From the Double Slit to the Schrödinger Equation
A Slow, Honest, Intuitive Derivation

Chase Martin

Contents

8 The Schrödinger Equation   **21**

How to Read This Document

This text is meant for motivated beginners and undergraduates. We assume only:

- Basic algebra,

- Some comfort with functions of a variable,

- A very light idea of derivatives (we explain when we use them).

Every time we introduce a new kind of math or notation, we pause in a yellow box and explain it in plain language.

Our goal is to get from:

- A real experiment with particles and a screen

to

$$i\hbar\frac{\partial\psi}{\partial t} \;=\; \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{x})\right)\psi(\boldsymbol{x},t) \tag{1}$$

without hand waving.

1 The Experimental Starting Point

1.1 The double slit with single particles

Imagine the classic double slit experiment:

- A source sends one particle at a time (electron, photon, etc.).

- The particle travels towards a wall with two narrow openings: left slit L and right slit R.

- Behind the slits there is a detection screen that can record where each particle arrives.

We repeat this experiment many times and look at the pattern of where the particles land.

We can do the experiment in three configurations:

1. Only the left slit is open. Right slit is blocked.

2. Only the right slit is open. Left slit is blocked.

3. Both slits are open at the same time.

For each configuration, and for each horizontal position x on the screen, we can measure how often a particle lands near that position.

Probability distribution in this experiment

If we send N particles (one at a time) and N_x of them land in a small bin around position x, then

$$\text{frequency at } x \approx \frac{N_x}{N}.$$

In the limit of very many particles this becomes a probability density.
We write:

- $P_L(x)$: the probability density of detection at x when only the left slit is open.

- $P_R(x)$: the same when only the right slit is open.

- $P_B(x)$: the same when *both* slits are open.

You can think of P_L, P_R, P_B as smooth curves over x that tell you how likely you are to detect a particle at each position.

1.2 What classical logic would predict

Suppose the particle is a tiny bullet that always goes through either L or R, but never both.
Let us define some events:

- $L = $ "particle went through the left slit",

- $R = $ "particle went through the right slit",

- $X_x = $ "particle arrived in a bin around position x on the screen".

If both slits are open in a classical world, we would say:

$$P_B(x) = P(X_x) = P(X_x \text{ and } L) + P(X_x \text{ and } R). \tag{2}$$

Basic probability rule used here

If two events are mutually exclusive, meaning they cannot both happen at once, then

$$P(\text{either } A \text{ or } B) = P(A) + P(B).$$

In our case, for a classical particle:

- It either goes through L or goes through R.

- These possibilities do not overlap.

So we add the probabilities of arriving at x given each case.

We can rewrite the terms in equation (2) using conditional probabilities:

$$P(X_x \text{ and } L) = P(L)P(X_x \mid L), \tag{3}$$
$$P(X_x \text{ and } R) = P(R)P(X_x \mid R). \tag{4}$$

So we get:

$$P_B(x) = P(L)P(X_x \mid L) + P(R)P(X_x \mid R). \tag{5}$$

Conditional probability

$P(X_x \mid L)$ reads as:

$$P(\text{arrive at } x \mid \text{the particle went through the left slit}).$$

That is: the probability of arriving at x given that we know the particle took the left path.

If the source is symmetric, we expect $P(L) = P(R) = \frac{1}{2}$, so classically we would predict:

$$P_B(x) = \frac{1}{2}P(X_x \mid L) + \frac{1}{2}P(X_x \mid R). \tag{6}$$

The next classical idea is:

> The pattern of detections from particles that go through the left slit does not care whether the right slit is open or closed.

In other words:

$$P(X_x \mid L) \stackrel{\text{classical}}{=} P_L(x), \qquad P(X_x \mid R) \stackrel{\text{classical}}{=} P_R(x). \tag{7}$$

If this is true, then

$$P_B(x) = \frac{1}{2}P_L(x) + \frac{1}{2}P_R(x). \tag{8}$$

Up to an overall factor, this says:

$$P_B(x) \propto P_L(x) + P_R(x). \tag{9}$$

What equation (9) really says
It says that when both slits are open, the pattern you expect is just the sum of the single slit patterns.
That is the classical picture: each slit contributes its own distribution, and when both are open, the screen just sees both contributions added together.

