Networks and Random Processes Assignment 1

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1 Question 1

This section considers a Simple Random Walk on $1, ..., L$ with probabilities $p \in [0, 1]$ and $q = 1 - p$ to jump right and left respectively.

Different boundary conditions are considered.

1.1 Part A

1.1.1 Case 1 - Periodic

Periodic boundary conditions, so $p(0, L) = q$ and $p(L, 0) = p$

The transition matrix is:

$$P = \begin{bmatrix} 0 & p & 0 & \dots & 0 & q \\ q & 0 & p & \dots & 0 & 0 \\ 0 & q & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & p \\ p & 0 & 0 & \dots & q & 0 \end{bmatrix}$$

The process is irreducible, i.e. every state can, eventually, reach every other state. And there is a finite state space, so it has 1 unique stationary distribution.

The states can be laid out on a circle, and are symmetrical, so the stationary distribution is where all states have equal probabilities.

$\pi = (1/L, 1/L, ..., 1/L)$

for all $p \in (0, 1)$

The stationary distribution is reversible only for the case $p = q = \frac{1}{2}$.

1.1.2 Case 2 - Closed

Transition matrix:

$$P = \begin{bmatrix} q & p & 0 & \dots & 0 & 0 \\ q & 0 & p & \dots & 0 & 0 \\ 0 & q & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & p \\ 0 & 0 & 0 & \dots & q & p \end{bmatrix}$$

If $p = 1$, there is an absorbing state at L, and the stationary dsitribution is $\pi = (0, 0, ..., 0, 1)$. This stationary distribution is reversible, all terms in the detailed balance equations are zero. This is not irreducible, as the walk can only move from s_x to s_{x+1}.

If $q = 1$, there is an absorbing state at 1, and the stationary dsitribution is $\pi = (1, 0, ..., 0, 0)$. This stationary distribution is reversible, all terms in the detailed balance equations are zero. This is not irreducible, as the walk can only move from s_x to s_{x-1}.

If $p = q = \frac{1}{2}$, the process is irreducible, as every state can, eventually, be reached from every other state. There is a finite state space and so there is onle 1 stationary distribution. The sum of all columns equal 1, so there is a constant left eigenvector, and so the stationary distribution is $\pi = (1/L, 1/L, ..., 1/L)$. This stationary distribution is reversible.

If $p, q \neq [0, 1/2, 1]$, then the process is irreducible, and it has a finite state space so there is only 1 stationary distribution. Looing at the detailed balance equations, we can find a recurrence relation of the form:

$\pi_{x-1}p = \pi_x p$ over $x = \{2, 3, ..., L-1\}$

$\pi_x = \pi_{x-1}\frac{p}{q}$

By induction, this suggests a solution like

$\pi_x = \pi_1(\frac{p}{q})^{x-1}$

This is a reversible distribution, and so must also be stationary. As the process is ergodic, this is the unique stationary distrubution.

We can check this distribution is staionary for a closed simple random walk with 4 states

$$\begin{bmatrix} \pi_1 \\ \pi_1\frac{p}{q} \\ \pi_1(\frac{p}{q})^2 \\ \pi_1(\frac{p}{q})^3 \end{bmatrix} \begin{bmatrix} q & p & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & q & p \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_1\frac{p}{q} \\ \pi_1(\frac{p}{q})^2 \\ \pi_1(\frac{p}{q})^3 \end{bmatrix}$$

We also need to normalise the distribution, so the stationary distribution will be

$$\pi_x = \frac{\pi_1(\frac{p}{q})^{x-1}}{\sum_i^L \pi_i}$$

1.2 Part B - Absorbing

$$P = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ q & 0 & p & \dots & 0 & 0 \\ 0 & q & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & p \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

The process is not irreducible, as no other states can be reached from state 1 or state L.

The (normalised) stationary distributions are

$\pi_1 = [1, 0, 0, ..., 0]$

$\pi_2 = [0, 0, 0, ..., 1]$
$\pi_3 = [a, 0, 0, .., 0, 1 - a]$ where $a \in [0, 1]$
These distributions are reversible, looking at the detailed balance conditions:
$\pi(x)p(x, y) = \pi(y)p(y, x)$
All terms for all equations are zero, therefore it is reversible.

