A More Robust Optimal Adjustment Set

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Goals and New Research

- Similar to how a trek is associated with covariance, can one find a graphical structure or function of one that defines the bias term.
- In sparse data situations, should researchers include forbidden terms if sample covariance meet certain constraints.

Previous Causal DAGs

- The optimal adjustment set is the unbiased adjustment set that has the *lowest asymptotic variance*.
- Henckel et al. [1] that for Structural Equation Models (SEMs), an optimal adjustment set for finding the causal effect of $T \to Y$ is:

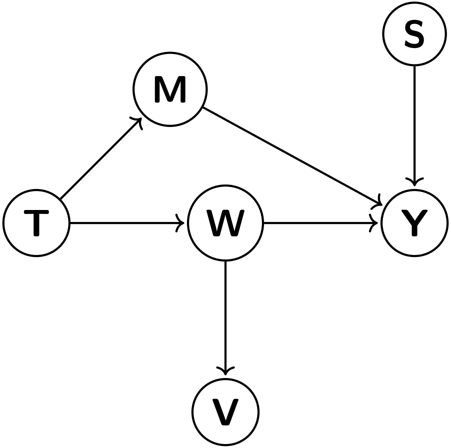
$$O_g(T \to Y) = pa_g(\mathrm{cn}_g(T \to Y)) \setminus (\mathrm{cn}_g(T \to Y) \cup \{T\}).$$

- It was also shown for the non-parametric case [2]

Example

So for instance in the graph below the

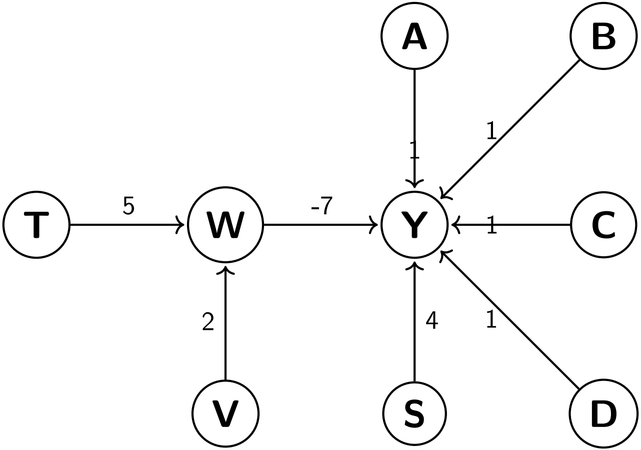
- $cn_G(T \to Y) = \{W, M, Y\}$
- $pa_g(cn_g(T \to Y)) = \{T, W, M, S\}$
- $\implies O_g(T \to Y) = \{S\}$



Graph Example

Consider the Structural Equation Model

- In the graph below, the $O_g(T \to Y) = \{A, B, C, D, V, S\}$



In R

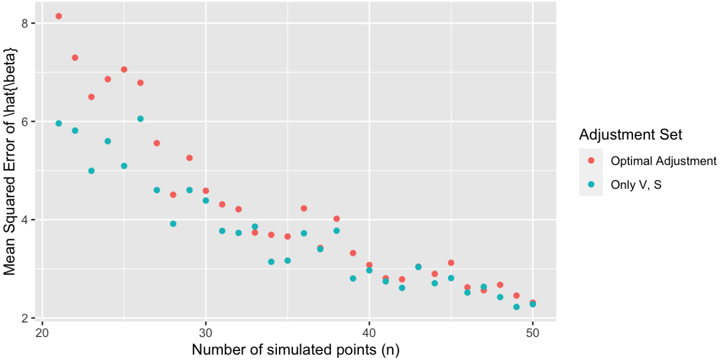
In R, this can be seen as

```
T <- rnorm(n)
V <- rnorm(n)
S <- rnorm(n)
A <- rnorm(n)
B <- rnorm(n)
C <- rnorm(n)
D <- rnorm(n)
E <- rnorm(n)

W <- 5 * T + 2 * V  +  + rnorm(n)
Y <- -7 * W  + A + B + 4 *S + C + D + rnorm(n)
```

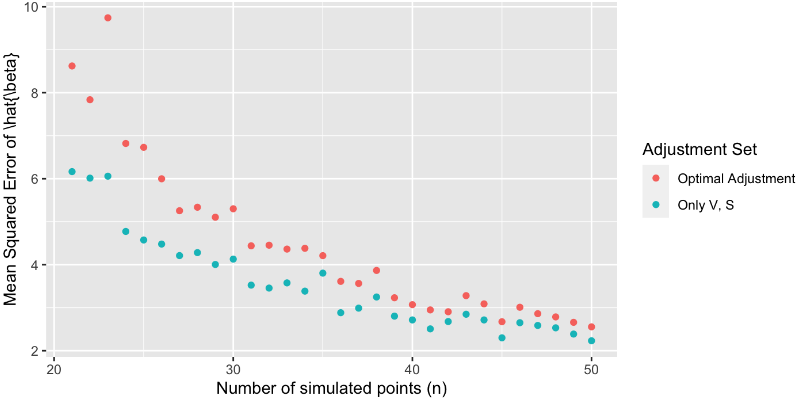
Results

I calculate the MSE of the $\hat{\beta}_T \mid \beta = -35$ over the optimal adjustment set and instead only using the strongly causal nodes V, S, I see the following:



