

Networks and Random Processes Assignment 2

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October 2018

1 Kingman's Coalescent

1.1 A

N_t is the number of particles at time t with $N_0 = L$. The process $(N_t : t \geq 0)$ has the state space $\{1, \dots, L\}$

1.1.1 Transition Rate of the process

$$r(n, n-1) = \binom{L}{2}, n \geq 2$$

QUESTION - WHAT ABOUT SAME STATE? $r(n, n) =$

QUESTION - WHAT ABOUT OTHER STATES - HOW TO WRITE IT?
 $r(n, y) = , y \neq n, n-1$

1.1.2 Generator

This is a jump process, so the generator is

$$(\mathcal{L}f)(x) = \int_{\mathcal{R}} r(x, y)[f(y) - f(x)]dy$$

For this process

$$(\mathcal{L}f)(n) = r(n, n-1)(f(n-1) - f(n))$$

$$(\mathcal{L}f)(n) = \binom{n}{2}(f(n-1) - f(n))$$

1.1.3 Master Equation

The master equation is

$$\frac{d}{dt}\pi_t(n) = \pi_t(n+1)r(n+1, n) - \pi_t(n)r(n, n-1)$$

$$\frac{d}{dt}\pi_t(n) = \pi_t(n+1)\binom{n+1}{2} - \pi_t(n)\binom{n}{2}$$

QUESTION - IS THIS RIGHT? QUESTION - IS THE NOTATION OKAY?
QUESTION - WHAT ABOUT EDGES?

1.1.4 Ergodicity

The process is ergodic.

1.1.5 Absorbing States

The unique absorbing state is $N = 1$.

1.1.6 Stationary Distributions

Let a distribution $\pi = [N = 1, N = 2, \dots, N = L]$

The unique stationary distribution is

$$\pi_0 = [1, 0, \dots, 0]$$

1.2 B - Mean Time to Asorption

The rate of coalescence, ie moving to the next state, for each state is

$$\lambda_n = r(n, n-1) = \binom{n}{2} = \frac{n(n-1)}{2}$$

The times in each state are expnentially dsitributed as

$$f_t(n) = \binom{n}{2} e^{-\binom{n}{2}t}$$

The expected time in each state, or the waiting time, is given by

$$\beta_n = \frac{1}{\lambda_n} = \frac{2}{n(n-1)}$$

The expected time to absorption is the sum of the expected waiting times in each of the states

$$E(T) = \sum_{n=2}^L \frac{2}{n(n-1)}$$

Bringing the 2 outside of the summand and splitting up into partial fractions

$$E(T) = 2 \sum_{n=2}^L \frac{1}{n-1} - \frac{1}{n}$$

$$E(T) = 2\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + \dots - \frac{1}{L-1} + \frac{1}{L+1} - \frac{1}{L}\right)$$

All but the first and last terms cancel giving

$$E(T) = 2\left(1 - \frac{1}{L}\right)$$

1.3 C - Rescaled process

Rescale the process to N_t/L

$$(\mathcal{L}^L f)(n/L) = \frac{1}{L} \binom{n}{2} \left(f\left(\frac{n-1}{L}\right) - f\left(\frac{n}{L}\right)\right)$$

Taylor expand and let $x = \frac{n}{L}$:

$$(\mathcal{L}^L f)(x) = \frac{1}{L} \frac{n(n-1)}{2} \left(f(x) - \frac{1}{L} f'(x) + \frac{1}{L^2} f''(x) + O\left(\frac{1}{L^3}\right) - f(x)\right)$$

Cancel terms, substitute $n = Lx$ and rearrange

$$(\mathcal{L}^L f)(x) = \left(\frac{x^2}{2} - \frac{x}{2L}\right) (-f'(x) + \frac{1}{L} f''(x) + O\left(\frac{1}{L^2}\right))$$

$$\lim_{L \rightarrow \infty} (\mathcal{L}^L f)(x) = \frac{-x^2}{2} f'(x)$$

QUESTION - STATE SPACE? QUESTION - INITIAL CONDITION?

