Chapter 9 Complex Sccalars

Fundamental Theorem of Algebra Every polynomial equation with coefficients in C has n solutions in C, where n is the degree of the polynomial and the solutions are counted with their algebraic multiplicity.

The modulus (or magnitude) of the complex number z = a + bi is $\mid z \mid = \sqrt{a^2+b^2}$ which is the length of the vector in Figure 9.1.

Geometric Representation of $z_1 z_2$

1. $\mid z_1 z_2 \mid = \mid z_1 \mid\mid z_2 \mid$
2. Arg(z_1) + Arg(z_2) is an argument of $z_1 z_2$.

Geometric Representation of z_1/z_2

1. $\mid z_1/z_2 \mid = \mid z_1 \mid / \mid z_2 \mid$
2. Arg(z_1) + Arg(z_2) is an argument of $z_1 z_2$.

nth roots of z = r(cos θ + i sin θ)
The nth roots of z are

$$r^{1/n}(cos(\frac{\theta}{n}+\frac{2k\pi}{n})+isin(\frac{\theta}{n}+\frac{2k\pi}{n}))$$

for k = 0, 1,2, ... , n - 1.

Chapter 9.2 Matrices and Vector Spaces with Complex scalars

Definition 9.1 Euclidean inner product Let u = $u_1, u_2, ..., u_n$ and v = $[v_1, v_2, ..., v_n]$ be vectors in C^n. The Euclidean inner product of u and v is

$$u, v = \bar{u_1}v_1 + \bar{u_2}v_2 + ... + \bar{u_n}v_n.$$

Theorem 9.2 Properties of the Euclidean Inner Product
Let u, v and w be vectors in c^n, and let z be a complex scalar.

1. $\langle u, u\rangle \leq 0$, and $\langle u, u\rangle = 0$ if and only if u = 0

2. $\langle u, v\rangle = \langle v,\bar{} u\rangle$,
3. $\langle (u+v), w\rangle = \langle u, w\rangle + \langle v, w\rangle$,
4. $\langle w, (u+v)\rangle = \langle w, u\rangle + \langle w, v\rangle$,
5. $\langle zu, v\rangle = \bar{z}\langle u, v\rangle, and\langle u, zv\rangle = z\langle u, v\rangle$

The Euclidean inner product in C^n is not commutative

Definition 9.2 Conjugate Transpose or Hermitian Adjoint Let A = $[a_{ij}]$ be an m X n matrix with complex scalar entries.

1. The conjugate of A is the m X n matrix $\bar{A} = [a\bar{i}j]$.

2. The conjugate transpose (or Hermitian adjoint) of A is the matrix $A^* = [\bar{a_i}j]^T$.

Theorem 9.3 Properties of the Conjugate Transpose
Let A and B be m X n matrices. Then
1. $(A^*)^* = A$
2. $(A+B)^* = A^* + B^*$,

3. $(zA)^* = \bar{z}A^*$ for any scalar $z \in C$,
4. If A and B are square matrices, $(AB)^* = B^*A^*$

Definition 9.3 Unitary Matrix

A square matrix U with complex entries in unitary if its column vectors are orthogonal unit vectors - that is, if $U^*U = I$

Eigenvalues and diagonalization

Every real symmetric matrix is diagonalizable by a real orthogonal matrix.

Every Hermitian matrix is diagonalizable by a unitary matrix

Theorem 9.4 Schur's Lemma

Let A be an n X n (complex) matrix. There is a unitary matrix U such that $U^{-1}AU$ is upper trangular.

Theorem 9.5 Spectral Theorem for Hermitian Matrices

If A is a Hermitian matrix, there exists a unitary matrix U such that $U^{-1}AU$ is a diagonal matrix. Furthermore, all eigenvalues of A are real.

Corollary Fundamental Theorem of Real Symmetric Matrices

Every n X n real symmetric matrix has n real eigenvalues, counted with their algebraic multiplicity, and is diagonalizable by a real orthogonal matrix.

Theorem 9.6 Orthogonality of Eigenspaces of a Hermitian Matrix

The eigenvectors of a Hermitian matrix corresponding to distinct eigenvalues are orthogonal.

Definition 9.5 Normal Matrix

A square matrix A is normal if it commutes with its conjugate transpose, so that $AA^* = A^*A$.

Theorem 9.7 Spectral Theorem for Normal Matrices

A square matrix A is unitarily diagonalizable if and only if it is a normal matrix.

Chapter 9.4 Jordan Canonical Form

Definition 9.6 Jordan Block

An m X m matrix is a Jordan block if it is structured as follows:
1. All diagonal entries are equal
2. Each entry immediately above a diagonal entry is 1.
3. All other entries are zero.

Theorem 9.8 Properties of a Jordan Block

Let J be an m X m Jordan block with diagonal entries all equal to λ. Then the following properties hold
1. (J - $\lambda I)e_i = e_{i-1}$ for $1 < i \leq m$, and $(J - \lambda I)e_1 = 0$.

2. (J - $\lambda I)^m = O, but (J - \lambda I)^i \neq O for i < m.$
3. $Je_i = \lambda e_i + e_{i-1} for 1 < i \leq m$, whereas $Je_1 = \lambda e_1$

Definition 9.7 Jordan Canonical Form An n X n matrix J is a Jordan canonical form if it consists of Jordan blocks, placed corner to corner along the main diagonal, as in matrix (4) with only zero entries outside these Jordan blocks.

Definition 9.8 Jordan Basis

Let A be an n X n matrix. An ordered basis B = $(b_1, b_2, ..., b_n)$ of C^n is a Jordan basis for A if, for $1 \leq j \leq n$, we have either $Ab_j = \lambda b_j$ or $Ab_j = \lambda b_j + b_{j-1}$, where

λ is an eigenvalue of A that we say is associated with b_j. If $Ab_j = \lambda b_j + b_{j-1}$, we require that the eigenvalue associated with b_{j-1} also be λ.

Theorem 9.9 Jordan Canonical Form of a Square Matrix

Let A be a square matrix. There exists an invertible matrix C such that the matrix J = $C^{-1}AC$ is a Jordan Canonical form. This Jordan canonical form is unique, except for the order of the Jordan bloacks of which it is composed.