

# Networks and Random Processes Assignment 1

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## 1 Question 1

This section considers a Simple Random Walk on  $1, \dots, L$  with probabilities  $p \in [0, 1]$  and  $q = 1 - p$  to jump right and left respectively.

Different boundary conditions are considered.

### 1.1 Part A

#### 1.1.1 Case 1 - Periodic

Periodic boundary conditions, so  $p(0, L) = q$  and  $p(L, 0) = p$

The transition matrix is:

$$P = \begin{bmatrix} 0 & p & 0 & \dots & 0 & q \\ q & 0 & p & \dots & 0 & 0 \\ 0 & q & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & p \\ p & 0 & 0 & \dots & q & 0 \end{bmatrix}$$

The process is irreducible, i.e. every state can, eventually, reach every other state. And there is a finite state space, so it has 1 unique stationary distribution.

The states can be laid out on a circle, and are symmetrical, so the stationary distribution is where all states have equal probabilities.

$$\pi = (1/L, 1/L, \dots, 1/L)$$

for all  $p \in (0, 1)$

The stationary distribution is reversible only for the case  $p = q = \frac{1}{2}$ .

#### 1.1.2 Case 2 - Closed

Transition matrix:

$$P = \begin{bmatrix} q & p & 0 & \dots & 0 & 0 \\ q & 0 & p & \dots & 0 & 0 \\ 0 & q & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & p \\ 0 & 0 & 0 & \dots & q & p \end{bmatrix}$$

If  $p = 1$ , there is an absorbing state at  $L$ , and the stationary distribution is  $\pi = (0, 0, \dots, 0, 1)$ . This stationary distribution is reversible, all terms in the detailed balance equations are zero. This is not irreducible, as the walk can only move from  $s_x$  to  $s_{x+1}$ .

If  $q = 1$ , there is an absorbing state at  $1$ , and the stationary distribution is  $\pi = (1, 0, \dots, 0, 0)$ . This stationary distribution is reversible, all terms in the detailed balance equations are zero. This is not irreducible, as the walk can only move from  $s_x$  to  $s_{x-1}$ .

If  $p = q = \frac{1}{2}$ , the process is irreducible, as every state can, eventually, be reached from every other state. There is a finite state space and so there is only 1 stationary distribution. The sum of all columns equal 1, so there is a constant left eigenvector, and so the stationary distribution is  $\pi = (1/L, 1/L, \dots, 1/L)$ . This stationary distribution is reversible.

If  $p, q \neq [0, 1/2, 1]$ , then the process is irreducible, and it has a finite state space so there is only 1 stationary distribution. Looking at the detailed balance equations, we can find a recurrence relation of the form:

$$\pi_{x-1}p = \pi_x q \text{ over } x = \{2, 3, \dots, L-1\}$$

$$\pi_x = \pi_{x-1} \frac{p}{q}$$

By induction, this suggests a solution like

$$\pi_x = \pi_1 \left(\frac{p}{q}\right)^{x-1}$$

This is a reversible distribution, and so must also be stationary. As the process is ergodic, this is the unique stationary distribution.

We can check this distribution is stationary for a closed simple random walk with 4 states

$$\begin{bmatrix} \pi_1 \\ \pi_1 \frac{p}{q} \\ \pi_1 \left(\frac{p}{q}\right)^2 \\ \pi_1 \left(\frac{p}{q}\right)^3 \end{bmatrix} \begin{bmatrix} q & p & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & q & p \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_1 \frac{p}{q} \\ \pi_1 \left(\frac{p}{q}\right)^2 \\ \pi_1 \left(\frac{p}{q}\right)^3 \end{bmatrix}$$

We also need to normalise the distribution, so the stationary distribution will be

$$\pi_x = \frac{\left(\frac{p}{q}\right)^{x-1}}{\sum_i^L \pi_i}$$

## 1.2 Part B - Absorbing

$$P = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ q & 0 & p & \dots & 0 & 0 \\ 0 & q & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & p \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

The process is not irreducible, as no other states can be reached from state 1 or state  $L$ .

The (normalised) stationary distributions are

$$\pi_1 = [1, 0, 0, \dots, 0]$$

$$\pi_2 = [0, 0, 0, \dots, 1]$$

$$\pi_3 = [a, 0, 0, \dots, 0, 1 - a] \text{ where } a \in [0, 1]$$

These distributions are reversible, looking at the detailed balance conditions:

$$\pi(x)p(x, y) = \pi(y)p(y, x)$$

All terms for all equations are zero, therefore it is reversible.

