Theory of Income II 2020 Technical Notes

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1 New Keynesian

I follow the New Keynesian construction of Galí, most closely with Chapter 3, in the second edition, though I cut out stuff that seems extraneous (like preference shocks).

1.1 Households

Households optimize discounted expected utility, subject to their period budget constraint, where they receive labor income, firm dividends (which are *not* generally zero because we will be introducing markups), and asset payoffs from yesterday, in order to buy bonds today, and consumption. Note that they buy a bundle of differentiated goods, where ϵ parameterizes the degree of differentiation/market power, so that low $\epsilon > 1$ means high monopolistic competition inefficiency, and as $\epsilon \to \infty$, we approach perfect competition.

$$\max_{C_t, N_t, B_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) \tag{1}$$

$$C_t \equiv \left(\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}} \mathrm{d}i \right)^{\frac{\epsilon}{\epsilon-1}} \tag{2}$$

$$\text{s.t.} \quad \underbrace{\int_0^1 P_t(i) C_t(i) \mathrm{d}i}_{\text{cons. today}} + \underbrace{Q_t B_t}_{\text{bond purchase today}} \leq \underbrace{B_{t-1}}_{\text{asset income}} + \underbrace{W_t N_t}_{\text{labor income}} + \underbrace{D_t}_{\text{firm profits}} \tag{3}$$

Consumers may view consumption as a two-step minimization: given total amount of consumption, choose how to allocate across firms, then use this to determine total consumption. From appendix 3.1 in Galí, let the total amount spent on consumption by X. Then, since U is presumably increasing in C, we seek to solve

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$$\max_{C(i)} \ C = \Big(\int_0^1 C(i)^{\frac{\epsilon-1}{\epsilon}} \mathrm{d}i\Big)^{\frac{\epsilon}{\epsilon-1}} \tag{4}$$

$$\text{s.t.} \ \int_0^1 P(i)C(i)\mathrm{d}i = X \tag{5}$$

We may consider this problem as a Lagrangian.

$$\mathcal{L}(C(j)_{j\in[0,1]}, \lambda) = \Big(\int_0^1 C(i)^{\frac{\epsilon-1}{\epsilon}} \mathrm{d}i\Big)^{\frac{\epsilon}{\epsilon-1}} + \lambda\Big[X - \int_0^1 P(i)C(i)\mathrm{d}i\Big] \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial C(j)} = \frac{\epsilon}{\epsilon-1} \cdot \Big(\int_0^1 C(i)^{\frac{\epsilon-1}{\epsilon}} \mathrm{d}i\Big)^{\frac{\epsilon}{\epsilon-1}-1} \cdot \frac{\epsilon-1}{\epsilon} C(j)^{-\frac{1}{\epsilon}} - \lambda P(j) = 0 \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = X - \int_0^1 P(i)C(i)\mathrm{d}i = 0 \tag{8}$$

All[1] FONCs share λ, and we may simplify the above equation to include C, so

$$\frac{1}{P(i)} C^{\frac{1}{\epsilon}} C(i)^{-\frac{1}{\epsilon}} = \frac{1}{P(j)} C^{\frac{1}{\epsilon}} C(j)^{-\frac{1}{\epsilon}} \qquad (i, j \in [0,1])$$

$$\Rightarrow C(i) = \Big(\frac{P(i)}{P(j)}\Big)^{-\epsilon} C(j) \tag{9}$$

$$\int_0^1 P(i)C(i)\mathrm{d}i = \int_0^1 P(i)\Big(\frac{P(i)}{P(j)}\Big)^{-\epsilon} C(j)\mathrm{d}i \tag{10}$$

$$X = C(j)P(j)^{\epsilon} P^{1-\epsilon} \qquad (P \equiv (\textstyle\int_0^1 P(i)^{1-\epsilon})^{\frac{1}{1-\epsilon}})$$

$$C(j) = \frac{X}{P}\Big(\frac{P(j)}{P}\Big)^{-\epsilon} \tag{11}$$

$$\Rightarrow C = \Big(\int_0^1 C(i)^{\frac{\epsilon-1}{\epsilon}} \mathrm{d}i\Big)^{\frac{\epsilon}{\epsilon-1}} \tag{12}$$

$$= \Big(\frac{X}{P}\Big)\frac{1}{P^{-\epsilon}}\Big(\int_0^1 P(i)^{1-\epsilon}\mathrm{d}i\Big)^{\frac{\epsilon}{\epsilon-1}} \tag{13}$$

$$\Rightarrow PC = X \tag{14}$$

$$\Rightarrow \boxed{C(j) = \Big(\frac{P(j)}{P}\Big)^{-\epsilon} C} \tag{15}$$

The above calculations give us some great info: the differentiated goods demands are simple downward sloping demands with respect to price, and are linear in the consumption bundle. Therefore, we may dramatically simplify the consumer's side of the problem, since they take prices/wages as given, so that they choose first their optimal bundle expenditures, then choose how to allocate across the bundle. This is a special property of the Blanchard-Kiyotaki

