Math 207B Partial Differential Equations: Ch 3: Nonlinear First-Order PDE

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Definition A C^2 function u = u(x;a) is called a complete integral in U x A provided
(i) u(x; a) solves the PDE (1) for each $a \in A$
and
(ii) $\text{rank}(D_a u, D^2_{xa}u) = n \qquad (x \in U, a \in A).$

Definition Let u = u(x;a) be a C^1 function of $x \in U, a \in A$, where $U \subset R^n$ and $A \subset R^m$ are open sets. Consider the vector equation

$$D_a u(x; a) = 0 \qquad (x \in U, a \in A)$$

Suppose that we can solve (10) for the paramter a as a C^1 function of x,

$$a = \phi(x)$$

thus

$$D_a u(x; \phi(x)) = 0 \qquad (x \in U)$$

We then call

$$v(x) := u(x; \phi(x)) = 0 \qquad (x \in U)$$

the envelope of the functions $\{u(\cdot; a)\}_{a \in A}$

Theorem 1 (Construction of new solutions) Suppose for each $a \in A$ as above that $u = u(\cdot; a)$ solves the partial differential equation (1). Assume further that the envelope v, defined by (12) and (13) above, exists and is a C^1 function. Then v solves (1) as well.
The envelope v defined above is sometimes called a singular integral of (1).

Definition The general integral (depending on h) is the envelope v'=v'(x) of the functions

$$u'(x; a') = u(x; a', h(a')) \qquad (x \in U, a' \in A')$$

provided this envelope exists and is C^1

Chapter 3.2 Characteristics

Theorem 1 (Structure of characteristic ODE) Let $u \in C^2(U)$ solve the nonlinear, first-order partial differential equation (1) in U. Assume $x(\cdot)$ solves the ODE (11)(c), where $p(\cdot) = Du(x(\cdots)), z(\cdots) = u(x(\cdot))$. Then $p(\cdot)$ solves the ODE (11)(a) and $z(\cdot)$ solves the ODE (11)(b), for those s such that $x(s) \in U$.

Theorem 2 (Local Existence Theorem) The function u defined above is C^2 and solves the PDE

$$F(Du(x), u(x), x) = 0 \qquad (x \in V)$$

with the boundary condition

$$u(x) = g(x) \qquad (x \in \Gamma \cap V)$$

Chapter 3.3 Introduction to Hamilton-Jacobi Equations

Theorem 1 (Euler-Lagrange equations) The function $x(\cdot)$ solves the system of Euler-Lagrange equations.

$$-\tfrac{d}{ds}(D_v L(\dot{x}, x(s))) + D_x L(\dot{x}(s), x(s)) = 0 \qquad (0 \leq s \leq t)$$

This is a vector equation, consisting of n coupled second-order equations.

Definition The Hamiltonian H associated with the Lagrangian L is

$$H(p, x) := p \cdot v(p, x) - L(v(p, x), x) \qquad (p, x \in R^n)$$

where the function $v(\cdot)$ is defined implicitly by (9).

Theorem 2 (Derivation of Hamilton's ODE) The functions $x(\cdot)$ and $p(\cdot)$ satisfy Hamilton's equations:

$$\{ \ \dot{x}(s) = D_p H(p(s), x(s))$$
$$\{ \ \dot{p}(s) = -D_x H(p(s), x(s))$$

for $0 \leq s \leq t$ Furthermore, the mapping $s \mapsto H(p(s), x(s))$ is constant.

Definition The Legendre transform of L is

$$L * (p) := sup_{q \in R^n} \{p \cdot v - L(v)\} \qquad (p \in R^n)$$

This is also referred to as the Fenchel transform.

Theorem 3 (Convex duality of Hamiltonian and Lagrangian) Assume L satisfies (11), (12) and define H by (13), (14)

(i) Then

$$\text{the mapping } p \mapsto H(p) \text{ is convex}$$

and

$$\lim_{|p| \to \infty} \tfrac{H(p)}{|p|} = +\infty$$

Thus H is the Legendre transform of L and vice versa

$$L = H*, H = L*$$

Theorem 4 (Hopf-Lax formula) If $x \in R^n$ and $t > 0$, then the solution u = u(x,t) of the minimization problem (17) is

$$u(x,t) = min_{y \in R^n}\{tL(\frac{x-y}{t}) + g(y)\}$$

Definition We call the expression on the right hand side of (21: the last theorem 4) the Hopf-Lax formula.