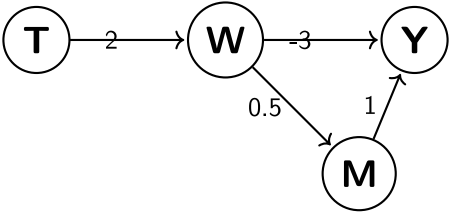
Results 2: Changing Parameters

If I make the variance of A, B, C, D, E small or V, S large, the difference is more pronounced



Forbidden Nodes

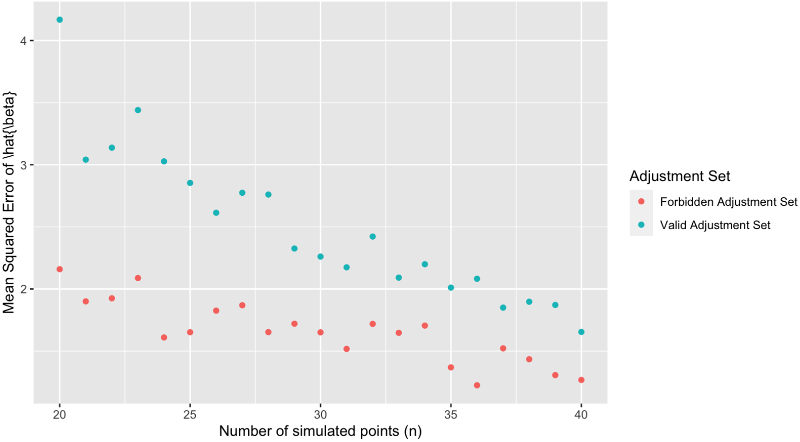
I've also been considering Forbidden Nodes and including them in adjustment sets.



Here M is a forbidden node and including it will give a biased estimate for $T \to Y$

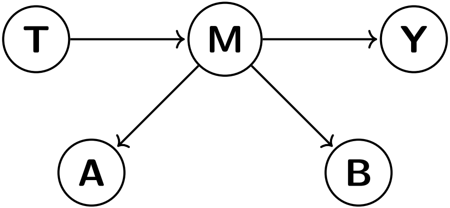
Variable Mediator

If data is sparse and M is quite a bit more variable than the other variables (5x), I found that the MSE of $\hat{\beta_T}$ is smaller when adjusting for M then for not adjusting for M



Adjusting for Two Forbidden Nodes

In the graph shown below, nodes A, B are forbidden as they are descendants of a causal node.



If we adjust for them, however, the bias is a ratio of determinants.

Proof

$$\beta_{ty\cdot ab} = \frac{\sigma_{ty\cdot ab}}{\sigma_{tt\cdot ab}} \tag{1}$$

$$= \frac{\sigma_{ty} - \Sigma_{t,AB}\Sigma_{AB}^{-1}\Sigma_{AB,y}}{|\Sigma_{AB\cdot t}|\sigma_{tt}/|\Sigma_{AB}|} \tag{2}$$

Focusing on the numerator, $\sigma_{ty} = \beta_{ty} * \sigma_{tt} = \beta_{tm}\beta_{my} * \sigma_{tt}$

$$= \frac{\beta_{tm}\beta_{my}\sigma_{tt} - \Sigma_{t,AB}\Sigma_{AB}^{-1}\Sigma_{AB,y}}{|\Sigma_{AB\cdot t}|\sigma_{tt}/|\Sigma_{AB}|} \tag{3}$$

Proof Cont.

Focusing on $\Sigma_{t,AB}\Sigma_{AB}^{-1}\Sigma_{AB,y}$, we get the following:

$$\Sigma_{t,AB}\Sigma_{AB}^{-1}\Sigma_{AB,y} = tr(\Sigma_{t,AB}\Sigma_{AB}^{-1}\Sigma_{AB,y}) \quad (4)$$

$$= tr(\Sigma_{t,AB}^{T}\Sigma_{AB,y}\Sigma_{AB}^{-1}) \quad (5)$$

$$= tr(\begin{bmatrix} \sigma_{ta} & \sigma_{tb} \end{bmatrix} \begin{bmatrix} \sigma_{ay} \\ \sigma_{by} \end{bmatrix} \Sigma_{AB}^{-1}) \quad (6)$$

We can use the trek rule to find that $\sigma_{ay} = \sigma_{mm}\beta_{ma}\beta_{my}$ and $\sigma_{by} = \sigma_{mm}\beta_{mb}\beta_{my}$

$$= \sigma_{mm}\beta_{my} tr(\begin{bmatrix} \sigma_{ta} & \sigma_{tb} \end{bmatrix} \begin{bmatrix} \beta_{ma} \\ \beta_{mb} \end{bmatrix} \Sigma_{AB}^{-1}) \quad (7)$$

