Math 207C Partial Differential Equations: Ch 6 Second-Order Eliptic Equations

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Chapter 6.1 Definitions

Chapter 6.1.1 Elliptic equations

In this chapter we will mostly study the boundary-value problem http://users.uoa.gr/ pjioannou/nonlin/Strogatz,

$$\{Lu = f \qquad \text{in U}$$
$$\{u = 0 \qquad \text{on } \partial U$$

where U is an open, bounded subset of R^n and $u : \bar{U} \to R$ is the unknown u = u(x). Here $f : U \to R$ is given and L denotes a second order partial differential operator having either the form

$$(2) \qquad Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{xj} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u$$

or else

$$(3) \qquad Lu = -\sum_{i,j=1}^{n} a^{ij}u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u,$$

for given coefficient functions $a^{ij}, b^i, c(i, j = 1, .., n)$.
We say that the PDE Lu = f is in divergence form if L is given by (2) and is in nondivergence form provided by (3). The requirement that u = 0 on ∂U in (1) is sometimes called Dirichlet's boundary condition.

Definition We say the partial differential operator L is (uniformly) elliptic if there exists a constant $\theta > 0$ such that

$$(4) \qquad \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$$

for a.e. $x \in U$ and all $\xi \in R^n$

Definitions (i) The bilinear form B[u,v] associated with the divergence form eliptic operator L defined by (2) is

$$(8) \qquad B[u, v] := \int_U \sum_{i,j=1}^{n} a^{ij}u_{x_i}, v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuvdx$$

for $u, v \in H_0^1(U)$

(ii) We say that $u \in H_0^1(U)$ is a weak solution of the boundary value problem (1) if

$$B[u,v] = (f,v)$$

for all $v \in H_0^1(U)$ where (,) {the book just leaves the interval blank btw } denotes the inner product in $L^2(U)$

Definition We say $u \in H_0^1(U)$ is a weak solution of problem (10) provided

$$B[u,v] = \langle f,v \rangle$$

for all $v \in H_0^1(U)$ where $\langle f,v \rangle = \int_U f^0 v + \sum_{i=1}^n f^i v_{x_i} dx$ and $\langle , \rangle$ is the pairing of $H^{-1}(U)$ and $H_0^1(U)$

Chapter 6.2 Existence of Weak Solutions

6.2.1 Lax-Milgram Theorem

We now introduce a fairly simple abstract principle from linear functional analysis, which will later in 6.2.2 provide in certain circumstances the existence and uniqueness of a weak solution to our boundary value problem.

Theorem 1 (Lax-Milgram Theorem) Assume that

$$B : HxH \to R$$

is a bilinear mapping, for which there exists constants $\alpha, \beta > 0$ such that

$$|B[u,v]| \leq \alpha ||u|| ||v|| \qquad (u, v \in H)$$

and

$$\beta ||u||^2 \leq B[u,u] \qquad (u \in H)$$

Finally, let $f : H \to R$ be a bounded linear functional on H.

Then there exists a unique element $u \in H$ such that

$$(1) \qquad B[u,v] = \langle f,v \rangle$$

for all $v \in H$

Theorem 2 (Energy estimates) There exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such taht

$$|B[u,v]| \leq \alpha ||u||_{H_0^1(U)} ||v||_{H_0^1(U)}$$

and

$$\beta ||u||_{H_0^1(U)}^2 \leq B[u,u] + \gamma ||u||_{L^2(U)}^2$$

for all $u, v \in H_0^1(U)$

Theorem 3 (First Existence Theorem for weak solutions) There is a number $\gamma \geq 0$ such that for each

$$(7) \qquad \mu \geq \gamma$$

and each function

$$f \in L^2(U)$$

there exists a unique weak solution $u \in H_0^1(U)$ of the boundary-value problem