1.3.1 Deterministic

The generator has no diffusion term, only drift. So there is no variance in the process and it must be entirely deterministic.

$$\frac{d}{dt} E(X_t) = E\left(-\frac{X_t^2}{2}\right)$$

X_t is deterministic, so

$$\frac{d}{dt} X_t = -\frac{X_t^2}{2}$$

$$X(t) = -\frac{X^2}{2} t$$

NOTATION OKAY?

1.4 D - Simulations

QUESTION - WHAT IS THE ADD THIS LINE BIT OF CODE IN EMMA'S WORKBOOK

2 Ornstein-Uhlenbeck process

2.1 A

The mean:

$$\frac{dm}{dt} = \frac{d}{dt}E(X_t) = E(-\alpha X_t) = -\alpha E(X_t)$$

$$\frac{dm}{dt} = -\alpha m(t)$$

X_t^2 :

$$\frac{d}{dt}E(X_t^2) = E(-\alpha 2X_t^2 + \sigma^2)$$

$$\frac{d}{dt}E(X_t^2) = -2\alpha E(X_t^2) + \sigma^2$$

The variance:

$$v(t) = E(X_t^2) - m(t)^2$$

$$\frac{dv}{dt} = \frac{d}{dt}E(X_t^2) - 2m \frac{dm}{dt}$$

$$\frac{dv}{dt} = -2\alpha E(X_t^2) + \sigma^2 + 2m^2$$

2.2 B

2.2.1 Solution of $m(t)$

Solving for $m(t)$, and considering the initial conditions, $c = x_0$

$$m(t) = x_0 e^{-\alpha t}$$

2.2.2 Solution of $v(t)$

Solving for $v(t)$

$$\frac{dv}{dt} = -2\alpha(v + m^2) + \sigma^2 + 2\alpha m^2$$

$$\frac{dv}{dt} + 2\alpha v = \sigma^2$$

Homogenous solution:

$$v_{hom} = c_2 e^{-2\alpha t}$$

Particular solution:

$$v_{part} = \frac{\sigma^2}{2\alpha}$$

Full solution:

$$v(t) = c_2 e^{-2\alpha t} + \frac{\sigma^2}{2\alpha}$$

From initial conditions, $v(0) = 0$:

$$v(t) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$$

2.2.3 Distribution

As a Gaussian process, the distribution is fully described by the mean and variance.

$$f(X_t) = \frac{1}{\sqrt{\frac{2\pi\sigma^2(1-e^{-2\alpha t})}{2\alpha}}} \exp\left(\frac{-\alpha(x - x_0 e^{-\alpha t})^2}{\sigma^2(1 - e^{-2\alpha t})}\right)$$

IS THAT RIGHT?

2.2.4 Stationary Distribution

Given enough time, the process will converge to a Gaussian stationary distribution.

$$\lim_{t \rightarrow \infty} m(t) = 0$$

$$\lim_{t \rightarrow \infty} v(t) = \frac{\sigma^2}{2\alpha}$$

The stationary distribution is $\sim N(0, \frac{\sigma^2}{2\alpha})$

$$f_0(X_t) = \frac{1}{\sqrt{\frac{\pi\sigma^2}{\alpha}}} \exp\left(\frac{-\alpha x^2}{\sigma^2}\right)$$

2.3 Simulation

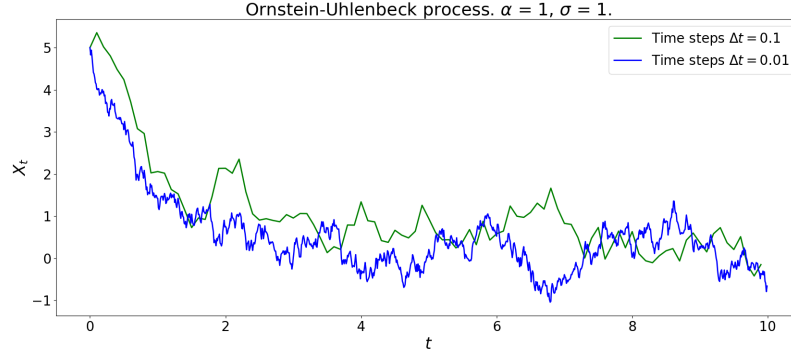


Figure 1: The Ornstein-Uhlenbeck process with $\alpha = 1$, $\sigma^2 = 1$ and $X_0 = 5$. Simulated for 10 seconds with timesteps $\Delta t = 0.1$ and 0.01

