Math 207A Ordinary Differential Equations: Ch.4 Stability Properties

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Chapter 4: Stability of Linear and Almost Linear Systems

Chapter 4.2 Defiitions of Stability

Definition 1 (See Figure 4.2) The equilibrium solution y_0 of (4.1) is said to be stable if for each number $\epsilon > 0$ we can find a number $\delta > 0$ (depending on ϵ) such that if $\Psi(t)$ is any solution of (4.1) having $||\Psi(t_0) - y_0|| < \delta$, then the solution $\Psi(t)$ exists for all $t \geq t_0$ and $||\Psi(t) - y_0|| < \epsilon$ for $t \geq t_0$ (where for convenience the norm is the Euclidean distance that makes neighborhoods spherical).

Definition 2 (See Figure 4.3) The equilibrium solution y_0 is said to be asymptotically stable if it is stable and if there exists a number $\delta_0 > 0$ such that if $\Psi(t)$ is any solution of (4.1) having $||\Psi(t_0) - y_0|| < \delta_0$, then $\lim_{t\to+\infty} \Psi(t) = y_0$.

Definition 3 A solution $\Phi(t)$ of (4.2) is said to be stable for every $\epsilon > 0$ and every $t_0 \geq 0$ there exists a $\delta > 0$ (δ now depends on both ϵ and possibly t_0) such that whenever $|\Phi(t_0) - y_0| < \delta$ the solution $\Psi(t, t_0, y_0)$ exists for $t > t_0$ and satisfies $|\Phi(t) - \Psi(t, t_0, y_0)| < \epsilon$ for $t \geq t_0$.

Definition 4 The solution $\Phi(t)$ of (4.2) is said to be asymptotically stable if it stable and if there exists $\delta_0 > 0$ such that whenever $|\Phi(t_0) - y_0| < \delta_0$ the solution $\Psi(t, t_0, y_0)$ approaches the solution $\Phi(t)$ as $t \to \infty$ (that is, $\lim_{t\to\infty} |\Psi(t, t_0, y_0) - \Phi(t)| = 0$)

Chapter 4.3 Linear Systems

Theorem 4.1 If all eigenvalues of A have nonpositive real parts and all those eigenvalues with zero real parts are simple, then the solution y = 0 of (4.3) is stable. If (and only if) all eigenvalues of A have negative real parts, the zero solution of (4.3) is asymptotically stable. In fact in this case, $\Psi(t, t_0)$ denotes the fundamental matrix of (4.3) which is the identity at $t = t_0$ $\Psi(t, t_0) = exp((t - t_0)A)$ and there exist constants $K > 0$ $\sigma > 0$ such that

$$|\Psi(t, t_0)| \leq Kexp(-\sigma(t - t_0)) \qquad (t_0 \leq t < \infty)$$

with $\sigma > 0$ in the case that all eigenvalues of A have negative real parts and $\sigma = 0$ if there are simple eigenvalues with zero real part. If one or more eigenvalues of A have a positive real part, the zero solution of (4.3) is unstable.

Theorem 4.2 Let all the eigenvalues of A have real parts negative and let B(t) be continuous for $0 \leq t < \infty$ with $\lim_{t\to\infty} B(t) = 0$. Then the zero solution of (4.5) is globally asymptotically stable

Chapter 4.4 Almost linear system

Theorem 4.3 Suppose all eigenvalues of A have negative real parts, f(t,y) and $(\partial f/\partial y_j)$ (t,y) (j=1,...,n) are continuous in (t,y) for $0 \leq t < \infty, |y| < k$ where $k > 0$ is a constant, and f is small in the sense that

$$\lim_{|y|\to 0} \frac{|f(t,y)|}{|y|} = 0$$

uniformly with respecct to t on $0 \leq t < \infty$. Then the solution y = 0 of (4.16) is asymptotically stable.

Theorem 4.4 If A and f satisfy the hypothesis of Theorem 4.3 and if B(t) is continuous for $0 \leq t < \infty$ with $\lim_{t\to\infty} B(t) = 0$, then the zero solution of (4.22) is asymptotically stable.

