EK381 HW8

Chase
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1a) $\hat{x}_{MAP}(y) = E[X] + \frac{cov[X,Y]}{var[Y]}(y - E[Y])$
$= -4/2$

b) $E[e^2] = Var[X] - \frac{cov[X,Y]^2}{Var[Y]} = 1/2$

2a) $P_X(x) = (.4, .3, .15)$ for $x = 1, 2, 3, 4$
$P_Y(y) = (.2, .25, .55)$ for $y = 1, 2, 3$

b) X & Y aren't independent b/c the table has zeroes where Σ any row // Σ any column isn't zero.

c) $P_{X|Y}(x|2) = \frac{P_{X,Y}(x,2)}{P_Y(2)} = (0, .8, .2, 0)$ for $x = 1, 2, 3, 4$

d) $P_{X|Y}(x=1|Y=3) = .3/.55 \approx .55$
$P_{X|Y}(x=2|Y=3) = .1/.55 \approx .182$
$P_{X|Y}(x=3|Y=3) = .1/.55 \approx .182$
$P_{X|Y}(x=4|Y=3) = .05/.55 \approx .091$

e) MAP estimate is the largest joint probability entry, which is $\hat{x}_{MAP}(2) = 2$
posterior probability of error is $.05/.25 = \underline{.2}$

f) $E[X|Y=2] = 1 \cdot 0 + 2 \cdot .8 + 3 \cdot .2 + 4 \cdot 0 = \underline{2.2}$

3a) $\hat{x}_{MAP}(y) \in \arg\max_{x \in [0,y]} 2(x+y) = y$

b) $E[e^2] = \int_0^1 \left(\int_0^y (x-y)^2 \cdot 2(x+y)\, dx \right) dy$
$= \int_0^1 \left(\int_0^y (x-y) \cdot 2(x^2-y^2)\, dx \right) dy$
$= \int_0^1 \left(\int_0^y (2x^3 - 2xy^2 - 2x^2y + 2y^3)\, dx \right) dy$
$= \int_0^1 \left(\int_0^y \left(\frac{y^4}{2} - y^4 - \frac{2y^4}{3} + 2y^4 \right) dx \right) dy$
$= \int_0^1 (5/6\, y^4)\, dy = 1/6$

3c) $f_{X|Y}(x|y) = \begin{cases} \frac{2x+2y}{f_Y(y)} & 0 \le x \le y \le 1 \\ 0 & \text{else} \end{cases}$

$f_Y(y) = \int_0^y (2x+2y)\,dx = y^2 + 2y^2 = 3y^2$

$E(X|Y=y) = \int_0^y x\left(\frac{2x+2y}{3y^2}\right)dx = \frac{2/3\,y^3 + y^3}{3y^2} = \underline{5/9\,y}$

d) $E[X] = \int_0^1 \int_0^y x\, f_{XY}(x,y)\,dx\,dy$

$= \int_0^1 \int_0^y (2x^2 + 2xy)\,dx\,dy$

$= \int_0^1 (2/3\,y^3 + y^3)\,dy = 5/12$

$E[Y] = \int_0^1 \int_0^y (2xy + 2y^2)\,dx\,dy = \int_0^1 3y^3\,dy = 3/4$

$E[X^2] = \int_0^1 \int_0^y (2x^3 + 2x^2 y)\,dx\,dy = \int_0^1 (y^4/2 + 2y^4/3)\,dy$

$= 7/30$

$Var[X] = E[X^2] - E[X]^2 = 7/30 - 25/144 = 43/720$

$E[Y^2] = \int_0^1 \int_0^y (2xy^2 + 2y^3)\,dx\,dy = \int_0^1 3y^4\,dy = 3/5$

$Var[Y] = E[Y^2] - E[Y]^2 = 3/5 - 9/16 = 3/80$

$E[XY] = \int_0^1 \int_0^y (2x^2 y + 2xy^2)\,dx\,dy = \int_0^1 (2y^4/3 + y^4)\,dy$

