Math 207B Partial Differential Equations: Ch 2: Equations with Explicit solutions

Charlie Seager

April 11, 2024

Part 1 Representation Formulas for Solutions: Ch 2: Equations with Explicit solutions: Four Important Linear PDE: Transport equation, Laplaces equation, heat equation, and wave equation

Here are the 4 equations

the transport equation	$u_t + b \cdot Du = 0$ (2.1)
laplaces equation	$\triangle u = 0$ (2.2)
the heat equation	$u_t - \triangle u = 0$ (2.3)
the wave equation	$u_{tt} - \triangle u = 0$ (2.4)

Chapter 2.1 Transport Equation

Chapter 2.2 Laplaces Equation

Among the most inportant of all partial differential equations are undoubtedly Laplace's equation $\triangle u = 0$
and Poisson's equation $-\triangle u = f$

Definition A C^2 function u satisfying (1) is called a harmonic function.

Physical interpretation Laplace's equation comes up in a wide variety of physical contexts. In a typical intepretation y denotes the density of some quality (e.g. a chemical concentration) in equilibrium. Then if V is any smooth subregion within U, the net flex of u through ∂V is zero

$$\int_{\partial V} F \cdot v ds = 0$$

F denoting the flux density and v the unit outer normal field. In view of the Gauss-Gree Theorem (C.2), we have

$$\int_V div F dx = \int_{\partial V} F \cdot v ds = 0$$

and so

$$\text{div F} = 0 \qquad \text{in U}$$

Definition The function

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} log|x| & \text{(n=2)} \\ \frac{2}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & (n \geq 3) \end{cases}$$

defined for $x \in R^n, x \neq 0$ is the fundamental solution of Laplace's equation.

Theorem 1 (Solving Poisson's equation) Define u by (8). Then

$$\text{(i) } u \in C^2(R^n)$$

and

$$\text{(ii) } -\triangle u = f \text{ in } R^n$$

Intepretation of Fundamental Solution We sometimes write

$$-\triangle \Phi = \delta_0 \text{ in } R^n$$

Theorem 2 (Mean-value formulas for Laplace's equation). If $u \in C^2(U)$ is harmonic, then

$$(16) \qquad u(x) = ⨍_{\partial B(x,r)} uds = ⨍_{B(x,r)} udy$$

for each ball $B(x,r) \subset U$

Theorem 3 (Converse to mean-value property) If $u \in C^2(U)$ satisfies

$$u(x) = ⨍_{\partial B(x,r)} uds$$

for each ball $B(x,r) \subset U$, then u is harmonic.

Theorem 4 (Strong Maximum Principle) Suppose $u \in C^2(U) \cap C(\hat{U})$ is harmonic within U

(i) Then

$$max_U u = max_{\partial U} U$$

(ii) Furthermore, if U is connected and there exists a point $x_0 \in U$ such that

$$u(x_0) = \max_U u$$

then

u is contant within U.

Theorem 5 (Uniqueness) Let $g \in C(\partial U), f \in C(U)$. Then there exists at most one solution $u \in C^2(U) \cap C(\hat{U})$ of the boundary-value problem

$$(17) \qquad \begin{array}{l} \{-\triangle u = f \text{ in U} \\ \{u = g \text{ on } \partial U \end{array}$$

Theorem 6 (Smoothness) If $u \in C(U)$ satisfies the mean-value property (16) for each ball $B(x,r) \subset U$ then

$$u \in C^\infty(U)$$

Note carefully, that u may not be smooth, or even continous, up to ∂U

Theorem 7 (Estimates on derivatives) Assume u is harmonic in U, Then

$$(18)\ |D^\alpha U(x_0)| \leq \frac{C_k}{r^{n+k}}||u||_{L^1(B(x_0,r))}$$

for each ball $B(x_0,r) \subset U$ and each multiindex α of order $|\alpha| = k$
Here

$$(19)\ C_0 = \frac{1}{\alpha(n)}, C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)}(k = 1, ...)$$

Liouville's Theorem We assert now that there are no nontrivial bounded harmonic function on all of R^n.

Theorem 8 (Liouvill's Theorem Suppose $u : R^n \to R$ is harmonic and bounded. Then u is constant.

