Math 207A Ordinary Differential Equations: Ch.6 Some Applications

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Chapter 6.2 The Undamped Oscillator

Theorem 6.1 Let (6.2) and (6.6) be satisfied. Then there exists a neighborhood N of the origin in the phase plane such that if $(\backslash_1, \backslash_2)$ is in N, then the solution of (6.3) through $(\backslash_1, \backslash_2)$ is periodic. Let $T(A) > 0$ be the least period of all periodic solutions that generate the solution curve through the point (A, 0) in N, with $A > 0$ and sufficiently small. Then

$$T(A) = 2\sqrt{2}\int_0^A \frac{d\sigma}{[G(A)-G(\sigma)]^{1/2}}$$

where $G(u) = \int_0^u g(\sigma)d\sigma$. Moreover,

$$T(A) = \frac{2\sqrt{2}}{[G(A)]^{1/2}} - \sqrt{2}\int_0^A \frac{[g(A)-g(\sigma)]}{[G(A)-G(\sigma)]^{3/2}}d\sigma$$

Corollary 1 If the hypotheses of Theorem 6.1 are satisfied and if in addition g(x) is monotone increasing in some neighborhood of x = 0 [for example, if $g'(0) > 0$], then we also have

$$T'(A) = -\frac{2\sqrt{2}g(A)}{G(A)}\int_0^A [\frac{G(\sigma)g'(\sigma)}{[g(\sigma)]^2} - \frac{1}{2}]\frac{d\sigma}{[G(A)-G(\sigma)]^{1/2}}$$

Corollary 2 Let the hypothesis of Corollary 1 be satisfied and assume in addition that g"(x) is continuous. If $g"(x) \geq 0$ on $0 \leq x \leq A$ and if g"(x) is not identically equal to zero, then $T'(A) < 0$. If $g"(x) \leq 0$ on $0 \leq x \leq A$ and if g"(x) is not identically zero, then $T'(A) > 0$.

Chapter 6.3 The Pendulum

Chapter 6.4 Self-Excited Oscillations-Periodic Solutions of the Lienard Equation

Theorem 6.2 Suppose
(i) $ug(u) > 0 \qquad (u \neq 0)$
(ii) $\lim_{|u|\to\infty} |F(u)| = +\infty$
and that for some $a, b > 0$
(iii) $F(u) < 0 \qquad (u < -a, 0 < u < b)$
$F(u) > 0 \qquad (-a < u < 0, u > b)$
Then (6.23) has a nontrivial periodic solution.

Theorem 6.3 Suppose
(i) $ug(u) > 0 \qquad (u \neq 0)$

(ii) $g(u) = -g(-u) \qquad f(u) = f(-u)$
and that for some $b > 0$
(iii) $F(u) < 0, \qquad (0 < u < b)$
$F(u) > 0, \qquad (u > b)$
(iv) F(u) is monotone increasing for $u > b$ and $\lim_{u\to\infty} F(u) = \infty$
Then the equation
$$u" + f(u)u' + g(u) = 0$$
has a unique nontrivial periodic solution p(t)

Chapter 6.5 The Regulator Problem

Definition The real symmetric n-by-n matrix B is said to be positive definite if and only if the quadratic form $y^T By$ is positive definite.

Sylvester's Theorem

Lyapunov's Theorem on Matrices Let A be a given constant stable matrix and let C be a given symmetric positive definite matrix. Then there exists a symmetric positive definite matrix B such that
$$A^T B + BA = -C$$

Corollary If $p > d^T C^{-1} d$ then $p \neq c^T A^{-1} B$

Chapter 6.6 Absolute Stability of the Regulator System

Theorem 6.4 Let A be a given stability matrix. Let C be any positive definite symmetric matrix and define B to be the positive definite symmetric matrix such that
$$A^T B + BA = -C$$
Define $d = Bb - \frac{1}{2}c$. Then if $p \neq c^T A^{-1} b$ and if $p > d^T C^{-1} d$, the system (6.33), is absolutely stable for all admissible characteristic functions. (If the system (6.37) is studied independent of the original system (6.33), the condition $p \neq c^T A^{-1} b$ is not needed).