Math 207A Ordinary Differential Equations: Ch.3 Existence Theory

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Chapter 3.1 Existence in the Scalar Case

Lemma 3.1 If ϕ is a solution of the initial value problem (3.1), (3.2) on an interval I, then ϕ satisfies (3.3) on I. Conversely, if y(t) is a solution of (3.3) on some interval J containing t_0, then y(t) satisfies (3.1) on J and also the initial condition (3.2)

Definition A function f that satisfies an inequality of the form (3.7) for all $(t, y_1), (t, y_2)$ in a region D is said to satisfy a Lipschitz condition in D.

Lemma 3.2 Define α to be the smaller of the positive numbers a and b/M. Then the successive approximations ϕ_j given by (3.8) are defined on the interval $I = \{t|t - t_0| \leq \alpha\}$ and on this interval

$$|\phi_j(t) - y_0| \leq M|t - t_0| \leq b, \qquad \text{(j=0,1,2,...)}$$

Theorem 3.1 Suppose f and $\partial f/\partial y$ are continuous on the closed rectangle R and satisfy the bounds (3.6). Then the successive approximation ϕ_j given by (3.8), converge (uniformly) on the interval $I = \{t|t - t_0| \leq \alpha\}$ to a solution ϕ of the differential equation (3.1) that satisfies the initial conditions (3.2)

Theorem 3.2 Suppose f is continuous on the rectangle R, and suppose $|f(t, y)| \leq M$ for all points (t,y) in R. Let α be the smaller of the positive numbers a and b/M. Then there is a solution ϕ of the differential equation (3.1) that satisfies the initial condition (3.2) existing on the interval $|t - t_0| < \alpha$

Chapter 3.2 Existence Theory for Systems of First-Order Equations

Theorem 3.3 Let f and $\partial f/\partial y_j$ (j = 1,...,n) be continuous on the box $B = \{(t, y)||t - t_0| \leq a, |y - \backslash| \leq b\}$, where a and b are positive numbers, and satisfying the bounds

$$|f(t, y)| \leq M, |\frac{\partial f(t,y)}{\partial y_j}| \leq K \qquad \text{(j = 1,...,n)}$$

for (t,y) in B. Let α be the smaller of the numbers a and b/M and define the successive approximations

$$\{\Phi_0(t) =$$

$$\{\Phi_n(t) = \ + \int_{t_0}^{t} f(s, \Phi_{n-1}(s))ds$$

Then the sequence $\{\Phi_j\}$ of successive approximations converges (uniformly) on the interval $|t - t_0| \leq \alpha$ to a solution $\Phi(t)$ of (3.16), that satisfies the initial condition $\Phi(t_0) =$

Chapter 3.3 Uniqueness of Solutions

Theorem 3.4 Suppose t and $\partial f / \partial y_j$ (j = 1,..,n) are continuous on the "box"

$$B = \{(t, y) | |t - t_0| \leq a, |y - | \leq b\}$$

Then there exists at most one solution of (3.22) satisfying the initial condition (3.23)

Theorem 3.5 Suppose f is continuous on the rectangle R = {(t,y) — $|t - t_0| \leq a, |y - y_0| \leq b\}$ and monotone nonincreasing in y for each fixed t on the rectangle R. Then the initial value problem

y' = f(t,y)

$$\phi(t_0) = y_0$$

has at most one solution on any interval J with t_0 as left end point.

Chapter 3.4 Continuation of Solutions

Lemma 3.3 Suppose f, $\partial f / \partial y_j$ (j = 1,..,n) are continuous in a domain D and suppose $|f|$ is bounded in D. Let Φ be a solution

y' = f(t,y)

$$\phi(t_0) =$$

existing on a finite interval $y < t < \delta$. Then $\lim_{t \to \delta^-} \phi(t)$ and $\lim_{t \to y^+} \phi(t)$ exist.

Theorem 3.6 Suppose that f and $\partial f / \partial y_j$ (j = 1,...,n) are continuous in a given region D and suppose f is bounded on D. Let $(t_0,)$ be a given point of D. Then the unique solution ϕ of the system y'=f(t,y) passing through the point $(t_0,)$ can be extended until its graph meets the boundary of D.

Chapter 3.5 Dependence on Initial Conditions and Parameters

Theorem 3.7 Suppose f and $\partial f / \partial y_j$ (j = 1,...,n) are continuous and bounded in a given region D. We assume that the bounds (3.19) are satisfied on D (rather than on the box B). Let Φ be the solution of the system (3.27) passing through the point $(t_0,)$ and let Ψ be the solution of (3.27) passing through the point $(\hat{t}_0, \hat{})$. Suppose that Φ and Ψ both exist on some interval $\alpha < t < \beta$. Then to each $\epsilon > 0$ there corresponds $\delta > 0$ such that if $|t - \hat{t}| < \delta$ and $| - \hat{}| < \delta$, then

$$|\Phi(t) - \Psi(\hat{t})| < \epsilon \qquad (\alpha < t < \beta, \qquad \alpha < \hat{t} < \beta)$$

Theorem 3.8 Let f, g be defined in a domain D and satisfy the hypotheses of Theorem 3.7. Let Φ and Ψ be solutions of y' = f(t,y), y' = g(t,y) respectively such that $\Phi(t_0) = , \Psi(t_0) = \hat{}$, existing on a common interval $\alpha < t < \beta$ suppose

$$|f(t, y) - g(t, y)| \leq \epsilon$$

for (t,y) in D. Then the solutions Φ, Ψ satisfy the estimate

$$|\Phi(t) - \Psi(t)| \leq | - \hat{}| exp(K|t - t_0|) + \epsilon(\beta - \alpha) exp(K|t - t_0|)$$

for all t in $\alpha < t < \beta$