The absorption probability in site L is
$h_k^L = P(X_n = L \text{ for some } n \geq 0 | X_0 = k)$
$h_k^L = P(X_n = L \text{ for some } n \geq 0 | X_0 = k)$
$h_k^L = P(X_n = L | X_0 = k)$
Considering k+1, k-1, and by law of total probability:
$h_k^L = P(X_n = L | X_1 = k+1, X_0 = k) * p + P(X_n = L | X_1 = k-1, X_0 = k) * q$
Using the Markov property:
$h_k^L = P(X_n = L | X_1 = k + 1) \times p + P(X_n = L | X_1 = k - 1) \times q$

$h_k^L = h_{k+1}^L \times p + h_{k-1}^L \times q$
And the boundary conditions are:
$h_1^L = 0$ and $h_L^L = 1$

If $p = q$ then this recursion relation becomes
$h_k^L = \dfrac{h_{k+1}^L + h_{k-1}^L}{2}$
This is linear interpolation between the two surrounding states, so the absorption probability is linear in k. Consdiering the boundary conditions, the solution is:
$h_k^L = \frac{k-1}{L-1}$

If $p \neq q$, we can solve the recursion realtion by considering the ansatz:
$h_k^L = \lambda^k$
$\lambda = p\lambda^2 + q$
This has roots:
$\lambda_1 = 1$ and $\lambda_2 = q/p$
The general solution is of the form:
$h_k^L = a\lambda_1 + b\lambda_2$
$h_k^L = a + b(\frac{q}{p})^k$
Looking at the boundary conditions:
$h_1^L = 0 = a + b(\frac{q}{p})$
$h_L^L = 1 = a + b(\frac{q}{p})^L$
Subtracting the first equation from the second equation, and solving for b:
$b = \dfrac{1}{(\frac{q}{p})^L - \frac{q}{p}})$
$a = \dfrac{-1}{(\frac{q}{p})^{L-1} - 1}$

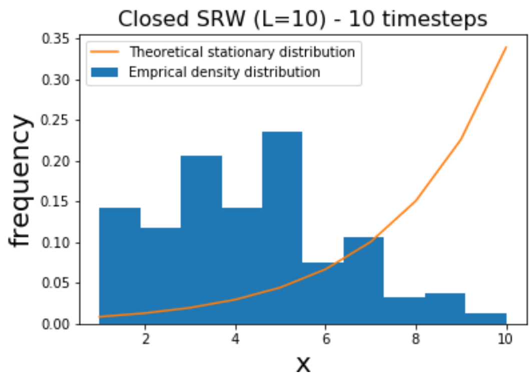

Figure 1. The frequency distribution over 500 different realisations after 10 timesteps.

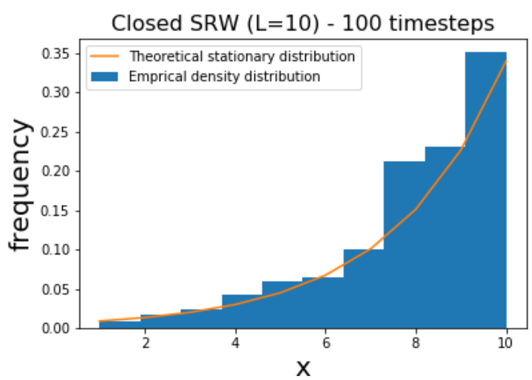

Figure 2. The frequency distribution over 500 different realisations after 100 timesteps.

1.3 Part C - simulations

A simple random walk with L=10 and closed boundary conditions was simulated 500 times, with a $p = 0.6$ and starting at x=1 at t=0.

After 10 time steps (Figure 1), the state of the process is still heavily influenced by the starting condition of x(0) = 1.

After 100 time steps (Figure 2), the empirical distribution is similar to the theoretical stationary distribution. Ergodic processes tend towards the stationary distribution after a large number of time steps.

After 500 time steps of 1 realisation(Figure 3), the empirical distribution is similar to the theoretical stationary distribution. This is jsut 1 realisation, so there is a lot of stochasticity in the specific distribution generated. This is a reasonably good representative example.

