Networks and Random Processes Assignment 2

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1 Kingman's Coalescent

1.1 A

N_t is the number of particles at time t with $N_0 = L$. The process $(N_t : t \geq 0)$ has the state space $\{1, ..., L\}$

1.1.1 Transition Rate of the process

$$r(n, n-1) = \binom{L}{2}, \; n \geq 2$$

QUESTION - WHAT ABOUT SAME STATE? $r(n, n) =$

QUESTION - WHAT ABOUT OTHER STATES - HOW TO WRITE IT? $r(n, y) = , \; y \neq n, n-1$

1.1.2 Generator

This is a jump process, so the generator is

$$(\mathcal{L}f)(x) = \int_{\Re} r(x, y)[f(y) - f(x)]dy$$

For this process

$$(\mathcal{L}f)(n) = r(n, n-1)(f(n-1) - f(n))$$

$$(\mathcal{L}f)(n) = \binom{n}{2}(f(n-1) - f(n))$$

1.1.3 Master Equation

The master equation is

$$\frac{d}{dt}\pi_t(n) = \pi_t(n+1)r(n+1, n) - \pi_t(n)r(n, n-1)$$

$$\frac{d}{dt}\pi_t(n) = \pi_t(n+1)\binom{n+1}{2} - \pi_t(n)\binom{n}{2}$$

QUESTION - IS THIS RIGHT? QUESTION - IS THE NOTATION OKAY? QUESTION - WHAT ABOUT EDGES?

1.1.4 Ergodicity

The process is ergodic.

1.1.5 Absorbing States

The unique absorbing state is $N = 1$.

1.1.6 Stationary Distributions

Let a distribution $\pi = [N = 1, N = 2, ..., N = L]$

The unique stationary distribution is

$$\pi_0 = [1, 0, ..., 0]$$

1.2 B - Mean Time to Asorption

The rate of coalescence, ie moving to the next state, for each state is

$$\lambda_n = r(n, n-1) = \binom{n}{2} = \frac{n(n-1)}{2}$$

The times in each state are expnentially dsitributed as

$$f_t(n) = \binom{n}{2} e^{-\binom{n}{2}t}$$

The expected time in each state, or the waiting time, is given by

$$\beta_n = \frac{1}{\lambda_n} = \frac{2}{n(n-1)}$$

The expected time to absorption is the sum of the expected waiting times in each of the states

$$E(T) = \sum_{n=2}^{L} \frac{2}{n(n-1)}$$

Bringing the 2 outside of the summand and splitting up into partial fractions

$$E(T) = 2\sum_{n=2}^{L} \frac{1}{n-1} - \frac{1}{n}$$

$$E(T) = 2(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + ... - \frac{1}{L-1} + \frac{1}{L+1} - \frac{1}{L})$$

All but the first and last terms cancel giving

$$E(T) = 2(1 - \frac{1}{L})$$

1.3 C - Rescaled process

Rescale the process to N_t/L

$$(\mathcal{L}^L f)(n/L) = \frac{1}{L}\binom{n}{2}(f(\frac{n-1}{L}) - f(\frac{n}{L}))$$

Taylor expand and let $x = \frac{n}{L}$:

$$(\mathcal{L}^L f)(x) = \frac{1}{L}\frac{n(n-1)}{2}(f(x) - \frac{1}{L}f'(x) + \frac{1}{L^2}f''(x) + O(\frac{1}{L^3}) - f(x))$$

Cancel terms, substitute $n = Lx$ and rearrange

$$(\mathcal{L}^L f)(x) = (\frac{x^2}{2} - \frac{x}{2L})(-f'(x) + \frac{1}{L}f''(x) + O(\frac{1}{L^2}))$$

$$\lim_{L\to\infty}(\mathcal{L}^L f)(x) = \frac{-x^2}{2}f'(x)$$

QUESTION - STATE SPACE? QUESTION - INITIAL CONDITION?

1.3.1 Deterministic

The generator has no diffusion term, only drift. So there is no variance in the process and it must be entireley deterministic.

$$\frac{d}{dt}E(X_t) = E(-\frac{X_t^2}{2})$$

X_t is deterministic, so

$$\frac{d}{dt}X_t = -\frac{X_t^2}{2}$$

$$X(t) = -\frac{X^2}{2}t$$

NOTATION OKAY?