Theorem 5 (Solving the Hamilton-Jacobi equation) Suppose $x \in R^n, t > 0$ and u defined by the Hopf-Lax formula (21) is differnetiable at a point $(x,t) \in R^n x(0,\infty)$. Then

$$u_t(x,t) + H(Du(x,t)) = 0$$

Theorem 6 (Hopf-Lax formula as solution) The function u defined by the Hopf-Lax formula (21) is Lipschitz continuous, is differentiable a.e. in $R^n x(0,\infty)$ and solves the initial-value problem

$$\{\ u_t + H(Du) = 0 \qquad \text{a.e. in } R^n x(0,\infty)$$
$$\{\qquad \text{u=g} \qquad \text{on } R^n x\{t=0\}$$

Definition A C^2 convex function $H : R^n \to R$ is called uniformly convex (with constant $\theta > 0$) if

$$\sum_{i,j=1}^n H_{p_i p_j}(p)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \text{for all } p,\xi \in R^n$$

Theorem 7 (Uniqueness of weak solutions) Assume H is C^2 and satisfies (19) and g satisfies (20). Then there exists at most one weak solution of the initial-value problem (38).

Theorem 8 (Hopf-Lax formula as weak solution) Suppose H is C^2 and satisfies (19) and g satisfies (20). If either g is semiconcave or H is uniformly convex, then

$$u(x,t) = min_{y \in R^n}\{tL(\frac{x-y}{t}) + g(y)\}$$

is the unique weak solution of the initial-value problem (38) for the Hamilton-Jacobi equation.

Chapter 3.4 Introduction to conservation laws

Definition We say that $u \in L^\infty(Rx(0,\infty))$ is an integral solution of (1), provided equality (4) holds for each test function v satsifying (2).

Theorem 1 (Lax-Oleinik formula) Assume $F : R \to R$ is smooth and uniformly convex and $g \in L^\infty(R)$

(i) For each time $t > 0$, there exists for all but at most countably many values of $x \in R$ a unique poitn y(x,t) such that

$$min_{y \in R}\{tL(\frac{x-y}{t}) + h(y)\} = tL(\frac{x-y(x,t)}{t}) + h(y(x,t)).$$

(ii) The mapping $x \mapsto y(x,t)$ is nondecreasing

(iii) For each time $t > 0$, the function u defined on (27) is

$$u(x,t) = G(\frac{x-y(x,t)}{t})$$

for a.e. x. In particular, formula (29) holds for a.e. $(x, t) \in Rx(0, \infty)$

Definition We call equation (29) the Lax-Oleinik formula for the solution (1) where h is defined by (23) and L by (25)

Theorem 2 (Lax-Oleinik formula as integral solution) Under the assumptions of Theorem 1, the function u defined by (29) is an integrla solution of the initial-value problem (1).

Definition We call inequality (36) the entropy condition.

Definition We say that a function $u \in L^{\infty}(Rx(0, \infty))$ is an entropy solution of the initial-value problem

$$\{u_t + F(u)_x = 0 \qquad \text{in } Rx(0, \infty)$$
$$\{ \qquad \text{u = g} \qquad \text{on } Rx\{t = 0\}$$

provided

$$\int_0^{\infty} \int_{-\infty}^{\infty} uv_t + F(u)v_x dxdt + \int_{-\infty}^{\infty} gvdx|_{t=0} = 0$$

for all test functions $v : Rx[0, \infty) \to R$ with compact support and

$$u(x + z, t) - u(x, t) \leq C(1 + \tfrac{1}{t})z$$

for some constant $C \geq 0$ and a.e. $x, z \in R, t > 0$ with $z > 0$

Theorem 3 (Uniqueness of entropy solutions) Assume F is convex and smooth. Then there exists-up to a set of measure zero-at most one entropy solution of (37).

Theorem 4 (Solution of Riemann's problem)

(i) If $u_l > u_r$, the unique entropy solution of the Riemann problem (1), (53) is

$$u(x, t) := \{u_l \qquad \text{if } \tfrac{x}{t} < \sigma$$
$$\{u_r \qquad \text{if } \tfrac{x}{t} > \sigma \qquad (x \in R, t > 0),$$

where

$$\sigma := \tfrac{F(u_l) - F(u_r)}{u_l - u_r}$$

(ii) If $u_l < u_r$, the unique entropy solution of the Riemann problem (1) (53) is

$$u(x, t) := \{ \qquad u_l \qquad \text{if } \tfrac{x}{t} < F'(u_l)$$
$$\{G(\tfrac{x}{t} \qquad \text{if } F'(u_l) < \tfrac{x}{t} < F'(u_r) \qquad (x \in R, t > 0)$$
$$\{ \qquad u_r \qquad \text{if } \tfrac{x}{t} > F'(u_r)$$

Theorem 5 (Asymptotics in L^{∞}-norm) There exists a constant C such that

$$|u(x, t)| \leq \tfrac{C}{t^{1/2}}$$

for all $x \in R, t > 0$

Theorem 6 (Asymptotics in L^1-norm). Assume that $p, q > 0$. Then there exists a constant C such that

$$\int_{-\infty}^{\infty} |u(\cdot, t) - N(\cdot, t)| dx \leq \tfrac{C}{t^{1/2}}$$

for all $t > 0$