Networks and Random Processes Assignment 1

Charlie Pilgrim - 1864704

October 2018

1 Question 1

This section considers a Simple Random Walk on $1, ..., L$ with probabilities $p \in [0, 1]$ and $q = 1 - p$ to jump right and left respectively.

Different boundary conditions are considered.

1.1 Part A

1.1.1 Case 1 - Periodic

Periodic boundary conditions, so $p(0, L) = q$ and $p(L, 0) = p$

The transition matrix is:

$$P = \begin{bmatrix} 0 & p & 0 & \dots & 0 & q \\ q & 0 & p & \dots & 0 & 0 \\ 0 & q & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & p \\ p & 0 & 0 & \dots & q & 0 \end{bmatrix}$$

The process is irreducible, i.e. every state can, eventually, reach every other state. And there is a finite state space, so it has 1 unique stationary distribution.

The states can be laid out on a circle, and are symmetrical, so the stationary distribution is where all states have equal probabilities.

$\pi = (1/L, 1/L, ..., 1/L)$

for all $p \in (0, 1)$

The stationary distribution is reversible only for the case $p = q = \frac{1}{2}$.

1.1.2 Case 2 - Closed

Transition matrix:

$$P = \begin{bmatrix} q & p & 0 & \dots & 0 & 0 \\ q & 0 & p & \dots & 0 & 0 \\ 0 & q & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & p \\ 0 & 0 & 0 & \dots & q & p \end{bmatrix}$$

If $p = 1$, there is an absorbing state at L, and the stationary dsitribution is $\pi = (0, 0, ..., 0, 1)$. This stationary distribution is reversible, all terms in the detailed balance equations are zero. This is not irreducible, as the walk can only move from s_x to s_{x+1}.

If $q = 1$, there is an absorbing state at 1, and the stationary dsitribution is $\pi = (1, 0, ..., 0, 0)$. This stationary distribution is reversible, all terms in the detailed balance equations are zero. This is not irreducible, as the walk can only move from s_x to s_{x-1}.

If $p = q = \frac{1}{2}$, the process is irreducible, as every state can, eventually, be reached from every other state. There is a finite state space and so there is onle 1 stationary distribution. The sum of all columns equal 1, so there is a constant left eigenvector, and so the stationary distribution is $\pi = (1/L, 1/L, ..., 1/L)$. This stationary distribution is reversible.

If $p, q \neq [0, 1/2, 1]$, then the process is irreducible, and it has a finite state space so there is only 1 stationary distribution. Looing at the detailed balance equations, we can find a recurrence relation of the form:

$\pi_{x-1}p = \pi_x p$ over $x = \{2, 3, ..., L-1\}$

$\pi_x = \pi_{x-1}\frac{p}{q}$

By induction, this suggests a solution like

$\pi_x = \pi_1(\frac{p}{q})^{x-1}$

This is a reversible distribution, and so must also be stationary. As the process is ergodic, this is the unique stationary distrubution.

We can check this distribution is staionary for a closed simple random walk with 4 states

$$\begin{bmatrix} \pi_1 \\ \pi_1\frac{p}{q} \\ \pi_1(\frac{p}{q})^2 \\ \pi_1(\frac{p}{q})^3 \end{bmatrix} \begin{bmatrix} q & p & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & q & p \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_1\frac{p}{q} \\ \pi_1(\frac{p}{q})^2 \\ \pi_1(\frac{p}{q})^3 \end{bmatrix}$$

We also need to normalise the distribution, so the stationary distribution will be

$$\pi_x = \frac{(\frac{p}{q})^{x-1}}{\sum_i^L \pi_i}$$

1.2 Part B - Absorbing

$$P = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ q & 0 & p & \dots & 0 & 0 \\ 0 & q & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & p \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

The process is not irreducible, as no other states can be reached from state 1 or state L.

The (normalised) stationary distributions are

$\pi_1 = [1, 0, 0, ..., 0]$

$\pi_2 = [0, 0, 0, ..., 1]$
$\pi_3 = [a, 0, 0, .., 0, 1 - a]$ where $a \in [0, 1]$
These distributions are reversible, looking at the detailed balance conditions:
$\pi(x)p(x, y) = \pi(y)p(y, x)$
All terms for all equations are zero, therefore it is reversible.