$= 1/3$

$Cov[X,Y] = E[XY] - E[X]E[Y] = 1/3 - 5/12 \cdot 3/4$

$= 1/48$

$\hat{x}_{LLSE}(y) = E[X] + \frac{Cov[X,Y]}{Var[X]}(y - E[Y])$

$= 5/12 + 1/48 \cdot 80/3\,(y - 3/4)$

$= 5/12 + 5/9\,(y - 3/4) = 5/9\,y$

e) $E[e^2] = Var[X] - \frac{Cov[X,Y]^2}{Var[Y]} = \frac{43}{720} - \left(\frac{1}{48}\right)\left(\frac{1}{48}\right)\left(\frac{80}{3}\right)$

$= \underline{13/270}$

4a) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \int_0^{\sqrt{1-y^2}} \frac{4}{\pi}\,dx & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

$f_{X|Y}(x|y) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_Y(y)} & f_Y(y) > 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{\sqrt{1-y^2}} & x,y \ge 0,\ x^2+y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$

$\hat{x}_{MMSE}(y) = E(X|Y=y) = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx$

$= \int_0^{\sqrt{1-y^2}} \frac{x}{\sqrt{1-y^2}}\,dx$

$= \frac{\sqrt{1-y^2}}{2}$

4b) $E[(X - \hat{x}_{MMSE}(Y))^2] = E[(X - \frac{\sqrt{1-Y^2}}{2})^2]$

$= E[X^2 - X\sqrt{1-Y^2} + (1-Y^2)/4]$

$= E[X^2] - E[X\sqrt{1-Y^2}] + 1/4 - \frac{E[Y^2]}{4}$

$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \int_0^1 y^2 \frac{4}{\pi}\sqrt{1-y^2}\,dy = 1/4$

$E[X^2] = 1/4$ (symmetry)

$E[X\sqrt{1-Y^2}] = \int_{-\infty}^{\infty} x\sqrt{1-y^2} f_{X,Y}(x,y)\,dx\,dy$

$= \int_0^1 \int_0^{\sqrt{1-y^2}} x\sqrt{1-y^2} \frac{4}{\pi}\,dx\,dy = 3/8$

$E[(X - \hat{x}_{MMSE}(Y))^2] = 1/4 - 3/8 + 1/4 - 1/4 \cdot 1/4 = \boxed{1/16}$

c) $\hat{x}_{LLSE}(y) = E[X] + \frac{\text{Cov}[X,Y]}{\text{Var}[Y]}(y - E[Y])$

$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^1 y \frac{4}{\pi}\sqrt{1-y^2}\,dy = \frac{4}{3\pi}$

$E[X] = 4/3\pi$ (by symmetry)

$\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 1/4 - (\frac{4}{3\pi})^2$

$E[XY] = \int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dx\,dy = \int_0^1 \int_0^{\sqrt{1-y^2}} xy \frac{4}{\pi}\,dx\,dy$

$= 1/2\pi$

$\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = 1/2\pi - (4/3\pi)^2$

$\hat{x}_{LLSE}(y) = 1/2\pi + \frac{1/2\pi - (4/3\pi)^2}{1/4 - (4/3\pi)^2}(y - 1/2\pi)$

d) $E[(X - \hat{x}_{LLSE}(Y))^2] = \text{Var}[X] - \frac{(\text{Cov}[X,Y])^2}{\text{Var}[Y]}$

$= 1/4 - (4/3\pi)^2 - \frac{(1/2\pi - (4/3\pi)^2)^2}{1/4 - (4/3\pi)^2}$

$\approx .0636$

5a) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$

$= \begin{cases} \int_y^1 \frac{1}{x}\,dx & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \ln(x)\Big|_y^1 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} -\ln(y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

$f_{X|Y}(x|y) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_Y(y)} & 0 \le x \le 1,\ 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} -1/x\ln(y) & 0 \le x \le 1,\ 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$

$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$

$= \int_y^1 x\left(-\frac{1}{x\ln(y)}\right)dx = -\frac{1}{\ln(y)}\int_y^1 1\,dx$

