

# Networks and Random Processes Assignment 2

Charlie Pilgrim - 1864704

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## 1 Kingman's Coalescent

### 1.1 A

$N_t$  is the number of particles at time  $t$  with  $N_0 = L$ . The process  $(N_t : t \geq 0)$  has the state space  $\{1, \dots, L\}$

#### 1.1.1 Transition Rate of the process

$$r(n, n-1) = \binom{L}{2}, n \geq 2$$

QUESTION - WHAT ABOUT SAME STATE?  $r(n, n) =$

QUESTION - WHAT ABOUT OTHER STATES - HOW TO WRITE IT?  
 $r(n, y) = , y \neq n, n-1$

#### 1.1.2 Generator

This is a jump process, so the generator is

$$(\mathcal{L}f)(x) = \int_{\mathcal{R}} r(x, y)[f(y) - f(x)]dy$$

For this process

$$(\mathcal{L}f)(n) = r(n, n-1)(f(n-1) - f(n))$$

$$(\mathcal{L}f)(n) = \binom{n}{2}(f(n-1) - f(n))$$

#### 1.1.3 Master Equation

The master equation is

$$\frac{d}{dt}\pi_t(n) = \pi_t(n+1)r(n+1, n) - \pi_t(n)r(n, n-1)$$

$$\frac{d}{dt}\pi_t(n) = \pi_t(n+1)\binom{n+1}{2} - \pi_t(n)\binom{n}{2}$$

QUESTION - IS THIS RIGHT? QUESTION - IS THE NOTATION OKAY?  
QUESTION - WHAT ABOUT EDGES?

#### 1.1.4 Ergodicity

The process is ergodic.

#### 1.1.5 Absorbing States

The unique absorbing state is  $N = 1$ .

#### 1.1.6 Stationary Distributions

Let a distribution  $\pi = [N = 1, N = 2, \dots, N = L]$

The unique stationary distribution is

$$\pi_0 = [1, 0, \dots, 0]$$

### 1.2 B - Mean Time to Asorption

The rate of coalescence, ie moving to the next state, for each state is

$$\lambda_n = r(n, n-1) = \binom{n}{2} = \frac{n(n-1)}{2}$$

The times in each state are expnentially dsitributed as

$$f_t(n) = \binom{n}{2} e^{-\binom{n}{2}t}$$

The expected time in each state, or the waiting time, is given by

$$\beta_n = \frac{1}{\lambda_n} = \frac{2}{n(n-1)}$$

The expected time to absorption is the sum of the expected waiting times in each of the states

$$E(T) = \sum_{n=2}^L \frac{2}{n(n-1)}$$

Bringing the 2 outside of the summand and splitting up into partial fractions

$$E(T) = 2 \sum_{n=2}^L \frac{1}{n-1} - \frac{1}{n}$$

$$E(T) = 2\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + \dots - \frac{1}{L-1} + \frac{1}{L+1} - \frac{1}{L}\right)$$

All but the first and last terms cancel giving

$$E(T) = 2\left(1 - \frac{1}{L}\right)$$

### 1.3 C - Rescaled process

Rescale the process to  $N_t/L$

$$(\mathcal{L}^L f)(n/L) = \frac{1}{L} \binom{n}{2} \left(f\left(\frac{n-1}{L}\right) - f\left(\frac{n}{L}\right)\right)$$

Taylor expand and let  $x = \frac{n}{L}$ :

$$(\mathcal{L}^L f)(x) = \frac{1}{L} \frac{n(n-1)}{2} \left(f(x) - \frac{1}{L} f'(x) + \frac{1}{L^2} f''(x) + O\left(\frac{1}{L^3}\right) - f(x)\right)$$

Cancel terms, substitute  $n = Lx$  and rearrange

$$(\mathcal{L}^L f)(x) = \left(\frac{x^2}{2} - \frac{x}{2L}\right) (-f'(x) + \frac{1}{L} f''(x) + O\left(\frac{1}{L^2}\right))$$

$$\lim_{L \rightarrow \infty} (\mathcal{L}^L f)(x) = \frac{-x^2}{2} f'(x)$$

QUESTION - STATE SPACE? QUESTION - INITIAL CONDITION?

#### 1.3.1 Deterministic

We have

$$\frac{d}{dt} E(f(x)) = E((\mathcal{L}^L f)(x))$$

$$\frac{d}{dt} E(f(x)) = E\left(\frac{-x^2}{2} \frac{dx}{dt}\right)$$

??? IS THAT RIGHT AT ALL?

### 1.4 D - Simulations

QUESTION - WHAT IS THE ADD THIS LINE BIT OF CODE IN EMMA'S WORKBOOK

## 2 Ornstein-Uhlenbeck process

### 2.1 A

$$\frac{d}{dt}E(X_t) = -\alpha x \frac{dx}{dt} + \frac{1}{2}\sigma^2 \frac{d^2x}{dt^2}$$

$$\frac{d}{dt}E(X_t^2) = -\alpha x \frac{d(x^2)}{dt} + \frac{1}{2}\sigma^2 \frac{d^2(x^2)}{dt^2}$$

???

### 2.2 B

Gaussian process - so fully characterised by mean and variance.

### 2.3 C

## 3 Moran Model and Wright-Fisher diffusion

### 3.1 A

### 3.2 B

### 3.3 C

### 3.4 D

### 3.5 E

### 3.6 F