Solving (Antras et al., 2021)

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May 2, 2021

1 Roadmap

We complete 4 steps in this document

(i) Setup the problem

(ii) Describe the solution in terms of equilibrium conditions

(iii) Translate the equilibrium conditions into a format a computer can solve

(iv) Report results for a few calibrations

2 Setup

There are two countries, each with two sectors. To ship a good from country i to country j in sector s, a firm must ship $\tau_{ij}^s \geq 1$ goods, where the equality always holds when $i = j$. Additionally, country i may charge ad-valorem taxes t_{ji}^s on goods coming from country j into country i in sector s, and may give subsidies v_{ij}^s to good coming from country i to country j in sector s.

In the upstream sector, there are a continuum of firms $\omega \in [0, M_i^u]$ in each country i, and production is constant returns to scale with labor as the only input, though there is a fixed cost to entry.

$$f_i^u + x_i^u(\omega) = A_i^u \ell_i^u(\omega)$$

Downstream there is a different continuum of firms $\omega \in [0, M_i^d]$ in each country i, and production is constant returns to scale, except now upstream goods are used as inputs via an aggregator[1]

$$\begin{aligned} f_i^d + x_i^d(\omega) &= A_i^d(\ell_i^d(\omega))^\alpha Q_i^u(\omega)^{1-\alpha} \\ Q_i^u(\omega) &= \left[\sum_j \left(\int_0^{M_j^u} q_{ji}^u(\overline{\omega}(\omega))^{\frac{\theta-1}{\theta}} \mathrm{d}\overline{\omega}\right)\right]^{\frac{\theta}{\theta-1}}, \quad \theta > 1 \end{aligned}$$

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[1]Note that I have added a clarification that the $\overline{\omega}$ is potentially a function of ω, otherwise the right side of the aggregator definition does not depend on ω. This technically makes the math messier, but it is immediately resolved once optimizing conditions are taken. If easier, we could also use $\omega(\overline{\omega})$.

In each country there is a representative household[2] who derives utility[3]

$$U_i = \left[\sum_j \left(\int_0^{M_j^d} q_{ji}^d(\omega)^{\frac{\sigma-1}{\sigma}} \mathrm{d}\omega \right) \right]^{\frac{\sigma}{\sigma-1}}, \quad \sigma > 1$$

All markets have free entry, and each country has exogenous total labor supply L_i.

3 Equilibrium

Here we present the full set of equilibrium conditions, leaving all endogenous variables as such, and in the next section reduce the dimensionality of the system.

The equilibrium requires

(i) Given their income, and taking prices as given, households maximize their utility.

(ii) Given downstream demand for their goods, and upstream prices for inputs, downstream firms maximize profits

(iii) Given upstream demand for their goods, upstream firms maximize profits.

(iv) Goods markets clear.

(v) Labor markets clear.

(vi) Free entry is imposed (supernormal profits cannot exist)

(vii) The government budget is balanced.

Taking first-order conditions and imposing free entry[4] and markets clearing, we are left with the below system, where all prices, wages, quantities, and masses are endogenous.

[2] It may be helpful to instead think of each country as having a continuum of identical households, with measure 1. This point about the measure helps clarify ideas, since this measure is exogenous, unlike the firm measures.

[3] I find it helpful to note $U_i = Q_i^d$.

[4] Technically, free entry by itself looks little less clean, but we can combine it with other conditions being used to get the nice expression below. The important point is that we are still imposing a unique new condition in part of the derivation.

