Numerical Analysis Section 4: Numerical Differentiation and integration, IVP and BVP

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January 29, 2024

Chapter 4 Numerical Differentiation and Integration
Chapter 4.1 Numerical Differentiation
Chapter 4.2 Richardson's Extrpolation
Chapter 4.3 Elements of Numerical Integration

Definition 4.1 The degree of accuracy or precission of a quadrature formula is the largest positive integer n such that the formula is exact for x^k for each k = 0,1,...,n.

Thoerem 4.2 Suppose that $\sum_{i=0}^{n} a_i f(x_i)$ denoted the (n+1)-point closed Newton-Cotes formula with $x_0 = a, x_n = b$ and $h = (b-a)/n$. There exists $\xi \in (a,b)$ for which

$$\int_a^b f(x)dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_0^n t^2(t-1)\dots(t-n)dt$$

if n is even and $f \in C^{n+2}[a,b]$ and

$$\int_a^b f(x)dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_0^n t(t-1)\dots(t-n)dt$$

if n is odd and $f \in C^{n+1}[a,b]$

Theorem 4.3 Suppose that $\sum_{i=0}^{n} a_i f(x_i)$ denotes the (n+1)-point open Newton-Cotes formula with $x_{-1} = a, x_{n+1} = b$, and $h = (b-a)/(n+2)$. There exists $\xi \in (a,b)$ for which

$$\int_a^b f(x)dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_{-1}^{n+1} t^2(t-1)\dots(t-n)dt$$

if n is even and $f \in C^{n+2}[a,b]$ and

$$\int_a^b f(x)dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_{-1}^{n+1} t(t-1)\dots(t-n)dt$$

if n is odd and $f \in C^{n+1}[a,b]$

Chapter 4.4 Composite Numerical Integration

Theorem 4.4 Let $f \in C^4[a,b]$, n be even, h = (b-a)/n, and $x_j = a + jh$ for each j = 0, 1,...,n. There exists a $\mu \in (a,b)$ for which the Composite Simpson's rule for n subintervals can be written with its error term as

$$\int_a^b f(x)dx = \frac{h}{3}[f(a)+2\sum_{j=1}^{(n/2)-1} f(x_{2j})+4\sum_{j=1}^{n/2} f(x_{2j-1})+f(b)] - \frac{b-a}{180}h^4 f^{(4)}(\mu)$$

Theorem 4.5 Let $f \in C^2[a,b], h = (b-a)/n$ and $x_j = a + jh$ for each j = 0,1,...,n. There exists a $\mu \in (a,b)$ for which the Composite Trapezoidal rule for n subintervals can be written with its error term as

$$\int_a^b f(x)dx = \frac{h}{2}[f(a) + 2\sum_{j=1}^{n-1} f(x_i) + f(b)] - \frac{b-a}{12}h^2 f''(\mu)$$

Theorem 4.6 Let $f \in C^2[a,b]$, n be even h = (b-a)/(n+2), and $x_i = a + (j+1)h$ for each j = -1,0,...,n+1. There exists a $\mu \in (a,b)$ for which the Composite Midpoint rule for n + 2 subintervals can be written with its error term as

$$\int_a^b f(x)dx = 2h\sum_{j=0}^{n/2} f(x_{2j}) + \frac{b-a}{6}h^2 f''(\mu)$$

Chapter 4.5 Romberg Integration

Chapter 4.6 Adaptive Quadrature Methods

Chapter 4.7 Gaussian Quadrature

Theorem 4.7 Suppose that $x_1, x_2, ..., x_n$ are the roots of the nth Legendre polynomial $P_n(x)$ and that for each i = 1,2,...,n, the numbers c_i are defined by

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j\neq 1}}^{n} \frac{x-x_j}{x_i-x_j}dx$$

If P(x) is any polynomial of degree less than 2n, then

$$\int_{-1}^{1} P(x)dx = \sum_{i=1}^{n} c_i P(x_i)$$

Chapter 4.8 Multiple Integrals

Chapter 4.9 Improper Integrals

Chapter 5 Initial-Value Problems for Ordinary Differential Equations

Chapter 5.1 The Elementary Theory of Initial-value Problems

Definition 5.1 A function f(t,y) is said to satisfy a Lipshitz condition in the variable y on a set $D \subset R^2$ if a constant ¿ 0 exists with

$$|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|,$$

whenever (t,y_1) and (t,y_2) are in D. The constant L is called a Lipschitz condition for f.

Definition 5.2 A set $D \subset R^2$ is said to be convex if whenever (t_1,y_1) and (t_2,y_2) belong to D, then $((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2)$ also belongs to D for every λ in [0,1].

Theorem 5.3 Suppose f(t,y) is defined on a convex set $D \subset R^2$. If a constant L ¿ 0 exists with

$$|\frac{\partial f}{\partial y}| \leq L, \text{ for all } (t,y) \in D$$

then f satisfies a Lipschitz condition on D in the variable y with Lipshitz constant L.

