Math 207B Partial Differential Equations: Ch 4 Techniques for Solving important examples: Other ways to represent solutions

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Turing instability

Chapter 4.2 Similarity Solutions

When investigating partial differential equations, it is often profitable to look for specific solutions u, the form of which reflects various symmetries in the structure of the PDE. We have already seen this idea in our derivation of the fundamental solutions for Laplace's equation and the heat equation in §2.2.1 and §2.3.1 and our discovery of rarefaction waves for conservation laws in §3.4.4. Following are some further applications of this important method.

Chapter 4.3 Tranformation Methods

In this section we develop some of the theory for the Fourier transform $\mathcal{F}$, the Radon transform $\mathcal{R}$ and the Laplace transform $\mathcal{L}$. These provide extremely powerful tools for converting certain linear partial differential equations intoeither algebraic equations or else differential equations involving fewer variables.

Chapter 4.3.1 Fourier Transform In this section all functions are complex-valued, and - denotes the complex conjugate

Definition If $u \in L^1(R^n)$, we define its Fourier transform $\mathcal{F}u = \hat{u}$ by

$$\hat{u}(y) := \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{-ixy} u(x)dx \qquad (y \in R^n)$$

and its inverse Fourier tranform $\mathcal{F}^{-1}u = \hat{u}$ by

$$\widehat{u}(y) := \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{ixy} u(x)dx \qquad (y \in R^n)$$

THeorem 1 (Plancherel's THeorem). Assume $u \in L^1(R^n) \cap L^2(R^n)$. Then $\hat{u}, \widehat{u} \in L^2(R^n)$ and

$$||\hat{u}||_{L^2(R^n)} = ||\widehat{u}||_{L^2(R^n)} = ||u||_{L^2(R^n)}$$

Theorem 2 (Properties of Fourier Transform) Assume $u, v \in L^2(R^n)$. Then

(i) $\int_{R^n} u, \hat{v} dx = \int_{R^n}, \hat{u}, \bar{\hat{v}} dy$
(ii) $(D^\alpha u)\hat{} = (iy)^\alpha \hat{u}$ for each multiindex α such that $D^\alpha u \in L^2(R^n)$.
(iii) If $u, v \in L^1(R^n) \cap L^2(R^n)$, then $(u * v)\hat{} = (2\pi)^{n/2} \hat{u}\hat{v}$

(iv) Furthermore, $u = (\hat{u})\check{}$. Assertion (iv) is the Fourier inversion formula which represents a function u in terms of the exponential plane waves e^{ixy}, provided $\hat{u} \in L^1(R^n)$

$$u(x) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{ixy} \hat{u}(y) dy$$

Definition The Radon tranform $\mathcal{R}u = \tilde{u}$ of a function $u \in C_c^\infty(R^n)$ is

$$\tilde{u}(x, \omega) := \int \prod(s, \omega) u dS \qquad (s \in R, \omega \in S^{n-1})$$

The term on the right is the integral over the plane $\prod(s\omega)$ with respect to (n-1)-dimensional surface measure

Theorem 3 (Properties of Radon tranform) Assume $u \in C_c^\infty(R^n)$ Then

(i) $\tilde{u}(-s, -\omega) = \tilde{u}(s, \omega)$

(ii) $(D^\alpha u)\tilde{} = \omega^\alpha \frac{\partial^{|\alpha|}}{\partial s^{|\alpha|}} \tilde{u}$ for each multiindex α

(iii) $(\triangle u)\tilde{} = \omega^\alpha \frac{\partial^2}{\partial s^2} \tilde{u}$

If u = 0 in $R^n - B(0, \mathcal{R})$, then $\tilde{u}(s, \omega) = 0$ for $|s| \geq \mathcal{R}$

Theorem 4 (Radon and Fourier tranforms) Assume that $u \in C_c^\infty(R^n)$. Then

$$\bar{u}(r, \omega) := \int_R \tilde{u}(s, \omega) e^{-irs} ds = (2\pi)^{n/2} \hat{u}(r\omega) \qquad (r \in R, \omega \in S^{n-1})$$

where $\hat{u} = \mathcal{F}u$ is the Fourier tranform.

Theorem 5 (Inverting the Radon tranform)

(i) We have

$$u(x) = \frac{1}{2(2\pi)^n} \int_R \int_{S^{n-1}} \bar{u}(r, \omega) r^{n-1} e^{ir\omega \cdot x} dS dr$$

the function $\bar{u}$ defined by (31)

(ii) If n = 2k + 1 is odd, then

$$u(x) = \int_{S^{n-1}} r(x \cdot \omega, \omega) dS$$

for

$$r(s, \omega) := \frac{(-1)^k}{2(2\pi)^{2k}} \frac{\partial^{2k}}{\partial s^{2k}} \tilde{u}(s, \omega).$$

Formulas (33) and (34) provide an elegant and useful decommpositions of u into plane waves.

Definition If $u \in L^1(R_+)$, we define its Laplace tranform $\mathcal{L}u = u\#$ to be

$$u\#(s) := \int_0^\infty e^{-st} u(t) dt \qquad (s \geq 0)$$

Chapter 4.4 Converting Nonlinear into Linear PDE

Chapter 4.5 Asymptotics

Chapter 4.6 Power Series

Definition We say the surface Γ is noncharacteristic for the partial differential equation (1) provided

$$\sum_{|\alpha|=k} a_\alpha \nu^\alpha \neq 0 \qquad \text{on } \Gamma$$

for all values of the arguments of the coefficients $a_\alpha(|\alpha| = k)$

Theorem 1 (Cauchy data and Noncharacteristic surfaces) Assume that Γ is noncharacteristic for the PDE (1). Then if u is a smooth solution of (1) and u satisfies the Cauchy conditions (2), we can uniquely compute all the partial differential equations of u along Γ in terms of Γ, the function $g_0, ..., g_{k-1}$ and the coefficients $a_\alpha(|\alpha| = k), a_0$

Definition A function $f : R^n \to R$ is called (real) analytic near x_0 if there exist $r > 0$ and constants $\{f_\alpha\}$ such that

$$f(x) \sum_\alpha f_\alpha (x - x_0)^\alpha \qquad (|x - x_0| < r),$$

the sum taken over all multiindex α

Definition Let

$$f = \sum_\alpha f_\alpha x^\alpha, g = \sum_\alpha g_\alpha x^\alpha$$

be two power series. We say g majorizes f, written

$$g >> f$$

provided

$$g_\alpha \leq |f_\alpha \qquad \text{for all multiindices } \alpha$$

Theorem 2 (Cauchy-Kovalevskaya Theorem) Assume $\{B_j\}_{j=1}^{n-1}$ and c are real analytics functions. Then there exist $r > 0$ and a real analytic function

$$u = \sum_\alpha u_\alpha x^\alpha$$

solving the boundary-value problem (15)