Chase
Mandld
U18719879

EK381 HW 9

1a) Markov's inequality: $P[X \geq c] \leq \frac{E[X]}{c}$

$\rightarrow P[X \geq 6] \leq 3/6 = \boxed{1/2}$

$P[X \geq 6] = 1 - (P(x=0) + P(x=1) + \ldots + P(x=5))$

$= 1 - 0.916082 = \boxed{0.08}$

The upper bound (1/2) > the exact value (0.08), satisfying the inequality.

b) $P[|X - E[X]| \geq c] \leq \frac{Var[X]}{c^2}$. Chebyshev's inequality

$\rightarrow P[|X - E[X]| \geq 2] \leq 3/2^2 = \boxed{3/4}$

$P[|X-3| \geq 2] = P[X \geq 5] = 1 - (P(x=0) + \ldots P(x=4))$

$= 1 - 0.815263$

$= 0.184$

$= P[X \leq 1] = P(x=0) + P(x=1)$

$= 0.199$

$P[X \geq 5] + P[X \leq 1] = \boxed{0.383}$

The upper bound (3/4) is closer to the exact value (0.383) in the case of Chebyshev's inequality compared to Markov's inequality.

c) Weak Law of Large Numbers: $P\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - E[X]\right| \geq c\right] \leq \frac{Var[X]}{nc^2}$

weak law population mean $\rightarrow$ sample mean

Chebyshev's inequality implies $P[|X - E[X]| \geq c] \leq \frac{Var[X]}{c^2}$

where $\rightarrow P[|X - E[X]| \geq c] \leq \frac{300}{60^2} = \boxed{0.0833}$

may serve as the upper bound

1d) Central Limit Theorem: $F_{W_n}(w) \approx \Phi\left(\frac{w - E(X)}{\sqrt{n Var(X)}}\right)$

where W_{100} is a Normal w/ mean 300 & Var = 100(3) = 300

$P[|W_{100} - 300| \geq 60] = P[W_{100} > 360] + P[W_{100} < 240]$
$= 0.000266 + 0.000266$
$\boxed{= 0.000532}$

2) N_i = # of flips until one head $i = 1, 2, \ldots, 100$

Given $N_i \sim$ Geometric ($\frac{1}{2}$)

$E[N_i] = \sum_{i=1}^{\infty} i/2^i = \sum_{i=1}^{\infty} ip^i$ where $p = \frac{1}{2}$

$\underline{p(1-p)^{-2}} = \frac{1}{2}(1-\frac{1}{2})^{-2} = \frac{1}{2}(\frac{1}{2})^{-2}$
$= (\frac{1}{2})^{-1}$
$\underline{E[N_i] = 2}$

$= p + 2p^2 + 3p^3 \ldots$
$= p(1 + 2p + 3p^2 \ldots)$
$\underline{= p(1-p)^{-2}}$

$E[N_i^2] = \sum_{i=1}^{\infty} i^2 \frac{1}{2^i} = \sum_{i=1}^{\infty} i^2 p^i$ where $p = \frac{1}{2}$
$= \sum_{i=1}^{\infty} (i(i-1) + i) p^i$
$= \sum_{i=1}^{\infty} (i(i-1)p^i) + \sum_{i=1}^{\infty} ip^i$
$= p^2 \sum_{i=1}^{\infty} (i(i-1)p^{i-2}) + E[N_i]$
$= p^2 \sum_{i=1}^{\infty} \frac{d^2}{dp^2} p^i + 2$
$= p^2 \frac{d^2}{dp^2}\left(\frac{p}{1-p}\right) + 2$
$= \frac{2p^2}{(1-p)^3} + 2 = \underline{6}$

By the Central Limit Theorem, $NT \sim N(E[NT], Var[NT])$
$\sim N(100, 200)$

$\rightarrow P[NT \leq 80] = P\left[\frac{NT - 100}{\sqrt{200}} \leq \frac{80 - 100}{\sqrt{200}}\right]$
$= P\left[\frac{NT - 100}{\sqrt{200}} \leq -1(414)\right]$
$\boxed{= \Phi(-1(414)) = 0.079}$

$Var[N_i] = E[N_i^2] - (E[N_i])^2 = 6 - 2^2 = \underline{2}$
$E[NT] = E[\sum_{i=1}^{100} N_i] - 100 = 200 - 100 = \underline{100}$
$Var[NT] = Var[\sum_{i=1}^{100} N_i - 100] = \sum_{i=1}^{100} Var[N_i] = \underline{200}$

3a) $P\left[\frac{1}{n}\left|\sum_{k=1}^{n} X_k - p\right| \geq c\right] \geq 1/4nc^2$

$\frac{1}{4nc^2} = 0.955$, $n = \frac{1}{4(0.02)^2(0.955)} = \boxed{654}$

b) $0.02 > Z\sqrt{\frac{0.5(1-0.5)}{n}} \Rightarrow \underline{n > 2500}$

b)

4a) $T_i \sim \text{exponential}(\lambda)$

$E[T_i] = 100$

$1/\lambda = 100 \Rightarrow \lambda = 1/100$

$Var[T_i] = 1/\lambda^2 = 1/(1/100)^2 = \boxed{10\,000}$

b) $T = \frac{1}{10000} \sum T_i$

$E[T] = 1/10000 \cdot E[\sum T_i]$

$= 1/10000 \cdot 10000 \cdot 100 = \boxed{100}$

c) $V[T] = \frac{1}{(10000)^2} \cdot V[\sum T_i]$

$= 1/(10000)^2 \cdot 10\,000 \cdot 10000 = 1$

By CLT, $T \sim N(E[T], V[T])$

$P[T > 110] = P\left[\frac{T - E[T]}{\sqrt{V[T]}} > \frac{110-100}{1}\right]$

$= P[Z > 10]$

$= 1 - P[Z < 10]$

$= \underline{1 - \phi(10)}$