[1] Technically, we only have agreement up to measure zero, but ya know how it be.

aggregation here, in that we basically introduce a markup without seriously screwing up the mechanics with tons of good/firm heterogeneity. Then we may write the budget more simply as

$$P_t C_t + Q_t B_t \leq B_{t-1} + W_t N_t + D_t \tag{16}$$

Now consumers solve the "simpler" [2] Lagrangian (note that we have a constraint for each period).

$$\begin{aligned}
\mathcal{L}(C_t, N_t, B_t, \lambda_t) &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) + \lambda_t [B_{t-1} + W_t N_t + D_t - P_t C_t - Q_t B_t] && (17)\\
\frac{\partial \mathcal{L}}{\partial C_t} &= U_{c,t} - \lambda_t P_t = 0 && (18)\\
\frac{\partial \mathcal{L}}{\partial N_t} &= U_{n,t} + \lambda_t W_t = 0 && (19)\\
\frac{\partial \mathcal{L}}{\partial B_t} &= -\lambda_t Q_t + \beta \mathbb{E}_t[\lambda_{t+1}] = 0 && (20)\\
& && (21)
\end{aligned}$$

Re-arranging we get the MRS equals relative prices of consumption and leisure result

$$\begin{aligned}
\lambda_t = \frac{U_{c,t}}{P_t} &= -\frac{U_{n,t}}{W_t} && (22)\\
\Rightarrow \boxed{-\frac{U_{c,t}}{U_{n,t}} = \frac{W_t}{P_t}} & && (23)\\
& && (24)
\end{aligned}$$

We also find the Euler equation

$$\begin{aligned}
Q_t &= \beta \mathbb{E}_t \Big[\frac{\lambda_{t+1}}{\lambda_t} \Big] && (25)\\
&\boxed{Q_t = \beta \mathbb{E}_t \Big[\frac{U_{c,t+1}}{U_{c,t}} \frac{P_t}{P_{t+1}} \Big]} && (26)
\end{aligned}$$

Now we take the stance that preferences are GHH

$$U(C, N) = \frac{C^{1-\sigma} - 1}{1-\sigma} - \frac{N^{1+\varphi}}{1+\varphi} \tag{27}$$

[2] Still choosing countably many controls

Now it is time to move to log-lin world. The MRS = prices equation is super easy because of the felicity functional form and the multiplicative structure

$$\boxed{\sigma c + \varphi n = w - p} \tag{28}$$

The Euler equation is tricker. First, we re-arrange it in terms of the logs of its variables.

$$\begin{aligned}
1 &= \mathbb{E}_t\Big[\frac{\beta}{Q_t}\frac{U_{c,t+1}}{U_{c,t}}\frac{P_t}{P_{t+1}}\Big] && (29)\\
&= \mathbb{E}_t\Big[\frac{\beta}{Q_t}\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\frac{P_t}{P_{t+1}}\Big] && (30)\\
&= \mathbb{E}_t[\exp(\log\beta - \log Q_t - \sigma(\log C_{t+1} - \log C_t) - \log \Pi_{t+1})] && (31)\\
&= \mathbb{E}_t[\exp(-\rho + i_t - \sigma(c_{t+1} - c_t) - \pi_{t+1})] && (32)
\end{aligned}$$

Now we linearize around the steady state. Note that in the constant inflation steady state, we must have

$$i = \rho + \pi \tag{33}$$

We log-linearize the inside of $\mathbb{E}_t[\cdot]$ around this state, letting $O(x^2)$ denote the errors from second-order Taylor approximations and beyond

$$\begin{aligned}
\exp(-\rho + i_t - \sigma(c_{t+1} - c_t) - \pi_{t+1}) &= 1 + (i_t - i) - \sigma(c_{t+1} - c) + \sigma(c_t - c) - (\pi_{t+1} - \pi) \\
&\quad + O(x^2) \\
&\approx 1 + i_t - \sigma\Delta c_{t+1} - \pi_{t+1} - \rho
\end{aligned}$$