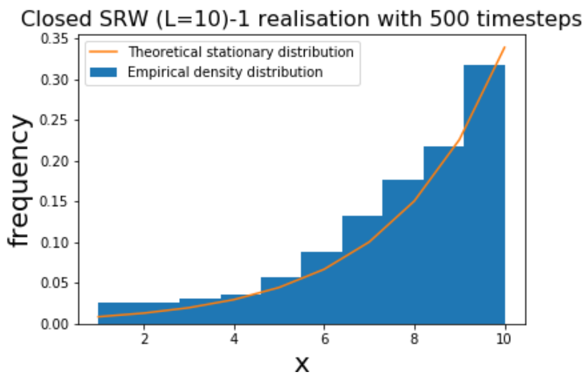

Figure 3. The frequency distribution of states over 500 timesteps of 1 realisation of the simulated simple random walk.

2 Question 2

$X_1, X_2, ...$ is a sequence of independent, identically distributed random variables (iirdvs) with

$X_i \sim N(\mu, \sigma^2)$

$\mu \in R$ and $\sigma^2 > 0$

Discrete time random walk on state space R

$(Y_n : n \geq 0)$ with $Y_{n+1} = Y_n + X_{n+1}$ and $Y_0 = 0$

2.1 Part A

The weak law of large numbers for Y_n:

$\frac{1}{n} Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k - > \mu$ as $n - > \infty$

The expected value of $\frac{Y_n}{n}$ will converge to μ with large n.

The central limit theorem for Y_n:

$\dfrac{Y_n - n\mu}{\sigma n^{\frac{1}{2}}} = \dfrac{1}{\sigma n^{\frac{1}{2}}} \sum_{k=1}^{n} (X_k - \mu) - > \xi$ as $n - > \infty$

where $\xi \sim N(0, 1)$

2.2 Part B - distribution of Y_n

$Y_n \sim N(n\mu, n\sigma^2)$

Y_n is approxiamtely normally distributed with mean equal to $n\mu$, where μ is the mean of the random variable X, and with variance $n\sigma^2$, where σ^2 is the variance of the random variable X.

2.3 Part C

Z_n has a recurive relationship given by

$Z_{n+1} = exp(Y_n + X_{n+1})$
$Z_{n+1} = Z_n exp(X_{n+1})$
TO DO - Derive the pdf, look at Matt's link on whatsapp
Y is distributed normally, and
$Y = ln(Z)$

Consider the cumulative distribution function:
$F(z) = P(Z \leq z) = P(Y \leq ln(z))$
$f_z(z) = \frac{d}{dz}F(z)$
by the chain rule, this gives
$f_z(z) = \dfrac{1}{z}\dfrac{1}{\sigma_y\sqrt{2\pi}}exp(\dfrac{-(ln(z)-\mu_y)^2}{2\sigma_y^2})$
Or, in terms of x
$f_z(z) = \dfrac{1}{z\sigma_x\sqrt{2n\pi}}exp(\dfrac{-(ln(z)-n\mu_x)^2}{2n\sigma_x^2})$
for $n \geq 1$ - WHY - why not on n=0?

$E(Z_n) = exp(n\mu + \frac{n\sigma^2}{2})$
$Var(Z_n) = exp(2n\mu + n\sigma^2)(exp(n\sigma^2) - 1)$
$Median(Z_n) = exp(n\mu)$
Where μ and σ^2 are the mean and variance of X.
TO DO - check all this

2.4 Part E

We want
$E(Z_n) = 1 foralln$
$E(Z_n) = exp(n\mu + \frac{n\sigma^2}{2}) = 1$
$\mu + \frac{\sigma^2}{2} = 0$
$\mu = -\frac{\sigma^2}{2}$
$\mu = -\frac{1}{50}$

3 Question 3

3.1 Part a

The state space is $X(i) \in X_0(j) \text{ where } i, j \in [1, L]$

It is not irreducible, as once a type goes extinct it stays extinct, and so not every state can be reached from every other state.

The stationary distributions are where all individuals are of one type

Q- is that right?

3.2 Part b

The future state of the process depends only on the current state at each time step, so yes it is a Markov process.

The state space is

$N_i \in [0, L]$ and $\sum_{i}^{L} N_i = L$

The transition probabilities are ???

The process is not irreducible, as once a state has $N_i = 0$ for any i, i.e. a type goes extinct, the state space where $N_i > 0$ is no longer accessible.

The stationary distributions are where

$N_i = \delta_{i,j} L$ where $j \in [1, L]$

Q. Limiting distribution?