Math 207C Partial Differential Equations: Ch 7 Linear evolution equations

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Chapter 7.1 Second-Order Parabolic Equations

Definition We say that the partial differential operator $\frac{\partial}{\partial t} + L$ is (uniformly) parabolic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2$$

for all $(x,t) \in U_T, \xi \in R^n$

Definition We say a function

$$u \in L^2(0,T;H_0^1(U)), \text{ with } u^{'} \in L^2(0,T;H^{-1}(U)),$$

is a weak solution of the parabolic initial/boundary-value problem (1) provided

(i) $\langle u^{'}, v\rangle + B[u,v;t] = (f,v)$

for each $v \in H_0^1(U)$ and a.e. time $0 \leq t \leq T$ and

(ii) u(0) = g

Theorem 1 (Construction of approximate solutions) For each integer m = 1,2,... there exists a unique function u_m of the form (14) satisfying (15), (16).

Theorem 2 (Energy estimates). There exists a constant C, depending only on U,T and the coefficients of L, such that

$$max_{0\leq t\leq T}||u_m(t)||_{L^2(U)} + ||u_m||_{L^2(0,T;H_0^1(U))} + ||u_m^{'}||_{L^2(0,T;H^{-1}(U))} \leq C(||f||_{L^2(0,T;L^2(U))} + ||g||_{L^2(U)}$$

for m = 1,2,...

Theorem 3 (Existence of weak solution) There exists a weak solution of (11).

Theorem 4 (Uniqueness of weak solutions) A weak solution of (11) is unique.

Theorem 5 (Improved regularity)

(i) Assume

$$g \in H_0^1(U), f \in L^2(0,T;L^2(U))$$

Suppose also $u \in L^2(0,T;H_0^1(U))$, with $u^{'} \in L^2(0,T;H^{-1}(U))$, is the weak solution of

$$\{u_t + Lu = f \qquad \text{in } U_T$$
$$\{\text{ u} = 0 \qquad \text{on } \partial Ux[0,T]$$
$$\{\text{ u} = \text{g} \qquad \text{on } Ux\{t=0\}$$

Then in fact

$$u \in L^2(0,T;H^2(U)) \cap L^\infty(0,T;H_0^1(U)), u^{'} \in L^2(0,T;L^2(U))$$

and we have the estimate

$$esssup_{0\leq t\leq T}||u(t)||_{H_0^1(U)} + ||u||_{L^2(0,T;H^2(U))} + ||u^{'}||_{L^2(0,T;L^2(U))} \leq C(||f||_{L^2(0,T;L^2(U))} + ||g||_{H_0^1(U)})$$

the constant C depending only on U,T and the coefficient of L

(ii) If, in addition,

$$g \in H^2(U), f^{'} \in L^2(0,T;L^2(U))$$

then

$$u \in L^\infty(0,T;H^2(U)), u^{'} \in L^\infty(0,T;L^2(U)) \cap L^2(0,T;H_0^1(U)), u^{"} \in L^2(0,T;H^{-1}(U))$$

with the estimate

$$esssup_{0\leq t\leq T}(||u(t)||_{H^2(U)} + ||u^{'}(t)||_{L^2(U)} + ||u^{'}||_{L^2(0,T;H_0^1(U))} + ||u^{"}||_{L^2(0,T;H^{-1}(U))} \leq C(||f||_{H^1(0,T;L^2(U))} + ||g||_{H^2(U)})$$

Theorem 6 (Higher regularity) Assume

$$g \in H^{2m+1}(U), \frac{d^k f}{dt^k} \in L^2(0,T;H^{2m-2k}(U)) \qquad (k=0,...,m)$$

Suppose also that the following mth-order compatibility conditions hold:

$$\{g_0 := g \in H_0^1(U), g_1 := f(0) - L_{g_0} \in H_0^1(U)$$
$$\{ \text{ ....}, g_m := \frac{d^{m-1}}{dt^{m-1}}(0) - L_{g_{m-1}} \in H_0^1(U)$$

