HW3

$c < \log_b a$ — $T(n) = O(n^{\log_b a})$
$c = \log_b a$ — $T(n) = O(n^c \log n)$
$c > \log_b a$ — $T(n) = O(f(n))$

Chase Manuald
BZ
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Recurrence Relations

1a) • subproblems - 4 size - $n/2$ time - linear $a=4, b=2$
$c=1$

$T(n) = 4T(n/2) + O(n)$ $\log_b a = 2$

$c < \log_b a \Rightarrow 1 < \log_2 4 \Rightarrow 1 < 2$

time complexity: $O(n^{\log_b a}) \Rightarrow O(n^2)$

• subproblems - 2 size - $n-1$ time - $O(k)$ - constant

$$\begin{aligned} T(n) &= 2T(n-1) + k \\ &= 2(2(T(n-2)) + k) + k \\ &= 2(2(2T(n-2) + k) + k) + k \\ &= \ldots \\ &= 2^n T(n-a) + k(2^a - 1) \end{aligned}$$

where $a = n$

$$\begin{aligned} T(n) &= 2^n T(0) + k(2^n - 1) \\ &= 2^n (T(0) + k) - k \end{aligned}$$

time complexity: $O(2^n)$

• subproblems - 9 size - $n/3$ time - $O(n^2)$ $a=9, b=3,$
$c=2$

$T(n) = 9T(n/3) + n^2$ $\log_b a = 2$

$c = \log_b a \Rightarrow 2 = 2$

time complexity: $O(n^2 \log n)$

1b) • $T(n) = 5T(n/3) + n^3$ $a=5$ $b=3$ $k=3$ $p=0$

$a < b^k \Rightarrow 5 < 3^3$ $\therefore T(n) = O(n^k \log^p n) = O(n^3)$

• $T(n) = 2T(n/4) + 3\sqrt{n}$ $a=2$ $b=4$ $k=1/2$ $p=0$

$2 = 4^{1/2} \Rightarrow a = b^k$

$a = b^k$ & $p > -1$, $T(n) = O(n^{\log_b a} \log^{p+1} n)$

$$\begin{aligned} &= O(n^{\log_4 2} \log n) \\ &= O(\sqrt{n} \log n) \end{aligned}$$

1b) • $T(n) = T(n-1) + \log n$

$T(n) = T(n-2) + \log(n-1) + \log n$

$= T(n-3) + \log(n-2) + \log(n-1) + \log n$

$= \dots$

$\to \log a + \log b = \log ab$

$= T(1) + \log(2) + \log(3) + \dots \log(n-1) + \log n$

$= 1 + \log n!$

$= O(\log n!)$

• $T(n) = n(T(n/2))^3$ $\rightarrow$ $T(n) = n\left(\frac{n}{2}\left(T\left(\frac{n}{2\cdot 2}\right)\right)^3\right)^3$

$T\left(\frac{n}{2}\right) = \frac{n}{2}\left(T\left(\frac{n}{2\cdot 2}\right)\right)^3$

$T\left(\frac{n}{2^k}\right) = \frac{n}{2^k}\left(T\left(\frac{n}{2^{k+1}}\right)\right)^3$

$T(n) = \frac{1}{2^3}\cdot n \cdot n^3 \cdot T\left(\frac{n}{2^2}\right)^{3^3}$

when $T\left(\frac{n}{2^{k+1}}\right) = T(1)$

$T(n) = n^{3(\log_2 n - 1) + 1}$

$T(n) = O\left(n^{3\log_2 n - 2}\right)$

• $T(n) = T\left(\frac{n}{2}\right) + 2^n$

$T\left(\frac{n}{2}\right) = T\left(\frac{n}{2^2}\right) + 2^{n/2}$

$T\left(\frac{n}{2^k}\right) = T\left(\frac{n}{2^{k+1}}\right) + 2^{n/2^k}$

$T\left(\frac{n}{2^{k+1}}\right) \Rightarrow T(1) = 1$

$\frac{n}{2^{k+1}} = 1$ $\quad n = 2^{k+1}$ $\quad k = \log_2 n - 1$

$T\left(\frac{n}{2^2}\right) = T\left(\frac{n}{2^{2+1}}\right) + 2^{n/2^2}$

$T(n) = T\left(\frac{n}{2^{2+1}}\right) + 2^{n/2^2} + 2^{n/2} + 2$

$T(n) = T\left(\frac{n}{2^2}\right) + 2^{n/2} + 2^n$

$= T\left(\frac{n}{2^{k+1}}\right) + 2^{n/2^k} + 2^n$

$T(n) = T(1) + 2^n\left(2^{\frac{1}{k+1}} + 2^{\frac{1}{k}} + 2^{\frac{1}{k-1}} \dots\right)$

$= 2^n \log n$

$\boxed{T(n) = O(2^n \log n)}$

2 a) Time complexity: $O(n)\cdot O(\log n)\cdot O(\log n)$

$= O(n \log^2 n)$

loops 1 & 2 → $\log n$

loop 3 → $O(n)$

b) $T(n) = 3\left(\frac{2}{3}n\right) + O(1)$ $\quad a=3$ $\quad b=\frac{3}{2}$ $\quad d=0$

$\log_{3/2} 3 = 2.7095$

$T(n) = O\left(n^{\log_b a}\right)$

$= O\left(n^{2.7095}\right)$