(8) $\{ Lu + \mu u = f$ in U
$\{$ u = 0 on ∂U

Definitions (i) The operator L*, the formal adjoint of L, is

$$L * v := -\sum_{i,j=1}^{n}(a^{ij}v_{x_j x_i} - \sum_{i=1}^{n} b^1 v_{x_i} + (c - \sum_{i=1}^{n} b^i_{x_i})v,$$

provided $b^i \in C^1(\bar{U})(i = 1, ..., n)$

(ii) The adjoint bilinear form

$$B* : H_0^1(U)xH_0^1(U) \to R$$

is defined by

$$B * [v, u] := B[u, v]$$

for all $u, v \in H_0^1(U)$

(iii) We say that $v \in H_0^1(U)$ is a weak solution of the adjoint problem

$\{L * v = f$ U
$\{$ v = 0 on ∂U

provided

$$B * [v, u] = (f, u)$$

for all $u \in H_0^1(U)$

Theorem 4 (Second Existence Theorem for weak solutions)

(i) Precisely one of the following statements holds either:

(α) {for each $f \in L^2(U)$ there exists a unique weak solution u of the boundary-value problem
(10) { {Lu = f in U
(10) { {u = 0 on ∂U

or else

(β) { there exists a weak solution $u \neq 0$ of the homogeneous problem
(11) { Lu = 0 in U
(11) { u = 0 on ∂U

(ii) futhermore, should assertion (β) hold, the dimention of the subspace $N \subset H_0^1(U)$ of weak solutions of (11) is finite and equals the dimensions of the subspace $N* \subset H_0^1(U)$ of weak solutions of

(12) { L*v = 0 in U
{ v = 0 on ∂U

Finally, the boundary-value problem (10) has a weak solution if and only if

$$(f, v) = 0 \qquad \text{for all } v \in N*$$

The dichotomy $(\alpha),(\beta)$ is the Fredholm alternative.

Theorem 5 (Third Existence THeorem for weak solutions)

(i) There exists an at most countable set $\sum \subset R$ such that the boundary-value problem

$$(24) \qquad \{ \text{Lu} = \lambda u + f \qquad \text{in U}$$
$$(24) \qquad \{ \text{u} = 0 \qquad \text{on } \partial U$$

has a unique weak solution for each $f \in L^2(U)$ if and only if $\lambda \notin \sum$

(ii) If $\sum$ is infinite, then $\sum = \{\lambda_k\}_{k=1}^{\infty}$, the values of a nondecreasing sequence with

$$\lambda_k \to +\infty$$

Definition We call $\sum$ the (real) spectrum of the operator L.

Theorem 6 (Boundedness of the inverse) If $\lambda \notin \sum$, there exists a constant C such that

$$(29) \qquad ||u||_{L^2(U)} \leq C||f||_{L^2(U)},$$

whenever $f \in L^2(U)$ and $u \in H_0^1(U)$ is the unique weak solution of

$$\{ \text{Lu} = \lambda u + f \qquad \text{in U}$$
$$\{ \text{u} = 0 \qquad \text{on } \partial U$$

The constant C depends only on λ, U and the coefficients of L.

This constant will blow up if λ approaches an eigenvalue .

Chapter 6.3 Regularity

Theorem 1 (Interior H^2-regularity) Assume

$$a^{ij} \in C^1(U), b^i, c \in L^\infty(U) \qquad (i,j = 1, ..., n)$$

and

$$f \in L^2(U)$$

Suppose furthermore that $u \in H^1(U)$ is a weak solution of the elliptic PDE

$$\text{Lu} = \text{f} \qquad \text{in U}$$

Then

$$u \in H^2_{loc}(U);$$

and for each open subset $V \subset\subset U$ we have the estimate

$$||u||_{H^2(V)} \leq C(||f||_{L^2(U)} + ||u||_{L^2(U)}).$$

the constant C depending only on V, U and the coefficients of L.

Theorem 2 (Higher Interior regularity) Let m be a nonnegative integer, and assume

$$a^{ij}.b^i, c \in C^{m+1}(U) \qquad (i,j = 1, ..., n)$$

and

$$f \in H^m(U)$$

Suppose $u \in H^1(U)$ is a weka solution of the elliptic PDE

$$\text{Lu = f} \qquad \text{in U}$$

Then

$$u \in H^{m+2}_{loc}(U)$$

and for each $V \subset\subset U$ we have the estimate

$$(28) \qquad ||u||_{H^{m+2}(V)} \leq C(||f||_{H^m(U)} + ||u||_{L^2(U)})$$

the constanc C depending only on m, U, V and the coefficients of L.

