Networks and Random Processes Assignment 2

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1 Kingman's Coalescent

1.1 A

N_t is the number of particles at time t with $N_0 = L$. The process $(N_t : t \geq 0)$ has the state space $\{1, ..., L\}$

1.1.1 Transition Rate of the process

$$r(n, n-1) = \binom{L}{2},\ n \geq 2$$

QUESTION - WHAT ABOUT SAME STATE? $r(n, n) =$

QUESTION - WHAT ABOUT OTHER STATES - HOW TO WRITE IT? $r(n, y) = ,\ y \neq n, n-1$

1.1.2 Generator

This is a jump process, so the generator is

$$(\mathcal{L}f)(x) = \int_{\Re} r(x, y)[f(y) - f(x)]dy$$

For this process

$$(\mathcal{L}f)(n) = r(n, n-1)(f(n-1) - f(n))$$

$$(\mathcal{L}f)(n) = \binom{n}{2}(f(n-1) - f(n))$$

1.1.3 Master Equation

The master equation is

$$\frac{d}{dt}\pi_t(n) = \pi_t(n+1)r(n+1, n) - \pi_t(n)r(n, n-1)$$

$$\frac{d}{dt}\pi_t(n) = \pi_t(n+1)\binom{n+1}{2} - \pi_t(n)\binom{n}{2}$$

QUESTION - IS THIS RIGHT? QUESTION - IS THE NOTATION OKAY? QUESTION - WHAT ABOUT EDGES?

1.1.4 Ergodicity

The process is ergodic.

1.1.5 Absorbing States

The unique absorbing state is $N = 1$.

1.1.6 Stationary Distributions

Let a distribution $\pi = [N = 1, N = 2, ..., N = L]$

The unique stationary distribution is

$$\pi_0 = [1, 0, ..., 0]$$

1.2 B - Mean Time to Asorption

The rate of coalescence, ie moving to the next state, for each state is

$$\lambda_n = r(n, n-1) = \binom{n}{2} = \frac{n(n-1)}{2}$$

The times in each state are expnentially dsitributed as

$$f_t(n) = \binom{n}{2} e^{-\binom{n}{2}t}$$

The expected time in each state, or the waiting time, is given by

$$\beta_n = \frac{1}{\lambda_n} = \frac{2}{n(n-1)}$$

The expected time to absorption is the sum of the expected waiting times in each of the states

$$E(T) = \sum_{n=2}^{L} \frac{2}{n(n-1)}$$

Bringing the 2 outside of the summand and splitting up into partial fractions

$$E(T) = 2\sum_{n=2}^{L} \frac{1}{n-1} - \frac{1}{n}$$

$$E(T) = 2(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + ... - \frac{1}{L-1} + \frac{1}{L+1} - \frac{1}{L})$$

All but the first and last terms cancel giving

$$E(T) = 2(1 - \frac{1}{L})$$

1.3 C - Rescaled process

Rescale the process to N_t/L

$$(\mathcal{L}^L f)(n/L) = \frac{1}{L}\binom{n}{2}(f(\frac{n-1}{L}) - f(\frac{n}{L}))$$

Taylor expand and let $x = \frac{n}{L}$:

$$(\mathcal{L}^L f)(x) = \frac{1}{L}\frac{n(n-1)}{2}(f(x) - \frac{1}{L}f'(x) + \frac{1}{L^2}f''(x) + O(\frac{1}{L^3}) - f(x))$$

Cancel terms, substitute $n = Lx$ and rearrange

$$(\mathcal{L}^L f)(x) = (\frac{x^2}{2} - \frac{x}{2L})(-f'(x) + \frac{1}{L}f''(x) + O(\frac{1}{L^2}))$$

$$\lim_{L\to\infty}(\mathcal{L}^L f)(x) = \frac{-x^2}{2}f'(x)$$

QUESTION - STATE SPACE? QUESTION - INITIAL CONDITION?

1.3.1 Deterministic

The generator has no diffusion term, only drift. So there is no variance in the process and it must be entireley deterministic.

WHAT IS THE CALCAULTION FOR X_T?

1.4 D - Simulations

QUESTION - WHAT IS THE ADD THIS LINE BIT OF CODE IN EMMA'S WORKBOOK

2 Ornstein-Uhlenbeck process

2.1 A

The mean:

$$\frac{dm}{dt} = \frac{d}{dt}E(X_t) = E(-\alpha X_t) = -\alpha E(X_t)$$

$$\frac{dm}{dt} = -\alpha m(t)$$

X_t^2:

$$\frac{d}{dt}E(X_t^2) = E(-\alpha 2X_t^2 + \sigma^2)$$

$$\frac{d}{dt}E(X_t^2) = -2\alpha E(X_t^2) + \sigma^2$$

The variance:

$$v(t) = E(X_t^2) - m(t)^2$$

$$\frac{dv}{dt} = \frac{d}{dt}E(X_t^2) - 2m\frac{dm}{dt}$$

$$\frac{dv}{dt} = -2\alpha E(X_t^2) + \sigma^2 + 2m^2$$

2.2 B

2.2.1 Solution of $m(t)$

Solving for m(t), and considering the initial conditions, $c = x_0$

$$m(t) = x_0 e^{-\alpha t}$$

2.2.2 Solution of $v(t)$

Solving for v(t)

$$\frac{dv}{dt} = -2\alpha(v + m^2) + \sigma^2 + 2\alpha m^2$$

$$\frac{dv}{dt} + 2\alpha v = \sigma^2$$

Homogenous solution:

$$v_h om = c_2 e^{-2\alpha t}$$

Particular solution:

$$v_p art = \frac{\sigma^2}{2\alpha}$$

Full solution:

$$v(t) = c_2 e^{-2\alpha t} + \frac{\sigma^2}{2\alpha}$$

From initial conditions, $v(0) = 0$:

$$v(t) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$$

2.2.3 Distribution

As a Gaussian process, the distribution is fully described by the mean and variance.

