Lesson 6: Number Systems: Natural Numbers, Integers, Rationals, Reals, and Complex Numbers

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December 29, 2025

We have now acquired the language of sets, which allows us to collect objects rigorously. The next essential step is to examine the various systems of numbers that physics employs. These systems are not arbitrary inventions; each arises to solve specific problems posed by measurement and calculation, and each extends the previous one to remove limitations.

We begin with the natural numbers, the simplest counting numbers. The set of natural numbers, often denoted $\mathbb{N}$, consists of the positive integers used for counting discrete objects: 1, 2, 3, and so on. Some conventions include 0 in $\mathbb{N}$, others begin at 1; for our purposes in physics, including 0 is convenient, so we take $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. These numbers arise naturally when we count items: the number of particles in a small system, the number of planets, the discrete energy levels in early quantum models.

Natural numbers support addition and multiplication perfectly. If you add or multiply two natural numbers, the result is again a natural number. However, subtraction is problematic: 3 - 5 is not a natural number. To resolve this, we extend the system to the integers.

The set of integers $\mathbb{Z}$ includes all natural numbers, their negatives, and zero: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. Integers allow unrestricted subtraction: the difference of any two integers is another integer. This extension is essential in physics for quantities that can be positive or negative, such as displacement (which has direction), electric charge (+ or -), or angular momentum quantum numbers.

Addition, subtraction, and multiplication are closed in the integers: the result of any of these operations on integers is an integer. Division, however, remains restricted. For example, 3 divided by 2 is not an integer. Measurement in the real world often produces ratios, so we need a system that accommodates division.

The rational numbers $\mathbb{Q}$ consist of all fractions p/q where p and q are integers and $q \neq 0$. We identify fractions that represent the same value, such as $1/2 = 2/4$. Rational numbers are dense: between any two rationals, there is another rational. All arithmetic operations except division by zero are closed in $\mathbb{Q}$.

Rationals suffice for many physical calculations involving exact ratios, such as gear ratios or stoichiometric coefficients in chemistry. Yet they have a profound limitation when confronted with geometry and continuous measurement. The length of the diagonal of a unit square is $\sqrt{2}$, which cannot be expressed as a ratio of integers. This discovery by the ancient Greeks revealed that not all lengths obtainable by ruler and compass are rational.

To describe continuous quantities such as position, time, or field strengths in classical physics, we require the real numbers $\mathbb{R}$. The reals include all rationals and all irrational numbers (those not expressible as fractions), such as $\sqrt{2}$, π, e, and countless others. The real numbers form a continuum: they are complete, meaning every nonempty subset bounded above has a least upper bound, and they are ordered seamlessly without gaps.

We construct the reals rigorously either as equivalence classes of Cauchy sequences of rationals or as Dedekind cuts, but the intuitive picture suffices for now: the real line is a continuous straight line with no holes, where every point corresponds to a length measurable in principle. All classical physical quantities—position, velocity, energy, field values—are modelled as real numbers.

The real numbers are closed under addition, subtraction, multiplication, and division (except by zero). They support limits, derivatives, and integrals, which we will develop later. However, certain physical equations, particularly in quantum mechanics and electromagnetism, lead to negative quantities under square roots. For example, the simple equation $x^2 + 1 = 0$ has no real solution.

To resolve this, we introduce the complex numbers $\mathbb{C}$. A complex number is of the form $a + bi$, where a and b are real numbers and i is a new object satisfying $i^2 = -1$. The real part is a, the imaginary part b. Complex numbers form a plane, with the real axis for a and imaginary axis for bi.

Addition and multiplication extend naturally: $(a + bi) + (c + di) = (a + c) + (b + d)i$, and multiplication follows the distributive law with $i^2 = -1$. Every nonzero complex number has a multiplicative inverse, so $\mathbb{C}$ is a field. Remarkably, every polynomial equation with complex coefficients has solutions in $\mathbb{C}$ (the fundamental theorem of algebra).

Complex numbers are indispensable in physics. They simplify the description of oscillations (using $e^{i\omega t}$), quantum states (wave functions are complex-valued), and electromagnetic waves. Although measurements yield real numbers, the underlying mathematical structure often requires complex numbers for elegance and completeness.

Each extension—from natural to integer to rational to real to complex—preserves the operations of the previous system while removing a limitation. The price is increased abstractness, but the reward is greater descriptive power for physical phenomena.

Summary

The natural numbers $\mathbb{N}$ serve counting. Integers $\mathbb{Z}$ allow signed quantities and subtraction. Rationals $\mathbb{Q}$ accommodate division and ratios. Reals $\mathbb{R}$ provide continuity for classical physics. Complex numbers $\mathbb{C}$ resolve square roots of negatives and underpin quantum mechanics and wave phenomena.

These systems form the numerical foundation upon which all subsequent mathematics and physics will be built.

Practice set

Checkpoint questions

1. To which number system does $\sqrt{2}$ belong, and why is it not rational?
2. Explain why -3 is an integer but not a natural number (including 0 in $\mathbb{N}$).
3. What is the imaginary part of the complex number $4 - 7i$?

Core exercises

1. Classify each of the following numbers by the smallest system containing it: 0, $-5/3$, π, $2 + 3i$, 4.
2. Explain why division by zero is undefined in any of these systems.
3. Give a physical quantity typically modelled by each system: natural, integer, rational, real, complex.

Deepening problems

1. Discuss why classical mechanics can often use real numbers alone, while quantum mechanics requires complex numbers.
2. Consider why the discovery that $\sqrt{2}$ is irrational was shocking to ancient mathematicians and what it implies for measurement.

Solutions

Solutions to checkpoint questions

1. $\sqrt{2}$ is real but irrational because assuming it equals p/q in lowest terms leads to contradiction (both p and q even, violating lowest terms).
2. -3 is negative, while natural numbers (including 0) are non-negative.
3. The imaginary part is -7.

Solutions to core exercises

Worked Example

Exercise 1: Classify the numbers.
0: natural (and integer, rational, real, complex).
$-5/3$: rational (and real, complex).
π: real (irrational) and complex.
$2 + 3i$: complex (not real).
4: natural (and all subsequent systems).
Sanity check: Each system contains the previous ones.

Worked Example

Exercise 2: Division by zero.
Division by zero would require a number x such that $0 \cdot x = 1$ (for $1/0$), but $0 \cdot x = 0$ always. No such x exists in any field.
Sanity check: Preserves consistency of arithmetic.

Worked Example

Exercise 3: Physical quantities.
Natural: number of particles in a discrete system.
Integer: net charge in units of e (considering sign).
Rational: gear ratio or stoichiometric coefficient.
Real: position, velocity, temperature.
Complex: quantum wave function amplitude or phasor in AC circuits.
Sanity check: Matches scale and nature of measurement.

Solutions to deepening problems

Worked Example

Deepening problem 1: Classical vs quantum.
Classical mechanics deals with real trajectories and observables. Quantum mechanics requires superposition and interference, naturally described by complex amplitudes whose magnitudes square to probabilities.
Sanity check: Schrödinger equation uses i.

Worked Example

Deepening problem 2: Irrationality of $\sqrt{2}$.
It showed that geometric lengths are not always commensurate with integer ratios, challenging the Pythagorean belief that all is number (rational). It implies that perfect measurement of certain lengths requires infinite information.
Sanity check: Forces acceptance of irrationals for accurate geometry.

Next Lesson

Lesson 7: Proof Techniques and Mathematical Thinking