Theorem 4.5 In equation (4.23) assume

(i) the eigenvalues of A all have negative real part;
(ii) $\lim_{|y|\to 0} |f(t,y)|/|y| = 0$ uniformly in t on $0 \leq t < \infty$;
(iii) $|h(t,y)| \leq \lambda(t)$ for $0 \leq t < \infty$, $|y| < k$ for some $k > 0$, where λ is a continuous nonnegative function on $0 \leq t < \infty$ such that
$\wedge(t) = \int_t^{t+1} \lambda(s)ds \to 0$ as $t \to \infty$

Then there exists $T_0 > 0$ such that every solution ϕ of (4.23) with $|\phi(T)|$ small enough for any $T \geq T_0$ remains small for $t \geq T$, and $\lim_{t\to\infty} \phi(t) = 0$.

Lemma 4.1 If λ satisfies (iii) and if $\omega > 0$, then there exists T_0 such that

$$\lim_{t\to\infty} \int_T^t e^{-\omega(t-s)} \lambda(s)ds = 0$$

for all $T \geq T_0$

Chapter 4.5 Conditional Stability

Theorem 4.6 Let g, $\partial g/\partial y_j$ (j = 1,2) be continuous for $|y| < k$ for some constant $k > 0$ (k can be small), and let g(0) = 0 and $\lim_{|y|\to 0} |\partial g/\partial y_j| = 0$(j=1,2). If the eigenvalues of A are $\lambda, -\mu$, with $\lambda, \mu > 0$, then there exists in y space a real curve C passing through the origin such that if ϕ is any solution of (4.30) with $\phi(0)$ (or $\phi(t_0)$) on C and $|\phi(0)|$ small enough, then $\phi(t) \to 0$ as $t \to \infty$. Moreover, no solution $\phi(t)$ with $|\phi(0)|$ small enough, but not on C, can remain small for $t \geq 0$; in particular, the zero solution of (4.30) is unstable.

Chapter 4.6 Asymptotic Equivalence

Definition We say that the systems (4.44) and (4.45) are asymptotically equivalent if to each solution x(t) of (4.44) with $|x(t_0)|$ sufficiently small there corresponds a solution y(t) of (4.45) such that

$$\lim_{t\to\infty} |y(t) - x(t)| = 0$$

and if to each solution $\hat{y}(t)$ of (4.45) with $|\hat{y}(t_0)|$ sufficiently small there corresponds a solution $\hat{x}(t)$ of (4.44) such that

$$\lim_{t\to\infty} |\hat{y}(t) - \hat{x}(t)| = 0$$

Theorem 4.7 Let A be a constant matrix such that all solutions of

$$x'=Ax$$

are bounded on $0 \leq t < \infty$. Let B(t) be a continuous matrix such that

$$\int_0^\infty |B(s)|ds < \infty$$

Then (4.48) and the system

$$y'=(A+B(t))y$$

are asymptotically equivalent.

Theorem 4.8 Let A be a real constant matrix satisfying the hypotheses of Theorem 4.7. Let g, $\partial g/\partial y_j$ (j =1,...,n) be continous for $0 \leq t < \infty$, $|y| < \infty$ for some $k > 0$ suppose

$$|g(t, y)| \leq \lambda(t)|y|$$

for $0 \leq t < \infty, |y| < k$ where $\int_0^\infty \lambda(t)dt < \infty$. Then (4.48) and (4.54) are asymptotically equivalent.

Chapter 4.7 Stability of Periodic Solutions

Theorem 4.9 Let g and $\partial g/\partial z_j$ (j=1,...,n) be periodic in t of period ω and continuous in (t,z) for $|z| < k_1$ ($k_1 > 0$ a constant). Let (4.58) be satisfied. Let A(t) be a continuous n-by-n periodic matrix of period ω in t. Let the multipliers $\lambda_1, \lambda_2, ..., \lambda_n$ (counting multiplicities) of the linear system x'=A(t)x have magnitude $|\lambda_k| < 1$ (k = 1,...,n). Then the zero solution of (4.59) is asymptotically stable.