$$mc_i^s = \frac{\bar{\alpha}_s}{A_i^s} w_i^{\alpha_s} (P_i^u)^{1-\alpha_s} \qquad \text{(Marginal Costs)}$$

$$\alpha_s = \begin{cases} \alpha & s = d \\ 1 & s = u \end{cases}$$

$$\bar{\alpha_s} = \left(\frac{1}{\alpha_s}\right)^{\alpha_s} \left(\frac{1}{1-\alpha_s}\right)^{1-\alpha_s}$$

$$p_{ij}^s = \frac{\mu_s \tau_{ij}^s mc_i^s}{1 + v_{ij}^s} \qquad \text{(Prices)}$$

$$q_{ji}^d = \left(\frac{(1+t_{ji}^d) p_{ji}^d}{P_i^d}\right)^{-\sigma} \frac{w_i L_i + T_i}{P_i^d} \qquad \text{(Downstream demand)}$$

$$q_{ji}^u = (1-\alpha) \frac{mc_i^d (f_i^d + y_i^d)}{P_i^u} \left(\frac{(1+t_{ji}^u) p_{ji}^u}{P_i^u}\right)^{-\theta} \qquad \text{(Upstream demand)}$$

$$\ell_i^d = \alpha \frac{mc_i^d (f_i^d + y_i^d)}{w_i} \qquad \text{(Downstream labor)}$$

$$\ell_i^u = \frac{f_i^u + y_i^u}{A_i^u} \qquad \text{(Upstream labor)}$$

$$P_i^d = \left[\sum_j \int_0^{M_j^d} \Big((1+t_{ji}^d) p_{ji}^d(\omega)\Big)^{1-\sigma} \mathrm{d}\omega\right]^{\frac{1}{1-\sigma}} \qquad \text{(Downstream price index)}$$

$$P_i^u = \left[\sum_j \int_0^{M_j^u} \big((1+t_{ji}^u) p_{ji}^u(\omega)\big)^{1-\theta} \mathrm{d}\omega\right]^{\frac{1}{1-\theta}} \qquad \text{(Upstream price index)}$$

$$y_i^d = (\sigma - 1) f_i^d \qquad \text{(Downstream free entry)}$$

$$y_i^u = (\theta - 1) f_i^u \qquad \text{(Upstream free entry)}$$

$$L_i = M_i^d \ell_i^d + M_i^u \ell_i^u \qquad \text{(Labor Market)}$$

$$y_i^d = \sum_j \tau_{ij}^d q_{ij}^d \qquad \text{(Downstream goods market)}$$

$$y_i^u = \sum_j M_j^u \tau_{ij}^u q_{ij}^u \qquad \text{(Upstream goods market)}$$

$$T_i = \sum_j \left[t_{ji}^d M_j^d p_{ji}^d q_{ji}^d + t_{ji}^u M_j^u M_i^d p_{ji}^u q_{ji}^u - v_{ij}^d M_i^d p_{ij}^d q_{ij}^d - v_{ij}^u M_i^u M_j^d p_{ij}^u q_{ij}^u\right] \qquad \text{(Govt Budget)}$$

4 Translation

The above system has 8 prices, 2 wages, 8 quantities, and 4 masses, for a total of 22 variables (and corresponding 22 equations), so in principle it could be directly solved by a computer up to scale. Instead, we take advantage of the structure of the system to reduce the system to only 9 variables (and 5 in a special case). Consider the following set of observations, where "pinned" means that a closed-form solution exists.

(i) Given wages and masses, upstream marginal costs are pinned.

(ii) Given marginal costs, their respective prices are pinned.

(iii) Given upstream prices and masses, upstream price indices are pinned.

(iv) Given wages and upstream price indices, downstream marginal costs are pinned.

(v) Given downstream marginal costs, upstream prices, and upstream price indices, upstream quantities are pinned.

(vi) Upstream and downstream total output is pinned.

(vii) **In the absence of taxes and subsidies**, $T_i = 0$, so, given downstream prices, downstream price indices, and wages, downstream quantities are pinned.

Concatenating these observations, we find the two characterizations below of the system. We believe this is the maximal *dimension* reduction possible.

(i) Given wages, masses, and downstream quantities, the rest of the system may be solved in closed-form. Hence solving the model amounts to solving a 9-dimensional nonlinear system of equations (one wage may be normalized).

(ii) Additionally, in the absence of taxes and subsidies, downstream quantities may also be solved in closed-form from wages and masses, hence the system is reduced to a 5-dimensional nonlinear system of equations.