Theorem 9 (Representation formula) Let $f \in C_c^2(R^n), n \geq 3$ Then any bounded solution of

$$-\triangle u = f \qquad \text{in } R^n$$

has the form

$$u(x) = \int_{R^n} \Phi(x-y)f(y)dy + C \qquad (x \in R^n)$$

for some constant C.

Theorem 10 (Analyticity) Assume u is harmonic in U. Then u is analytic in U.

Theorem 11 (Harnack's inequality) For each connected open set V $\subset\subset$ U, there exists a positive constant C, depending only on V, such that

$$sup_V u \leq Cinf_V u$$

for all nonnegative harmonic function u in U.

Definition Green's function for the region U is

$$G(x,y) := \Phi(y-x) - \phi^x(y) \qquad (x,y \in U, x \neq y)$$

Theorem 12 (Representation formula using Green's function) If $u \in C^2(\bar{U})$ solves problem (29), then

$$(30) \qquad u(x) = -\int_{\partial U} g(y)\frac{\partial G}{\partial v}(x,y)dS(y) + \int_U f(y)G(x,y)dy \qquad (x \in U)$$

Theorem 13 (Symmetry of Green's function) For all $x, y \in U, x \neq y$ we have

$$\text{G(y,x)} = \text{G(x,y)}$$

Definition If $x = (x_1, .., x_{n-1}, x_n) \in R^n_+$, its reflection in the plane ∂R^n_+ is the point

$$\bar{x} = (x_1, \ldots, x_{n-1}, -x_n)$$

Definition Green's function for the half space R^n_+ is

$$G(x,y) := \Phi(y-x) - \Phi(y-\hat{x}) \qquad (x,y \in R^n_+, x \neq y)$$

Then

$$G_{y_n}(x,y) = \Phi_{y_n}(y-x) - \Phi_{y_n}(y-\tilde{x}) = \frac{-1}{n\alpha(n)}[\frac{y_n-x_n}{[y-x]^n} - \frac{y_n+x_n}{|y-\tilde{x}|^n}]$$

Theorem 14 (Poisson's formula for half-space) Assume $g \in C(R^{n-1}) \cap L^{\infty}(R^{n-1})$, and define u by (33). Then

$$\text{(i) } u \in C^{\infty}(R^n_+) \cap L^{\infty}(R^n_+)$$
$$\text{(ii) } \triangle u = 0 \text{ in } R^n_+$$

and

$$\text{(iii) } \lim x \to x^0_{n x \in R^n_+} u(x) = g(x^0) \text{ for each point } x^0 \in \partial R^n_+$$

Definition If $x \in R^n - \{0\}$, the point

$$\tilde{x} = \frac{x}{|x|^2}$$

is called the point dual to x with respect to ∂B(0,1). The mapping $x \mapsto \tilde{x}$ is inversion through the unit sphere $\partial B(0,1)$

Definition Green's function for the unit ball is

$$G(x,y) := \Phi(y-x) - \Phi(|x|(y-\tilde{x})) \qquad (x, y \in B(0,1), x \neq y) \tag{41}$$

Theorem 15 (Poisson's formula for ball) Assume $g \in C(\partial B(0,r))$ and define u by (45). Then

$$\text{(i) } u \in C^{\infty}(B^0(0,r))$$
$$\text{(ii) } \triangle u = 0 \text{ in } B^0(0,r)$$

and

$$(iii) \lim x \to x^0{}_{x \in B^0(0,r)} u(x) = g(x^0) \text{ for each point } x^0 \in \partial B(0,r)$$

Theorem 16 (Uniqueness) There exists at most one solution $u \in C^2(\tilde{U})$ of (46).

Theorem 17 (Dirichlet's principle) Assume $u \in C^2(\tilde{U})$ solves (46) Then

$$f|u| = min_{\omega \in \mathcal{A}} I|\omega|$$

conversely, if $u \in \mathcal{A}$ satisfies (47), then u solves the boundary-value problem (46).

Chapter 2.3 Heat equation Physical intepretation: The heat equation, also known as the diffusion equation describes in typical applications the evolution in time of the density u of some quantity such as heat, chemical concentration, etc.