Then

$$\frac{d^k u}{dt^k} \in L^2(0,T;H^{2m+2-2k}(U)) \qquad (k=0,...,m+1);$$

and we have the estimate $\sum_{k=0}^{m+1}||\frac{d^k u}{dt^k}||_{L^2(0,T;H^{2m+2-2k}(U))} \leq C(\sum_{k=0}^{m}||\frac{d^k f}{dt^k}||_{L^2(0,T;H^{2m-2k}(U))} + ||g||_{H^{2m+1}(U)})$ the constant C depending only on m, U, T and the coefficients of L.

Theorem 7 (Infinite differentiability). Assume

$$g \in C^\infty(\bar{U}), f \in C^\infty(\bar{U}_T)$$

and the mth-order compatibility conditions hold for m=0,1,... Then the parabolic initial/boundary value problem (11) has a unique solution

$$u \in C^\infty(\bar{U}_T)$$

Theorem 8 (Weak maximum principle) Assume $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ and

$$c = 0 \qquad \text{in } U_T$$

(i) If

$$u_t + Lu \leq 0 \qquad \text{in } U_T$$

then

$$max_{\bar{U}_T} u = max_{\Gamma_T} u$$

(ii) Likewise, if

$$u_t + Lu \geq 0 \qquad \text{in } U_T$$

then

$$min_{\bar{U}_T} u = min_{\Gamma_T} u$$

Theorem 9 (Weak maximum principle for $c \geq 0$) Assume $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ and

$$c \geq 0 \qquad \text{in } U_T$$

(i) If

$$u_t + Lu \leq 0 \qquad \text{in } U_T,$$

then

$$max_{\bar{U}_T} u \leq max_{\Gamma_T} u^+$$

(ii) If

$$u_t + Lu \geq 0 \qquad \text{in } U_T$$

then

$$min_{\bar{U}_T} \geq -max_{\Gamma_T} u^-.$$

Theorem 10 (Parabolic Harnack inequality) Assume $u \in C_1^2(U_T)$ solves

$$u_t + Lu = 0 \qquad \text{in } U_T \tag{68}$$

and

$$u \geq 0 \qquad \text{in } U_T$$

Suppose $V \subset\subset U$ is connected. Then for each $0 \leq t_1 \leq t_2 \leq T$, there exists a constant C such that

$$sup_V u(\cdot, t_1) \leq C inf_V u(\cdot, t_2) \tag{69}$$

The constant C depends only on V, t_1, t_2 and the coefficients of L.

Theorem 11 (Strong maximum principle) Assume $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ and

$$c = 0 \qquad \text{in } U_T$$

Suppose also U is connected

(i) If

$$u_{t_{Lu}} \leq 0 \qquad \text{in } U_T$$

and u attains its maximum over $\bar{U}_T$ at a point $(x_0, t_0) \in U_T$, then

$$\text{u is constant on } U_{t_0}$$

(ii) Likewise, if

$$u_t + Lu \geq 0 \qquad \text{in } U_T$$

and u attains its minimum over $\bar{U}_T$ at a point $(x_0, t_0) \in U_T$, then

$$\text{u is constant on } U_{t_0}$$

Theorem 12 (Strong maximum principle for $c \geq 0$) Assume $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ and

$$c \geq 0 \qquad \text{in } U_T$$

Suppose also U is connected

(i) If

$$u_{t_{Lu}} \leq 0 \qquad \text{in } U_T$$

and u attains a nonnegative maximum over $\bar{U}_T$ at a point $(x_0, t_0) \in U_T$, then

$$\text{u is constant on } U_{t_0}$$

(ii) Similarly, if

$$u_t + Lu \geq 0 \qquad \text{in } U_T$$

and u attains a nonpositive minimum over $\bar{U}_T$ at a point $(x_0, t_0) \in U_T$, then

$$\text{u is constant on } U_{t_0}$$

Chapter 7.2 Second Order Hyperbolic equations

Second-order hyperbolic equations are natural generalizations of the wave equation (2.4) We will build in this section appropiately defined weak solutions and study their uniqueness, smoothness and other properties. It is interesting, given the utterly different physical character of second-order parabolic and hyperbolic PDE, that we can provide rather similar functional analytic constructions of weak solutions.