Now we plug back in

$$\begin{aligned}
1 &\approx \mathbb{E}_t[1 + i_t - \sigma\Delta c_{t+1} - \pi_{t+1} - \rho] && (34)\\
0 &= \mathbb{E}_t[i_t - \sigma c_{t+1} - \pi_{t+1} - \rho] + \sigma c_t && (35)\\
\Rightarrow & \boxed{c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho)} && (36)
\end{aligned}$$

Take a moment and enjoy him [3]. The interpretation is that an optimizing consumer, in equilibrium, will choose consumption to be equal to expected consumption tomorrow, adjusting for changes in expected real returns on bonds bought today.

[3] OR HER!

1.2 Firms

Firms face possibly decreasing returns to scale ($\alpha = 0$ is CRS from class).

$$Y_t(i) = A_t N_t(i)^{1-\alpha} \tag{37}$$

They also face the second friction in the model: Calvo price-setting. Note the distinction: the market structure creates a wedge due to monopolistic competition, whereas the Calvo price-setting introduces a wedge because not all firms are re-optimizing each period.

What should the firms be optimizing? Profits today, tomorrow? Recall that the dividends will be returned to the consumers, and at the end of the day the consumer own the firms, and this distinction just helps with modeling. Then the firms maximize their expected discounted stream of profits, but act in their owners (the consumers) interest, so must discount as such. Therefore firms optimize

$$\max_{\{P_t^*\}} \mathbb{E}_t \sum_{k=0}^{\infty} \Lambda_{t+k,t} \frac{D_{t+k}}{P_{t+k}} \tag{38}$$

$$\Lambda_{t,t+k} \equiv \beta^k \frac{U_{c,t+k}}{U_{c,t}} \tag{39}$$

where $\{P_t^*\}$ indicates all possible sets of choice for reoptimizing today and in the future (it's what you would call big). Note that Λ is the *real* stochastic discount factor[4], and the other term is the nominal divided by price $\rightarrow$ *real* profits (of course, all inside the expectation).

Now recognize that firms have no control over aggregate prices[5], including the evolution of aggregate prices, therefore they will merely need to choose optimal price based on the expectation for how long this price will last. Once a reset hits, the problem gets re-solved anyway. Now note there is $(1-\theta)\theta^{k-1}$ probability of getting to reset in k periods exactly. So we ignore the $(1-\theta)\theta$ term, and solve

$$\max_{P_t^*} \mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} \frac{1}{P_{t+k}} [P_t^* Y_{t+k|t} - \mathcal{C}_{t+k}(Y_{t+k|t})] \tag{40}$$

where $Y_{t+k|t}$ denotes the optimal output chosen in period k, given that the firms is stuck using price from t, and $\mathcal{C}(\cdot)$ is the price to produce such output. Note that we are making the assumption that in any state of the world, firms find it profitable to produce, and not exit i.e. no one ever optimizes with a super low price, then lives in a world where wages prices are systematically to high to be profitable.
Now recall how consumers demanded things. They considered aggregate price, good price, and aggregate consumption, all of which will be fixed for a firm when they cannot re-optimize, so

[4] I belive first edition has a nominal SDF Q. This is *different*!
[5] They are monopolistically competitive, not oligopolistic.

$$Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} C_{t+k} \tag{41}$$

FONCs then imply, using the above condition for $Y_{t+k|t}$, and re-arranging a la the homework.

$$\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} \frac{Y_{t+k|t}}{P_{t+k}} [P_t^* - \mathcal{M}\Psi_{t+k|t}] = 0 \tag{42}$$

where $\mathcal{M} = \frac{\epsilon}{\epsilon-1}$ is the markup up flexy prices, and $\Psi_{t+k|t} = \mathcal{C}'_{t+k}(Y_{t+k|t})$ denotes marginal cost[6]
Note that if *all*[7] firms get to reset prices every period, they will simply follow the markup

$$P_t^* = \mathcal{M}\Psi_{t|t} \tag{43}$$

Now gotta log-lin this big boy around the zero inflation steady state, where $\Lambda_{t,t+k} = \beta^k$, $Y_{t+k|t} = Y_t$, $P_t = P_{t+k} = P_t^*$. There are multiple ways to log-lin. One is to take the log, then linearize, which I do below, though I often find jumping to the approximation $\frac{X}{\bar{X}} \approx 1 + x - \bar{x}$ works quite well, and might even be simpler, here.

