Eigenvalues and eigenvectors *Basic definitions and methods for determining eigenvectors and eigenvalues, diagonalization, applications to computing powers of matrices and systems of linear differential equations.*

Computations of $A^k x$ arise in any process in which information given by a column vector gives rise to analogous information at a later time by multiplying the vector by a matrix A.

Definition 5.1 Eigenvalues and Eigenvectors Let A be an n X n matrix. A scalar λ is an eigenvalue of A if there is a nonzero column vector v in R^n such that Av = λv. The vector v is then an eigenvector of A corresponding to λ. (The terms characteristic vector and characteristic value or proper vector and proper value are also used in place of eigenvector and eigenvalue, respectively.)

Theorem 5.1 Properties of Eigenvalues and Eigenvectors
Let A be an n X n matrix.
1. If λ is an eigenvalue of A with v as a corresponding eigenvector, then λ^k is an eigenvalue of A^k, again with v as a corresponding eigenvector, for any positive integer k.
2. If λ is an eigenvalue of A of an invertible matrix A with v as a corresponding eigenvector, then $\lambda \neq 0$ and $1/\lambda$ is an eigenvalue of A^-1, again with v as a corresponding eigenvector.
3. If λ is an eigenvalue of A, then the set E_λ consisting of the zero vector together with all eigenvectors of A for this eigenvalue λ is a subspace of n-space, the eigenspace of λ

Definition 5.2 Eigenvalues and eigenvectors Let T be a linear transformation of vector space V into itself. A scalar λ is an eigenvalue of T if there is a nonzero vector v in V such that T(v)=λv. The vector v is then an eigenvector of T corresponding to λ.

5.2 Diagonalization Let A be an n X n matrix and let $\lambda_1, \lambda_2, ...\lambda_n$ be (possibly complex scalars and $v_1, v_2, ..., v_n$ be nonzero vectors in n-space. Let C be the n X n matrix having v_j as the jth column vector and let *OK OK this is my first latex matrix*

D =

$$\begin{vmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{vmatrix}$$

Then AC = CD if and only if $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of A and v_j is an eigenvector of A corresponding to λ_j for j = 1,2,...n.

Definition 5.3 Diagonalizable Matrix An n X n matrix A is diagonalizale if there exists an invertible matrix C such that C^{-1}AC = D, a diagonal matrix. The matrix C is said to diagonalize the matrix A.

Corollary 1 A Criterion for Diagonaiization An n X n matrix A is diagonalizable if and only if n-space has a basis consisting of eigenvectors of A.

Corollary 2 Computation of A^k Let an n X n matrix A have n eigenvectors and eigenvalues, giving rise to the matrices C and D so that AC = CD, as described in Theorem 5.2. If the n eigenvectors are independent, then C is an invertible matrix and $C^{-1}AC = D$. Under these conditions, we have

$$A^k = CD^kC^{-1}$$

Theorem 5.3 Independence of Eigenvectors Let A be an n X n matrix. If v_1, v_2,..., v_n are eigenvectors of A corresponding to distinct eigenvalues λ_1 , λ_2 , ... , λ_n , respectively, the set v_1, v_2, ...,v_n is linearly independent and A is diagonalizable.

Definition 5.4 Similar Matrices An n X n matrix P is similar to an n X n matrix Q if there exists an invertible n X n matrix C such that $C^{-1}PC = Q$.

The geometric multiplicity of an eigenvalue of a matrix A is less than or equal to its algebraic multiplicity.

Theorem 5.4 A Criterion for Diagonalization An n X n matrix A is diagonalizable if and only if the algebraic multiplicity of each (possibly complex) eigenvalue is equal to its geometric multiplicity.

Theorem 5.5 Diagonalization of Real Symmetric Matrices Every real symmetric matrix is real diagonalizable. That is, if A is an n X n symmetric matrix with real number entries, then each eigenvalue of A is a real number, and its algebraic multiplicity equals its geometric multiplicity.

5.3 Two Applications