The absorption probability in site L is
$h_k^L = P(X_n = L \text{ for some } n \geq 0 | X_0 = k)$
$h_k^L = P(X_n = L \text{ for some } n \geq 0 | X_0 = k)$
$h_k^L = P(X_n = L | X_0 = k)$
Considering k+1, k-1, and by law of total probability:
$h_k^L = P(X_n = L | X_1 = k+1, X_0 = k) * p + P(X_n = L | X_1 = k-1, X_0 = k) * q$
Using the Markov property:
$h_k^L = P(X_n = L | X_1 = k + 1) \times p + P(X_n = L | X_1 = k - 1) \times q$

$h_k^L = h_{k+1}^L \times p + h_{k-1}^L \times q$
And the boundary conditions are:
$h_1^L = 0$ and $h_L^L = 1$

If $p = q$ then this recursion relation becomes
$h_k^L = \dfrac{h_{k+1}^L + h_{k-1}^L}{2}$
This is linear interpolation between the two surrounding states, so the absorption probability is linear in k. Consdiering the boundary conditions, the solution is:
$h_k^L = \frac{k-1}{L-1}$

If $p \neq q$, we can solve the recursion realtion by considering the ansatz:
$h_k^L = \lambda^k$
$\lambda = p\lambda^2 + q$
This has roots:
$\lambda_1 = 1$ and $\lambda_2 = q/p$
The general solution is of the form:
$h_k^L = a\lambda_1 + b\lambda_2$
$h_k^L = a + b(\frac{q}{p})^k$
Looking at the boundary conditions:
$h_1^L = 0 = a + b(\frac{q}{p})$
$h_L^L = 1 = a + b(\frac{q}{p})^L$
Subtracting the first equation from the second equation, and solving for b:
$b = \dfrac{1}{(\frac{q}{p})^L - \frac{q}{p}})$
$a = \dfrac{-1}{(\frac{q}{p})^{L-1} - 1}$

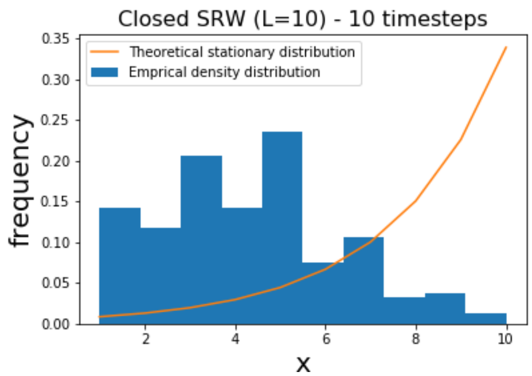

Figure 1. The frequency distribution over 500 different realisations after 10 timesteps.

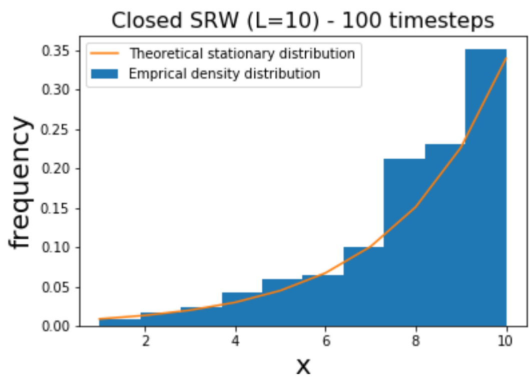

Figure 2. The frequency distribution over 500 different realisations after 100 timesteps.

1.3 Part C - simulations

A simple random walk with L=10 and closed boundary conditions was simulated 500 times, with a $p = 0.6$ and starting at x=1 at t=0.

After 10 time steps (Figure 1), the state of the process is still heavily influenced by the starting condition of x(0) = 1.

After 100 time steps (Figure 2), the empirical distribution is similar to the theoretical stationary distribution. Ergodic processes tend towards the stationary distribution after a large number of time steps.

After 500 time steps of 1 realisation(Figure 3), the empirical distribution is similar to the theoretical stationary distribution. This is jsut 1 realisation, so there is a lot of stochasticity in the specific distribution generated. This is a reasonably good representative example.

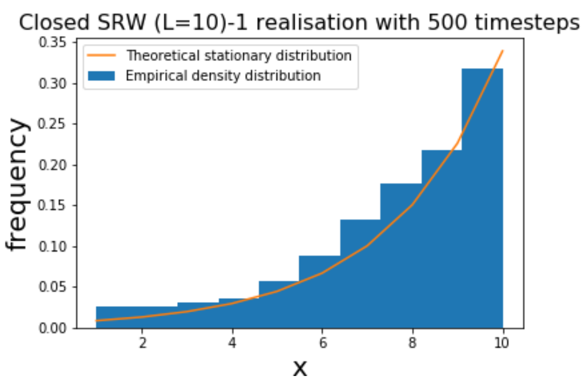

Figure 3. The frequency distribution of states over 500 timesteps of 1 realisation of the simulated simple random walk.