1.4 D - Simulations

QUESTION - WHAT IS THE ADD THIS LINE BIT OF CODE IN EMMA'S WORKBOOK

2 Ornstein-Uhlenbeck process

2.1 A

The mean:

$$\frac{dm}{dt} = \frac{d}{dt}E(X_t) = E(-\alpha X_t) = -\alpha E(X_t)$$

$$\frac{dm}{dt} = -\alpha m(t)$$

X_t^2:

$$\frac{d}{dt}E(X_t^2) = E(-\alpha 2X_t^2 + \sigma^2)$$

$$\frac{d}{dt}E(X_t^2) = -2\alpha E(X_t^2) + \sigma^2$$

The variance:

$$v(t) = E(X_t^2) - m(t)^2$$

$$\frac{dv}{dt} = \frac{d}{dt}E(X_t^2) - 2m\frac{dm}{dt}$$

$$\frac{dv}{dt} = -2\alpha E(X_t^2) + \sigma^2 + 2m^2$$

2.2 B

2.2.1 Solution of $m(t)$

Solving for m(t), and considering the initial conditions, $c = x_0$

$$m(t) = x_0 e^{-\alpha t}$$

2.2.2 Solution of $v(t)$

Solving for v(t)

$$\frac{dv}{dt} = -2\alpha(v + m^2) + \sigma^2 + 2\alpha m^2$$

$$\frac{dv}{dt} + 2\alpha v = \sigma^2$$

Homogenous solution:

$$v_h om = c_2 e^{-2\alpha t}$$

Particular solution:

$$v_part = \frac{\sigma^2}{2\alpha}$$

Full solution:

$$v(t) = c_2 e^{-2\alpha t} + \frac{\sigma^2}{2\alpha}$$

From initial conditions, $v(0) = 0$:

$$v(t) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$$

2.2.3 Distribution

As a Gaussian process, the distribution is fully described by the mean and variance.

$$f(X_t) = \frac{1}{\sqrt{\frac{2\pi\sigma^2(1-e^{-2\alpha t})}{2\alpha}}} exp(\frac{-\alpha(x - x_0 e^{-\alpha t})^2}{\sigma^2(1 - e^{-2\alpha t})}$$

IS THAT RIGHT?

2.2.4 Stationary Distribution

Given enough time, the process will converge to a Gaussian stationary distribution.

$$\lim_{t\to\infty} m(t) = 0$$

$$\lim_{t\to\infty} v(t) = \frac{\sigma^2}{2\alpha}$$

The stationary distribution is $\sim N(0, \frac{\sigma^2}{2\alpha})$

$$f_0(X_t) = \frac{1}{\sqrt{\frac{\pi\sigma^2}{\alpha}}} exp(\frac{-\alpha x^2}{\sigma^2})$$

2.3 Simulation

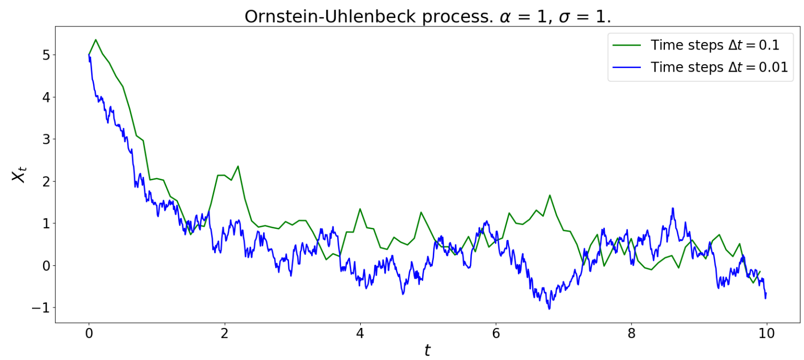

Figure 1: The Ornstein-Uhlenbeck process with $\alpha = 1$, $\sigma^2 = 1$ and $X_0 = 5$. Simulated for 10 seconds with timsteps $\Delta t = 0.1$ and 0.01

Figure 1 shows the Ornstein-Uhlenbeck process simulated. The process begins at $X_0 = 5$, and expereinces a "force" pulling it towards zero. As time progresses, the process moves towards $X_t = 0$ and the noise term begins to dominate the behaviour. Both choices of timestep give a similar result.

3 Moran Model and Wright-Fisher diffusion

3.1 A

3.1.1 State space

Let the total possible L types be

$$T = \{1, 2, ..., L\}$$

Each of the L individuals can have any of those types, so the state space is

$$S = \{1, 2, 3, ..., L\}^L$$

3.1.2 Irreducibility

It is not irreducible because there are absorbing states.

3.1.3 Stationary distributions

The absorbing states are where all individuals have the same type

$$x_k = [k, k, ..., k] \quad \forall k \in \{1, 2, ..., L\}$$

The stationary distributions are any linear combination of the absorbing states that sum to 1.

$$\pi(y) = \sum_{k=1}^{L} \alpha_k \pi_k(y)$$

$$\sum_{k=1}^{L} \alpha_k = 1$$

The coefficients, α_k, are determined by the initial conditions and can be thought of as the probability of each type "winning" and taking over all individuals.

3.2 B

3.2.1 Markov process

$N_t : t \geq 0$ is a Markov process. It's future distribution is determined only by it's current state, not the specific history.

