Chapter 5.5 Stationary Measures

Theorem 5.5.5 Let μ be a stationary measure and suppose X_0 has "distribution" μ. Then $Y_m = X_{n-m}, 0 \leq m \leq n$ is a Markov chain with initial measure μ and trnasition probability

$$q(x,y) = \mu(y)p(y,x)/\mu(x)$$

q is called the dual transition probability. If μ is a reversible measure then q = p.

Theorem 5.5.6 Kolmogorov's cycle condition Suppose p is irreducible. A necessary and sufficient condition for the existence of a reversible measure is that (i) $p(x,y) > 0$ implies $p(y,x) > 0$ and (ii) for any loop $x_0, x_1, ..., x_n = x_0$ with $\prod_{1 \leq i \leq n} p(x_i, x_{i-1}) > 0$

$$\prod_{i=1}^{n} \frac{p(x_{i-1},x_i)}{p(x_i,x_{i-1})} = 1$$

Theorem 5.5.7 Let x be a recurrent state and let $T = inf\{n \geq 1 : X_n = x\}$. Then

$$\mu_x(y) = E_x(\sum_{n=0}^{T-1} 1_{\{X_n=y\}}) = \sum_{n=0}^{\infty} P_x(X_n = y, T > n)$$

defines a stationary measure.

Theorem 5.5.9 if p isirreducible and recurrent (i.e. all states are) then the stationary measure is unique up to constant multiples.

Theorem 5.5.10 If there is a stationary distribtuion then all states that have $\pi(y) > 0$ are recurrent.

Theorem 5.5.11 if p is irreducible and has stationary distribution π, then

$$\pi(x) = 1/E_x T_x$$

Theorem 5.5.12 If p is irreducible then the following are equivalent

(i) Some x is positive recurrent

(ii) There is a stationary distribution

(iii) All states are positive recurrent

This result shows that being positive recurrent is a class property. If it holds for one state in an irreducible set, then it is true for all.

Theorem 5.5.15 If p is irreducible and has a stationary distribution π then any other stationary measure is a multiple of π

Chapter 5.6 Asymptotic behavior

Theorem 5.6.1 Suppose y is recurrent. For any $x \in S$, as $n \to \infty$

$$\frac{N_n(y)}{n} \to \frac{1}{E_y T_y} 1_{\{T_y < \infty\}} \qquad P_x - a.s.$$

Here $1/\infty = 0$

lemma 5.6.4 If $p_{xy} > 0$ then $d_y = d_x$

Lemma 5.6.5 If $d_x = 1$ then $p^m(x, x) > 0$ for $m \geq m_0$

Theorem 5.6.6 Convergence theorem Suppose p is irreducible aperiodic (i.e. all states have $d_x = 1$), and has stationary distribution π. Then, as $n \to \infty, p^n(x, y) \to \pi(y)$

Chapter 5.7 Periodicity, Tail $\sigma - field$* Lemma 5.7.1 Suppose p is irreducible, recurrent, and all states have period d. Fix $x \in S$, and for each $y \in S$, let $K_y = \{n \geq 1 : p^n(x, y) > 0\}$ (i) There is an $r_y \in \{0, 1, .., d - 1\}$ so that if $n \in K_y$ then $n = r_y$ mod d, i.e. the difference $n - r_y$ is a multiple of d. (ii) Let $S_r = \{y : r_y = r\}$ for $0 \leq r < d.$ If $y \in S_i, z \in S_j$ and $p^n(y, z) > 0$, then $n = (j - i)$ mod d. (iii) $S_0, S_1, ..., S_{d-1}$ are irreducible classes for p^d, and all states have period 1.

Theorem 5.7.2 Convergence theorem, periodic case Suppose p is irreducible, has a stationary distribution π, and all states have period d. Let $x \in S$ and let $S_0, S_1, .., S_{d-1}$ be the cyclic decomposition of the state space with $x \in S_0$ If $y \in S_r$ then

$$lim_{m\to\infty} p^{md+r}(x, y) = \pi(y)d$$

Theorem 5.7.3 Suppose p is irreducible, recurrent, and all states have period d, $\mathcal{T} = \sigma(\{X_0 \in S_r\} : 0 \leq r < d)$

Theorem 5.7.4 Suppose X_0 has initial distribution μ. The equations

$$h(X_n, n) = E_\mu(Z|\mathcal{F}_n) \qquad \text{and} \qquad Z = lim_{n\to\infty} h(X_n, n)$$

set up a 1-1 correspondence between bounded $Z \in \mathcal{T}$ and bounded space-time harmonic functions, i.e. bounded $h : Sx\{0, 1, ...\} \to R$, so that $h(X_n, n)$ is a martingale.

Theorem 5.7.6 For d-dimensional simple random walk

$$\mathcal{T} = \sigma(\{X_0 \in L_i\}, i = 0, 1)$$

Chapter 5.8 General State Space*

Lemma 5.8.4 $v\bar{p} = \bar{p}$ and $\bar{p}v = p.$

Lemma 5.8.5 Let Y_n be an inhomogeneous Markov chain with $p_{2k} = v$ and $p_{2k+1} = \bar{p}$. Then $\bar{X}_n = Y_{2n}$ is a Markov chian with transition probability $\bar{p}$ and $X_n = Y_{2n+1}$ is a Markov chain with transition probaiblity p.

Lemma 5.8.6 If μ is a probability measure on $(S, \mathcal{S})$ then

$$E_\mu f(X_n) = E_\mu \bar{f}(\bar{X}_n)$$

Chapter 5.8.1 Recurrence and Transience

Theorem 5.8.8 Let $\lambda(C) = \sum_{n=1}^{\infty} 2^{-n} p^{-n}(\alpha, C)$. In the recurrent case if $\lambda(C) > 0$ then $P_\alpha(\bar{X}_n \in C \text{ i.o.}) = 1$. For $\lambda - a.e.x, P_x(R < \infty) = 1$

Chapter 5.8.2 Stationary Measures

Theorem 5.8.9 in the recurrent case, there is a $\sigma - finite$ stationary measure $\bar{\mu} << \lambda$

Lemma 5.8.10 If v is a $\sigma - finite$ stationary measure for p, then $v(A) < \infty$ and $\bar{v} = v\bar{p}$ is a stationary measure for $\bar{p}$ with $\bar{v}(\alpha) < \infty$

Theorem 5.8.11 Suppose p is recurrent. If v is a $\sigma - finite$ stationary measure then $v = \bar{v}(\alpha)\mu$, where μ is the measure constructed in the proof of theorem 5.8.9.

Theorem 5.8.12 Let X_n be an aperiodic recurrent Harris chain with stationary distribution π. If $P_x(R < \infty) = 1$ then as $n \to \infty$

$$||p^n(x, \cdot) - \pi(\cdot)|| \to 0$$

Chapter 5.8.4 G1/G/1 queue

Lemma 5.8.13 Suppose $X, Y \geq 0$ are independent and $P(X > x) = e^{-\lambda x}$. Show that $P(X - Y > x) = ae^{-\lambda x}$. Show that $P(X - Y > x) = ae^{-\lambda x}$, where $a = P(X - Y > 0)$.