The absorption probability in site L is

$$h_k^L = P(X_n = L \text{ for some } n \geq 0 | X_0 = k)$$

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Considering  $k+1$ ,  $k-1$ , and by law of total probability:

$$h_k^L = P(X_n = L | X_1 = k+1, X_0 = k) * p + P(X_n = L | X_1 = k-1, X_0 = k) * q$$

Using the Markov property:

$$h_k^L = P(X_n = L | X_1 = k+1) \times p + P(X_n = L | X_1 = k-1) \times q$$

$$h_k^L = h_{k+1}^L \times p + h_{k-1}^L \times q$$

And the boundary conditions are:

$$h_1^L = 0 \text{ and } h_L^L = 1$$

If  $p = q$  then this recursion relation becomes

$$h_k^L = \frac{h_{k+1}^L + h_{k-1}^L}{2}$$

This is linear interpolation between the two surrounding states, so the absorption probability is linear in  $k$ . Considering the boundary conditions, the solution is:

$$h_k^L = \frac{k-1}{L-1}$$

If  $p \neq q$ , we can solve the recursion relation by considering the ansatz:

$$h_k^L = \lambda^k$$

$$\lambda = p\lambda^2 + q$$

This has roots:

$$\lambda_1 = 1 \text{ and } \lambda_2 = q/p$$

The general solution is of the form:

$$h_k^L = a\lambda_1 + b\lambda_2$$

$$h_k^L = a + b\left(\frac{q}{p}\right)^k$$

Looking at the boundary conditions:

$$h_1^L = 0 = a + b\left(\frac{q}{p}\right)$$

$$h_L^L = 1 = a + b\left(\frac{q}{p}\right)^L$$

Subtracting the first equation from the second equation, and solving for  $b$ :

$$b = \frac{1}{\left(\frac{q}{p}\right)^L - \frac{q}{p}}$$

$$a = \frac{-1}{\left(\frac{q}{p}\right)^{L-1} - 1}$$

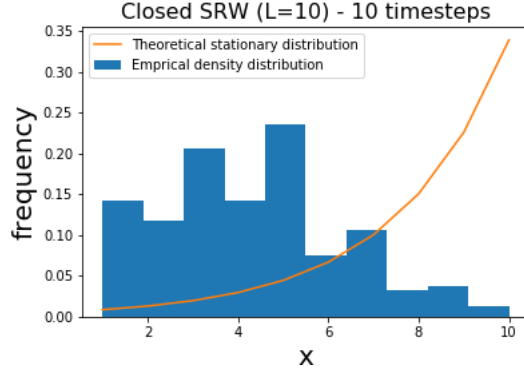


Figure 1. The frequency distribution over 500 different realisations after 10 timesteps.

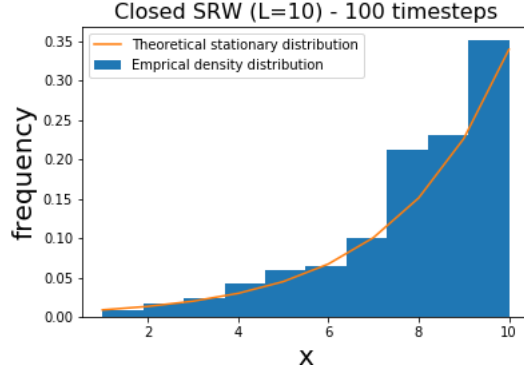


Figure 2. The frequency distribution over 500 different realisations after 100 timesteps.

### 1.3 Part C - simulations

A simple random walk with  $L=10$  and closed boundary conditions was simulated 500 times, with a  $p = 0.6$  and starting at  $x=1$  at  $t=0$ .

After 10 time steps (Figure 1), the state of the process is still heavily influenced by the starting condition of  $x(0) = 1$ .

After 100 time steps (Figure 2), the empirical distribution is similar to the theoretical stationary distribution. Ergodic processes tend towards the stationary distribution after a large number of time steps.

After 500 time steps of 1 realisation (Figure 3), the empirical distribution is similar to the theoretical stationary distribution. This is just 1 realisation, so there is a lot of stochasticity in the specific distribution generated. This is a reasonably good representative example.

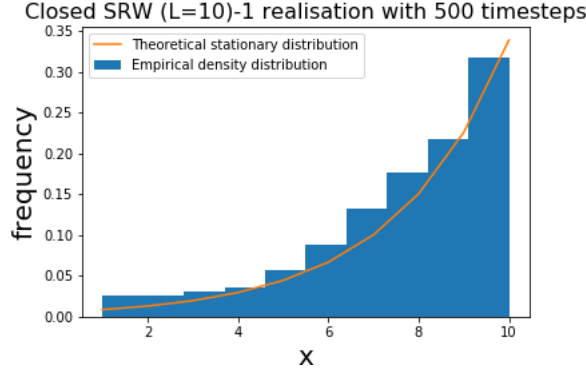


Figure 3. The frequency distribution of states over 500 timesteps of 1 realisation of the simulated simple random walk.

