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801 Homework 4

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Problem 1:

(a) Show that the F test developed in the first part of this section is equivalent to the (generalized) likelihood ratio test for the reduced versus full models. (b) Find an F test for $H_0\colon X\beta = X\beta_0$ where β_0 is known. (c) Construct a full versus reduced model test when σ^2 has a known value σ_0^2.

Solution: (a) Let the full model be $Y = X\beta + \epsilon$ and the reduced model be $Y = X_0\gamma + \epsilon$, where $\epsilon \sim N(0, \sigma^2 I)$. Denote the likelihood under the full model L_F and the likelihood under the reduced model L_R. Then, the likelihood ratio is

$$r = \frac{\sup L_R(\sigma^2, \gamma)}{\sup L_F(\sigma^2, \beta)} = \frac{(\widehat{\sigma}_R^2)^{-n/2} \exp\{-(Y - X_0\widehat{\gamma})'(Y - X_0\widehat{\gamma})/2\widehat{\sigma}_R^2\}}{(\widehat{\sigma}_F^2)^{-n/2} \exp\{-(Y - X\widehat{\beta})'(Y - X\widehat{\beta})/2\widehat{\sigma}_F^2\}}.$$

First, note that the estimates for σ^2 under the full and reduced model is the MSE under those models, i.e.

$$\widehat{\sigma}_R^2 = \frac{Y'(I - M_0)Y}{n} \quad \text{and} \quad \widehat{\sigma}_F^2 = \frac{Y'(I - M)Y}{n}.$$

Then, we see that we can rewrite the exponentials as

$$\begin{aligned}
\exp\{-(Y - X_0\widehat{\gamma})'(Y - X_0\widehat{\gamma})/2\widehat{\sigma}_R^2\} &= \exp\left\{-\frac{n}{2} \cdot \frac{(Y - M_0Y)'(Y - M_0Y)}{Y'(I - M_0)Y}\right\} \\
&= \exp\left\{-\frac{n}{2} \cdot \frac{Y'Y - Y'M_0Y}{Y'Y - Y'M_0Y}\right\} \\
&= \exp\left\{-\frac{n}{2}\right\}.
\end{aligned}$$

and

$$\exp\{-(Y - X\widehat{\beta})'(Y - X\widehat{\beta})/2\widehat{\sigma}_F^2\} = \exp\left\{-\frac{n}{2}\right\}.$$

Therefore, the ratio becomes

$$r = \left(\frac{\widehat{\sigma}_R^2}{\widehat{\sigma}_F^2}\right)^{-n/2} = \left(\frac{Y'(I - M_0)Y}{Y'(I - M)Y}\right)^{-n/2}.$$

Recall the F statistic is

$$F = \frac{Y'(M - M_0)Y/r(M - M_0)}{Y'(I - M)Y/r(I - M)}.$$

Therefore, we see the two test tests are equivalent.
(c) Recall from section 2.6 that

$$\frac{Y'(I-M)Y}{\sigma^2} \sim \chi^2\big(r(I-M)\big).$$

Then, under H_0, we calculate the test statistic

$$\chi_0^2 = \frac{Y'(I-M)Y}{\sigma_0^2}.$$

Therefore, reject H_0 if $\chi_0^2 < \chi^2\big(\alpha, r(I-M)\big)$ or if $\chi_0^2 > \chi^2\big(1-\alpha, r(I-M)\big)$.

Problem 2:

Redo the tests in Exercise 2.2 using the theory of Section 3.2. Write down the models and explain the procedure. **Exercise 2.2:** Let $y_{11}, y_{12}, ..., y_{1r}$ be $N(\mu_1, \sigma^2)$ and $y_{21}, y_{22}, ..., y_{2s}$ be $N(\mu_2, \sigma^2)$ with all y_{ij}'s independent. Write this as a linear model. Find estimates of $\mu_1, \mu_2, \mu_1 - \mu_2$, and σ^2. Form an $\alpha = .01$ test for $H_0\colon \mu_1 = \mu_2$. Similarly, form 95% confidence intervals for $\mu_1 - \mu_2$ and μ_1. What is the test for $H_0\colon \mu_1 = \mu_2 + \Delta$, where Δ is some known fixed quantity? How do these results compare with the usual analysis for two independent samples?

Solution:

Problem 3:

Redo the tests in Exercise 2.3 using the procedures of Section 3.2. Write down the models and explain the procedure. Hints: (a) Let A be a matrix of zeros, the generalized inverse of A, A^-, can be anything at all because $AA^-A = A$ for any choice of A^-. (b) There is no reason why X_0 cannot be a matrix of zeros. **Exercise 2.3:** Let $y_1, ..., y_n$ be independent $N(\mu, \sigma^2)$. Write a linear model for these data. Form an $\alpha = .01$ test for $H_0\colon \mu = \mu_0$, where μ_0 is some known fixed number and form a 95% confidence interval for μ. How do these results compare with the usual analysis for one sample?

Solution:

Problem 4:

Show that $\beta' X' M_{MP} X\beta = 0$ if and only if $\Lambda'\beta = 0$.

Solution:

Problem 5:

Consider a set of seemingly unrelated regression equations

$$Y_i = X_i\beta_i + e_i, \qquad e_i \sim N(0, \sigma^2 I),$$

$i = 1, ..., r$, where X_i is an $n_i \times p$ matrix and the e_is are independent. Find the test for $H_0\colon \beta_1 = ... = \beta_r$.

Solution: