Math 207A Ordinary Differential Equations: Ch.5 Lyapunov's Theorems and Methods

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Lyapunov's Theorems

Lemma 5.1 If \ is any point of D that is not a critical point of (5.6), then through the point \ there passes at most one orbit of (5.6)

Lemma 5.2 If any orbit C of (5.6) passes through some ordinary point of D, then C cannot reach any critical point α in D in finite time. (More precisely, if C is generated by a solution ϕ and if $\lim_{t \to a} \phi(t) = \alpha, \alpha$ in D, then $a = + - \infty$).

Lemma 5.3 An orbit C of (5.6) that passes through at least one ordinary point of D cannot cross itself, unless, it is a closed curve in D. In this case, C corresponds to a periodic solution of (5.6)

Definition 1 The scalar function V(y) is said to be positive definite on the set Ω if and only if V(0) = 0 and $V(y) > 0$ for $y \neq 0$ and y in Ω.

Definition 2 The scalar function V(y) is negative definite on the set Ω if and only if -V(y) is positive definite on Ω.

Definition 3 The derivative of V with respect to the system y'=f(y) is the scalar product

$$V*(y) = gradV(y) \cdot f(y) = \frac{\partial V}{\partial y_1}(y) f_1(y) + \frac{\partial V}{\partial y_2}(y) f_2(y) + \cdots + \frac{\partial V}{\partial y_n}(y) f_n(y)$$

Theorem 5.1 If there exists a scalar function V(y) that is positive definite and for which $V*(y) \leq 0$ (that is, the derivative (5.7) with respect to y'=f(y) is nonpositive) on some region Ω containing the origin, then the zero solution of y'=f(y) is stable.

Theorem 5.2 If there exists a scalar function V(y) that is positive definite and for which V*(y) is negative definite on some region Ω containing the origin, then the zero solution of y'=f(y) is asymptotically stable.

Theorem 5.3 If there exists a scalar function V(y), V(0) = 0, such that V*(y) is either positive definite or negative definite on some region Ω containing the origin and if there exists in every neighborhood N of the origin, $N \subset \Omega$ at least one point $a \neq 0$ such that V(a) has the same sign as V*(y), then the zero solution of y'=f(y) is unstable.

Theorem 5.4 If there exists a scalar function V such that in a region Ω containing the origin,

$$V* = \lambda V + W$$

where $\lambda > 0$ is a constant and W is either identically zero or W is a nonnegative or a nonpositive function such that in every neighborhood N of the origin, $N \subset \Omega$, there is at least one point a such that $V(a) \cdot W(a) > 0$, then the zero solution of y'=f(y) is unstable.

Chapter 5.3 Proofs of Lyapunov's Theorems

Chapter 5.4 Invariant Sets and Stability

Definition 1 A set Γ of points in E_n is (positively) invariant with respect to the system (5.6) if every solution of (5.6) starting in Γ remains in Γ for all furture time*.

Definition 2 A point p in E_n is said to lie in the positive limit set L(C*) (or is said to be a limit point of the orbit C*) of the solution $\phi(t)$ if and only if there exists a sequence $\{t_n\} \to +\infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \phi(t_n) = p$.

Lemma 5.4 If the solution $\phi(t, y_0)$ is bounded for $0 \leq t < \infty$ (that is, if there exists a constant M such that $||\phi(t, y_0)|| \leq< M$ for $0 \leq t < \infty$) then its positive limit set $L(C^+)$ is a nonempty invariant set (with respect to (5.6)). Moreover, the solution $\phi(t, y_0)$ approaches the set $L(C^+)$ as $t \to +\infty$ (in the sense that for each $\epsilon > 0$ there exists a $T > 0$ such that for every $t > T$ there exists a point p in $L(C^+)$ (possibly depending on t) such that $||\phi(t, y_0) - p|| < \epsilon$; that is, for t sufficiently large the semiorbit of the solution $\phi(t, y_0)$ lies arbitrarily close to points of $L(C^+)$).

Lemma 5.5 Let V be continuously differentiable in a set Ω containing the origin and let $V * (y) \leq 0$ at all points of Ω. Let $y_0 \in \Omega$ and let $\phi(t, y_0)$ be a bounded solution of (5.6) whose positive semiorbit C^+ lies in Ω for all $t \geq 0$ and let the positive limit set $L(C^+)$ of $\phi(t, y_0)$ lie in Ω. Then V*(y) = 0 at all points of $L(C^+)$.