1.3 But the experiment disagrees

In reality, $P_B(x)$ does not look like the sum of two broad bumps.
  Instead:

- $P_L(x)$ is a broad single hump.

- $P_R(x)$ is a similar hump shifted sideways.

- $P_B(x)$ shows many narrow bright and dark fringes.

At some positions x_0, you find:

$$P_L(x_0) > 0, \qquad P_R(x_0) > 0, \qquad P_B(x_0) \approx 0.$$

At other positions you find $P_B(x) > P_L(x) + P_R(x)$.
So equation (9) is *empirically* false.

This is the core mystery

If a particle were just a tiny bullet that chose one slit or the other, and if each slit behaved independently, the both-open pattern would be the sum of the single slit patterns.

The fact that this is not what happens is the main reason quantum mechanics exists at all.

Any explanation must account for:

- Positions where P_B is smaller than $P_L + P_R$,

- Positions where P_B is larger than $P_L + P_R$,

- The fact that changing something on one slit (for example adding a thin piece of glass) can change $P_B(x)$ everywhere.

2 We Need Something More Fundamental Than Probability

2.1 Introducing "pre probabilities"

The fact that $P_B \neq P_L + P_R$ tells us:

> We cannot get the both open pattern by simply adding the probabilities of the two alternatives.

So we try a new idea.

For each path (through L or through R), and for each detection position x, we assign a new quantity:

$$A_L(x), \qquad A_R(x).$$

At this stage, we do not assume what kind of mathematical object A_L or A_R is. We only decide:

- When both slits are open, we first combine these objects:

$$A_{\text{total}}(x) = A_L(x) + A_R(x).$$

- Then we extract the actual probability $P_B(x)$ from $A_{\text{total}}(x)$ somehow.

So we want a rule of the form

$$P_B(x) = F(A_L(x), A_R(x)), \tag{10}$$

and we hope the rule is simple and the same everywhere.

> **What we are doing conceptually**
> We are saying: the thing that we should add is not probability. Probabilities are derived from something deeper, which we call *amplitudes* (we will justify the name later).
> This deeper thing carries both size and some additional information that allows for cancellation and enhancement.

2.2 A very simple and powerful guess

We want:

- $P_L(x)$ to depend only on $A_L(x)$,

- $P_R(x)$ to depend only on $A_R(x)$,

- and when both are present:

$$A_{\text{total}}(x) = A_L(x) + A_R(x).$$

We also want $P(x)$ to be positive and continuous, and for the pattern to show the kind of oscillations seen in experiments.

The simplest choice that works is:

$$P_L(x) = |A_L(x)|^2, \qquad P_R(x) = |A_R(x)|^2, \tag{11}$$

and when both paths are open,

$$P_B(x) = |A_L(x) + A_R(x)|^2. \tag{12}$$

Magnitude and why we square it

Here $|A|$ means the *size* or *length* of the object A.

If A is just a number on a line, $|A|$ is the usual absolute value. If A is a 2D arrow, $|A|$ is its length.

Squaring $|A|$ makes sure the result is always non negative, which is what we need for probability.

Now let us see how this actually produces interference.

2.3 The cross term that saves the day

Expand the square in (12):

$$P_B(x) = |A_L + A_R|^2 \tag{13}$$
$$= |A_L|^2 + |A_R|^2 + \text{cross term}. \tag{14}$$

We already know that

$$|A_L|^2 = P_L(x), \qquad |A_R|^2 = P_R(x).$$

So

$$P_B(x) = P_L(x) + P_R(x) + \text{cross term}. \tag{15}$$

It is this cross term that allows P_B to be sometimes bigger, sometimes smaller than $P_L + P_R$. Its exact form depends on the internal structure of A_L and A_R.

