Lesson 5: Sets: Elements, Membership, Operations, Relations, and Functions

Chase Martin

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We have established a precise logical language in previous lessons. We now introduce the most basic mathematical object that underlies everything else in the curriculum: the set.

A set is a well-defined collection of distinct objects, regarded as a single entity. The objects contained in the set are called its elements or members. The requirement that the collection be well-defined is crucial: for any object we might consider, there must be a clear, unambiguous way to determine whether that object belongs to the set or does not belong to it. There is no room for vagueness or partial membership. Furthermore, sets do not recognise order among their elements, and repetition of an element does not alter the set; listing the same element multiple times has no effect.

To illustrate, consider the collection consisting of the eight planets currently recognised as orbiting the Sun: Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, and Neptune. This forms a set. We can decide definitively whether Pluto belongs (by current astronomical convention, it does not) or whether the Moon belongs (it does not, as it orbits Earth). The order in which we name the planets is irrelevant; naming Jupiter first or last changes nothing about the collection itself.

The standard notation for asserting that an object a is an element of a set A is $a \in A$, read aloud as "a is an element of A" or "a belongs to A." The negation, that a is not an element of A, is written $a \notin A$. Thus, we write Earth $\in$ {Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune}, while Pluto $\notin$ that same set under present definitions.

Sets are denoted by enclosing a description of their elements within curly braces {}. When the elements can be listed explicitly, we use roster notation. For example, the set containing the numbers one, two, and three is written $\{1, 2, 3\}$. The set of primary additive colours of light is {red, blue, green}. Observe that $\{1, 2, 3\}$ is identical to $\{3, 2, 1\}$ or $\{1, 1, 2, 3\}$; order and repetition are disregarded.

When explicit listing is impractical because the set is infinite or simply too large, we employ set-builder notation. This takes the form $\{x \mid P(x)\}$, read "the set of all x such that the property P holds for x." The vertical bar separates the placeholder variable from the condition it must satisfy. For instance, the set of all positive even integers is $\{n \mid n \text{ is positive and } n \text{ is even }\}$. In a physical context, the set of all position vectors lying inside or on the surface of a sphere of radius R centred at the origin is $\{\mathbf{r} \mid |\mathbf{r}| \leq R\}$.

One particularly important set is the empty set, which contains no elements whatsoever. It is denoted $\emptyset$ or simply $\{\}$. Examples include the set of all living dinosaurs, the set of human beings taller than ten metres, or (as we shall see later) the set of real solutions to the equation $x^2 + 1 = 0$. There is only one empty set; its uniqueness follows from the fact that any two sets with no elements must be equal.

In many contexts, we work relative to a larger collection called the universal set for that discussion, often denoted U. This universal set contains every object we are currently considering. For example, when studying properties of real numbers, the universal set is typically the set of all real numbers. The complement of a set A with respect to U, written $\overline{A}$ or A^c, consists of every element in U that is not in A.

We now examine operations that produce new sets from existing ones.

The union of two sets A and B, denoted $A \cup B$ and read "A union B," is the set containing every object that belongs to A or to B or to both. It corresponds precisely to the inclusive logical "or." If A is the set of all massive particles and B the set of all charged particles, then $A \cup B$ contains essentially every known elementary particle except neutrinos and photons in certain classifications.

The intersection of A and B, denoted $A \cap B$ and read "A intersection B," contains only those objects that belong to both A and B simultaneously. It corresponds to the logical "and." The intersection of the set of massive particles and the set of charged particles includes protons, electrons, muons, and most quarks.

The set difference $A \setminus B$ (sometimes written $A - B$) consists of those elements that belong to A but not to B. For instance, the set of real numbers minus the set of integers yields all non-integer real numbers.

Inclusion relations are also fundamental. We write $A \subseteq B$ and say "A is a subset of B" if every element of A also belongs to B; equality is allowed. We write $A \subset B$ for proper subset if $A \subseteq B$ and A is not equal to B. The set of rational numbers is a proper subset of the set of real numbers.

Having established basic sets and operations, we extend the idea to relations and functions, which are indispensable for expressing physical laws.

A relation from a set A to a set B is any collection of ordered pairs (a, b) where the first component a comes from A and the second b from B. Formally, it is a subset of the Cartesian product $A \times B$, which consists of all possible ordered pairs.