Figure 1 shows the Ornstein-Uhlenbeck process simulated. The process begins at $X_0 = 5$, and experiences a "force" pulling it towards zero. As time progresses, the process moves towards $X_t = 0$ and the noise term begins to dominate the behaviour. Both choices of timestep give a similar result.

3 Moran Model and Wright-Fisher diffusion

3.1 A

3.1.1 State space

Let the total possible L types be

$$T = \{1, 2, \dots, L\}$$

Each of the L individuals can have any of those types, so the state space is

$$S = \{1, 2, 3, \dots, L\}^L$$

3.1.2 Irreducibility

It is not irreducible because there are absorbing states.

3.1.3 Stationary distributions

The absorbing states are where all individuals have the same type

$$x_k = [k, k, \dots, k] \quad \forall k \in \{1, 2, \dots, L\}$$

The stationary distributions are any linear combination of the absorbing states that sum to 1.

$$\pi(y) = \sum_{k=1}^L \alpha_k \pi_k(y)$$

$$\sum_{k=1}^L \alpha_k = 1$$

The coefficients, α_k , are determined by the initial conditions and can be thought of as the probability of each type "winning" and taking over all individuals.

3.2 B

3.2.1 Markov process

$N_t : t \geq 0$ is a Markov process. It's future distribution is determined only by it's current state, not the specific history.

3.2.2 State space

Each type can have any integer between 0 and L types, and there are L types, so the state space is

$$S = \{0, 1, 2, \dots, L\}^L$$

3.2.3 Generator

For each type, we assume that the number of individuals of that type, n , can only increase and decrease by 1 at one moment in time, i.e only one event happens at a time. The rates of gain and loss can be described as:

$$r(n, n+1) = \frac{n(L-n)}{L-1}$$

$$r(n, n-1) = \frac{n(L-n)}{L-1}$$

These are symmetrical.

This is a jump process, so the generator is

$$(\mathcal{L}f)(x) = \int_{\mathbb{R}} r(x, y)[f(y) - f(x)]dy$$

For this process

$$(\mathcal{L}f)(n) = r(n, n-1)(f(n-1) - f(n)) + r(n, n+1)(f(n+1) - f(n))$$

$$(\mathcal{L}f)(n) = \frac{(L-n)n}{L-1}(f(n-1) + f(n+1) - 2f(n))$$

3.2.4 Irreducibility

The process is not irreducible because there are absorbing states.

3.2.5 Stationary distributions

The absorbing states are where one type has $N_k = L$ and the rest $N_k = 0$. i.e

$$\pi_k(y) = \delta_{y,k} = \begin{cases} L & , \quad y = k \\ 0 & , \quad \text{otherwise} \end{cases}$$

?? IS THAT RIGHT?

3.2.6 Limiting distribution

As $t \rightarrow \infty$ with initial condition $N_0 = 1$. By symmetry each type has the same chance of "winning", therefore the limiting distribution is

$$\pi_\infty = [\frac{L-1}{L}, 0, 0, \dots, 0, \frac{1}{L}]$$

3.3 C

3.3.1 $m_1(t)$

$$\frac{d}{dt}(E(N_t)) = E(\frac{n(L+n)}{L-1}(n+1+n-1-2n))$$

$$\frac{d}{dt}(E(N_t)) = 0$$

$N_0 = n$, therefore $E(N_t) = n$.