This is what we fix next.

3 Why Amplitudes Must Be Arrows (Complex Numbers)

3.1 The need for direction and angle

To match experiment, the cross term in (15) must:

- Be positive in some regions (enhanced probability),

- Be negative in others (suppressed probability),

- Vary smoothly when we change something in one path (for example, insert a thin piece of glass in front of the left slit).

If A_L and A_R were just ordinary real numbers, then

$$|A_L + A_R|^2 = (A_L + A_R)^2,$$

and the interference behavior would be very limited. You could get some cancellation if A_L and A_R are opposite in sign, but you do not get the rich pattern of many fringes as you move x along the screen.

We need each A to have:

- A size (how big the contribution is),

- A direction or angle (which will encode how it combines with other contributions).

This is exactly what a 2D arrow has.

Arrows (vectors) in the plane
An arrow in the plane can be specified by:

$$\text{length} = r, \qquad \text{angle} = \theta,$$

or by its horizontal and vertical components (x, y).
The length is

$$r = \sqrt{x^2 + y^2}.$$

When you add two arrows, you add their components or use the usual head-to-tail picture.
The key point is that *direction matters*. Adding arrows pointing in the same direction makes a bigger arrow; adding opposite arrows cancels.

So we model each amplitude as an arrow:

$$A_L(x) = \text{arrow with length } r_L(x) \text{ and angle } \theta_L(x),$$
$$A_R(x) = \text{arrow with length } r_R(x) \text{ and angle } \theta_R(x).$$

When we add them, the squared length of the sum is:

$$|A_L + A_R|^2 = r_L^2 + r_R^2 + 2r_L r_R \cos(\theta_L - \theta_R). \qquad (16)$$

Interference from arrow addition

The term

$$2r_L r_R \cos(\theta_L - \theta_R)$$

is the cross term.

- If θ_L and θ_R are equal, $\cos(\theta_L - \theta_R) = 1$, and the contributions add.

- If they differ by 180°, the cosine is -1, and the contributions cancel as much as possible.

- As the angle difference changes smoothly, the cosine oscillates between $+1$ and -1.

This gives exactly the alternating bright and dark regions we observe in $P_B(x)$.

This shows that amplitudes must behave like 2D arrows.

3.2   Complex numbers as arrows

A complex number is simply a convenient way to represent an arrow:

$$z = x + iy,$$

where x and y are real numbers, and i is a symbol with the property $i^2 = -1$.

Complex number as an arrow

The complex number $z = x + iy$ corresponds to the arrow from the origin to the point (x, y) in the plane.

- Real part $\mathrm{Re}(z) = x$ is the horizontal component.

- Imaginary part $\mathrm{Im}(z) = y$ is the vertical component.

- The length is
$$|z| = \sqrt{x^2 + y^2}.$$

Two complex numbers add exactly like arrows: you add their components.

We can also write complex numbers in polar form:

$$z = re^{i\theta},$$

where $r = |z|$ is the length and θ is the angle.

The key fact is:

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}.$$

Why this is useful

Multiplying by $e^{i\theta}$ rotates arrows by angle θ without changing their length.

This makes it easy to describe how phases (angles) change as you change the physical setup, for example adding extra distance in one path.

So from now on, we think of amplitudes as complex numbers:

$$A_L(x), \quad A_R(x) \in \mathbb{C},$$

and probability as

$$P(x) = |A(x)|^2.$$

This is the core of the *Born rule*: the probability is the squared magnitude of the complex amplitude.

4 The Wave Function as a Field of Amplitudes

4.1 From discrete paths to a continuous screen

In the double slit example, we already indexed amplitudes by position x on the screen.

More generally, a quantum particle in space has a complex amplitude associated with each point in space.

We call this function the wave function:

$$\psi(x, t)$$

in one dimension, or

$$\psi(\boldsymbol{x}, t)$$

in three dimensions.