Using the above equations to find the corresponding **closed-form** expressions of prices and upstream quantities, define the following system (we include the normalization as a condition for ease of exposition, but of course it is really a given, so the 10-D system is really 9-D).

$$
\begin{aligned}
\mathbf{x} &= \{w_i, q_{ji}^d, M_i^s\} \\
g_1(\mathbf{x}) &= w_i - 1 && (i = 1) \\
g_k(\mathbf{x}) &= q_{ji}^d - \left(\frac{(1 + t_{ji}^d) p_{ji}^d(\mathbf{x})}{P_i^d(\mathbf{x})} \right)^{-\sigma} \frac{w_i L_i + T_i(\mathbf{x})}{P_i^d(\mathbf{x})} && (k \in \{2, 3, 4, 5\},\ j, i \in \{1, 2\}) \\
g_k(\mathbf{x}) &= y_i^d - \sum_j \tau_{ij}^d q_{ij}^d && (k \in \{6, 7\},\ i \in \{1, 2\}) \\
g_k(\mathbf{x}) &= y_i^u - \sum_j M_j^d \tau_{ij}^u q_{ij}^u && (k \in \{8, 9\},\ i \in \{1, 2\}) \\
g_k(\mathbf{x}) &= L_i - M_i^d \ell_i^d(\mathbf{x}) - M_i^u \ell_i^u(\mathbf{x}) && (k = 10,\ i = 2)
\end{aligned}
$$

The idea is that we normalize wages in the home sector, use labor market clearing in the foreign sector, use downstream goods demand, and goods market clearing as our nonlinear equations that determine equilibrium $\mathbf{x}$. To solve the system, we merely search for an $\mathbf{x}$ satisfying

$$\mathbf{g}(\mathbf{x}) = \mathbf{0}$$

with $\mathbf{x}$ including wages, masses and downstream quantities.

In order to use an optimizer instead of a nonlinear solver, we instead tell the computer to solve

$$\min_{\mathbf{x}} \|\mathbf{g}(\mathbf{x})\|^2$$

which will give the same result.

5 Results

First we lay out some of the computational notes and lessons we learned along the way, then display some results found when solving the model.

5.1 Notes and Trickiness

Naively, we assumed that since the problem is smooth and comparatively small, a basic Newton-Raphson solver might work, so we coded our own. Much of our time and confusion was spent trying to get this to work, because resorting to a black-box solver was somewhat unappealing[5]. This endeavor returned no fruits however, so we eventually switched to using `Ipopt` inside of `JuMP` for `Julia`, and may in the future switch to `KNITRO` as well. Our hope is that, even if our minimal system turns out to be identical to Agus', at the very least we have implemented it in a new ecosystem.

In terms of transformation, even with `Ipopt`, the solver runs into errors since it attempts to search/test negative parameter values, which leads to complex values. The natural choice for a transformation is the exponential function, which allows the solver to search over $\mathbb{R}$ to effectively search the positive reals. Unfortunately, this choice often leads to overflow errors, both because the value and derivative explode and implode at an exponential rate. Using a quadratic transformation loses the bijectivity of the transformation, but dramatically slows the growth in value (quadratic) and derivative (linear), and keeps the parameter values non-negative. This transformation is what we found to work.

5.2 Simple Symmetry

For all current calibrations, we use

$$\begin{aligned} \sigma &= 4 \\ \theta &= 4 \\ f_i^s &= 1 \\ \alpha &= 0.5483 \\ t_{ij}^s &= 0 \\ v_{ij}^s &= 0 \end{aligned}$$

For our first test, we want to be sure that if everything is symmetric in terms of calibration, then our equilibrium should also be symmetric. We calibrate

$$\begin{aligned} L_i &= 1 \\ A_i^s &= 1 \\ \tau_{ij}^s &= 1 \end{aligned}$$

Our solution is then

[5] I (Chase) should clarify that I believe we had the same system here functional last week, but I was foolishly intent on home-growing a solver. Lesson learned.