$$\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \frac{Y_{t+k|t}}{P_{t+k}} P_t^* = \mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \frac{Y_{t+k|t}}{P_{t+k}} \mathcal{M}\Psi_{t+k|t}$$
$$\log\left(\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \frac{Y_{t+k|t}}{P_{t+k}} P_t^*\right) = \log\left(\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \frac{Y_{t+k|t}}{P_{t+k}} \mathcal{M}\Psi_{t+k|t}\right)$$

Now take first order Taylor, noting that the first step in the chain rule is just a nasty constant which will be on both sides. We call it K.

$$\frac{1}{K}[\frac{1}{1-\beta\theta} p_t^*] = \frac{1}{K}[\sum_{k=0}^{\infty} (\beta\theta)^k] \mathbb{E}_t[\psi_{t+k|t} + \mu]$$
$$\Rightarrow \boxed{p_t^* = \mu + (1-\beta\theta) \sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t[\psi_{t+k|t}]}$$

[6]Again, this second edition, so does not match first edition notation.
[7]I emphasize because if only one firm got to reset, but others were stuck, there would be a distributional effect, allowing the special flexy firm to get profits even above markups.

1.3 Ekwilibryum

Goods market gotta clear for all firms and all time

$$Y_t(i) = C_t(i) \tag{44}$$

Then we can throw output into the Euler

$$\boxed{y_t = \mathbb{E}_t[y_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho)}$$

Now we want to relate the logs of output, technology and labor. Properly, we need to aggregate labor *and* output in meaningful ways and address the fact that differentiated profits are being made across firms which could potentially screw up a simply log-linearization. However, Galí shows in the appendix that these profit concerns can be considered second order, and ignores them[8], so we "naively" log-linearize the production function.

$$\boxed{y_t = a_t + (1-\alpha)n_t}$$

Now note that marginal costs are the product of wages and the reciprocal of the marginal product of labor. Therefore in logs

$$\psi_{t+k|t} = w_{t+k} - (a_{t+k} - \alpha n_{t+k|t} + \log(1-\alpha))$$

where the second term comes from logging the derivative of the production function with respect to labor. Now with more sloppy approximation arguing, we consider the above relationship as holding in aggregate,

$$\psi_t = w_t - (a_a - \alpha n_t + \log(1-\alpha))$$

and derive the relationship between firm and aggregate

$$\begin{aligned}
\psi_{t+k|t} &= \psi_{t+k} + \alpha(n_{t+k|t} - n_t) && (45)\\
&= \psi_{t+k} + \frac{\alpha}{1-\alpha}(y_{t+k|t} - y_t) && \text{(Log of Prod.)}\\
&= \psi_{t+k} - \frac{\alpha\epsilon}{1-\alpha}(p_t^* - p_{t+k}) && \text{(Demand with Goods clearing)}
\end{aligned}$$

Now we need to contextualize these cost in our above optimal price-setting equation, which will lead us to define marginal costs in terms of markup gaps, which we can then use to define output gaps

[8] I am personally skeptical of this dubious technique, because it only works when price dispersion is small, and we might want to consider a larger ϵ.

$$
\begin{aligned}
p_t^* &= \mu + (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t[\psi_{t+k|t}] && (46)\\
&= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t[\psi_{t+k} - \frac{\alpha\epsilon}{1-\alpha}(p_t^* - p_{t+k}) + \mu] && (47)\\
p_t^*(1+\frac{\alpha\epsilon}{1-\alpha}) &= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t[\psi_{t+k} + \frac{\alpha\epsilon}{1-\alpha}p_{t+k} + \mu] && (48)\\
p_t^* &= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t[\frac{1-\alpha}{1-\alpha+\alpha\epsilon}(\psi_{t+k} + \mu + \frac{\alpha\epsilon}{1-\alpha}p_{t+k})] && (49)\\
&= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t[\frac{1-\alpha}{1-\alpha+\alpha\epsilon}(\psi_{t+k} + \mu - p_{t+k} + \frac{1-\alpha+\alpha\epsilon}{1-\alpha}p_{t+k})] && (50)\\
&= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t[p_{t+k} - \Theta(p_{t+k} - \psi_{t+k} - \mu)] && (51)\\
&= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t[p_{t+k} - \Theta(\mu_t - \mu)] && \text{(Markup is price minus cost)}\\
&= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t[p_{t+k} - \Theta\hat{\mu}_t] && (52)\\
& && (53)
\end{aligned}
$$