What the wave function is

You can think of $\psi(x, t)$ as a field of arrows over space:

- At each position x, there is a complex number $\psi(x, t)$.

- The squared magnitude $|\psi(x, t)|^2$ is the probability density of finding the particle near x at time t.

- Integrating $|\psi|^2$ over all space gives 1 (total probability).

The word "wave" is historical. The wave function is not a material wave, but a bookkeeping field of complex amplitudes.

4.2 Total probability

We require that the particle exists somewhere with probability 1:

$$\int_{-\infty}^{\infty} |\psi(x, t)|^2 \, dx = 1 \quad \text{for all times } t. \tag{17}$$

This is the statement that the total probability is always 100 percent.
Any rule of time evolution we invent must preserve this property.

5 How Time Evolution Must Look

5.1 First: a single amplitude

Before tackling the full wave function, let us return to the simpler case of a single complex amplitude $c(t)$.

We want a rule for how $c(t)$ changes in time that is:

- linear in c,

- preserves $|c|^2$.

The simplest linear equation is:

$$\frac{dc}{dt} = Kc, \tag{18}$$

where K is some constant (possibly complex).

What equation (18) says
This is a first order linear differential equation. It says:

$$\text{rate of change of } c = \text{some constant} \times c.$$

If c gets larger, its rate of change scales with it. This is the simplest possible time dependence that is smooth and respects superposition.

5.2 Solving the simple equation

Equation (18) has the solution

$$c(t) = e^{Kt}c(0). \tag{19}$$

Exponential solution
If $\frac{dy}{dt} = ky$, then the solution is $y(t) = e^{kt}y(0)$.
We apply the same idea here, but c and K can be complex.

Now we ask: what does probability do?
Probability is $|c(t)|^2$.
Using (19):

$$|c(t)|^2 = |e^{Kt}|^2\,|c(0)|^2.$$

We want $|c(t)|^2$ to be constant in time, that is:

$$|e^{Kt}|^2 = 1 \quad \text{for all } t.$$

So we must analyze $|e^{Kt}|$.

5.3 For probability conservation, the generator must be imaginary

Write K in terms of its real and imaginary parts:

$$K = a + ib,$$

with a, b real numbers.
Then

$$e^{Kt} = e^{(a+ib)t} = e^{at} \cdot e^{ibt}.$$

> **Magnitude of a complex exponential**
> For e^{at} with real a, the magnitude is $|e^{at}| = e^{at}$.
> For e^{ibt}, this is a point moving around a circle in the complex plane, so $|e^{ibt}| = 1$.
> So in general,
>
> $$|e^{Kt}| = |e^{at}||e^{ibt}| = e^{at} \cdot 1 = e^{at}.$$

To keep $|e^{Kt}|$ equal to 1 for all t, we need

$$e^{at} = 1 \quad \Rightarrow \quad a = 0.$$

So K must be of the form

$$K = ib, \qquad b \in \mathbb{R}. \tag{20}$$

> **Why the time evolution generator must be imaginary**
> If there were any real part a in K, the amplitude would grow or shrink exponentially as e^{at}, and the probability $|c|^2$ would not be conserved.
> A purely imaginary K gives a pure rotation in the complex plane: the arrow for c spins around with constant length. That is exactly what we need for probability conservation.

So the only probability preserving linear time evolution for a single amplitude is

$$\frac{dc}{dt} = ib\,c. \tag{21}$$

This is already the seed of the i in the Schrödinger equation.

5.4 Generalizing to a full wave function

Now let $\psi(x, t)$ be a wave function. Conceptually, it is like a vector with infinitely many components (one for each position x).
We want a linear evolution equation of the form

$$\frac{\partial \psi}{\partial t} = K\psi, \tag{22}$$

where K is now some operator that can act on the function ψ. We want the total probability

$$\int |\psi(x,t)|^2 \, dx$$

to be constant in time, just like $|c(t)|^2$ was constant in the simple case.