3.2.2 State space

Each type can have any integer between 0 and L types, and there are L types, so the state space is

$$S = \{0, 1, 2, ..., L\}^L$$

3.2.3 Generator

For each type, we assume that the number of individuals of that type, n, can only increase and decrease by 1 at one moment in time, i.e only one event happens at a time. The rates of gain and loss can be described as:

$$r(n, n+1) = \frac{n(L-n)}{L-1}$$

$$r(n, n-1) = \frac{n(L-n)}{L-1}$$

These are symmetrical.
This is a jump process, so the generator is

$$(\mathcal{L}f)(x) = \int_{\Re} r(x, y)[f(y) - f(x)]dy$$

For this process

$$(\mathcal{L}f)(n) = r(n, n-1)(f(n-1) - f(n)) + r(n, n+1)(f(n+1) - f(n))$$

$$(\mathcal{L}f)(n) = \frac{(L-n)n}{L-1}(f(n-1)+f(n+1)-2f(n))$$

3.2.4 Irreducibility

The process is not irreducible because there are absorbing states.

3.2.5 Stationary dsitributions

The absorbing states are where one type has $N_k = L$ and the rest $N_k = 0$. i.e

$$\pi_k(y) = \delta_{y,k} = \begin{cases} L & , \quad y = k \\ 0 & , \quad \text{otherwise} \end{cases}$$

?? IS THAT RIGHT?

3.2.6 Limiting distribution

As $t \to \infty$ with initial condition $N_0 = 1$. By symmetry each type has the same chance of "winning", therefore the limiting distribution is

$$\pi_\infty = [\frac{L-1}{L}, 0, 0, ..., 0, \frac{1}{L}]$$

3.3 C

3.3.1 $m_1(t)$

$$\frac{d}{dt}(E(N_t)) = E(\frac{n(L+n)}{L-1}(n+1+n-1-2n)$$

$$\frac{d}{dt}(E(N_t)) = 0$$

$N_0 = n$, therefore $E(N_t) = n$.

3.3.2 $m_2(t)$

$$\frac{d}{dt}(E(N^2)) = E(\frac{n(L+n)}{L-1}((n+1)^2+(n-1)^2-2n^2)$$

$$\frac{d}{dt}(E(N^2)) = E(\frac{n(L+n)}{L-1}(n^2+2n+1+n^2-2n+1-2n^2)$$

$$\frac{d}{dt}(E(N^2)) = E(\frac{2n(L+n)}{L-1})$$

$$\frac{d}{dt}(E(N^2)) = \frac{2L}{L-1}n + \frac{2}{L-1}E(N^2)$$

Solution in the form:

$$E(N^2) = Ae^{\frac{-2t}{L-1}} + B$$

From initial condition, $A = N_0^2$
Substitute everything into the original differential equation:

$$\frac{-2n}{L-1}e^{\frac{-2t}{L-1}} = \frac{2Ln}{L-1} + \frac{2n}{L-1}e^{\frac{-2t}{L-1}} + \frac{2B}{L-1}$$

$$B = Ln(1 - ne^{\frac{-2t}{L-1}})$$

$$E(N^2) = n^2e^{\frac{-2t}{L-1}} + Ln(1 - e^{\frac{-2t}{L-1}})$$

QUESTION - I HAVE A FACTOR OF n in front of exponential in the bracket for B - I don't think that I should have

3.3.3 Absorption probabilities

By symmetry, each individual has an equal chance of winning?? NOT SURE ABOUT THIS

3.3.4 Absorption time scales with system size L

The expected value of $E(N_t) = n$, and the variance grows with time. As the variance grows, the probability of being absorbed at either $N = 0$ or $N = L$ also grows.

The variance is given by

$$E(N^2) = n^2e^{\frac{-2t}{L-1}} + Ln(1 - ne^{\frac{-2t}{L-1}}) - n^2$$

$$E(N^2) = (Ln - n^2)(1 - e^{\frac{-2t}{L-1}})$$

3.4 D

3.4.1 Rescaling limit

$$(\mathcal{L}f)(n) = \frac{(L-n)n}{L-1}(f(n+1) + f(n-1) - 2f(n))$$

Rescale by dividing state space by L, multiplying time by L^α and changing the variable to $x = \frac{n}{L}$:

$$(\mathcal{L}f)(x) = \frac{(L-xL)xL}{L-1}L^\alpha(f(x+\frac{1}{L}) + f(x+\frac{1}{L}) - 2f(x))$$

Taylor expand and simplify:

$$(\mathcal{L}f)(x) = \frac{(L-xL)xL}{L-1}L^\alpha(\frac{1}{L^2}f''(x) - O(\frac{1}{L^3})$$

$$(\mathcal{L}f)(x) = \frac{L^\alpha}{L}\frac{xL^2 - x^2L^2}{1-\frac{1}{L}}(\frac{1}{L^2}f''(x) - O(\frac{1}{L^3})$$

Set $\alpha = 1$ to get a non-trivial scaliming limit:

$$(\mathcal{L}f)(x) = (\frac{x - x^2}{1-\frac{1}{L}})f''(x) - O(\frac{1}{L})$$

Take limit $L \to \infty$:

$$(\mathcal{L}^L f)(x) = (\lim_{L\to\infty}\mathcal{L}f)(x) = (x - x^2)f''(x)$$

$$(\mathcal{L}^L f)(x) = x(1-x)f''(x)$$

3.4.2 Fokker-Planck Equation

3.5 C

3.6 D

3.7 E

3.8 F