Theorem 5.5 Let V(y) be a nonnegative scalar function defined on some set $\Omega \subset R_n$ containing the origin. Let V be continuously differentiable on Ω, let $V * (y) \leq 0$ at all points of Ω, and let V(0) = 0. For some real constant $\lambda \geq 0$ let C_λ be the component of the set $S_\lambda = \{y | V(y) \leq \lambda\}$ which contains the origin. Suppose that C_λ is a closed bounded subset of Ω. Let E be the subset of Ω defined by $E = \{y | V * (y) = 0\}$. Let M be the largest positively invariant subset of E (with respect to (5.6)). Then every solution of (5.6) starting in C_λ at t = 0 approaches the set M as $t \to +\infty$

Chapter 5.5 The Extent of Asymptotic Stability- Global Asymptotic Stability

Theorem 5.6 Let there exist a scalar function V(y) such that:
(i) V(y) is positive definite on E_n and $V(y) \to \infty$ as $||y|| \to \infty$;
(ii) with respect to (5.6), $V * (y) \leq 0$ on R_n;
(iii) the origin is the only invariant subset of the set $E = \{y | V * (y) = 0\}$
Then the zero solution of v'=f(y) is gloablly asmptotically stable.

Corollary 1 Let there exist a scalar function V(y) that satisfies (i) above and that has V*(y) negative definite. Then the zero solution of y'=f(y) is globally asymptotically stable.

Corollary 2 If only (i) and (ii) of Theorem 5.6 are satisfied, then all solutions of y'=f(y) are bounded for $t \geq 0$ (that is, (5.6) is Lagrange stable).

Chapter 5.6 Nonautonomous Systems

Definition 1 The scalar function V(t,y) is said to be positive definite on the set Ω if and only if V(t,0) = 0 and there exists a scalar function W(y) independent of t, with $V(t,y) \geq W(y)$ for (t,y) in Ω and such that W(y) is positive definite in the sense of Definition 1 (Section 5.2).

Definition 2 The scalar function V(t,y) is negative definite on Ω if and only if -V(t,y) is positive on Ω

Definition 3 The derivative of V(t,y) with respect to the system y'=f(t,y) is

$$V*(t,y) = \frac{\partial V}{\partial t}(t,y) + \sum_{j=1}^{n} \frac{\partial V}{\partial y_j}(t,y) f_j(t,y)$$

If ϕ is any solution of (5.41), we have

$$\frac{d}{dt} V(t, \phi(t)) = V*(t, \phi(t))$$

Theorem 5.7 If there exists a scalar function V(t,y) that is positive definite and for which $V*(t,y) \leq 0$ (that is, the derivative (5.42) with respect to the system (5.41) is nonpositive) on some region Ω that contains the set $H = \{(t,0) | t \geq 0\}$, then the zero solution of y'=f(t,y) is stable.

Definition 4 A scalar function U(t,y) is said to satisfy an infinitesimal upper bound if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|U(t,y)| < \epsilon \qquad \text{on } \{(t,y) | t \geq 0, |y| \leq \delta\}$$

Theorem 5.8 If there exists a scalar function V(t,y) that is positive definite, satisfies an infinitesimal upper bound, and for which V*(t,y) is negative definite, then the zero solution y'=f(t,y) is asymptotically stable.

Definition The system
y'=f(t,y)
is said to be asymptotically autonomous on the set Ω if and only if (a) $\lim_{t\to\infty} f(t,y) = h(y)$ for $y \in \Omega$ and this convergence is uniform for y in closed bounded subsets of Ω (b) For every $\epsilon > 0$ and every $y \in \Omega$ there exists a $\delta(\epsilon, y) > 0$ such that $|f(t,x) - f(t,y)| < \epsilon$, whenever $|x - y| < \delta$ for $0 \leq t < \infty$.

Theorem 5.9 (See p.214 and compare with theorem 5.5) Suppose the system (5.41) is asymptotically autonomous on some set Ω in y-space. Suppose f(t,y) is bounded for $0 \leq t < \infty$ whenever y lies in a closed bounded set $Q = \{y | |y| \leq K, K > 0\}$. Suppose there exists a nonnegative scalar function V(t,y) such that $V*(t,y) \leq -W(y)$ where $W(y) \geq 0$ with W(y) = 0 only for $y \in \Omega$ (this defines Ω, that is $\Omega = \{y | W(y) = 0\}$). Let M be the largest positively invariant subset of Ω with respect to the limiting autonomous system
y'= h(y)
Then every bounded solution of (5.41) approaches M as $t \to \infty$. In particular if all solutions of (5.41) are bounded, then every solution of (5.41) approaches M.