The same kind of argument (but with integrals and more indices) shows that K must be of the form

$$K = -\frac{i}{\hbar}H,$$

where H is an operator that plays the role of a generalized energy.

> **What an operator is here**
>
> An operator H is a rule that takes a function $\psi(x)$ and produces a new function $H\psi(x)$. Examples:
>
> - Multiply by a function: $(H\psi)(x) = V(x)\psi(x)$.
>
> - Differentiate: $(H\psi)(x) = \frac{d\psi}{dx}$.
>
> - Combine several operations.
>
> You can think of H as an infinite dimensional matrix and ψ as an infinite dimensional column vector.

Putting $K = -\frac{i}{\hbar}H$ into (22) gives:

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}H\psi. \tag{23}$$

Multiply both sides by $i\hbar$:

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi. \tag{24}$$

This is the general form of the time dependent Schrödinger equation. We still have to figure out what H is for a particle in space.

6 Spatial Translations and the Momentum Operator

6.1 How a small shift in position acts on the wave function

Consider a wave function $\psi(x)$ at some fixed time. If we shift the whole system to the right by a small distance a, the new wave function is:

$$\psi_{\text{shifted}}(x) = \psi(x - a). \tag{25}$$

If a is small, we can approximate this using a Taylor expansion:

$$\psi(x - a) \approx \psi(x) - a\frac{\partial \psi}{\partial x}(x). \tag{26}$$

> **Taylor expansion in simple words**
> For a smooth function $f(x)$, when a is small,
>
> $$f(x - a) \approx f(x) - af'(x).$$
>
> This is the first term of the Taylor series around x. It says that near x, the function is roughly a straight line, and moving a small step a changes the value by slope times step.

So the effect of a small shift is:

$$\psi \to \psi - a\frac{\partial \psi}{\partial x}. \tag{27}$$

This suggests that the operator $\frac{\partial}{\partial x}$ is the generator of spatial translations.

6.2 Making the shift preserve probabilities

Just like time evolution, we want spatial translations to preserve total probability:

$$\int |\psi(x)|^2 \, dx$$

should not change under a change of coordinate origin.

To make translations act like rotations in amplitude space (so they preserve $|\psi|^2$), we introduce a factor of $-i\hbar$ in front of the derivative.

This leads us to define the momentum operator

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}. \tag{28}$$

Why define momentum like this

- We already know from classical mechanics that momentum generates spatial motion.

- Here we see that derivatives with respect to x generate spatial shifts.

- The factor $-i$ makes the effect a rotation in complex amplitude space, which preserves norms.

- The factor $\hbar$ fixes units and connects this with measured values of momentum from experiment.

In multiple dimensions, the gradient becomes

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right),$$

and the momentum operator becomes

$$\hat{\boldsymbol{p}} = -i\hbar\boldsymbol{\nabla}.$$

7 Constructing the Hamiltonian from Classical Energy

7.1 Classical energy for a particle in a potential

For a nonrelativistic particle of mass m moving in a potential $V(x)$, the classical energy is

$$E = \frac{p^2}{2m} + V(x). \tag{29}$$

> **Pieces of the classical energy**
>
> - Kinetic energy $T = \frac{p^2}{2m}$ depends on the momentum p and the mass m.
>
> - Potential energy $V(x)$ depends on position x and describes forces like gravity, electric fields and so on.

In quantum mechanics, we want an operator H that plays the role of energy and reduces to (29) in the classical limit.

7.2 Promote p to an operator

The standard rule of quantization is to replace p by the momentum operator $\hat{p}$ from (28):

$$p \to \hat{p} = -i\hbar \frac{\partial}{\partial x}.$$

Then we form

$$H = \frac{\hat{p}^2}{2m} + V(x). \tag{30}$$

Compute $\hat{p}^2$:

$$\hat{p}^2 = \left(-i\hbar \frac{\partial}{\partial x}\right)^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}. \tag{31}$$

So in one dimension,

$$H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x). \tag{32}$$

> **Second derivative and kinetic energy**
> The second derivative $\frac{\partial^2}{\partial x^2}$ measures how curved a function is.
>
> - If ψ is very wiggly (many oscillations), its second derivative is large.
>
> - If ψ is smooth and flat, its second derivative is small.
>
> Large curvature corresponds to large kinetic energy. This matches the idea that high momentum states have rapidly oscillating wave functions.

