HW1

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1.) Sums

a) $\sum_{i=1}^{330} \left(\frac{1}{3}\right)^i \rightarrow \left(\text{Geom}, r=\frac{1}{3}, a_i = \left(\frac{1}{3}\right)^i\right)$

$$a_i \frac{1-r^i}{1-r} \rightarrow \frac{1}{3} \cdot \frac{1-\left(\frac{1}{3}\right)^{330}}{1-1/3} = \frac{1-\left(\frac{1}{3}\right)^{330}}{3\cdot(2/3)} = \underline{\frac{1}{2}}$$

b) $\sum_{i=0}^{\infty} \left(\frac{2}{9}\right)^i \rightarrow \text{Geom}, r = 2/9, a_i = \left(\frac{2}{9}\right)^i$

$$a_i \frac{1-r^i}{1-r} \rightarrow 1 \cdot \frac{1-\left(\frac{2}{9}\right)^i}{1-2/9} = \lim_{i\to\infty} \frac{9\left(1-\left(\frac{2}{9}\right)^i\right)}{7} = \underline{\frac{9(1)}{7}}$$

c) $\sum_{i=1}^{N} (i^3 + 3i^2 - 5i + 7) \rightarrow \sum_{i=1}^{N} i^3 + \sum_{i=1}^{N} 3i^2 - \sum_{i=1}^{N} 5i + \sum_{i=1}^{N} 7$

- $\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$
- $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$
- $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$
- $\sum_{k=1}^{n} a = a \cdot n$

$$\rightarrow \frac{1}{4}N^2(N+1)^2 + \frac{3}{6}N(N+1)(2N+1) - \frac{5}{2}N(N+1) + 7N$$

$$= \frac{N^2(N+1)^2}{4} + \frac{N(N+1)(2N+1)}{2} - \frac{5N(N+1)}{2} + 7N$$

$$= \frac{N^2(N+1)^2}{4} + \frac{N(N+1)(2N+1)\cdot 2}{4} - \frac{10N(N+1)}{4} + 7N$$

$$= \underline{\frac{N^4 + 6N^3 - 3N^2 - 8N}{4} + 7N}$$

2) Exponents & Logs

a) $x^1 \cdot x^2 \cdot x^3 \cdots x^{330} \rightarrow \prod_{k=1}^{330} x^k = \underline{x^{54615}}$

b) $\log_x x^{330x} \rightarrow 330x \log_x(x) \rightarrow 330x(1) = \underline{330x}$

- $\log_a(x^b) = b \cdot \log_a(x)$ if $x \geq 0$
- $\log_a(a) = 1$

c) $\log_{330}(330^{330} \cdot 330) \rightarrow \log_{330}(330^{331}) \rightarrow 331 \log_{330}(330)$
$= \underline{331}$

- $\log_a(a) = 1$
- $\log_a(x^b) = b \cdot \log_a(x)$ if $x \geq 0$

3) Combinatorics

a) Considering only 12 hexadecimal digits, 0-9 + E + F, there are $(12)^{33} = 4.102 \times 10^{35}$ combinations in 33 digits.

b) Integral sol. for $x_1 + x_2 + x_3 = 33$ (*) w/ conditions $x_1 \geq 5$, $x_2 \geq 2$, & $x_3 \geq -3$

$k=3$ → $y_1 = x_1 - 5$ $y_2 = x_2 - 2$ $y_3 = x_3 + 3$

An integral sol. of (*) corresponds to an integral sol. of $y_1 + y_2 + y_3 = 29$

$\{(y_1, y_2, y_3, y_4) : y_1 \ldots y_4 \in \mathbb{Z}_+ \text{ and } y_1 + \ldots + y_4 = 29\} = \binom{29+k-1}{k-1} = \binom{31}{2}$

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)} = 465$$

4.) Fibonacci

a) Let $F_n = F_{n-1} + F_{n-2}$, $F_0 = 1$, $F_1 = 1$

Prove $F_n \geq 2^{0.5n}$ for $n \geq 2$

Base case $\xrightarrow{n=2}$ $F_2 = F_{n-1} + F_{n-2} \rightarrow F_1 + F_0 = 2$

$2 \geq 2^{(.5 \cdot 2)} \rightarrow 2 \geq 2^1$ TRUE

Induction step → Show $F_k \geq 2^{0.5k}$

↳ assume true for F_k,

$F_{k-1} \geq 2^{0.5(k-1)}$

$F_{k+1} \geq 2^{0.5(k+1)}$

$F_{k+1} \geq 2^{0.5k} + 2^{0.5k-0.5}$

$$\frac{2^{0.5k} + 2^{0.5k-0.5}}{2^{0.5}} \geq 2^{0.5k+0.5}$$

$2^{0.5}(2^{0.5k}) + 2^{0.5k} \geq 2(2^{0.5k})$

$2^k + 2^{0.5k} \geq 2^{0.5k+1}$

b) Let $F_n = F_{n-1} + F_{n+2}$, $F_0 = 1$, $F_1 = 1$, $F_{n+2} = F_n + F_{n+1}$

Prove $F_n \leq 2^{cn}$ for all $c \geq 0$ where $c < 1$

Base case $\rightarrow$ $F_2 = F_1 + F_0 = 2 \rightarrow 2 \leq 2^{c(2)}$

$\rightarrow \frac{1 \leq 2c}{2} = \frac{1}{2} \leq c$

✓ TRUE $\rightarrow \frac{1}{2} \leq c < 1$

Induction step $\rightarrow$ Assume for $k \geq 2$, $F_k \leq 2^{ck}$, $F_{k+1} \leq 2^{c(k+1)}$, $F_{k+2} \leq 2^{c(k+2)}$

$F_{k+2} = F_k + F_{k+1} \leq 2^{ck} + 2^{c(k+1)} = 2^{ck}(1+2^c) = (1+2^c)\, 2^{c(k+2)} = 2^{c(k+2)}$

$\frac{1+2^c}{2^{2c}} = \frac{2^{c(k+2)}}{2^{c(k+2)}} \rightarrow (1+2^c)\,2^{-2c} = 1$ $\quad u = 2^c \quad 2^c = \frac{1+\sqrt{5}}{2}$

$(1+u)(u^{-2}) = 1$ $\quad c = \frac{\ln\left|\frac{1+\sqrt{5}}{2}\right|}{\ln|2|} < 1$

$\hookrightarrow u = \frac{1+\sqrt{5}}{2}, \cancel{\frac{1-\sqrt{5}}{2}}$

TRUE ✓

5.) Program Understanding

Value of sum at the end of the loop?

```
int sum = 0;

L1 → for(int i = n; i>0; i--) {
L2 →    for (int j = n-i; j<n; j++) {
            sum = sum + j*j;
        }
}
```

$L_1 \rightarrow = (n-1)\times$

$L_2 \rightarrow = (n-i-1)\times$ $\quad$ ~~L2 × L1~~

$\hookrightarrow j = 0,\ (n-1)\times \rightarrow L2 \times L1 \rightarrow (n-1)(n-1) = n^2 - 2n + 1$