A function f from A to B, denoted $f : A \to B$, is a special kind of relation with the crucial property that each element of A appears as the first component of exactly one pair. The set A is called the domain, B the codomain, and the set of all second components that actually appear is the range or image of f. Functions provide the mathematical structure for any situation where a unique output is associated with each input. In physics, velocity as a function of time $v(t)$, gravitational potential as a function of position $V(\mathbf{r})$, or the quantum state as a function of position and time $\psi(\mathbf{r}, t)$ are all functions in this sense.

A function is injective (one-to-one) if distinct inputs produce distinct outputs; no two different elements of the domain map to the same element in the codomain. It is surjective (onto) if every element of the codomain is reached by at least one input. A function that is both injective and surjective is bijective, establishing a perfect one-to-one correspondence between domain and codomain. Bijective functions play a central role later when we discuss changes of basis or coordinate transformations.

Functions capture the essence of physical laws: given complete input data (position, time, state), the law specifies a unique outcome or value.

Summary

Sets provide the rigorous foundation for collecting and organising objects without ambiguity. Core notation includes $\in$ and $\notin$ for membership, curly braces for description, set-builder form for properties, $\emptyset$ for the empty set, $\cup$ for union, $\cap$ for intersection, $\setminus$ for difference, and $\subseteq$, $\subset$ for inclusion. Relations are arbitrary pairings; functions require unique outputs, with domain, codomain, range, injectivity, and surjectivity describing their behaviour.

These tools enable precise formulation of the collections and mappings central to physics.

Practice set

Checkpoint questions

1. Explain in words what $\{x \mid x^2 < 4\}$ represents (assuming real x).

2. What is $A \cap A$ for any set A?
3. Is $\emptyset \subseteq$ every set? Why?

Core exercises

1. Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$. Compute $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$.
2. Describe the domain and codomain of the function that gives gravitational potential energy as a function of height near Earth.
3. Give an example of a physical quantity that is a function of position and explain why it is injective or not.

Deepening problems

1. Discuss why physical laws are almost always functions (unique output for given input) rather than general relations.
2. Consider the set of all possible velocities in special relativity. Why is it not the entire real line?

Solutions

Solutions to checkpoint questions

1. The set $\{x \mid x^2 < 4\}$ (for real x) is the open interval from -2 to 2 on the real line, all real numbers whose square is less than 4.
2. The intersection $A \cap A$ is always A itself.
3. Yes, the empty set is a subset of every set because there are no elements in $\emptyset$ that could violate the subset condition (vacuously true).

Solutions to core exercises

Worked Example

Exercise 1: Compute the set operations.
Given $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$.
$A \cup B = \{1, 2, 3, 4, 5\}$ (all elements from either).
$A \cap B = \{3, 4\}$ (common elements).
$A \setminus B = \{1, 2\}$ (in A but not B).
$B \setminus A = \{5\}$ (in B but not A).
Sanity check: The operations are symmetric except for differences, and union contains everything.

Worked Example

Exercise 2: Gravitational potential energy near Earth.
The function is $U(h) = mgh$ (taking zero at ground), where h is height above ground.
Domain: $h \geq 0$ (or all real h if allowing below ground).
Codomain: typically real numbers (or non-negative if $h \geq 0$).
Range: $[0, \infty)$ for $h \geq 0$.
Sanity check: This is a simplified model; actual gravitational potential is $-GMm/r$.

Worked Example

Exercise 3: Physical quantity as function of position.
Example: Electric potential $V(\mathbf{r})$ due to a point charge.
It is a function of position $\mathbf{r}$.
It is not injective: different positions at the same distance have the same potential (spherical symmetry).
Sanity check: Equipotential surfaces are spheres.

Solutions to deepening problems

Worked Example

Deepening problem 1: Why physical laws are functions rather than general relations.
Physical laws are deterministic in classical physics: given complete input (state), there is exactly one output (future state or measured value). General relations would allow multiple or no outputs, violating reproducibility. Even in quantum mechanics, the evolution is unitary (deterministic on states), and measurements have unique probability distributions.
Sanity check: Non-deterministic laws would make prediction impossible.

Worked Example

Deepening problem 2: Velocities in special relativity.
The set of possible velocities for massive particles is $(-c, c)$, open interval excluding $\pm c$. Nothing with mass reaches light speed. Photons have exactly c (or $-c$ directionally).
Sanity check: This follows from Lorentz transformation and energy-momentum relation.

Next Lesson

Lesson 6: Number Systems: Natural Numbers, Integers, Rationals, Reals, and Complex Numbers