Theorem 3 (Infinite differentiability in the interior) Assume

$$a^{ij}, b^i, c \in C^\infty(U) \qquad (i, j = 1, ..., n)$$

and

$$f \in C^\infty(U)$$

Suppose $u \in H^1(U)$ is a weak solution of the elliptic PDE

$$\text{Lu = f} \qquad \text{in U}$$

Then

$$u \in C^\infty(U)$$

Theorem 4 (Boundary H^2-regularity) Assume

$$a^{ij} \in C^1(\bar{U}), b^i, c \in L^\infty(U) \qquad (i, j = 1, ..., n)$$

and

$$f \in L^2(U)$$

Suppose that $u \in H^1_0(U)$ is a weak solution of the elliptic boundary-value problem

$$(40) \qquad \{ \text{ Lu = f} \qquad \text{in U}$$
$$(40) \qquad \{ \text{ u = 0} \qquad \partial U$$

Assume finally

$$(41) \qquad \partial U \text{ is } C^2$$

Then

$$u \in H^2(U)$$

and we have the estimate

$$(42) \qquad ||u||_{H^2(U)} \leq C(||f||_{L^2(U)} + ||u||_{L^2(U)})$$

the constant C depending only on U and the coefficients of L.

Theorem 5 (Higher boundary regularity) Let m be a nonnegative integer, and assume

$$(72) \qquad a^{ij}, b^i, c \in C^{m+1}(\hat{U}) \qquad (i,j = 1, ..., n)$$

and

$$(73) \qquad f \in H^m(U)$$

Suppose that $u \in H_0^1(U)$ is a weak solution of the boundary-value problem

$$(74) \qquad \begin{cases} \text{Lu} = \text{f} & \text{in U} \\ \text{u} = 0 & \text{on } \partial U \end{cases}$$

Assume finally

$$(75) \qquad \partial U \text{ is } C^{m+2}$$

Then

$$(76) \qquad u \in H^{m+2}(U)$$

and we have the estimate

$$(77) \qquad ||u||_{H^{m+2}(U)} \leq C(||f||_{H^m(U)} + ||u||_{L^2(U)}$$

the constant C depending only on m, U and the coefficients of L.

Theorem 6 (Infinite differentiability up to the boundary) Assume

$$a^{ij}, b^i, c \in C^\infty(\bar{U}) \qquad (i,j = 1, ..., n)$$

and

$$f \in C^\infty(\bar{U})$$

Suppose $u \in H_0^1(U)$ is a weak solution of the boundary-value problem

$$\begin{cases} Lu = f & \text{in U} \\ \text{u} = 0 & \text{on } \partial U \end{cases}$$

Assume also that ∂U is C^∞. Then

$$u \in C^\infty(\bar{U})$$

Chapter 6.4 Maximum principles

Theorem 1 (Weak maximum principle) Assume $u \in C^2(U) \cap C(\bar{U})$ and

$$\text{c} = 0 \qquad \text{in U}$$

(i) If

$$Lu \leq 0 \qquad \text{in U}$$

then

$$max_{\bar{U}} u = max_{\partial U} u$$

(ii) If

$$Lu \geq 0 \qquad \text{in U}$$

then

$$min_{\bar{U}} u = min_{\partial U} u$$

Theorem 2 (Weak maximum principle for $c \geq 0$) Assume $u \in C^2(U) \cap C(\bar{U})$ and

$$c \geq 0 \qquad \text{in U}$$

(i) If

$$Lu \leq 0 \qquad \text{in U}$$

then

$$(11) \qquad max_{\bar{U}} u \leq max_{\partial U} u^+.$$

(ii) Likewise, if

$$Lu \geq 0 \in U$$

then

$$(12) \qquad min_{\bar{U}} u \geq -max_{\partial U} u^-.$$

Lemma (Hopf's Lemma) Assume $u \in C^2(U) \cap C^1(\bar{U})$ and

$$\text{c} = 0 \qquad \text{in U}$$

Suppose further

$$Lu \leq 0 \qquad \text{in U}$$

and there exists a point $x^0 \in \partial U$ such that

$$(14) \qquad u(x^0) > u(x) \qquad \text{for all } x \in U$$

Assume finally that U satisfies the interior ball condition at x^0; that is, there exists an open ball $B \subset U$ with $x^0 \in \partial B$

