Sums of Random Variables

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Sum of Independent and Identically Distributed Random Variables:
Let Y be the sum of a sequence of independent and identically distributed X_i's, that is, $Y = \sum_{i=1}^{k} X_i$. The table gives the distribution based on the distribution of X_i.

Distribution	**Sum of Distribution**
$X_i \sim$ Bernoulli(p)	$Y \sim$ Binomial$(k{,}p)$
$X_i \sim$ Binomial(n, p)	$Y \sim$ Binomial$(\sum_{i=1}^{k} n_i, p)$
$X_i \sim$ Poisson(λ)	$Y \sim$ Poisson$(\sum_{i=1}^{k} \lambda_i)$
$X_i \sim$ Geometric(p)	$Y \sim$ Negative Binomial(k, p)
$X_i \sim$ Negative Binomial(r, p)	$Y \sim$ Negative Binomial$(\sum_{i=1}^{k} r_i, p)$
$X_i \sim$ Normal(μ, σ)	$Y \sim$ Normal$(\sum_{i=1}^{k} \mu_i, \sum_{i=1}^{k} \sigma_i^2)$
$X_i \sim$ Exponential(λ)	$Y \sim$ Gamma$(\alpha = k, \beta = \lambda)$
$X_i \sim$ Gamma(α, β)	$Y \sim$ Gamma$(\sum_{i=1}^{k} \alpha_i, \beta)$
$X_i \sim$ Chi-squared(υ)	$Y \sim$ Chi-squared$(\sum_{i=1}^{k} \upsilon_i)$

I will prove a few of the results in the table by the use of the fact that moment generating functions are unique. Thus, if we find the moment generating function of Y, it will reveal the distribution of Y.

Proof 1: $X_i \sim$ Bernoulli(p) $\rightarrow Y \sim$ Binomial(k,p)

$$
\begin{aligned}
M_Y &= E[e^{Yt}] = E[e^{(\sum_{i=1}^{k} X_i)t}] \\
&= E[e^{(X_1+X_2+\cdots+X_k)t}] \\
&= E[e^{(X_1)t}e^{(X_2)t}\cdots e^{(X_k)t}] \\
&= E[e^{(X_1)t}]E[e^{(X_2)t}]\cdots E[e^{(X_k)t}] \\
&= M_{X_1}M_{X_2}\cdots M_{X_k} \\
&= (M_{X_1})^k \\
&= \{(1-p)+pe^t\}^k
\end{aligned}
$$

Thus, $Y \sim$ Binomial(k, p).

Proof 2: $X_i \sim$ Binomial(k, p) $\rightarrow Y \sim$ Binomial($\sum_{i=1}^{k} n_i, p$)

$$
\begin{aligned}
M_Y &= E[e^{Yt}] = E[e^{(\sum_{i=1}^{k} X_i)t}] \\
&= E[e^{(X_1+X_2+\cdots+X_k)t}] \\
&= E[e^{(X_1)t}e^{(X_2)t}\cdots e^{(X_k)t}] \\
&= E[e^{(X_1)t}]E[e^{(X_2)t}]\cdots E[e^{(X_k)t}] \\
&= M_{X_1}M_{X_2}\cdots M_{X_k} \\
&= (M_{X_1})^k \\
&= \{[(1-p)+pe^t]^n\}^k \\
&= [(1-p)+pe^t]^{nk}
\end{aligned}
$$

Thus, $Y \sim$ Binomial($\sum_{i=1}^{k} n_i, p$).

Proof 3: $X_i \sim \text{Poisson}(\lambda) \rightarrow Y \sim \text{Poisson}(\sum_{i=1}^{k} \lambda_i)$

$$\begin{aligned}
M_Y &= E[e^{Yt}] = E[e^{(\sum_{i=1}^{k} X_i)t}] \\
&= E[e^{(X_1+X_2+\cdots+X_k)t}] \\
&= E[e^{(X_1)t} e^{(X_2)t} \cdots e^{(X_k)t}] \\
&= E[e^{(X_1)t}] E[e^{(X_2)t}] \cdots E[e^{(X_k)t}] \\
&= M_{X_1} M_{X_2} \cdots M_{X_k} \\
&= (M_{X_1})^k \\
&= e^{(\lambda k)(e^t-1)}
\end{aligned}$$

Thus, $Y \sim \text{Poisson}(\sum_{i=1}^{k} \lambda_i)$.