$= -\frac{1}{\ln(y)}(x)\Big|_y^1 = \boxed{\frac{y-1}{\ln(y)}}$

b) $\hat{x}_{LLSE}(y) = E[X] + \frac{\mathrm{Cov}[X,Y]}{\mathrm{Var}[Y]}(y - E[Y])$

$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \int_0^1 (-y\ln(y))\,dy$

$= \left(-y^2/2\,\ln(y)\right)\Big|_0^1 - \int_0^1\left(-\frac{y^2}{2}\cdot\frac{1}{y}\right)dy$

$= 0 + y^2/4\Big|_0^1 = 1/4$

$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \int_0^1 (-y^2\ln(y))\,dy$

$= \left(-y^3/3\,\ln(y)\right)\Big|_0^1 - \int_0^1\left(-\frac{y^3}{3}\cdot\frac{1}{y}\right)dy$

$= 0 + y^3/9\Big|_0^1 = 1/9$

$\mathrm{Var}[Y] = E[Y^2] - (E[Y])^2 = 1/9 - (1/4)^2 = 7/144$

$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \begin{cases} \int_0^x \frac{1}{x}\,dy & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} y/x\Big|_0^x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

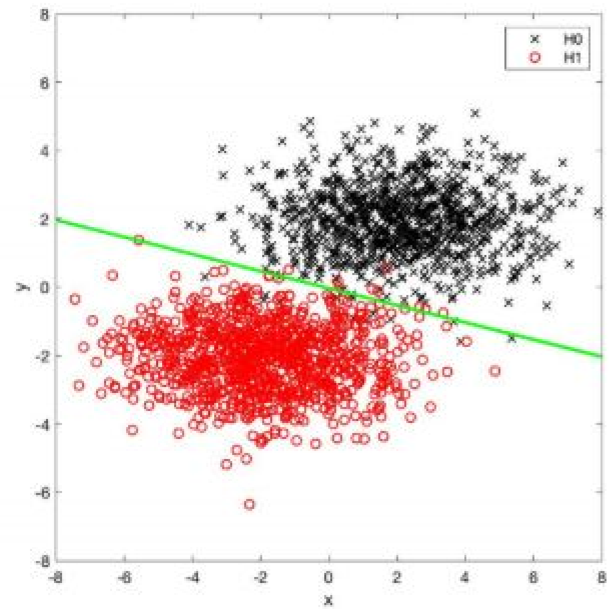
$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_0^1 x\,dx = x^2/2\Big|_0^1 = 1/2$

$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dy\,dx = \int_0^1\int_0^x y\,dy\,dx = \int_0^1 \frac{y^2}{2}\Big|_0^x dx$

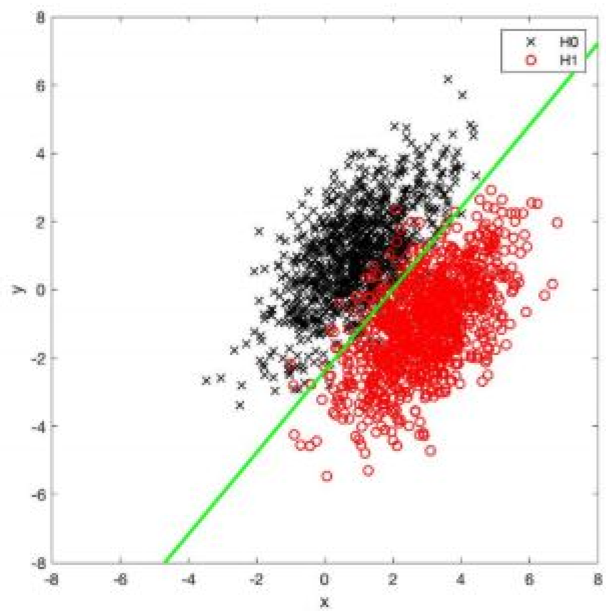
$= \int_0^1 x^2/2\,dx = (x^3/6)\Big|_0^1 = 1/6$