$$
\begin{aligned}
p_{ij}^d &= [3.78, 3.78; 3.78, 3.78] \\
p_{ij}^u &= [1.33, 1.33; 1.33, 1.33] \\
w &= [1.0, 1.0] \\
q_{ij}^d &= [1.5, 1.5; 1.5, 1.5] \\
q_{ij}^u &= [17.02, 17.02; 17.02, 17.02] \\
M_i^s &= [0.08, 0.13; 0.10, 0.10]
\end{aligned}
$$

We see that the solution is mostly symmetric, except the masses are a little off. Nonetheless, at this solution our objective has value

$$
\|\mathbf{g}(\mathbf{x})\|^2 = 1.29 \times 10^{-18}
$$

so the mass error is “quite small” in terms of its implications for equilibrium residuals.

To verify our solution, we note that the domestic labor market clearing condition was never used in solving, but by Walras’ Law should be satisfied. Indeed, at our solution we find the labor residual sufficiently small, verifying our solution.

$$
L_H - M_H^d \ell_H^d(\mathbf{x}) - M_H^u \ell_H^u(\mathbf{x}) = -1.08 \times 10^{-10}
$$

5.3 Symmetric Autarky

As iceberg costs increase, we should approach two autarkic economies instead of a single open trading economy. The issue, however, is that in this case we would also need to normalize the wage in the foreign economy as well. To alleviate this issue, we choose iceberg costs large enough to effectively discourage trade, but not too large so that we need to normalize the foreign wage. We set

$$
\begin{aligned}
L_i &= 0.4531 \\
A_i^s &= 1 \\
\tau_{ij}^s &= 1 + \mathbf{1}\{i \neq j\} \times 4
\end{aligned}
$$

Our solution is

$$
\begin{aligned}
p_{ij}^d &= [4.72, 23.62; 23.62, 4.72] \\
p_{ij}^u &= [1.33, 6.67; 6.67, 1.33] \\
w &= [1.0, 1.0] \\
q_{ij}^d &= [2.98, 0.0; 0.0, 2.98] \\
q_{ij}^u &= [93.07, 0.15; 0.15, 93.07] \\
M_i^s &= [0.03, 0.05; 0.03, 0.05]
\end{aligned}
$$

As hoped, the prices, quantities, and masses within each economy roughly match those in the closed economy, for which we have closed-form solutions for the entire system.

Additionally, at our solution, the objective is

$$\|\mathbf{g}(\mathbf{x})\|^2 = 3.95 \times 10^{-15}$$

and we verify

$$L_H - M_H^d \ell_H^d(\mathbf{x}) - M_H^u \ell_H^u(\mathbf{x}) = 2.06 \times 10^{-8}$$

5.4 Paper Calibration

Lastly, we use the parameters given in the paper

$$\begin{aligned}
L_i &= [0.4531, 9.5469] \\
A_i^s &= [1.0, 1.0; 0.2752, 0.1121] \\
\tau_{ij}^s &= \begin{cases} 3.0301 & i \neq j,\ s = d \\ 2.6039 & i \neq j,\ s = u \\ 1 & \text{else} \end{cases}
\end{aligned}$$

We find

$$\begin{aligned}
p_{ij}^d &= [4.67, 14.13; 15.71, 5.19] \\
p_{ij}^u &= [1.33, 3.47; 3.98, 1.53] \\
w &= [1.0, 0.13] \\
q_{ij}^d &= [2.84, 0.05; 0.02, 2.93] \\
q_{ij}^u &= [84.44, 1.38; 1.06, 36.83] \\
M_i^s &= [0.03, 0.05; 0.08, 0.12]
\end{aligned}$$

The objective at the solution is

$$\|\mathbf{g}(\mathbf{x})\|^2 = 1.28 \times 10^{-13}$$

We verify

$$L_H - M_H^d \ell_H^d(\mathbf{x}) - M_H^u \ell_H^u(\mathbf{x}) = -7.00 \times 10^{-8}$$

6 Small comments

On page 36, Appendix A Open Economy Equilibrium with taxes, last line in the first paragraph it says “Nota” and it should say “Note”.

References

Antras, Pol et al. (2021). "Import Tariffs and Global Sourcing". In: *Working Paper*.