Definition We say the partial differential operator $\frac{\partial^2}{\partial t^2} + L$ is (uniformly) hyperbolic if there exists a constant $\theta > 0$ such that

$$(4) \qquad \sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2$$

for all $(x,t) \in U_T, \xi \in R^n$

Definition We say a function

$$u \in L^2(0,T;H_0^1(U)), \text{ with } u^{'} \in L^2(0,T;L^2(U)), u^{"} \in L^2(0,T;H^{-1}(U))$$

is a weak solution of the hyperbolic initial/boundary-value problem (1) provided

(i) $\langle u^{"}, v\rangle + B[u,v;t] = (f,v)$

for each $v \in H_0^1(U)$ and a.e. time $0 \leq t \leq T$ and

(ii) $u(0) = g, u^{'}(0) = h$

Theorem 1 (Construction of approximate solutions) For each integer m = 1,2,..., there exists a unique function u_m of the form (13) satisfying (14)-(16).

Theorem 2 (Energy estimates) There exists a constant C, depending only on U, T and the coefficients of L , such that

(19)

$$max_{0\leq t\leq T}(||u_m(t)||_{H_0^1(U)} + ||u_m^{'}(t)||_{L^2(U)}) + ||u_m^{"}||_{L^2(0,T;H^{-1}(U))} \leq C(||f||_{L^2(0,T;L^2(U))} + ||g||_{H_0^1(U)} + ||h||_{L^2(U)})$$

for m = 1,2,...

Theorem 3 (Existence of weak solution) There exists a weak solution of (1).

Theorem 4 (Uniqueness of weak solution) A weak solution of (1) is unique.

Theorem 5 (Improved regularity)

(i) Assume

$$g \in H_0^1(U), h \in L^2(U), f \in L^2(0,T;L^2(U))$$

and suppose also $u \in L^2(0,T;H_0^1(U))$ with $u^{'} \in L^2(0,T;L^2(U)), u^{"} \in L^2(0,T;H^{-1}(U))$, is the weak solution of the problem

$$\begin{array}{ll} \{u_{tt} + Lu = f & \text{in } U_T \\ \{ \qquad \text{u=0} & \text{on } \partial U x[0,T] \\ \{u = g, u_t = h & \text{on } Ux\{t=0\} \end{array}$$

Then in fact

$$u \in L^\infty(0,T;H_0^1(U)), u^{'} \in L^\infty(0,T;L^2(U))$$

and we have the estimate

$$\text{ess } sup_{0\leq t\leq T}(||u(t)||_{H_0^1(U)} + ||u^{'}(t)||_{L^2(U)}) \leq C(||f||_{L^2(0,T;L^2(U))} + ||g||_{H_0^1(U)} + ||h||_{L^2(U)}$$

(ii) If, in addition,

$$g \in H^2(U), h \in H_0^1(U), f^{'} \in L^2(0,T;L^2(U)),$$

then

$$u \in L^\infty(0,T;H^2(U)), u^{'} \in L^\infty(0,T;H_{0^1(U)}),$$
$$u^{"} \in L^\infty(0,T;L^2(U)), u^{"'} \in L^2(0,T;H^{-1}),$$

with the estimate

$$esssup_{0\le t\le T}(||u(t)||_{H^2(U)}+||u^{'}(t)||_{H_0^1(U)}+||u^{"}(t)||_{L^2(U)})+||u^{"'}||_{L^2(0,T;H^{-1}(U))} \le C(||f||_{H^1(0,T;L^2(U))}+||g||_{H^2(U)}+||h||_{H^1(U)})$$

