Calculus of Variations Ch 8 Direct Methods in The Calculus of Variations

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Section 39 Minimizing Sequences

Theorem If $\{y_n\}$ is a minimizing sequence of the functional J[y], with limit function $\hat{y}$ and if J[y] is lower semicontinuous at $\hat{y}^2$ then

$$J[\hat{y}] = \lim_{n\to\infty} J[y_n]$$

Section 40: The Ritz Method and the Method of Finite Differences

Section 40.1 First, we describe the Ritz method, one of the most widely used direct variational methods. Suppose we are looking for the minimum of a functional J[y] defined on some space $\mathcal{M}$ of admissible functions, which for simplicity we take to be a normed lienar space. Let

$$\varphi_1, \varphi_2, ...$$

be an infinite sequence of functions in $\mathcal{M}$ and let $\mathcal{M}_n$ be the n-dimensional linear subspace of $\mathcal{M}$ spanned by the first n of the functions (8).

Definition The sequence (8) is said to be complete (in $\mathcal{M}$) if given any $y \in \mathcal{M}$ and any $\epsilon > 0$, there is a linear combination $\backslash_n$ of the form (9) such that $||\backslash_n - y|| < \epsilon$ (where n depends on ϵ

Theorem If the functional J[y] is continuous, and if the sequence (8) is complete, then

$$\lim_{n\to\infty} \mu_n = \mu,$$

where,

$$\mu = inf_y J[y]$$

Section 41 The Sturm-Liouville Problem In this section, we illustrate the application of direct variational methods to differential equations (cf. the remarks on p.192), by studying the following boundary value problem, known as the Sturm-Liouville problem: Let $P = P(x) > 0$ and $Q = Q(x)$ be two given functions, where Q is continous and P is continously differentiable.

Theorem The Sturm-Liouville problem (14), (15) has an infintie sequence of eigenvalues $\lambda^{(1)}, \lambda^{(2)}, ...$, and to each eigenvalue $\lambda^{(n)}$ there corresponds an eigenfunction $y^{(n)}$ which is unique to within a constant factor

Lemma 1 The sequence $\{y_n^{(1)}(x)\}$ contains a uniformly convergent subsequence.

Lemma 2 Let y(x) be continuous in $[0, \pi]$ and let

$$\int_0^\pi [-(Ph')' + Q_1 h] y dx = 0$$

for every function $h(x) \in \mathcal{D}_2(0, \pi)$ satisfying the boundary conditions

$$h(0) = h(\pi) = 0 \qquad h'(0) = h'(\pi) = 0$$

Then y(x) also belongs to $\mathcal{D}_2(0, \pi)$ and

$$-(Py')' + Q_1 y = 0$$