(i) Then

$$\frac{\partial u}{\partial v}(x^0) > 0$$

where v is the outer unit normal to B at x^0

(ii) If

$$c \geq 0 \qquad \text{in U}$$

the same conclusion holds provided

$$u(x^0) \geq 0$$

Theorem 3 (Strong maximum principle) Assume $u \in C^2(U) \cap C(\bar{U})$ and

$$\text{c} = 0 \qquad \text{in U}$$

Suppose also U is connected, open and bounded

(i) If

$$Lu \leq 0 \qquad \text{in U}$$

and u attains its maximum over $\bar{U}$ at an interior point, then

$$\text{u is constant within U}$$

(ii) Similarly, if

$$Lu \geq 0 \qquad \text{in U}$$

and u attains its minimum over $\bar{U}$ at an interior point, then

$$\text{u is constant within U.}$$

Theorem 4 (Strong maximum principle with $c \geq 0$) Assume $u \in C^2(U) \cap C(\bar{U})$ and

$$c \geq 0 \qquad \text{in U}$$

Suppose also U is connected

(i) If

$$Lu \leq 0 \qquad \text{in U}$$

and u attains a nonnegative maximum over $\bar{U}$ at an interior point, then

$$\text{u is constant within U}$$

(ii) SImilarly, if

$$Lu \geq 0 \qquad \text{in U}$$

and u attains a nonpositive minimum over $\bar{U}$ at an interior point, then

$$\text{u is constant within U}$$

Theorem 5 (Harnack's inequality) Assume $u \geq 0$ is a C^2 solution of

$$\text{Lu} = 0 \qquad \text{in U}$$

and suppose $V \subset\subset U$ is connected. Then there exists a constant C suhc that

$$sup_V u \leq C inf_V u \tag{18}$$

The constant C depends only on V and the coefficients of L

Chapter 6.5 Eignevalues and Eigenfunctions

Theorem 1 (Eigenvalues of symmetric elliptic operators)

(i) Each eigenvalue of L is real

(ii) Furthermore, if we repeat each eigenvalue according to its (fintie) multiplicity, we have

$$\sum = \{\lambda_k\}_{k=1}^{\infty}$$

where

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

and

$$\lambda_k \to \infty \qquad \text{as } k \to \infty$$

(iii) Finally, there exists an orthonormal basis $\{w_k\}_{k=1}^{\infty}$ of $L^2(U)$, where $w_k \in H_0^1(U)$ is an eigenfunction corresponding to λ_k

$$(4) \qquad \{\ Lw_k = \lambda_k w_k \qquad \text{in U}$$
$$(4) \qquad \{w_k = 0 \qquad \text{on } \partial U$$

for k = 1,2,...

Definition We call $\lambda_1 > 0$ the principal eigenvalue of L.

Theorem 2 (Variational principle for the principal eigenvalue)

(i) We have

$$(5) \qquad \lambda_1 = min\{B[u,u] | u \in H_0^1(U), ||u||_{L^2} = 1\}$$

(ii) Furthermore, the above minimum is attained for a function w_1, positive within U, which solves

$$\{Lw_1 = \lambda_1 w_1 \qquad \text{in U}$$
$$\{\ w_1 = 0 \qquad \text{on } \partial U$$

(iii) Finally, if $u \in H_0^1(U)$ is any weak solution of

$$\{Lu = \lambda_1 u \qquad \text{in U}$$
$$\{\ \text{u} = 0 \qquad \text{on } \partial U$$

then u is a multiple of w_1

Theorem 3 (Principle eigenvalue for nonsymmetric elliptic operators)

(i) There exists a real eigenvalue λ_1 for the operators L, taken with zero boundary conditions, such that if $\lambda \in \mathcal{C}$ is any other eigenvalue, we have

$$Re(\lambda) \geq \lambda_1$$

(ii) There exists a corresponding eigenfunction w_1, which is positive within U

(iii) The eignevalue λ_1 is simple; that is, if u is any solution of (1), then u is a multiple of w_1