Theorem 6 (Higher regularity) Assume

$$\{\ g \in H^{m+1}(U), h \in H^m(U)$$
$$\{\frac{d^k f}{dt^k} \in L^2(0,T;H^{m-k}(U)) \qquad (k=0,...,m)$$

Suppose also that the following mth-order compatibility conditions hold:

$$\{\ g_0 := g \in H_0^1(U), \qquad h_1 := h \in H_0^1(U),...,$$
$$\{\ g_{2l} := \frac{d^{2l-2}f}{dt^{2l-2}}(\cdot,0) - Lg_{2l-2} \in H_0^1(U) \qquad \text{(if m = 2l)}$$
$$\{\ h_{2l+1} := \frac{d^{2l-2}f}{dt^{2l-1}}(\cdot,0) - Lh_{2l-1} \in H_0^1(U) \qquad \text{(if m=2l+1)}$$

Then

$$\frac{d^k u}{dt^k} \in L^\infty(0,T;H^{m+1-k}(U)) \qquad \text{(k=0,...,m+1)},$$

and we have the estimate

$$esssup_{0\le t\le T}\sum_{k=0}^{m+1}||\frac{d^k u}{dt^k}||_{H^{m+1-k}(U)} \le$$
$$C(\sum_{k=0}^{m}||\frac{d^k f}{dt^k}||_{L^2(0,T;H^{m-k}(U))}+||g||_{H^{m+1}(U)}+||h||_{H^m(U)})$$

Theorem 7 (Infinite differentiability) Assume

$$g,h \in C^\infty(\bar{U}), f \in C^\infty(\bar{U}_T)$$

and the mth-order compatibility conditions hold for m=0,1,...

Then the hyperbolic initial/boundary value problem (1) has a unique solution

$$u \in C^\infty(\bar{U}_T)$$

Theorem 8 (Finite propogation speed) Assume u is a smooth solution of the hyperbolic equation (72). If $u = u_t = 0$ on K_0, then u = 0 within K.

Chapter 7.3 Hyperbolic systems of first-order equations

Definition The system of PDE (1) is called hyperbolic if the m x n matrix B(x,t;y) is diagonalizable for each $x, y \in R^n, t \ge 0$

Definition (i) We say (1) is a symmetric hyperbolic system if $B_j(x,t)$ is a symmetric m x m matrix for each $x \in R^n, t \ge 0 (j = 1,...,m)$

(ii) The system (1) is strictly hyperbolic if for each $x, y \in R^n, y \ne 0$ and each $t \ge 0$, the matrix B(x,t;y) has m distinct real eigenvalues.

$$\lambda_1(x,t;y) < \lambda_2(x,t;y) < \cdots < \lambda_m(x,t;y)$$

Definition We say

$$u \in L^2(0,T;H^1(R^n;R^m)), \text{ with } u^{'} \in L^2(0,T;L^2(R^n;R^m))$$

is a weak solution of the initial-value problem (5) for the symmetric hyperbolic system provided

(i) $(u^{'}, v) + B[u, v; t] = (f, v)$

for each $v \in H^1(R^n;R^m)$ and a.e. $0 \leq t \leq T$ and

(ii) u(0) = g Here and afterwards (,) denotes the inner product in $L^2(R^n;R^m)$

Theorem 1 (Existence of approximate solutions) For each $\epsilon > 0$, there exists a unique solution u^ϵ of (11), with

$$u^\epsilon \in L^2(0,T;H^3(R^n;R^m)), u^{\epsilon'} \in L^2(0,T;H^1(R^n;R^m))$$

Theorem 2 (Energy estimates) There exists a constant C, depending only on n and the coefficients, such that

$$max_{0 \leq t \leq T}(||u^\epsilon(t)||_{H^1(R^n;R^m)} + ||u^{\epsilon'}(t)||_{L^2(R^n;R^m)} \leq$$
$$C(||g||_{H^1(R^n;R^m)} + ||f||_{L^2(0,T;H^1(R^n;R^m))} + ||f^{'}||_{L^2(0,T;L^2(R^n;R^m))})$$

for each $0 < \epsilon \leq 1$

Theorem 3 (Existence of weak solution) There exists a weak solution of the initial value problem (5).

