II.1 Discrete Random Variables

- A **random variable** is a mapping that assigns real numbers to outcomes in the sample space.
- Random variables are denoted by capital letters (such as X) and their specific values are denoted by lowercase letters (such as x).
- The **range** of a random variable X is denoted by S_X.

II.1.1 Probability Mass Function

- The **probability mass function (PMF)** specifies the probability that a discrete random variable X takes the value x:
$$P_X(x) = \mathsf{P}[X = x].$$
- The PMF satisfies the following basic properties:
  1. **Non-negativity:** $P_X(x) \geq 0$ for all x.
  2. **Normalization:** $\sum_{x \in S_X} P_X(x) = 1$.
  3. **Additivity:** For any event $B \subset S_X$, the probability that X falls in B is
  $$\mathsf{P}[B] = \sum_{x \in B} P_X(x).$$
  Note that $\mathsf{P}[B]$ implicitly refers to the event $\mathsf{P}[B] = \mathsf{P}\big[\{X \in B\}\big]$.

II.1.2 Cumulative Distribution Function

- The **cumulative distribution function (CDF)** returns the probability that a random variable X is less than or equal to a value x:
$$F_X(x) = \mathsf{P}[X \leq x].$$
- The CDF satisfies the following basic properties:
  - **Normalization:** $F_X(-\infty) = 0$ and $F_X(\infty) = 1$.
  - **Non-negativity:** $F_X(x)$ is a non-decreasing function of x.
  - For $b \geq a$, $F_X(b) - F_X(a) = \mathsf{P}[a < X \leq b]$.
  - $F_X(x)$ is piecewise constant and jumps at points x where $P_X(x) > 0$ by height $P_X(x)$.

II.2 Expectation

II.2.1 Expected Value

- The **expected value** of a discrete random variable X is
$$\mathsf{E}[X] = \sum_{x \in S_X} x\, P_X(x).$$
- This is also known as the **mean** or **average.**
- Sometimes denoted as $\mu_X = \mathsf{E}[X]$.

II.2.2 Variance

- The **variance** measures how spread out a random variable is around its mean,
$$\mathsf{Var}[X] = \mathsf{E}\Big[\big(X - \mathsf{E}[X]\big)^2\Big] = \sum_{x \in S_X} (x - \mu_X)^2 P_X(x).$$
- Alternate formula: $\mathsf{Var}[X] = \mathsf{E}[X^2] - \big(\mathsf{E}[X]\big)^2$.
- **Standard Deviation:** $\sigma_X = \sqrt{\mathsf{Var}[X]}$.
- The variance is sometimes written as $\sigma_X^2 = \mathsf{Var}[X]$.

II.2.3 Moments

- n^{th} **Moment:** $\mathsf{E}[X^n] = \sum_{x \in S_X} x^n P_X(x)$.
- n^{th} **Central Moment:** $\mathsf{E}\Big[\big(X - \mathsf{E}[X]\big)^n\Big] = \sum_{x \in S_X} (x - \mu_X)^n P_X(x)$.

II.3 Functions of a Random Variable

- A **function** $Y = g(X)$ of a discrete random variable X is itself a discrete random variable.
- Sometimes referred to as a **derived random variable.**
- Range: $S_Y = \{g(x) : x \in S_X\}$.
- PMF: $P_Y(y) = \sum_{x : g(x) = y} P_X(x)$.
- Expected Value: $\mathsf{E}[Y] = \sum_{y \in S_Y} y\, P_Y(y) = \sum_{x \in S_X} g(x)\, P_X(x)$.
- **Linearity of Expectation:** $\mathsf{E}[aX + b] = a\,\mathsf{E}[X] + b$.
- **Variance of a Linear Function:** $\mathsf{Var}[aX + b] = a^2\,\mathsf{Var}[X]$.

