Numerical Analysis 206B Approximation Theory Chapter 8

Charlie Seager

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Chapter 8 Approximation Theory

Chapter 8.1 Discrete Least Squares Approximation

Chapter 8.2 Orthogonal Polynomials and Least Squares Approximation

Definition 8.1 The set of functions $\{\phi_0 \ldots\ldots \phi_n\}$ is said to be linearly independent on [a,b] if, whenever

$$c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) = 0 \qquad \text{for all } x \in [a, b]$$

we have $c_0 = c_1 = \cdots = c_n = 0$. Otherwise the set of functions is said to be linearly dependent.

Theorem 8.2 Suppose that, for each j =0,1,...,n, $\phi_j(x)$ is a polynomial of degree j. Then $\{\phi_0 \ldots \phi_n\}$ is linearly independent on any interval [a,b].

Theorem 8.3 Suppose that $\{\phi_0(x), \phi_1(x) \ldots \phi_n(x)\}$ is a collection of linearly independent polynomials in $\prod_n$. Then any polynomial in $\prod_n$ can be written uniquely as a linear combination of $\phi_0(x), \phi_1(x), \ldots \phi_n(x)$.

Definition 8.4 An integrable function ω is called a weight function on the interval I if $\omega(x) \geq 0$ for all x in I, but $\omega(x) \neq 0$ on any subinterval of I.

Definition 8.5 $\{\omega_0, \omega_1, \ldots \phi_n\}$ is said to be an orthogonal set of functions for the interval [a,b] with respect to the weight function ω if

$$\int_a^b \omega(x)\phi_k(x)\phi_j(x)dx \, \{^{0}_{\alpha_j>0,} \qquad ^{when j \neq k}_{when j = k}$$

If, in addition $\alpha_j = 1$ for each j = 0,1,...,n the set is said to be orthonormal.

Theorem 8.6 If $\{\phi_0, \ldots \phi_n\}$ is an orthogonal set of functions on an interval [a,b] with respect to the weight function ω, then the least squares approximation to f on [a,b] with respect to ω is

$$P(x) = \textstyle\sum_{j=0}^{n} a_j\phi_j(x)$$

where for each j = 0,1,...,n

$$a_j = \frac{\int_a^b \omega(x)\phi_j(x)f(x)dx}{\int_a^b \omega(x)[\phi_j(x)]^2 dx} = \frac{1}{\alpha_j}\omega(x)\phi_j(x)f(x)dx.$$

Theorem 8.7 The set of polynomial functions $\{\phi_0, \phi_1, ..., \phi_n\}$ defined in the following way is orthogonal on [a,b] with respect to the weight function ω

$$\phi_0(x) = 1 \qquad \phi_1(x) = x - B_1, \qquad \text{for each x in [a,b]}$$

where

$$B_1 = \frac{\int_a^b x\omega(x)[\phi_0(x)]^2 dx}{\int_a^b \omega(x)[\phi_0(x)]^2 dx}$$

and when $k \geq 2$

$$\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k\phi_{k-2}(x), \qquad \text{for each x in [a,b]}$$

where

$$B_k = \frac{\int_a^b x\omega(x)[\phi_{k-1}(x)]^2 dx}{\int_a^b \omega(x)[\phi_{k-1}(x)]^2 dx}$$

and

$$C_k = \frac{\int_a^b x\omega(x)\phi_{k-1}(x)\phi_{k-2}(x) dx}{\int_a^b \omega(x)[\phi_{k-2}(x)]^2 dx}$$

Corollary 8.8 For any $n > 0$ the set of polynomial functions $\{\phi_0, ..., \phi_n\}$ given in Theorem 8.7 is linearly independent on [a,b] and

$$\int_a^b \omega(x)\phi_n(x)Q_k(x)dx = 0$$

for any polynomial $Q_k(x)$ of degree $k < n$

Chapter 8.3 Chebyshev Polynomials and Economization of Power Series

Theorem 8.9 The Chebyshev polynomial $T_n(x)$ of degree $n \geq 1$ has n simple zeros in [-1,1] at

$$\bar{x}_k = cos(\frac{2k-1}{2n}\pi), \qquad \text{for each k = 1,2,...,n}$$

Moreover $T_n(x)$ assumes its absolute extrema at

$$\bar{x}'_k = cos(\frac{k\pi}{n}) \text{ with } T_n(\bar{x}'_k) = (-1)^k, \qquad \text{for each k = 0,1,...,n}$$

Theorem 8.10 The polynomials of the form $\bar{T}_n(x)$, when $n \geq 1$ have the property that

$$\frac{1}{2^{n-1}} = \max_{x\in[-1,1]} |P_n(x)|, \qquad \text{for all } P_n(x) \in \tilde{\prod}_n$$

Moreover, equality occurs only if $P_n = \tilde{T}_n$

Corollary 8.11 Suppose that P(x) is the interpolating polynomial of degree at most n with nodes at the zeros of $T_{n+1}(x)$. Then

$$\max_{x\in[-1,1]} |f(x) - P(x)| \leq \frac{1}{2^n(n+1)!} \max_{x\in[-1,1]} |f^{n+1}(x)|, \qquad \text{for each } f \in C^{n+1}[-1,1]$$

Chapter 8.4 Rational Function Approximation

Chapter 8.5 Trigonometric Polynomial Approximation

Lemma 8.12 Suppose that the integer r is not a multiple of 2m. Then

$$\sum_{j=0}^{2m-1} cosrx_j = 0 \text{ and } \sum_{j=0}^{2m-1} sinrx_j = 0$$

Moreover, if r is not a multiple of m, then

$$\sum_{j=0}^{2m-1}(cosrx_j)^2 = m \text{ and } \sum_{j=0}^{2m-1}(sinrx_j)^2 = m$$

Theorem 8.13 The constants in the summation

$$S_n(x) = \frac{a_0}{2} + a_n cosnx + \sum_{k=1}^{n-1}(a_k coskx + b_k sinkx)$$

that minimize the least squares sum

$$E(a_0, ..., a_n, b_1, ..., b_{n-1}) = \sum_{j=0}^{2m-1}(y_j - S_n(x_j))^2$$

are

$$a_k = \frac{1}{m}\sum_{j=0}^{2m-1} y_j coskx_j \qquad \text{for each k = 0,1,...,n}$$

and

$$b_k = \frac{1}{m}\sum_{j=0}^{2m-1} y_j sinkx_j \qquad \text{for each k = 1,2,...,n-1}$$

Chapter 8.6 Fast Fourier Transforms