Math 206B Introduction to Numerical Analysis notes based off "Numerical Analysis by R.L. Burden and J.D. Faires" Section 1 which is Numerical Solution of Nonlinear Equations

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2.1 The Bisection Method

Bisection Technique The first technique based on the Intermediate Value Theorem, is called the Bisection or Binary Search method.

Theorem 2.1 Suppose that $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$|p_n - p| \leq \frac{b-a}{2^n}, \text{ when } n \geq 1$$

Chapter 2.2 Fixed Point Iteration A fixed point for the function is a number at which the value of the function does not change when the function is applied.

Defintion 2.2 The number p is a fixed point for a given function g if g(p) = p.

Theorem 2.3 (i) If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has at least one fixed point in [a,b].
(ii) If, in addition $g'(x)$ exists on (a,b) and a positive constnant k ¡ 1 exists with

$$|g'(x) \leq k, \text{ for all } x \in (a,b)$$

then there is exactly one fixed point in [a,b].

Theorem 2.4 (Fixed Point Theorem) Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$ for all x in [a,b]. Suppose in addition that g' exists on (a,b) and that a constant 0 ¡ k ¡ 1 exists with

$$|g'(x)| \leq k, \text{ for all } x \in (a,b)$$

Then for any number p_0 in [a,b], the sequence defined by

$$p_n = g(p_{n-1}), n \geq 1.$$

converges to the unique fixed point p in [a,b].

Corollary 2.5 If g satisfies the hypotheses of Th. 2.4, then bounds for the error involved in using p_n to approximate p are given by

$$|p_n - p| \leq k^n max\{p_0 - a, b - p_0\}$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|, \text{ for all } n \geq 1$$

Chapter 2.3 Newtons Method and Its Extensions
Newtons (or the Newton-Raphson) method is one of the most powerful and well known numerical methods for solving a root-finding problem. There are many ways of introducing Newton's Method.

Theorem 2.6 Let $f \in C^2[a,b]$. If $p \in (a,b)$ is such that f(p) = 0 and $f'(p) \neq 0$ then there exists a $\delta > 0$ such that Newton's Method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximations $p_0 \in [p-\delta, p+\delta]$.

Chapter 2.4 Error Analysis for Iterative Methods In this section we investigate the order of convergence of functional iteration schemes and as a means of obtaining rapid convergence, rediscover Newton's method. We also consider ways of accelerating the convergence of Newton's method in special circumstances. First, however, we need a new procedure for measuring how rapidly a sequence converges.

Defintion 2.7 Order of convergence Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$

then $\{p_n\}_{n=0}^{\infty}$ converges to p of order α with asymptotic error constant λ.

Theorem 2.8 Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose, in addition that g' is continous on (a,b) and a positive constant k ¡ 1 exists with

$$|g'(x) \leq k, \text{ for all } x \in (a,b)$$

If $g'(p) \neq 0$, then for any number $p_0 \neq p$ in [a,b], the sequence

$$p_n = g(p_{n-1}) \text{ for } n \geq 1$$

converges only linearly to the unique fixed point p in [a,b]

Theorem 2.9 Let p be a solution of the equation x = g(x). Suppose that $g'(p) = 0$ and g'' is continous with $|g''(x)| < M$ on an open interval I containing p. Then there exists a $\delta > 0$ such that for $p_0 \in [p-\delta, p+\delta]$, the sequence defined by $p_n = g(p_{n-1})$ when $n \geq 1$, converges at least quadratically to p. Moreover for sufficiently large values of n.

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2$$

Defintion 2.10 A solution p of f(x) = 0 is a zero of multiplicity m of f if for $x \neq p$ we can write $f(x) = (x-p)^m q(x)$, where $\lim_{x\to p} q(x) \neq 0$.

Theorem 2.11 The function $f \in C^1[a,b]$ has a simple zero at p in (a,b) if and only if f(p) = 0, but $f'(p) \neq 0$

Theorem 2.12 The function $f \in C^m[a,b]$ has a zero of multiplicity m at p in (a,b) if and only if

$$0 = f(p) = f'(p) = f''(p) = \ldots = f^{(m-1)}(p), \text{ but } f^{(m)}(p) \neq 0$$

Chapter 2.5 Accelerating Convergence Therem 2.8 indicates that it is rare to have the luxury of quadratic convergence. We now consider a technique called Aitken's Δ^2 method that can be used to accelerate the convergence of a seqeuence that is linearly convergent, regardless of its origin or application.

Defintion 2.13 For a given sequence $\{p_n\}_{n=0}^{\infty}$ the forward difference Δp_n (read "$deltap_n$") is defined by

$$\Delta p_n = p_{n+1} - p_n \text{ for } n \geq 0$$

Higher powers of the operator Δ are defined recursively by

$$\Delta^k p_n = \Delta(\Delta^{k-1} p_n) \text{ for } k \geq 2$$

Theorem 2.14 Suppose that $\{p_n\}_{n=0}^{\infty}$ is a seqeunce that converges linearly to the limit p and that

$$\lim_{n \to \infty} \frac{p_{n+1} - p}{p_n - p} < 1$$

Then the Aitkeyn's Δ^2 seqeunce $\{\hat{p}_n\}_{n=0}^{\infty}$ converges to p faster than $\{p_n\}_{n=0}^{\infty}$ in the sense that

$$\lim_{n \to \infty} \frac{\hat{p}_n - p}{p_n - p} = 0.$$

Theorem 2.15 Suppose that x = g(x) has the solution p with $g'(p) \neq 1$. If there exists a $\delta > 0$ such that $g \in C^3[p - \delta, p + \delta]$, then Steffensen's method gives quadratic convergence for any $p_0 \in [p - \delta, p + \delta]$

Chapter 2.6 Zeros of Polynomials and Mullers Method

Theorem 2.16 (Fundamental Theorem of Algebra)
If P(x) is a polynomial of degree $n \geq 1$ with real or complex coefficients, the P(x) = 0 has at least one (possibly complex) root.

Corollary 2.17 If P(x) is a polynomail of degree $n \geq 1$ with real or complex coefficients, then there exist unique constants $x_1, x_2, .., x_k$ possibly complex and unique positive integers $m_1, m_2, ..., m_k$ such that $\Sigma_{i=1}^{k} m_i = n$ and

$$P(x) = a_n (x - x_1)^{m_1} (x - x_2)^{m_2} ... (x - x_k)^{m_k}$$

Corollary 2.18 Let P(x) and Q(x) be polynomials of degree at most n. If $x_1, x_2, ..., x_k$ with k ¿ n, are distinct numbers with $P(x_i) = Q(x_i)$ for $i = 1, 2, ..., k$ then P(x) = Q(x) for all values.

Theorem 2.19 (Horners Method)
Let

$$P(x) = a_n X^n + a_{n-1} X^{n-1} + ... + a_1 x + a_0$$

Define $b_n = a_n$ and

$$b_k = a_k + b_{k+1} x_0. \text{ for k = n - 1, n - 2,..., 1,0.}$$

Then $b_0 = P(x_0)$. Moreover if

$$Q(x) = b_n X^{n-1} + b_{n-1}x^{n-2} + ... + b_2 x + b_1.$$

then

$$P(x) = (x - x_0)Q(x) + b_0$$

Theorem 2.20 If $z = bi$ is a complex zero of multiplicity m of the polynomial P(x) with real coefficients then $\bar{z} = a - bi$ is also a zero of multiplicty m of the polynomial P(x) and $(x^2 - 2ax + a^2 + b^2)^m$ is a factor of P(x).

Chapter 2.7 Survey of Methods and Software