Definition The function

$$\Phi(x,t) := \{ \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} (x \in R^n, t > 0)$$
$$\{ \; 0 \qquad (x \in R^n, t < 0)$$

is called the fundamental solution of the heat equation.

Lemma (Integral of fundamental solution) For each time $t > 0$

$$\int_{R^n} \Phi(x,t)dx = 1$$

Theorem 1 (Solution of initial-value problem) Assume $g \in C(R^n) \cap L^\infty(R^n)$ and define u by (9). Then

$$\text{(i) } u \in C^\infty(R^n x(0,\infty))$$
$$\text{(ii) } u_t(x,t) - \triangle u(x,t) = 0 (x \in R^n, t > 0)$$

and

$$\text{(iii) } lim_{(x,t)\to(x^0,0)_{x\in R^n, t>0}} u(x,t) = g(x^0) \text{ for each point } x^0 \in R^n$$

Theorem 2 (Solution of nonhomogeneous problem) Define u by (13). Then
(i) $\quad u \in C_1^2(R^n x(0,\infty))$
(ii) $\quad u_t(x,t) - \triangle u(x,t) = f(x,t) \qquad (x \in R^n, t > 0)$
and
(iii) $\lim_{(x,t)\to(x^0,0)} x \in R^n, t > 0 u(x,t) = 0$ for each point $x^0 \in R^n$

Definitions

(i) We define the parabolic cylinder

$$U_T := Ux(0,T].$$

(ii) The parabolic boundary of U_T is

$$\Gamma_T := \tilde{U}_T - U_T$$

Definition For fixed $x \in R^n, t \in R, r > 0$, we define

$$E(x,t;r) := \{(y,s) \in R^{n+1} | s \le t, \Phi(x-y,t-s) \ge \tfrac{1}{r^n}\}$$

Theorem 3 (A mean-value property for the heat equation) Let $u \in C_1^2(U_T)$ solve the heat equation. Then

$$u(x,t) = \frac{1}{4r^n} \int\int_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds \tag{19}$$

for each $E(x,t;r) \subset U_T$.

Theorem 4 (Strong maximum principle for the heat equation) Assume $u \in C_1^2(U_T) \cap C(\tilde{U}_T)$ solves the heat equation in U_T

(i) Then

$$max_{\tilde{U}_T} u = max_{\Gamma_T} u$$

(ii) Furthermore, if U is connected and there exists a point $(x_0, t_0) \in U_T$ such that

$$u(x_0, t_0) = max_{\tilde{U}_T} u$$

then

$$\text{u is constant in } \tilde{U}_{t_0}$$

Theorem 5 (Uniqueness on bounded domains) Let $g \in C(\Gamma_T), f \in C(U_T)$. Then there exists at most one solution $u \in C_1^2(U_T) \cap C(\tilde{U}_T)$ of the initial/boundary-value problem

(22) { $u_t - \triangle u = f$ in U_T
{ u = g on Γ_T

Theorem 6 (Maximum principle for the Cauchy problem) Suppose $u \in C_1^2(R^n x(0,T]) \cap C(R^n x[0,T])$ solves

(23) { $u_t - \triangle u = 0$ in $R^n x(0,T)$
{ u = g on $R^n x\{t=0\}$

and satisfies the growth estimate

$$u(x,t) \leq Ae^{a|x|^2} \qquad (x \in R^n, 0 \leq t \leq T) \tag{24}$$

for constants $A, a > 0$. Then

$$sup_{R^n x[0,T]} u = sup_{R^n} g$$

Theorem 7 (Uniqueness for cauchy problem) Let $g \in C(R^n), f \in C(R^n x[0,T])$. Then there exists at most one solution $u \in C_1^2(0,T] \cap C(R^n x[0,T])$ of the initial value problem