The infinite series nature of this equation means that we can write p_t^* recursively

$$
p_t^* = \beta\theta\mathbb{E}_t[p_{t+1}^*] + (1-\beta\theta)(p_t - \Theta\hat{\mu}_t)
$$

Now we need to jump waaay back and consider price dynamics. The Calvo system implies that $(1-\theta)$ firms get to re-optimize, but the rest are stuck at P_{t-1}, so the price index becomes

$$
\begin{aligned}
P_t &= ((1-\theta)(P_t^*)^{1-\epsilon} + \theta P_{t-1}^{1-\epsilon})^{\frac{1}{1-\epsilon}}\\
P_t^{1-\epsilon} &= (1-\theta)(P_t^*)^{1-\epsilon} + \theta P_{t-1}^{1-\epsilon}\\
\left(\frac{P_t}{P_{t-1}}\right)^{1-\epsilon} &= \theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\epsilon}
\end{aligned}
$$

Now we have another adventure in log-linearization.

$$
\begin{aligned}
\log \Pi^{1-\epsilon} &= \log(\theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\epsilon})\\
\frac{1}{K}[(1-\epsilon)\pi_t] &= \frac{1}{K}[(1-\theta)(1-\epsilon)(p_t^* - p_{t-1})]\\
\pi_t &= (1-\theta)(p_t^* - p_{t-1})
\end{aligned}
$$

We combine the inflation equation with the recursive pricing equation to find

$$
\begin{aligned}
p_t^* &= \beta\theta\mathbb{E}_t[p_{t+1}^*] + (1-\beta\theta)(p_t - \Theta\hat{\mu}_t) && (54)\\
\frac{\pi_t}{1-\theta} + p_{t-1} &= \beta\theta\mathbb{E}_t[\frac{\pi_{t+1}}{1-\theta} + p_t] + (1-\beta\theta)(p_t - \Theta\hat{\mu}_t) && (55)\\
\pi_t &= \beta\theta\mathbb{E}_t[\pi_{t+1}] + (1-\theta)(p_t - p_{t-1}) + (1-\theta)(1-\beta\theta)(-\Theta\hat{\mu}_t) && (56)\\
\theta\pi_t &= \beta\theta\mathbb{E}_t[\pi_{t+1}] - (1-\theta)(1-\beta\theta)\Theta\hat{\mu}_t && (57)\\
&\boxed{\pi_t = \beta\mathbb{E}_t[\pi_{t+1}] - \lambda\hat{\mu}_t} && (58)
\end{aligned}
$$

This is almost the Phillips equation, but we have inflation only in terms of markup deviations, not output deviations.
We now use the definition of μ_t and μ to find the output and natural output rates, then relate them accordingly. This is pretty mechanical, but if we did it all in one step it would look like magic, and splitting it up increases the economic intuition.

$$
\begin{aligned}
\mu_t &= p_t - \psi_t\\
&= -(w_t - p_t) + (w_t - \psi_t)\\
&= -(\sigma c_t + \varphi n_t) + (a_t - \alpha n_t + \log(1-\alpha))\\
&= -(\sigma y_t + \frac{\varphi}{1-\alpha}(y_t - a_t)) + (a_t - \frac{\alpha}{1-\alpha}(y_t - a_t) + \log(1-\alpha))\\
&= -\Big(\sigma + \frac{\varphi+\alpha}{1-\alpha}\Big)y_t + \Big(\frac{1+\varphi}{1-\alpha}\Big)a_t + \log(1-\alpha)
\end{aligned}
$$

In flexy price world $\mu_t = \mu$, and $y_t^n = y_t$. Then

$$
\mu_t - \mu = -\Big(\sigma + \frac{\varphi+\alpha}{1-\alpha}\Big)(y_t - y_t^n)
$$

While we might have guessed this was true, we now have an expression proving that the markup gap is proportional to the output gap ($\tilde{y}_t$). Now if we plug this into our above inflation equation we finally have the **New Keynesian Phillips Curve**

$$
\boxed{\pi_t = \beta\mathbb{E}_t[\pi_{t+1}] + \kappa\tilde{y}_t} \tag{59}
$$

$$
\kappa \equiv \lambda\Big(\sigma + \frac{\varphi+\alpha}{1-\alpha}\Big) \tag{60}
$$

This equation relates (in equilibrium), how inflation depends on expected inflation and the output gap. Importantly, an understanding of this relationship is instructive for making automatic economic policies *only*. If we try to use the Phillips curve as a diagnostic tool for discretionary policy, we are immediately subject to the Lucas critique, and probably break the relationship, but including a policy block which uses only observables, such as an interest rate or money supply rule, will fit neatly into the context of our model.