Theorem 4 (Uniqueness of weak solution) A weak solution of (5) is unique.

Theorem 5 (Existence of solution) Assume

$$g \in H^s(R^n;R^m) \qquad (s > \tfrac{n}{2} + m)$$

Then there is a unique solution $u \in C^1([0,\infty);R^m)$ of the initial value problem (32), (33).

Chapter 7.4 Semigroup Theory

Semigroup theory is the abstract study of first-order ordinary differential equations with values in Banach spaces, driven by linear, but possibly unbounded operators. In this section, we outline the basics of the theory and present as well two applicatoins to linear PDE . This approach provides an elegant alternative to some of the existence theory for evolution equations set forth in Ch 7.1-7.3

Definitions Write

$$(8) \qquad D(A) := \{u \in X | \lim_{t \to 0+} \tfrac{S(t)u-u}{t} \text{ exists in X } \}$$

and

$$(9) \qquad Au := \lim_{t \to 0+} \tfrac{S(t)u-u}{t} (u \in D(A))$$

We call $A : D(A) \to X$ the (infinitesimal) generator of the semigroup $\{S(t)\}_{t \geq 0}; D(A)$ is the domain of A.

Theorem 1 (Differential properties of semigroups). Assume $u \in D(A)$. Then

(i) $S(t)u \in D(A)$ for each $t \geq 0$
(ii) $AS(t)u = S(t)Au$ for each $t \geq 0$
(iii) The mapping $t \mapsto S(t)u$ is differentiable for each $t > 0$
(iv) $\frac{d}{dt}S(t)u = AS(t)u \qquad (t > 0)$

Theorem 2 (Properties of generators)
(i) The domain D(A) is dense in X
and
(ii) A is a closed operator.

Definitions (i) We say a real number λ belongs to p(A), the resolvent set of A, provided the operator

$$\lambda I - A : D(A) \to X$$

is one to one and onto
(ii) If $\lambda \in p(A)$, the resolvent operator $R_\lambda u := (\lambda I - A)^{-1}u$

Theorem 3 (Properties of resolvent operators)
(i) If $\lambda, \mu \in p(A)$, we have

$$(12) \qquad R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu \qquad \text{(resolvent identity)}$$

and

$$(13) \qquad R_\lambda R_\mu = R_\mu R_\lambda$$

(ii) If $\lambda > 0$ then $\lambda \in p(A)$,

$$(14) \qquad R_\lambda u = \int_0^\infty e^{-\lambda t} S(t)u dt \qquad (u \in X)$$

and so $||R_\lambda|| \leq \frac{1}{\lambda}$
Thus the resolvent operator is the Laplace transform of the semigroup (cf. Example 8 in ch. 4.3.3).

Theorem 4 (Hille-Yoshida Theorem) Let A be a closed, densely-defined linear operator on X. Then A is the generator of a contraction semigroup $\{S(t)\}_{t\geq 0}$ if and only if

$$(0, \infty) \subset p(A) \text{ and } ||R_\lambda|| \leq \frac{1}{\lambda} \text{ for } \lambda > 0$$

Theorem 5 (Second-Order parabolic PDE as semigroups) The operator A generates a γ-contraction semigroup $\{S(t)\}_{t\geq 0}$ on $L^2(U)$

Theorem 6 (Second-Order hyperbolic PDE as semigroups) The operator A generates a contraction semigroup $\{S(t)\}_{t\geq 0}$ on $H_0^1(U)xL^2(U)$.