3.3.2 $m_2(t)$

$$\frac{d}{dt}(E(N^2)) = E(\frac{n(L+n)}{L-1}((n+1)^2 + (n-1)^2 - 2n^2))$$

$$\frac{d}{dt}(E(N^2)) = E(\frac{n(L+n)}{L-1}(n^2 + 2n + 1 + n^2 - 2n + 1 - 2n^2))$$

$$\frac{d}{dt}(E(N^2)) = E(\frac{2n(L+n)}{L-1})$$

$$\frac{d}{dt}(E(N^2)) = \frac{2L}{L-1}n + \frac{2}{L-1}E(N^2)$$

Solution in the form:

$$E(N^2) = Ae^{\frac{-2t}{L-1}} + B$$

From initial condition, $A = N_0^2$

Substitute everything into the original differential equation:

$$\frac{-2n}{L-1}e^{\frac{-2t}{L-1}} = \frac{2Ln}{L-1} + \frac{2n}{L-1}e^{\frac{-2t}{L-1}} + \frac{2B}{L-1}$$

$$B = Ln(1 - ne^{\frac{-2t}{L-1}})$$

$$E(N^2) = n^2e^{\frac{-2t}{L-1}} + Ln(1 - e^{\frac{-2t}{L-1}})$$

QUESTION - I HAVE A FACTOR OF n in front of exponential in the bracket for B - I don't think that I should have

3.3.3 Absorption probabilities

By symmetry, each individual has an equal chance of winning?? NOT SURE ABOUT THIS

3.3.4 Absorption time scales with system size L

The expected value of $E(N_t) = n$.

The variance is given by

$$Var(N_t) = n^2e^{\frac{-2t}{L-1}} + Ln(1 - ne^{\frac{-2t}{L-1}}) - n^2$$

$$Var(N_t) = (Ln - n^2)(1 - e^{\frac{-2t}{L-1}})$$

If we assume that we start in the middle, so $n = \frac{L}{2}$

$$Var(N_t) = \frac{L^2}{4}(1 - e^{\frac{-2t}{L-1}})$$

As the standard deviation grows, the probability of being absorbed at either $N = 0$ or $N = L$ also grows. The rate at which the standard deviation grows with time scales approximately with the length of the state space.

3.4 D

3.4.1 Rescaling limit

$$(\mathcal{L}f)(n) = \frac{(L-n)n}{L-1}(f(n+1) + f(n-1) - 2f(n))$$

Rescale by dividing state space by L , multiplying time by L^α and changing the variable to $x = \frac{n}{L}$:

$$(\mathcal{L}f)(x) = \frac{(L-xL)xL}{L-1} L^\alpha (f(x + \frac{1}{L}) + f(x - \frac{1}{L}) - 2f(x))$$

Taylor expand and simplify:

$$(\mathcal{L}f)(x) = \frac{(L-xL)xL}{L-1} L^\alpha (\frac{1}{L^2} f''(x) - O(\frac{1}{L^3}))$$

$$(\mathcal{L}f)(x) = \frac{L^\alpha}{L} \frac{xL^2 - x^2L^2}{1 - \frac{1}{L}} (\frac{1}{L^2} f''(x) - O(\frac{1}{L^3}))$$

Set $\alpha = 1$ to get a non-trivial scaling limit:

$$(\mathcal{L}f)(x) = (\frac{x-x^2}{1-\frac{1}{L}}) f''(x) - O(\frac{1}{L})$$

Take limit $L \rightarrow \infty$:

$$(\mathcal{L}^L f)(x) = \lim_{L \rightarrow \infty} (\mathcal{L}f)(x) = (x-x^2) f''(x)$$

$$(\mathcal{L}^L f)(x) = x(1-x) f''(x)$$

3.4.2 Fokker-Planck Equation

$$\frac{\partial}{\partial t} p_t(x, y) = \frac{1}{2} \frac{\partial^2}{\partial y^2} (y(1-y) p_t(x, y))$$

WHAT DOES THIS MEAN?

3.5 E

3.5.1 $m(t)$

$$\frac{d}{dt} E(M) = 0$$

No change in $E(M_t)$ with time. It's a diffusion process with no drift, so $E(M_t) = 0$

3.5.2 $v(t)$

$$v(t) = E(M_t^2)$$

Let $f(m) = m^2$:

$$\frac{d}{dt} E(M^2) = E(2M - 2M^2) = -2E(M^2)$$

$$v(m) = E(M^2) = c_1 e^{-2t}$$

3.6 F