$\{u_t - \triangle u = f$ in $R^n x(0,T)\{$ $u = g$ in $R^n x\{t=0\}$

satisfying the growth estimate

$$|u(x,t)| \leq Ae^{a|x|^2} \qquad (x \in R^n, 0 \leq t \leq T)$$

for constants $A, a > 0$

Theorem 8 (Smoothness). Suppose $u \in C_1^2(U_T)$ solves the heat equation in U_T. Then

$$u \in C^\infty(U_T)$$

Theorem 9 (Estimates on derivatives) There exists for each pair of integers k, l = 0,1,... a constant $C_{k,l}$ such that

$$maxC(x,t;r/2)|D_x^k D_t^l u| \leq \tfrac{C_{kl}}{r^{k+2l+n+2}}||u||_{L^1(C(x,t;r))}$$

for all cylinders $C(x,t;r/2) \subset C(x,t;r) \subset U_T$ and all solutions u of the heat equaiton in U_T

Theorem 10 (Uniqueness) There exists only one solution $u \in C_1^2(\tilde{U}_T)$ of the initial/boundary-value problem.

Theorem 11 (Backwards uniqueness) Suppose $u, \tilde{u} \in C^2(\tilde{U}_T)$ solve (42), (43). If

$$u(x,T) = \tilde{u}(x,T) \qquad (x \in U)$$

then

$$u = \tilde{u} \text{ within } U_T$$

Chapter 2.4 Wave Equation: Physical interpretation: The wave equaiton is a simplified model for a vibrating string (n=1), membrane (n=2), or elastic solid (n=3). in these physical interpretations u(x,t) represents the displacement in some direction of the point x at time $t \geq 0$

Theorem 1 (Solution of wave equation, n = 1) Assume $g \in C^2(R), h \in C^1(R)$ and define u by d'Alembert's formula (8). Then

(i) $u \in C^2(Rx[0, \infty))$

(ii) $u_{tt} - u_{xx} = 0$ in $Rx(0, \infty)$,

and

(iii) $lim_{(x,t) \to (x^0,0)_{t>0}} u(x,t) = g(x^0), lim_{(x,t)\to(x^0,0)_{t>0}} u_t(x,t) = h(x^0)$ for each point $x^0 \in R$

Theorem 2 (Solution of wave equation in odd dimension) Assume n is an odd integer, $n \geq 3$ and suppose also $g \in C^{m+1}(R^n), h \in C^m(R^n)$ for $m = \frac{n+1}{2}$. Define u by (31). Then

(i) $u \in C^2(R^n x[0, \infty))$,

(ii) $u_{tt} - \triangle u = 0$ in $R^n x(0, \infty)$

and

(iii) $lim_{(x,t)\to(x^0,0)_{x \in R^n, t>0}} u(x,t) = g(x^0), lim_{(x,t)\to(x^0,0)_{x\in R^n,t>0}} u_t(x,t) = h(x^0)$

Theorem 3 (Solution of wave equation in even dimensions) Assume n is an even integer, $n \geq 2$, and suppose also $g \in C^{m+1}(R^n), h \in C^m(R^n)$ for $m = \frac{n+1}{2}$. Define u by (38). Then

(i) $u \in C^2(R^n x[0, \infty))$

(ii) $u_{tt} - \triangle u = 0$ in $R^n x(0, \infty)$

and

(iii) $lim_{(x,t)\to(x^0,0)_{x\in R^n,t>0}} u(x,t) = g(x^0) lim_{(x,t)\to(x^0,0)_{x\in R^n,t>0}} u_t(x,t) = h(x^0)$

Theorem 4 (Solution of nonhomogeneous wave equation) Assume that $n \geq 2$ and $f \in C^{|n/2|+1}(R^n x[0, \infty))$ Define u by (41). Then

(i) $u \in C^2(R^n x[0, \infty))$,

(ii) $u_{tt} - \triangle u = f$ in $R^n x(0, \infty)$

and

(iii) $lim_{(x,t)\to(x^0,0)_{x\in R^n,t>0}} u(x,t) = 0 \qquad lim_{(x,t)\to(x^0,0)_{x\in R^n,t>0}} u_t(x,t) = 0$ for each point $x^0 \in R^n$

Theorem 5 (Uniqueness for wave equation). There exists at most one function $u \in C^2(\tilde{U}_T)$ solving (45).

Theorem 6 (Finite propogation speed). If $u = u_t = 0$ on $B(x_0, t_0)x\{t = 0\}$ then u=0 within the cone $K(x_0, t_0)$