$\mathrm{Cov}[X,Y] = E[XY] - E[X]E[Y] = 1/6 - \frac{1}{2}\cdot\frac{1}{4} = 1/24$

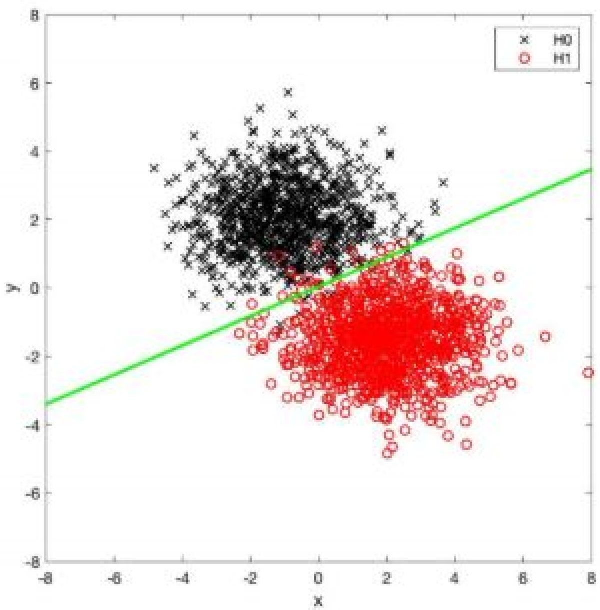
$\hat{x}_{LLSE}(y) = \frac{1}{2} + \frac{1/24}{7/144}(y - 1/4) = \boxed{\frac{6}{7}y + \frac{2}{7}}$



Pfa = .016
Pmd = .01
Pe = .013



Pfa = .045
Pmd = .046
Pe = .0455



Pfa = .045
Pmd = .046
Pe = .0455

```
close all;
load gaussiandata

n = length(partc_H0_data);

for i = 1:3
   if(i==1),
      H0_data = parta_H0_data;
      H1_data = parta_H1_data;
   elseif(i == 2),
      H0_data = partb_H0_data;
      H1_data = partb_H1_data;
   elseif(i == 3),
      H0_data = partc_H0_data;
      H1_data = partc_H1_data;
   end

   mX0 = 1/n*sum(H0_data(:,1));
   mY0 = 1/n*sum(H0_data(:,2));
   mX1 = 1/n*sum(H1_data(:,1));
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```
    mY1 = 1/n*sum(H1_data(:,2));
    VarX = (1/(2*n)) * (sum((H0_data(:,1) - mX0.^2) + sum((H1_data(:,1) - mX1).^2)));
    VarX = (1/(2*n)) * (sum((H0_data(:,2) - mX0.^2) + sum((H1_data(:,2) - mY1).^2)));
    CovXY = (1/(2*n)) * (sum((H0_data(:,1) - mX0).*(H0_data(:,2) - mY0))...
        + sum((H1_data(:,1) - mX1).*(H1_data(:,2) - mY1)));

    Sigma = [VarX CovXY; CovXY VarY];
    m0 = [mX0;mY0];
    m1 = [mX1;mY1];
    iSigma = inv(Sigma);
    d = (m1-m0)' * iSigma;
    a = d(1); b = d(2);
    c = 0.5*(m0' * iSigma * m0 - m1' * iSigma * m1);
    count = 0;
    H0errors = (a*H0_data(:,1)+b*H0_data(:,2)+c > 0);
    H1errors = (a*H1_data(:,1)+b*H1_data(:,2)+c < 0);

    PFA = sum(H-errors)/n
    PMD = sum(H1errors)/n
    Pe = 0.5*(PFA + PMD)
    x = linspace(-8,8,500);
    y = -a/b * x - c/b;

    figure()
    hold off
    plot(H0_data(:,1),H0_data(:,2),'x','Color','k','MarkerSize',6)
    hold on
    plot(H1_data(:,1),H1_data(:,2),'x','Color','r','MarkerSize',6)
    plot(x,y,'g','linewidth',2)
    xlabel('x')
    ylabel('y')
    legend('H0','H1')
    axis([-8 8 -8 8])
    axis square
end
```