$\sigma_{ta} = \beta_{ta}\sigma_{tt}, \sigma_{tb} = \beta_{tb}\sigma_{tt}$

Proof Cont.

$$= \sigma_{mm}\sigma_{tt}\beta_{my} tr(\begin{bmatrix}\beta_{ta} & \beta_{tb}\end{bmatrix}\begin{bmatrix}\beta_{ma} \\ \beta_{mb}\end{bmatrix}\Sigma_{AB}^{-1}) \tag{8}$$

$$\tag{9}$$

Further, $\beta_{ta} = \beta_{tm}\beta_{ma}, \beta_{tb} = \beta_{tm}\beta_{mb}$

$$= \sigma_{mm}\sigma_{tt}\beta_{my}\beta_{tm} tr(\begin{bmatrix}\beta_{ma} & \beta_{mb}\end{bmatrix}\begin{bmatrix}\beta_{ma} \\ \beta_{mb}\end{bmatrix}\Sigma_{AB}^{-1}) \tag{10}$$

$$= \sigma_{mm}\sigma_{tt}\beta_{my}\beta_{tm}(\beta_{ma}^2 + \beta_{mb}^2) tr(\Sigma_{AB}^{-1}) \tag{11}$$

$$= \sigma_{tt}\beta_{my}\beta_{tm}(\sigma_{ma}^2 + \sigma_{mb}^2) tr(\Sigma_{AB}^{-1})/\sigma_{mm} \tag{12}$$

$$= \sigma_{tt}\beta_{my}\beta_{tm}(\sigma_{ma}^2 + \sigma_{mb}^2)(\frac{1}{\sigma_{aa\cdot b}} + \frac{1}{\sigma_{bb\cdot a}})/\sigma_{mm} \tag{13}$$

$$\tag{14}$$

Proof Cont.

Returning to the original equation

$$= \frac{\beta_{tm}\beta_{my}\sigma_{tt} - \sigma_{tt}\beta_{my}\beta_{tm}(\sigma_{ma}^2 + \sigma_{mb}^2)(\frac{1}{\sigma_{aa\cdot b}} + \frac{1}{\sigma_{bb\cdot a}})/\sigma_{mm}}{|\Sigma_{AB\cdot t}|\sigma_{tt}/|\Sigma_{AB}|} \tag{15}$$

$$= \frac{\beta_{tm}\beta_{my}|\Sigma_{AB}|\left(1 - (\sigma_{ma}^2 + \sigma_{mb}^2)(\frac{1}{\sigma_{aa\cdot b}} + \frac{1}{\sigma_{bb\cdot a}})/\sigma_{mm}\right)}{|\Sigma_{AB\cdot t}|} \tag{16}$$

$$= \frac{\beta_{tm}\beta_{my}|\Sigma_{AB}|\frac{1}{\sigma_{mm}}\left(\sigma_{mm} - (\sigma_{ma}^2 + \sigma_{mb}^2)(\frac{1}{\sigma_{aa\cdot b}} + \frac{1}{\sigma_{bb\cdot a}})\right)}{|\Sigma_{AB\cdot t}|} \tag{17}$$

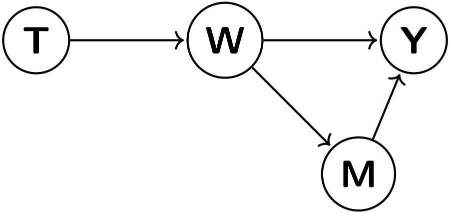
$$= \frac{\beta_{tm}\beta_{my}|\Sigma_{AB}|\frac{1}{\sigma_{mm}}\sigma_{mm\cdot AB}}{|\Sigma_{AB\cdot t}|} \tag{18}$$

$$\sigma_{mm\cdot AB}|\Sigma_{AB}| = |\Sigma_{AB\cdot m}||\sigma_{mm}|$$

$$= \beta_{tm}\beta_{my}\frac{|\Sigma_{AB\cdot m}|}{|\Sigma_{AB\cdot t}|}$$

Adjusting for A Mediator

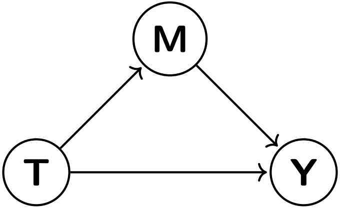
In the graph shown below, nodes M is a mediator, and the bias term is a ratio of schur complements minus a path from $W \to Y$ through only M.



$$\beta_{ty \cdot M} = \beta_{tw}\beta_{wy}\left(\frac{\sigma_{mm \cdot w}}{\sigma_{mm \cdot t}} - \beta_{wm}\sigma_{mm}\beta_{my}\right)$$

Future Work

After talking with Dr. Evans, I plan to look into the simple linear scenario to come up with constraints about whether to include M as a confounding factor.



Perhaps, if one is uncertain if M is a mediator or not, one should still include it if it meets a certain criterion.

References

Leonard Henckel, Emilija Perković, and Marloes H. Maathuis.
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