**Chapter 3.1 Interpolation and the Lagrange Polynomial**

**Theorem 3.1 (Weiestrass Approximation Theorem)**
Suppose that f is defined and continuous on [a,b]. For each $\epsilon > 0$, there exists a polynomial P(x), with the property that

$$|f(x) - P(x)| < \epsilon, \text{ for all x in [a,b]}$$

**Theorem 3.2** If $x_o, x_1, ..., x_n$ are n + 1 distinct numbers and f is a function whose values are given at these numbers, then a unique polynomial P(x) of degree at most n exists with

$$f(x_k) = P(x_k), \text{ for each k = 0,1,...,n}$$

**Theorem 3.3** Suppose $x_o, x_1, ..., x_n$ are distinct numbers in the interval [a,b] and $f \in C^{n+1}$[a,b]. Then, for each x in [a,b] exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_o)(x - x_1) \dots (x - x_n),$$

where P(x) is the interpolating polynomial given in Eq. (3.1)

**Chapter 3.2 Data approximation and Nevilles Method**

**Definition 3.4** Let f be a function defined at $x_o, x_1, x_2, ..., x_n$ and suppose that $m_1, m_2, ..., m_k$ are k distinct integers with $0 \leq m_i \leq n$ for each i. The lagrange polynomial that agrees with f(x) at the k points $x_{m1}, x_{m2}, ..., x_{mk}$ is denoted $P_{m1,m2,...,mk}(x)$

**Theorem 3.5** Let f be defined at $x_0, x_1, ..., x_k$ and let $x_j$ and $x_i$ be two distinct numbers in this set. Then

$$P(x) = \frac{(x-x_j)P_{0,1\dots j-1,j+1,\dots,k}(x) - (x-x_i)P_{0,1\dots i-1,i+1,\dots,k}(x)}{(x_i - x_j)}$$

is the kth lagrange polynomial that interpolates f at the k + 1 points $x_0, x_1, ..., x_k$.

**Chapter 3.3 Divided Differences**

**Theorem 3.6** Suppose that $f \in C^n[a, b]$ and $x_0, x_1, ..., x_n$ are distinct numbers in [a,b]. Then a number $\xi$ exists in (a,b) with

$$f[x_0, x_1, ..., x_n] = \frac{f^{(n)}(\xi)}{n!}$$

**Definition 3.7** Given the sequence $\{p_n\}_{n=0}^{\infty}$ define the backward difference $\nabla p_n$ (read nabla $p_n$) by

$$\nabla p_n = p_n - p_{n-1}, \text{ for } n \geq 1$$

Higher powers are defined recursively by

$$\nabla^k p_n = \nabla(\nabla^{k-1} p_n) \text{ for } k \geq 2$$

**Chapter 3.4 Hermite Interpolation**

**Definition 3.8** Let $x_0, x_1, ..., x_n$ be n + 1 distinct numbers in [a,b] and for i = 0,1,...,n let $m_i$ be a nonnegative integer. Suppose that $f \in C^m[a, b]$, where m = $max_{0 \leq i \leq n} m_i$.

**Theorem 3.9** If $f \in C^1[a, b]$ and $x_0, ..., x_n \in [a, b]$ are distinct, the unique polynomial of least degree agreeing with f and $f'$ at $x_0, ..., x_n$ is the Hermite polynomial of degree at most 2n + 1 given by

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_i)H_{n,j}(x) + \sum_{j=0}^{n} f^{'}(x_j)\hat{H}_{n,j}(x)$$

where, for $L_{n,j}(x)$ denoting the jth lagrange coefficient polynomial of degree n, we have

$$H_{n,j}(x) = [1 - 2(x - x_j)L_{n,j}^{'}(x_j)]L_{n,j}^{2}(x) \text{ and } \hat{H}_{n,j}(x) = (x - x_j)L_{n,j}^{2}(x).$$

Moreover, if $f \in C^{2n+2}[a, b]$ then

$$f(x) = H_{2n+1}(x) + \frac{(x-x_0)^2 \ldots (x-x_n)^2}{(2n+2)!} f^{2n+2}(\xi(x)),$$

for some (generally unknown) $\xi(x)$ in the interval (a,b).

**Chapter 3.5 Cubic Spline Interpolation**

**Definition 3.10** Given a function f defined on [a,b] and a set of nodes $x_0 < x_1 < \ldots < x_n = b$, a cubic spline interpolant S for f is a function that satisfies the following conditions:

(a) S(x) is a cubic polynomial, denoted $S_j(x)$ on the subinterval $[x_j, x_{j+1}]$ for each j = 0,1,...,n-1.
(b) $S_j(x_j) = f(x_{j+1})$ for each j = 0,1,...,n-1
(c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1}$ for each j = 0,1,...,n-2; (implied by (b))
(d) $S_{j+1}^{'}(x_{j+1}) = S_j^{'}(x_{j+1})$ for each j = 0,1,...,n-2;
(e) $S_{j+1}^{''}(x_{j+1}) = S_j^{''}(x_{j+1}$ for each j = 0,1,...,n-2;
(f) One of the following sets of boundary conditions is satisfied.
(i) $S^{''}(x_0) = S^{''}(x_n) = 0$ (natural or free boundary)
(ii) $S^{"}(x_0) = f^{'}(x_0)$ and $S^{'}(x_n) = f^{'}(x_n)$ (clamped boundary).

**Theorem 3.11** If f is defined at $a = x_0 < x_1 < \ldots < x_n = b$, then f has a unique natural spline interpolant S on the nodes $x_0, x_1, \ldots x_n$; that is a spline interpolant that satisfies the natural boundary conditions $S^{''}(a) = 0$ and $S^{''}(b) = 0$

**Theorem 3.12** If f is defined at $a = x_0 < x_1 < \ldots < x_n = b$ and differentiable at a and b, then f has a unique clamped spline interpolant S on the nodes $x_0, x_1, ..., x_n$; that is, a spline interpolant that satisfies the clamped boundary conditions $S^{'}(a) = f^{'}(a)$ and $S^{'}(b) = f^{'}(b)$

**Theorem 3.13** Let $f \in C^4[a, b]$ with $max_{a \leq x \leq b}|f^{(4)}(x)| = M$. If S is the unique clamped cubic spline interpolant to f with respect to the nodes $a = x_0 < x_1 < \ldots x_n = b$, then for all x in [a,b]

$$|f(x) - S(x)| \leq \frac{5M}{384} max_{0 \leq j \leq n-1}(x_{j+1} - x_j)^4$$