The last thing we need to do is find the dynamic IS[9] equation, which serves something like a demand equation across time. We just start with the Euler and find it almost immediately.

$$\begin{aligned}
c_t &= \mathbb{E}_t[c_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) \\
y_t &= \mathbb{E}_t[y_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) \qquad \text{(Goods)} \\
y_t - y_t^n &= \mathbb{E}_t[y_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) - y_t^n \\
&= \mathbb{E}_t[y_{t+1} - y_{t+1}^n] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) - y_t^n + \mathbb{E}_t[y_{t+1}^n] \\
&\boxed{\tilde{y}_t = \mathbb{E}_t[\tilde{y}_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^n)} \\
r_t^n &\equiv \rho - \sigma(y_t^n - \mathbb{E}_t[y_{t+1}^n]
\end{aligned}$$

Note that the natural real interest rate is equal to the flexy natural real interest rate (ρ) corrected by the the expected change in the natural rate of output, scaled accordingly to inverse elasticity of substitution/risk aversion for consumption.
Now we have found the two key NK equations that define equilibrium outside of policy. With these two equations alone, there will be indeterminacy issues[10]. Notably, these are pretty much the same in any NK model, they just sometimes have more bells and whistles. The policy block is a little more open.

1.4 Rotemberg price adjustment costs

This section is based on my own derivations, so be cautious about all conclusions and algebra.
Consider again the firm above, but now replace Calvo price-setting with quadratic price adjustment. Now the firm is able to reoptimize every period, but faces a cost for how differently it sets its price from the previous period. The cost of adjustment is rebated to consumers as a lump-sum subsidy, merely as a way to close model (i.e. goods market clears). This is actually a key point, as the taxes are distortionary on the firms, but non-distorting for the consumers, and therefore do not effect consumer optimizing behavior. Note that this approach removes heterogeneity across firms in pricing because any shock hitting one firm necessarily hits all the firms. Therefore we may restrict ourselves to the symmetric equilibria[11], where $P_t(i) = P_t$ for all i and t. Then firms solved exactly as above,

[9] Interest-Savings
[10] A good understanding check is to take just use the NKPC and IS to try to close the model. What variable can you not pin down?
[11] Though we have not ruled out the existence of asymmetric equilibria.

$$\max_{\{P_t^*\}} \mathbb{E}_t \sum_{k=0}^{\infty} \Lambda_{t+k,t} \frac{D_{t+k}}{P_{t+k}} \tag{61}$$

$$\Lambda_{t,t+k} \equiv \beta^k \frac{U_{c,t+k}}{U_{c,t}} \tag{62}$$

but now we have

$$D_t = P_t^* Y_t - \mathcal{C}(Y_t) - \frac{\theta}{2} P_t C_t \Big(\frac{P_t^*}{P_{t-1}^*} - 1 \Big)^2$$

$$Y_t(i) = \Big(\frac{P_t^*}{P_t} \Big)^{-\epsilon} C_t$$

We take the FONC, and find that for all $k \geq 0$

$$0 = \mathbb{E}_t \Bigg[\Lambda_{t+k,t} \frac{1}{P_{t+k}} \Big[(1-\epsilon) \Big(\frac{P_{t+k}^*}{P_{t+k}} \Big)^{-\epsilon} C_{t+k} - \Psi_{t+k}(-\epsilon) \Big(\frac{1}{P_{t+k}^*} \Big) \Big(\frac{P_{t+k}^*}{P_{t+k}} \Big)^{-\epsilon} C_{t+k} \tag{63}$$

$$- \theta P_{t+k} C_{t+k} \Big(\frac{P_{t+k}^*}{P_{t-1+k}^*} - 1 \Big) \frac{1}{P_{t-1+k}^*} \Big] \tag{64}$$

$$- \Lambda_{t+1+k,t} \frac{1}{P_{t+1+k}} \Big[\theta P_{t+1+k} C_{t+1+k} \Big(\frac{P_{t+1+k}^*}{P_{t+k}^*} - 1 \Big) \Big(- \frac{P_{t+1+k}^*}{(P_{t+k}^*)^2} \Big) \Big] \Bigg] \tag{65}$$