II.4 Important Families of Discrete Random Variables

II.4.1 Bernoulli Random Variables

- X is a **Bernoulli**(p) random variable if it has PMF
$$P_X(x) = \begin{cases} 1 - p & x = 0, \\ p & x = 1 \end{cases} .$$
- Range: $S_X = \{0, 1\}$.
- Expected Value: $\mathsf{E}[X] = p$.
- Variance: $\mathsf{Var}[X] = p(1 - p)$.
- Interpretation: single trial with success probability p.

II.4.2 Geometric Random Variables

- X is a **Geometric**(p) random variable if it has PMF

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

- Range: $S_X = \{1, 2, \ldots\}$.
- Expected Value: $\mathsf{E}[X] = \dfrac{1}{p}$.
- Variance: $\mathsf{Var}[X] = \dfrac{1-p}{p^2}$.
- Interpretation: # of independent Bernoulli(p) trials until first success.

II.4.3 Binomial Random Variables

- X is a **Binomial**(n, p) random variable if it has PMF

$$P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

- Range: $S_X = \{0, 1, \ldots, n\}$.
- Expected Value: $\mathsf{E}[X] = np$.
- Variance: $\mathsf{Var}[X] = np(1-p)$.
- Interpretation: # of successes in n independent Bernoulli(p) trials.

II.4.4 Discrete Uniform Random Variables

- X is a **Discrete Uniform**(k, ℓ) random variable if it has PMF

$$P_X(x) = \begin{cases} \dfrac{1}{\ell - k + 1} & x = k, k+1, \ldots, \ell, \\ 0 & \text{otherwise.} \end{cases}$$

- Range: $S_X = \{k, k+1, \ldots, \ell\}$.
- Expected Value: $\mathsf{E}[X] = \dfrac{k + \ell}{2}$.
- Variance: $\mathsf{Var}[X] = \dfrac{(\ell - k)(\ell - k + 2)}{12} = \dfrac{(\ell - k + 1)^2 - 1}{12}$.
- Interpretation: equally likely to take any value between k and ℓ.

II.4.5 Poisson Random Variables

- X is a **Poisson**(α) random variable if it has PMF

$$P_X(x) = \begin{cases} \dfrac{\alpha^x}{x!} e^{-\alpha} & x = 0, 1, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

- Range: $S_X = \{0, 1, \ldots\}$.
- Expected Value: $\mathsf{E}[X] = \alpha$.
- Variance: $\mathsf{Var}[X] = \alpha$.
- Interpretation: # of arrivals in a fixed time window.

II.5 Conditional Probability Models

- The **conditional PMF** of X given an event B is

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{\mathsf{P}[B]} & x \in B \\ 0 & \text{otherwise.} \end{cases}$$

  The conditional PMF satisfies the basic PMF properties of non-negativity, normalization, and additivity.

- The **conditional expected value** of X given an event B is

$$\mathsf{E}[X|B] = \sum_{x \in S_X} x P_{X|B}(x) \ .$$

- The **conditional expected value of a function** $g(X)$ given an event B is

$$\mathsf{E}\big[g(X)|B\big] = \sum_{x \in S_X} g(x) P_{X|B}(x) \ .$$

- We can interpret our conditional probability tricks in the context of the conditional PMF:
  - **Multiplication Rule:** For a random variable X and event B,

$$\mathsf{P}\big[\{X = x\} \cap B\big] = P_{X|B}(x)\,\mathsf{P}[B] = \begin{cases} P_X(x) & x \in B \\ 0 & \text{otherwise.} \end{cases}$$

  - **Total Probability Theorem:** For a partition $B_1, \ldots, B_n$, we can write the PMF as a weighted sum of conditional PMFs,

$$\mathsf{P}_X(x) = \sum_{i=1}^{n} P_{X|B_i}(x) \mathsf{P}[B_i] \ .$$

  - **Bayes' Theorem:** We can "flip" the conditioning,

$$\mathsf{P}\big[B\big|\{X = x\}\big] = \frac{P_{X|B}(x)\mathsf{P}[B]}{P_X(x)} \ .$$