$$f(X_t) = \frac{1}{\sqrt{\frac{2\pi\sigma^2(1-e^{-2\alpha t})}{2\alpha}}} exp(\frac{-\alpha(x - x_0 e^{-\alpha t})^2}{\sigma^2(1 - e^{-2\alpha t})}$$

IS THAT RIGHT?

2.2.4 Stationary Distribution

Given enough time, the process will converge to a Gaussian stationary distribution.

$$\lim_{t\to\infty} m(t) = 0$$

$$\lim_{t\to\infty} v(t) = \frac{\sigma^2}{2\alpha}$$

The stationary distribution is $\sim N(0, \frac{\sigma^2}{2\alpha})$

$$f_0(X_t) = \frac{1}{\sqrt{\frac{\pi\sigma^2}{\alpha}}} exp(\frac{-\alpha x^2}{\sigma^2})$$

CHECK IF THAT IS RIGHT

2.3 Simulation

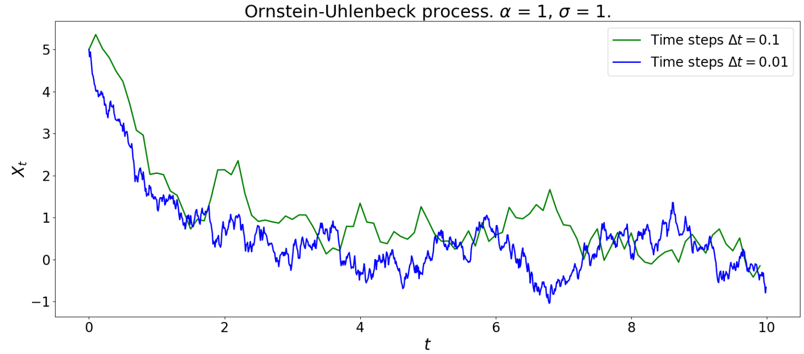

Figure 1: The Ornstein-Uhlenbeck process with $\alpha = 1$, $\sigma^2 = 1$ and $X_0 = 5$. Simulated for 10 seconds with timsteps $\Delta t = 0.1$ and 0.01

Figure 1 shows the Ornstein-Uhlenbeck process simulated. The process begins at $X_0 = 5$, and expereinces a "force" pulling it towards zero. As time progresses, the process moves towards $X_t = 0$ and the noise term begins to dominate the behaviour. Both choices of timestep give a similar result.

3 Moran Model and Wright-Fisher diffusion

3.1 A

3.1.1 State space

Let the total possible L types be

$$T = \{1, 2, ..., L\}$$

Each of the L individuals can have any of those types, so the state space is

$$S = \{1, 2, 3, ..., L\}^L$$

3.1.2 Irreducibility

It is not irreducible because there are absorbing states.

3.1.3 Stationary distributions

The absorbing states are where all individuals have the same type

$$x_k = [k, k, ..., k] \quad \forall k \in \{1, 2, ..., L\}$$

The stationary distributions are any linear combination of the absorbing states that sum to 1.

$$\pi(y) = \sum_{k=1}^{L} \alpha_k \pi_k(y)$$

$$\sum_{k=1}^{L} \alpha_k = 1$$

The coefficients, α_k, are determined by the initial conditions and can be thought of as the probability of each type "winning" and taking over all individuals.

3.2 B

3.2.1 Markov process

$N_t : t \geq 0$ is a Markov process. It's future distribution is determined only by it's current state, not the specific history.

3.2.2 State space

Each type can have any integer between 0 and L types, and there are L types, so the state space is

$$S = \{0, 1, 2, ..., L\}^L$$

3.2.3 Generator

For each type, we assume that the number of individuals of that type, n, can only increase and decrease by 1 at one moment in time, i.e only one event happens at a time. The rates of gain and loss can be described as:

$$r(n, n+1) = \frac{n(L-n)}{L-1}$$

$$r(n, n-1) = \frac{n(L-n)}{L-1}$$

These are symmetrical.

??? WHERE TO GO FROM HERE???

3.2.4 Irreducibility

The process is not irreducible because there are absorbing states.

3.2.5 Stationary dsitributions

The absorbing states are where one type has $N_k = L$ and the rest $N_k = 0$. i.e

$$\pi_k(y) = \delta_{y,k} = \begin{cases} L & , \quad y = k \\ 0 & , \quad \text{otherwise} \end{cases}$$

?? IS THAT RIGHT?

3.2.6 Limiting distribution

As $t \to \infty$ with initial condition $N_0 = 1$, the limiting distribution is

$$[\frac{L-1}{L}, 0, 0, ..., 0, \frac{1}{L}]$$

3.3 C

3.4 D

3.5 E

3.6 F