## 2 Question 2

$X_1, X_2, \dots$  is a sequence of independent, identically distributed random variables (iirdvs) with

$$X_i \sim N(\mu, \sigma^2)$$

$$\mu \in R \text{ and } \sigma^2 > 0$$

Discrete time random walk on state space  $R$

$$(Y_n : n \geq 0) \text{ with } Y_{n+1} = Y_n + X_{n+1} \text{ and } Y_0 = 0$$

### 2.1 Part A

The weak law of large numbers for  $Y_n$ :

$$\frac{1}{n} Y_n = \frac{1}{n} \sum_{k=1}^n X_k \rightarrow \mu \text{ as } n \rightarrow \infty$$

The expected value of  $\frac{Y_n}{n}$  will converge to  $\mu$  with large  $n$ .

The central limit theorem for  $Y_n$ :

$$\frac{Y_n - n\mu}{\sigma n^{\frac{1}{2}}} = \frac{1}{\sigma n^{\frac{1}{2}}} \sum_{k=1}^n (X_k - \mu) \rightarrow \xi \text{ as } n \rightarrow \infty$$

where  $\xi \sim N(0, 1)$

### 2.2 Part B - distribution of $Y_n$

$$Y_n \sim N(n\mu, n\sigma^2)$$

$Y_n$  is approximately normally distributed with mean equal to  $n\mu$ , where  $\mu$  is the mean of the random variable  $X$ , and with variance  $n\sigma^2$ , where  $\sigma^2$  is the variance of the random variable  $X$ .

### 2.3 Part C

$Z_n$  has a recursive relationship given by

$$Z_{n+1} = \exp(Y_n + X_{n+1})$$

$$Z_{n+1} = Z_n \exp(X_{n+1})$$

TO DO - Derive the pdf, look at Matt's link on whatsapp

$Y$  is distributed normally, and

$$Y = \ln(Z)$$

Consider the cumulative distribution function:

$$F(z) = P(Z \leq z) = P(Y \leq \ln(z))$$

$$f_z(z) = \frac{d}{dz} F(z)$$

by the chain rule, this gives

$$f_z(z) = \frac{1}{z} \frac{1}{\sigma_y \sqrt{2\pi}} \exp\left(-\frac{(\ln(z) - \mu_y)^2}{2\sigma_y^2}\right)$$

Or, in terms of  $x$

$$f_z(z) = \frac{1}{z \sigma_x \sqrt{2n\pi}} \exp\left(-\frac{(\ln(z) - n\mu_x)^2}{2n\sigma_x^2}\right)$$

for  $n \geq 1$  - WHY - why not on  $n=0$ ?

$$E(Z_n) = \exp(n\mu + \frac{n\sigma^2}{2})$$

$$\text{Var}(Z_n) = \exp(2n\mu + n\sigma^2)(\exp(n\sigma^2) - 1)$$

$$\text{Median}(Z_n) = \exp(n\mu)$$

Where  $\mu$  and  $\sigma^2$  are the mean and variance of  $X$ .

TO DO - MEDIAN TO DO - check all this

## 2.4 Part E

We want

$$E(Z_n) = 1 \text{ for all } n$$

$$E(Z_n) = \exp(n\mu + \frac{n\sigma^2}{2}) = 1$$

$$\mu + \frac{\sigma^2}{2} = 0$$

$$\mu = -\frac{\sigma^2}{2}$$

$$\mu = -\frac{1}{50}$$

## 3 Question 3

### 3.1 Part a

The state space is  $X(i) \in X_0(j)$  where  $i, j \in [1, L]$

It is not irreducible, as once a type goes extinct it stays extinct, and so not every state can be reached from every other state.

The stationary distributions are where all individuals are of one type

Q- is that right?

### 3.2 Part b

The future state of the process depends only on the current state at each time step, so yes it is a Markov process.

The state space is

$$N_i \in [0, L] \text{ and } \sum_i^L N_i = L$$

The transition probabilities are ???

The process is not irreducible, as once a state has  $N_i = 0$  for any  $i$ , i.e. a type goes extinct, the state space where  $N_i > 0$  is no longer accessible.

The stationary distributions are where

$$N_i = \delta_{i,j} L \text{ where } j \in [1, L]$$

Q. Limiting distribution?