Theorem 5.4 Suppose that $D = \{(t,y)|a \leq t \leq b \text{ and } -\infty < y < \infty\ \}$ and that f(t,y) is continous on D. If f satisfies a Lipshitz condition on D in the variable y, then the initial-value problem

$$y^{'}(t) = f(t, y), a \leq t \leq b, y(a) = \alpha$$

has a unique solution y(t) for $a \leq t \leq b$

Definition 5.5 The initial-value problem

$$\frac{dy}{dt} = f(t, y) a \leq t \leq b, y(a) = \alpha$$

is said to be a well posed problem if: 1. A unique solution y(t), to the problem exists, and
2.There exists constants $\epsilon_0 > 0$ and k ¿ 0 such that for any ϵ, with $\epsilon_0 > \epsilon > 0$ whenever $\delta(t)$ is continous with $|\delta(t)| < \epsilon$ for all t in [a,b], and when $\delta_0| < \epsilon$, the initial value problem

$$\frac{dz}{dt} = f(t, z) + \delta(t), a \leq t \leq b, z(a) = \alpha + \delta_0$$

has a unique solution z(t) that satisfies

$$|z(t) - y(t)| < k\epsilon \text{ for all t in [a,b]}$$

Theorem 5.6 Suppose $D = \{(t, y)|a \leq t \leq b \text{ and } -\infty < y < \infty\}$. If f is continous and satisfies a Lipshitz condition in the variable y on the set D, then the initial-value problem

$$\frac{dy}{dt} = f(t, y)\ a \leq t \leq b\ y(a) = \alpha$$

is well-posed.

Chapter 5.2 Euler's Method

Theorem 5.9 Suppose f is continous and satisfies a Lipshitz condition with constant L on

$$D = \{(t, y)|a \leq t \leq b \text{ and } -\infty < y < \infty\}$$

and that a constant M exists with

$$|y^{''}(t)| \leq M, \text{ for all } t \in [a, b]$$

where y(t) denotes the unique solution to the initial value problem

$$y^{'} = f(t, y), a \leq t \leq b, y(a) = \alpha$$

Let $\omega_0, \omega_1, ..., \omega_N$ be the approximations generated by Euler's method for some integer N. Then, for each i = 0,1,2,..,N

$$|y(t_i) - \omega_i| \leq \frac{hM}{2L}[e^{L(t_i - a)} - 1]$$

Theorem 5.10 Let y(t) denote the unique solution to the initial-value problem

$$y^{'} = f(t, y), a \leq t \leq by(a) = \alpha$$

and $u_0, u_1, ..., u_N$ be the approximation obtained using (5.11). If $|\delta_i| < \delta$ for each i = 0,1,...,N and the hypotheses of Theorem 5.9 hold for (5.12), then

$$|y(t_i) - u_i| \leq \frac{1}{L}(\frac{hM}{2} + \frac{\delta}{h})[e^{L(t_i-a)} - 1] + |\delta_0|e^{L(t_i-a)},$$

for each i = 0,1,...,N.

Chapter 5.4 Runge-Kutta Methods

Theorem 5.13 Suppose that f(t,y) and all its partial derivatives of order less than or equal to n + 1 are continous on $D = \{(t,y)|a \leq t \leq b, c \leq y \leq d\}$ and let $(t_0, y_0) \in D$. For every $(t,y) \in D$, there exists ξ between t and t_0 and μ between y and y_0 with

$$f(t,y) = P_n(t,y) + R_n(t,y)$$

where

$$\begin{aligned} P_n(t,y) &= f(t_0,y_0) + [(t-t_0)\frac{\partial f}{\partial t}(t_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(t_0,y_0)] \\ &+[\frac{(t-t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0) + (t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t \partial y}(t_0,y_0) \\ &+\frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)] \ldots \\ &+[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t-t_0)^{n-j}(y-y_0)^j\frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0,y_0)] \end{aligned}$$

and

$$R_n(t,y) = \frac{1}{(n+1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}(t-t_0)^{n+1-j}(y-y_0)^j\frac{\partial^{n+1}}{\partial t^{n+1-j}\partial y^j}(\xi,\mu)$$

The function $P_n(t,y)$ is called the nth Taylor polynomial in two variables for the function f about (t_0, y_0) and $R_n(t,y)$ is the remainder term associated with $P_n(t,y)$.

Chapter 11 Boundary-Value Problems for Ordinary Differential Equations

Chapter 11.1 The Linear Shooting Method

Theorem 11.1 Suppose the function f in the boundary-value problem

$$y'' = f(x,y,y'), \text{ for } a \leq x \leq b \text{ with y(a) } = \alpha \text{ and y(b) } = \beta$$

is continous on the set

$$D = \{(x,y,y')| \text{ for } a \leq x \leq b \text{ with } -\infty < y < \infty \text{ and } -\infty < y' < \infty\}$$

and that the partial derivatives f_y and $f_{y'}$ are also continous on D. If
(i) $f_y(x,y,y') > 0$, for all $(x,y,y') \in D$, and
(ii) a constant M exists with

$$|f_{y'}(x,y,y')| \leq M, \text{ for all } (x,y,y') \in D$$

then the boundary-value problem has a unique solution.

Corollary 11.2 Suppose the linear boundary-value problem

$$y'' = p(x)y' + q(x)y + r(x), \text{ for } a \leq x \leq b \text{ with } y(a) = \alpha \text{ and } y(b) = \beta.$$

satisfies
(i) p(x), q(x) and r(x) are continous on [a,b]
(ii) $q(x) > 0$ on [a,b]
Then the boundary-value problem has a unique solution.

Theorem 11.3 Suppose that p, q and r are continous on [a,b]. If $q(x) \geq 0$ on [a,b], then the tridiagonal linear system (11.19) has a unique solution provided that $h < 2/L$ where $L = max_{a \leq x \leq b}|p(x)|$

Theorem 11.4 Let $p \in C^1[0, 1], q, \in C[0, 1]$ and

$$p(x) \geq \delta > 0 \qquad q(x) \geq 0 \qquad \text{for } 0 \leq x \leq 1$$