Multiply by P_{t+k}^*, and divide by $\Lambda_{t,t+k}(1-\epsilon)\Gamma$, where $\Gamma = \Big(\frac{P_{t+k}^*}{P_{t+k}} \Big)^{-\epsilon} C_{t+k}$.

$$0 = \mathbb{E}_t \Bigg[\frac{1}{P_{t+k}} \Big[P_{t+k}^* - \mathcal{M} \Psi_{t+k} \tag{66}$$

$$- \frac{\theta}{(1-\epsilon)\Gamma} P_{t+k} C_{t+k} \Big(\frac{P_{t+k}^*}{P_{t-1+k}^*} - 1 \Big) \frac{P_{t+k}^*}{P_{t-1+k}^*} \Big] \tag{67}$$

$$- \beta \frac{U_{c,t+k+1}}{U_{c,t+k}} \frac{1}{P_{t+1+k}} \Big[\frac{\theta}{(1-\epsilon)\Gamma} P_{t+1+k} C_{t+1+k} \Big(\frac{P_{t+1+k}^*}{P_{t+k}^*} - 1 \Big) \Big(- \frac{P_{t+1+k}^*}{P_{t+k}^*} \Big) \Big] \Bigg] \tag{68}$$

We need to log-lin this, around the perfect foresight steady state. Again, use K for denominator crap.

$$\log \mathbb{E}_t\left[\frac{1}{P_{t+k}}\left[P^*_{t+k}\right]\right] = \log \mathbb{E}_t\left[\frac{1}{P_{t+k}}\left[\mathcal{M}\Psi_{t+k}\right.\right. \tag{69}$$

$$+ \frac{\theta}{(1-\epsilon)\Gamma}P_{t+k}C_{t+k}\Big(\frac{P^*_{t+k}}{P^*_{t-1+k}} - 1\Big)\frac{P^*_{t+k}}{P^*_{t-1+k}}\Big] \tag{70}$$

$$\left.- \beta\frac{U_{c,t+k+1}}{U_{c,t+k}}\frac{1}{P_{t+1+k}}\Big[\frac{\theta}{(1-\epsilon)\Gamma}P_{t+1+k}C_{t+1+k}\Big(\frac{P^*_{t+1+k}}{P^*_{t+k}} - 1\Big)\Big(\frac{P^*_{t+1+k}}{P^*_{t+k}}\Big)\Big]\right] \tag{71}$$

$$\Rightarrow \frac{1}{K}[p^*_t] = \frac{1}{K}[\mu + \psi_{t+k} + \frac{\theta}{1-\epsilon}\pi^*_{t+k} - \frac{\beta\theta}{1-\epsilon}\pi^*_{t+1+k}] \tag{72}$$

$$\pi^*_{t+k} = \beta\mathbb{E}_{t+k}[\pi^*_{t+1+k}] + \frac{\epsilon - 1}{\theta}(\mu + \psi_{t+k} - p^*_{t+k}) \tag{73}$$

$$\boxed{\pi_t = \beta\mathbb{E}_t[\pi_{t+1}] - \frac{\epsilon-1}{\theta}\hat{\mu}_t} \tag{74}$$

Now, obviously, I jumped a few steps when doing the log-lin, but I promise not too many, because I was able to do it in my head. The key is to get comfortable with what approximations come from what types of operations, recalling that we are evaluating around deterministic steady-state. I'll be honest, there's also a bit of hand-waving on my part, but I believe I am correct[12].

Now take a look at the mark-up precursor to Phillips curve in the Calvo calculations. It has the same form as the above equation! This tells us that, up to a first approximation, appropriate parameter choices in the Rotemberg framework yield the same Phillips curve (and vice versa), and therefore same equilibrium. This result does not hold if we consider higher order approximations, and it does not imply that the mechanics are the same, merely the aggregate dynamics. This is easily demonstrated by noting that in Calvo's model, a firm exhibits, on average the "same level of stickiness" as in the Rotemberg model (with proper parameterization), but in Calvo this is due to stochastic costless adjust, which might be larger adjustment, whereas Rotemberg is due to constant, costly adjustment.

[12]Because I match the Rotemberg part of these notes: https://www3.nd.edu/∼ esims1/new_keynesian_2016.pdf