In three dimensions, the kinetic part uses the Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

so the Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{x}). \tag{33}$$

8 The Schrödinger Equation

8.1 Putting the pieces together

We had already reached the general time evolution law:

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi. \qquad\text{(24 revisited)}$$

We have now constructed H for a single nonrelativistic particle in a potential:

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{x}).$$

Substituting this into the general law gives the explicit Schrödinger equation:

$$i\hbar\frac{\partial\psi(\boldsymbol{x},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{x})\right]\psi(\boldsymbol{x},t). \qquad\text{(34)}$$

What this equation says in words

- The wave function $\psi(\boldsymbol{x},t)$ is a field of complex arrows over space and time.

- Its time rate of change is determined by:

    - The curvature of ψ in space (through ∇^2), which represents kinetic energy,
    - The potential function $V(\boldsymbol{x})$, which represents external forces.

- The factor i guarantees that the evolution is a rotation in the space of amplitudes, not a stretch or shrink, so the total probability stays equal to 1.

8.2 Logical chain from experiment to equation

Let us summarize the logical steps with no mystery:

1. **Experiment:** Double slit with single particles shows that the both open pattern $P_B(x)$ is not equal to the sum of single slit patterns $P_L(x) + P_R(x)$.

2. **Conclusion:** Classical probability addition fails. We need a deeper quantity whose combinations produce the observed probabilities.

3. **Introduce amplitudes:** For each alternative path and each detection point, we introduce amplitudes $A_L(x)$ and $A_R(x)$. We require

$$P(x) = |A(x)|^2,$$

and when both paths contribute,

$$P_B(x) = |A_L(x) + A_R(x)|^2.$$

4. **Need for angles:** The interference pattern (regions where P_B is larger or smaller than $P_L + P_R$) forces amplitudes to have a size and a direction (angle). This means amplitudes are like arrows.

5. **Complex numbers:** Arrows in the plane are represented efficiently by complex numbers. Multiplying by $e^{i\theta}$ rotates amplitudes by angle θ.

6. **Wave function:** For a particle in space, we assign a complex amplitude to each point in space and time. This gives the wave function $\psi(\boldsymbol{x}, t)$ with probability density $|\psi|^2$.

7. **Probability conservation in time:** We demand that the total probability $\int |\psi|^2 d^3x$ stays equal to 1. For a simple amplitude $c(t)$, this forces the time evolution generator to be purely imaginary: $dc/dt = ibc$. Generalizing this, the evolution equation must be of the form

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi.$$

8. **Spatial translations and momentum:** A small shift of the wave function in space is generated by derivatives with respect to position. To preserve norms, the generator takes the form

$$\hat{\boldsymbol{p}} = -i\hbar \boldsymbol{\nabla}.$$

9. **Hamiltonian from classical energy:** Classical energy is $E = \frac{p^2}{2m} + V(\boldsymbol{x})$. In quantum mechanics, we replace p by $\hat{p}$ and get

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{x}).$$

10. **Final equation:** Plugging this H into the time evolution law yields the Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{x})\right)\psi.$$

Nothing was guessed out of thin air

- The need for amplitudes comes directly from experiment.

- The need for angles (phases) comes from the structure of interference.

- Complex numbers are the simplest way to represent arrows with angles.

- The factor i in time evolution comes from demanding that probabilities do not change with time in a linear theory.

- The form of H comes from importing the classical energy expression and replacing p with the derivative based momentum operator.

The Schrödinger equation is therefore not a magical postulate, but the simplest consistent dynamical equation for complex probability amplitudes that matches experiments.