Networks and Random Processes Assignment 2

Charlie Pilgrim - 1864704

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1 Kingman's Coalescent

1.1 A

N_t is the number of particles at time t with $N_0 = L$. The process $(N_t : t \geq 0)$ has the state space $\{1, ..., L\}$

1.1.1 Transition Rate of the process

$$r(n, n-1) = \binom{L}{2}, \; n \geq 2$$

QUESTION - WHAT ABOUT SAME STATE? $r(n, n) =$

QUESTION - WHAT ABOUT OTHER STATES - HOW TO WRITE IT? $r(n, y) =, \; y \neq n, n-1$

1.1.2 Generator

This is a jump process, so the generator is

$$(\mathcal{L}f)(x) = \int_{\Re} r(x, y)[f(y) - f(x)]dy$$

For this process

$$(\mathcal{L}f)(n) = r(n, n-1)(f(n-1) - f(n))$$

$$(\mathcal{L}f)(n) = \binom{n}{2}(f(n-1) - f(n))$$

1.1.3 Master Equation

The master equation is

$$\frac{d}{dt}\pi_t(n) = \pi_t(n+1)r(n+1, n) - \pi_t(n)r(n, n-1)$$

$$\frac{d}{dt}\pi_t(n) = \pi_t(n+1)\binom{n+1}{2} - \pi_t(n)\binom{n}{2}$$

QUESTION - IS THIS RIGHT? QUESTION - IS THE NOTATION OKAY? QUESTION - WHAT ABOUT EDGES?

1.1.4 Ergodicity

The process is ergodic.

1.1.5 Absorbing States

The unique absorbing state is $N = 1$.

1.1.6 Stationary Distributions

Let a distribution $\pi = [N = 1, N = 2, ..., N = L]$

The unique stationary distribution is

$$\pi_0 = [1, 0, ..., 0]$$

1.2 B - Mean Time to Asorption

The rate of coalescence, ie moving to the next state, for each state is

$$\lambda_n = r(n, n-1) = \binom{n}{2} = \frac{n(n-1)}{2}$$

The times in each state are expnentially dsitributed as

$$f_t(n) = \binom{n}{2} e^{-\binom{n}{2}t}$$

The expected time in each state, or the waiting time, is given by

$$\beta_n = \frac{1}{\lambda_n} = \frac{2}{n(n-1)}$$

The expected time to absorption is the sum of the expected waiting times in each of the states

$$E(T) = \sum_{n=2}^{L} \frac{2}{n(n-1)}$$

Bringing the 2 outside of the summand and splitting up into partial fractions

$$E(T) = 2\sum_{n=2}^{L} \frac{1}{n-1} - \frac{1}{n}$$

$$E(T) = 2(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + ... - \frac{1}{L-1} + \frac{1}{L+1} - \frac{1}{L})$$

All but the first and last terms cancel giving

$$E(T) = 2(1 - \frac{1}{L})$$

1.3 C - Rescaled process

Rescale the process to N_t/L

$$(\mathcal{L}^L f)(n/L) = \frac{1}{L}\binom{n}{2}(f(\frac{n-1}{L}) - f(\frac{n}{L}))$$

Taylor expand and let $x = \frac{n}{L}$:

$$(\mathcal{L}^L f)(x) = \frac{1}{L}\frac{n(n-1)}{2}(f(x) - \frac{1}{L}f'(x) + \frac{1}{L^2}f''(x) + O(\frac{1}{L^3}) - f(x))$$

Cancel terms, substitute $n = Lx$ and rearrange

$$(\mathcal{L}^L f)(x) = (\frac{x^2}{2} - \frac{x}{2L})(-f'(x) + \frac{1}{L}f''(x) + O(\frac{1}{L^2}))$$

$$\lim_{L\to\infty}(\mathcal{L}^L f)(x) = \frac{-x^2}{2}f'(x)$$

QUESTION - STATE SPACE? QUESTION - INITIAL CONDITION?

1.3.1 Deterministic

We have

$$\frac{d}{dt}E(f(x)) = E((\mathcal{L}^{\mathcal{L}} f)(x))$$

$$\frac{d}{dt}E(f(x)) = E(\frac{-x^2}{2}\frac{dx}{dt}))$$

??? IS THAT RIGHT AT ALL?

1.4 D - Simulations

QUESTION - WHAT IS THE ADD THIS LINE BIT OF CODE IN EMMA'S WORKBOOK

2 Ornstein-Uhlenbeck process

2.1 A

$$dm = dE(X_t) = -\alpha x dt$$

$$dE(X_t^2) = (-\alpha 2x^2 + \sigma^2)dt$$

$$dv(t) = dE(X_t^2) - dm(t)^2$$

$$dv(t) = (-\alpha 2x^2 + \sigma^2 + 2m(t)\alpha x)dt$$

???

2.2 B

Gaussian process - so fully characterised by mean and variance.

2.3 C

3 Moran Model and Wright-Fisher diffusion

3.1 A

3.2 B

3.3 C

3.4 D

3.5 E

3.6 F