Lesson 7: Proof Techniques and Mathematical Thinking

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We have constructed the basic language of sets and the number systems that populate them. With these tools in hand, we now turn to the activity that distinguishes mathematics from mere calculation: proving statements rigorously.

A mathematical proof is a sequence of logical steps that establishes the truth of a statement beyond any doubt, starting from accepted axioms or previously proven results. Proofs are the mechanism by which we build the edifice of mathematics and ensure that physical theories rest on solid ground.

We begin with direct proof, the most straightforward method. To prove a statement of the form "if P then Q," we assume P is true and show, step by step using definitions and known facts, that Q must follow. For example, to prove that the sum of two even integers is even, let the integers be $2m$ and $2n$ for integers m and n. Their sum is $2m + 2n = 2(m + n)$, which is manifestly even.

Proof by contrapositive offers an equivalent but sometimes easier approach. The contrapositive of "if P then Q" is "if not Q then not P." Since these are logically equivalent, proving the contrapositive establishes the original. Consider proving "if n^2 is even, then n is even." The contrapositive is "if n is odd, then n^2 is odd." If $n = 2k + 1$, then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd.

Proof by contradiction assumes the negation of the conclusion and derives a falsehood. To prove $\sqrt{2}$ is irrational, assume it is rational, so $\sqrt{2} = p/q$ in lowest terms. Then $2q^2 = p^2$, so p^2 even implies p even, $p = 2r$, then $2q^2 = 4r^2$, $q^2 = 2r^2$, q even, contradicting lowest terms.

Proof by induction is essential for statements about natural numbers. To prove a property holds for all natural numbers $n \geq n_0$, show the base case (usually $n = n_0$ or $n = 1$) and the inductive step: if it holds for k, then for $k + 1$. For example, the sum of the first n naturals is $n(n + 1)/2$.

We also use existence proofs (construct an example or show one must exist) and uniqueness proofs (show at most one object satisfies the conditions).

Mathematical thinking involves more than technique. We conjecture by observing patterns, attempt counterexamples to test claims, and refine definitions for clarity. We distinguish necessary from sufficient conditions and recognise when a statement is true by definition versus requiring proof.

In physics, proofs ensure consistency of theories. Conservation laws follow from symmetries via Noether's theorem, proven rigorously. Solutions to differential equations are shown unique under suitable conditions, justifying physical predictions.

Rigorous proof prevents errors that intuition might miss. We proceed carefully, verifying each step.

Summary

Proofs establish truth through direct argument, contrapositive, contradiction, induction, and other methods. They rely on precise definitions and logical deduction. Mathematical thinking involves conjecturing, testing, and refining. Proofs underpin the reliability of physical theories.

These techniques will be applied throughout the curriculum.

Practice set

Checkpoint questions

1. What is the contrapositive of "if a number is divisible by 6, then it is divisible by 3"?
2. Explain why proof by contradiction works.
3. What are the two parts of a proof by induction?

Core exercises

1. Prove directly that the sum of two odd integers is even.
2. Use contradiction to show that there is no integer n such that $n^2 = 2$.
3. Prove by induction that $1 + 2 + \cdots + n = n(n+1)/2$ for $n \geq 1$.

Deepening problems

1. Discuss why physicists sometimes accept plausibility arguments while mathematicians demand full proof.
2. Consider how uniqueness proofs ensure that physical solutions are well-defined.

Solutions

Solutions to checkpoint questions

1. "If a number is not divisible by 3, then it is not divisible by 6."
2. Assuming the negation leads to contradiction implies the original must be true.
3. Base case and inductive step.

Solutions to core exercises

Worked Example

Exercise 1: Sum of two odd integers.
Let the integers be $2m + 1$ and $2n + 1$.
Sum $= 2m + 1 + 2n + 1 = 2(m + n + 1)$, even.
Sanity check: Direct from definition.

Worked Example

Exercise 2: No integer square equals 2.
Assume $n^2 = 2$, $n = p/q$ lowest terms.
$p^2 = 2q^2$, p even, $p = 2r$, $4r^2 = 2q^2$, $q^2 = 2r^2$, q even, contradiction.
Sanity check: Similar to $\sqrt{2}$ irrational.

Worked Example

Exercise 3: Sum formula by induction.
Base $n = 1$: $1 = 1(2)/2$.
Assume for k: sum $= k(k + 1)/2$.
For $k + 1$: sum $= k(k + 1)/2 + (k + 1) = (k + 1)(k/2 + 1) = (k + 1)(k + 2)/2$.

Sanity check: Holds for small n.

Solutions to deepening problems

Worked Example

Deepening problem 1: Plausibility vs proof.
Physics often uses approximation and empirical validation where full proof is intractable (many-body systems). Mathematics requires absolute certainty.
Sanity check: Complementary approaches.

Worked Example

Deepening problem 2: Uniqueness.
Picard-Lindelöf theorem ensures unique solutions to ODEs given initial conditions, justifying deterministic classical trajectories.
Sanity check: Prevents ambiguity in predictions.

Next Lesson

Lesson 8: Introduction to Geometry as Structure and Invariance