Chapter 6.1 Projections

Projection p of b on sp(a) in R^R $p = \frac{b*a}{a*a}a$

Definition 6.1 Orthogonal Complement Let W be a subspace of R^n. The set of all vectors in R^n that are orthogonal to every vector in W is the orthogonal complement of W, and is denoted by $W^\perp$.

Theorem 6.1 Properties of $W^\perp$ The orthogonal complement $W^\perp$ of a subspace W of R^n has the following properties:
1. $W^\perp$ is a subspace of R^n
2. dim($W^\perp$ = n - dim(W).
3. $(W^\perp)^\perp$ = W; that is the orthogonal complement of $W^\perp$ is W
4. Each vector b in R^n can be expressed uniquely in the form $b = b_W + b_{W^\perp}$ for b_W in W and $b_W^\perp$ in $W^\perp$.

Definition 6.2 Projection of b on W Let b be a vector in R^n, and let W be a subspace of R^n. Let

$$\text{b} = b_W + b_W^\perp$$

as described in Theorem 6.1. Then b_W is the projection of b on W.

Chapter 6.2 The Gram-Schmidt Process

Theorem 6.2 Orthogonal Bases Let $v_1, v_2, ..., v_k$ be an orthogonal set of nonzero vectors in R^n. Then this set is independent and consequently is a basis for the subspace sp($v_1, v_2, ..., v_k$).

Theorem 6.3 Projection Using an Orthogonal Basis Let $v_1, v_2, ..., v_k$ be an orthogonal basis for a subspace W of R^n, and let b be any vector in R^n. The projection of b on W is

$$b_W = \frac{b*v_1}{v_1*v_1}v_1 + \frac{b*v_2}{v_2*v_2}v_2 + ... + \frac{b*v_k}{v_k*v_k}v_k$$

Definition 6.3 Orthonormal Basis Let W be a subspace of R^n. A basis $q_1, q_2, ..., q_k$ for W is orthonormal if
1. $q_i * q_j = 0$ for i neq j Mutually perpendicular

2. $q_i * q_i = 1$ Length 1

Projection of b on W with orhtonormal basis $q_1, q_2, ..., q_k$

$$b_w = (b * q_1)q_1 + (b * q_2)q_2 + ... + (b * q_k)q_k$$

Theorem 6.4 Orthonormal Basis (Gram-Schmidt) Theorem Let W be a subspace of R^n, let $a_1, a_2, ..., a_k$ be any basis for W, and let

$$W_j = sp(a_1, a_2, ..., a_j)forj = 1, 2, ..., k$$

Then there is an orthonormal basis $q_1, q_2, ..., q_k$ for W such that $W_j = sp(q_1, q_2, .., q_j)$

General Gram-Schmidt Formula

$$v_j = a_j - (\frac{a_j*v_1}{v_1*v_1}v_1 + \frac{a_j*v_2}{v_2*v_2}v_2 + ... + \frac{a_j*v_{j-1}}{v_{j-1}*v_{j-1}j_{v-1}}$$

Normalized Gram-Schmidt Formula

$$v_j = a_j - ((a_j * q_1)q_1 + (a_j * q_2)q_2) + ... + (a_j * q_{j-1})q_{j-1})$$

Corollary 1 QR-Factorization Let A be an n X k matrix with independent column vectors in R^n. There exists an n X k matrix Q with orthonormal column vectors and an upper triangular invertible k X k matrix R such that A = QR.

6.3 Orthogonal Matrices

Definition 6.4 Orthogonal Matrx An n X n matrix A is orthogonal if $A^T A = L$

Theorem 6.5 Characterizing Properties of an Orthogonal Matrix Let A be an n X n matrix. The following conditions are equivalent:

1. The rows of A form an orthonormal basis for R^N

2. The columns of A form an orthonormal basis for R^n

3. The matrix A is orthogonal-that is,invertible with $A^{-1} = A^T$

Theorem 6.6 Properties of Ax for an Orthogonal Matrix A Let A be an orthogonal n X n matrix and let x and y be any column vectors in R^n
1. (Ax) * (Ay) = x * y Preservation of dot product
2. ——Ax—— = ——x—— Preservation of length
3. The angle between nonzero vectors x and y equeals the angle betweem Ax and Ay Preservation of angle

Multiplication by orthogonal matrices is a stable operation

Theorem 6.7 Orthogonality of Eigenspaces of a Real Symmetric Matrix
Eigenvectors of a real symmetric matrix that correspond to different eigenvalues are orthogonal. That is, the eigenspaces of real symmetryic matrix are orthogonal.

Theorem 6.8 Fundamental Theorem of Real Symmetric Matrices Every real symmetric matrix A is diagonalizable. The diagonalization $C^{-1}AC = D$ can be achieved by using a real orthogonal matrix C.

Definition 6.5 Orthogonal Linear Transformation A linear transformation T: $R^n --> R^n$ is orthoganal if it satisfies T(v) * T(w) = v * w for all vectors v and w in R^n.

Theorem 6.9 Orthogonal Tranformations vis-a-vis matrices A linear transformation T of R^n into itself is orthogonal if and only if its standard matrix representation A is an orthogonal matrix.

Chapter 6.5 The Projection Matrix

Theorem 6.10 The Rank of (A^T)A Let A be an m X n matrix of rank r. Then the n X n symmetric matrix (A^T)A also has rank r.

Properties of the Projection p of Vector b on the Subspace W.
1. The vector p must lie in the subspace W
2. The vector b - p must be perpendicular to every vector in W.

Projoection b_w of b on subspace W Let W = sp($a_1, a_2, ..., a_k$) be a k-dimensional subspace of R^n, and let A have as columns the vectors $a_1, a_2, ..., a_k$. The projection of b in R^n on W is given by

$$b_w = A(A^TA)^{-1}A^Tb$$

The Projection Matrix P for the Subspace W Let W = sp($a_1, a_2, ..., a_k$) be a k-dimensional subspace of R^n, and let A have as columns the vectors $a_1, a_2, ..., a_k$. The projection matrix for the subspace W is given by

$$P = A(A^TA)^{-1}A^T$$

Theorem 6.11 Projection Matrix Let W be a subspace of R^n. There is a unique n X n matrix P such that for each column vector b in R^n, the vector P_b is the projection of b on W. This projection matrix P can be found by selecting any basis $a_1, a_2, ..., a_k$ for W and computing $P = A(A^TA)^{-1}A^T$, where A is the n X k matrix having column vectors $a_1, a_2, ..., a_k$.

Properties of a Projection Matrix P
1. P^2 = P P is idempotent
2. P^T = P P is symmetric

Theorem 6.12 Characterization of Projection Matrices The projection matrix P for a subspace W of R^n is both idempotent and symmetric. Conversely, every n X n matrix that is both idempotent and symmetric is a projection matrix: specifically, it is the projection matrix for its column space.

Projection Matrix: Orthonormal Case Let $a_1, a_2, ..., a_k$ be an orthonormal basis for a subspace W of R^n. The projection matrix for W is

$$\mathrm{P} = AA^T,$$

where A is the n X k matrix having column vectors $a_1, a_2, ..., a_k$

Chapter 6.5 The method of least squares

Least Squares Solution of Ar ≈ b Let A be a matrix with independent column vectors. The least-squares solution $\bar{r}$ of Ar ≈ b can be computed in either of the follwing ways:
1. Compute r = $(A^TA)^{-1}A^Tb$
2. Solve $(A^TA)r = A^Tb$

When a computer is being used, the second method is more efficient.