Calculus of Variations Ch 6: Fields, Sufficient Condition for a Strong Extremum

Charlie Seager

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Section 31 Consistent Boundary Conditions. General Definition of a Field

Definition 1 The boundary conditions

$$y_i' = \Psi_i^{(1)}(y_1, ..., y_n) \qquad (i = 1, ..., n)$$

prescribed for $x = x_1$ and the boundary conditions

$$y_i' = \Psi_i^{(2)}(y_1, ..., y_n) \qquad (i = 1, ..., n)$$

prescribed for $x = x_2$ are said to be (mutually) consistent if every solution of the system (1) satisfying the boundary conditions (3) at $x = x_1$ also satisfies the boundary conditions (4) at $x = x_2$ and conversely

Definition 2 Suppose the boundary conditions

$$y_i' = \Psi_i(x, y_1, ..., y_n) \qquad (i = 1, ..., n)$$

(where the Ψ_i are continously differentiable functions) are prescribed for every x in the interval [a,b] and suppose they are consistent for every pair of points x_1, x_2 in [a,b]. Then the family of mutually consistent boundary conditions (5) is called a field (of directions) for the given system(1).

Theorem The first-order system

$$y_i' = \Psi_i(x, y_1, ..., y_n) \qquad (a \leq x \leq b; 1 \leq i \leq n)$$

is a field for the second-order system

$$y"_i = f_i(x, y_i, ..., y_n, y_1', ..., y_n')$$

if and only if the functions $\Psi_i(x, y_1, ..., y_n)$ satisfy the following system of partial differential equations, called the Hamilton-Jacobi system for the original

$$\frac{\partial\Psi_i}{\partial x} + \sum_{k=1}^{n} \frac{\partial\Psi_i}{\partial y_k}\Psi_k = f_i(x, y_1, ..., y_n, \Psi_1, ..., \Psi_n)$$

Thus, every solution of the Hamilton-Jacobi system (8) gives a field for the original system (7).

Section 32 The Field of a Functional

Definition 1 Given a functional

$$\int_a^b F(x, y, y')dx$$

with momenta (28), the boundary conditions (30), prescribed for x = a are said to be self adjoint if there exists a function g(x,y) such that

$$p_i[x, y, \Psi(y)]|_{x=a} \equiv g_{yi}(x, y)|_{x=a} \qquad (i = 1, ..., n)$$

Theorem 1 The boundary conditions (30) are self-adjoint if and only if they satisfy the conditions

$$\frac{\partial p_i[x,y,\Psi(y)]}{\partial y_k}|_{x=a} = \frac{\partial p_k[x,y,\Psi(y)]}{\partial y_i}|_{x=a} \qquad (i, k = 1, ..., n)$$

called the self-adjointness conditions

Definition 2 Given a functional

$$\int_a^b F(x, y, y')dx$$

with the system of Euler equations

$$F_{yi} - \frac{d}{dx}F_{yi'} = 0 \qquad (i = 1, ..., n)$$

we say that the boundary conditions

$$y_i' = \Psi_i^{(1)}(y) \qquad (i = 1, ..., n)$$

prescribed for $x = x_1$ and the boundary conditions

$$y_i' = \Psi_i^{(2)}(y) \qquad (i = 1, ..., n)$$

prescribed for $x = x_2$ are (mutually consistent with respect to the functional (33) if they are consistent with respect to the system (34), i.e. if every extremal satisfying the boundary conditions (35) at $x = x_1$ also satisfies the boundary conditions (36) at $x = x_2$ and conversely

Definition 3 The family of boundary conditions

$$y_i' = \Psi_i(x, y) \qquad (i = 1, ..., n)$$

prescribed for every x in the interval [a,b] is said to be a field of the functional (33) if
1. The conditions (37) are self-adjoint for every x in [a,b];
2. The conditions (37) are consistent for every pair of points x_1, x_2 in [a,b]

Theorem 2 A necessary and sufficient condition for the family of boundary conditions (37) to be a field of the functional (33) is that the self-adjointness conditions

$$\frac{\partial p_i[x,y,\Psi(x,y)]}{\partial y_k} = \frac{\partial H[x,y,\Psi(x,y)]}{\partial y_i}$$

and the consistency conditions

$$\frac{\partial p_i[x,y,\Psi(x,y)]}{\partial x} = -\frac{\partial H[x,y,\Psi(x,y)]}{\partial y_i}$$

be satisfied at every point x in [a,b] where

$$p_i(x, y, y') = F_{yi;}(x, y, y')$$

and H is the Hamiltonian corresponding to the functional (33)

$$y_i' = \Psi_i(x, y) \qquad (i = 1, ..., n)$$

Theorem 3 The expression

$$\frac{\partial p_i(x,y,y')}{\partial y_k} - \frac{\partial p_k(x,y,y')}{\partial y_i}$$

has a constant value along each extremal

Theorem 4 The boundary conditions (49) defined by the relations (50) are consistent if and only if the function g(x,y) satisfies the Hamilton-Jacobi equation

$$\frac{\partial g}{\partial x} + H(x, y_1, ..., y_n, \frac{\partial g}{\partial y_1}, ..., \frac{\partial g}{\partial y_n}) = 0$$