<Additional exercise>

① Schur complement

(a) $f(u,v) = u^T A u + 2v^T B^T u + v^T C v$

$$f_u(u,v) = 2Au + 2Bv = 0$$

$$u = -A^{-1}Bv$$

$$\rightarrow g(v) = (-A^{-1}Bv)^T A(-A^{-1}Bv) + 2v^T B^T(-A^{-1}Bv) + v^T C v$$

$$v^T B^T (A^{-1})^T Bv - 2v^T (B^T A^{-1} B)v + v^T C v$$

(A is symmetric)

$$= v^T(\underbrace{C - B^T A^{-1} B}_{S})v = v^T S v$$

(b-1) $X \succ 0 \iff A \succ 0$ and $S \succ 0$

$X \succ 0 \rightarrow f(u,v) > 0 \rightarrow f(u,0) \succ 0 \Rightarrow A \succ 0$

u or $v \neq 0$ $\searrow$ $v^T(C - B^T A^{-1} B)v \succ 0 \Rightarrow S \succ 0$

If $v \neq 0$, $A \succ 0$, $S \succ 0$,

$\rightarrow f(u,v) \geq v^T S v \succ 0$

If $v = 0$, $A \succ 0$, $S \succ 0$,

$\rightarrow u^T A u \succ 0 \rightarrow A \succ 0$

(b-2) $A \succ 0$ then $X \succeq 0 \iff S \succeq 0$

If $A \succ 0$, $X \succeq 0 \rightarrow f(u,v) \geq \inf f(u,v) = v^T S v \geq 0$

$\rightarrow S \succeq 0$

If $A \succ 0$, $S \succeq 0 \rightarrow \quad 0 \leq \inf f(u,v) \leq f(u,v)$

$\rightarrow X \succeq 0$

(b-3) $X \succeq 0 \iff A \succeq 0,\ B^T(I - AA^+) = 0,\ C - B^T A^+ B \succeq 0$

① ② ③

<intuitive explanation>

: $I - AA^+$: projection matrix to $N(A)$

$f(u,v) = u^T A u + 2v^T B u + v^T C v$ 에서

✓ $X \succ 0$ 일 땐, $A \succ 0, S \succ 0$ 이면 필요충분.

✓ $X \succeq 0$ 일 땐 psedo inverse 사용.

① $A \prec 0$ 이면 $-\infty$ 발산 가능.

② $u^T A u = 0$ 으로 만들면서 $2v^T B u \neq 0$ 인 u가 존재하면,
(같은 말로, A의 Null space 성분이 B에도 존재하면,)

음수 부분이 존재가능. $-\infty$으로 발산하지 않음. $v^T C v$ 때메.

③ Schur - complement 계산하면, $X = 0$ 인 경우에도 같음.

$\rightarrow \inf f(u,v) = g(v) = v^T \underbrace{(C - B^T A^+ B)}_{} v$.

↳ 이 부분이 $\succeq 0$ 이어야겠지.

(C)

$(P_0 + x_1 P_1 + \cdots x_n P_n)$: linear comb of P_i

$$f(x) \le t \iff \begin{bmatrix} t & Ax+b \\ (Ax+b)^T & P_0 + \cdots x_n P_n \end{bmatrix} \succeq 0$$

② $F_i \in S^m$, $F(x)$ is symmetric, $F(x) \succ 0$

(a)

$\langle$ minimize $c^T F^{-1}(x) c$ $\rangle$

$c^T F^{-1} c \leq t$, minimize t

$$\Rightarrow \left[\begin{array}{ll} \text{minimize} & t \\ \text{s.t} & \begin{bmatrix} t & c \\ c^T & F(x) \end{bmatrix} \succeq 0 \end{array} \right.$$

(b)

$\langle$ minimize $\max\limits_{i=1\sim k} c_i^T F(x)^{-1} c_i$ $\rangle$

$$\Rightarrow \left[\begin{array}{ll} \text{minimize} & t \\ \text{s.t} & \begin{bmatrix} F(x) & c_i \\ c_i^T & t \end{bmatrix} \succeq 0, \quad i = 1 \ldots K \end{array} \right.$$

(c)

$\langle$ minimize $\sup\limits_{\|c\|_2 \leq 1} c^T F^{-1}(x) c$ $\rangle$ $\Rightarrow \left[\begin{array}{ll} \min & t \\ & \begin{bmatrix} F(x) & I \\ I & tI \end{bmatrix} \succeq 0 \end{array} \right.$

$\mathcal{E} = \{ y = F^{-1/2} c \mid \|c\|_2 \leq 1 \}$ 이 타원.

$y^T y = c^T F^{-1}(x) c$, $\sup\limits_{\|c\|_2 \leq 1} c^T F^{-1}(x) c$ 은

타원의 가장 긴 방향의 제곱. $\Rightarrow \lambda_{max}(F(x)^{-1})$

(d)

$$f(x) = \bar{c}^T F(x)^{-1} \bar{c}$$

$$E(c^T F^{-1} c) = E(tr(F^{-1} c c^T))$$

$$= tr(F^{-1} E(cc^T)) = tr(F^{-1}(S + \bar{c}\bar{c}^T))$$

$$\because \quad Cov = E(cc^T) - \bar{c}\bar{c}^T$$

$$= \bar{c}^T F^{-1} \bar{c} + tr(F^{-1} S)$$

$$\Rightarrow f(x) \quad \bar{c}^T F(x)^{-1} \bar{c} + tr(F(x)^{-1} S)$$

$$tr(F^{-1} S) = \sum_k c_k^T F^{-1} c_k$$

$$\rightarrow \begin{cases} \min \quad t_0 + \sum_k t_k \\ \\ \text{s.t} \quad \begin{bmatrix} F & \bar{c} \\ \bar{c}^T & t_0 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} F & c_k \\ c_k^T & t_k \end{bmatrix} \succeq 0 \end{cases}$$

③ optimality conditions and dual for log-optimal investment

(a) with 1st optimality

j번째 Scenario 의 i번째 asset 의 수익률 : P_{ij}

$$\sum_{j=1}^{m} \pi_j \frac{p_{ij}}{p_j^T x} : \sum_{j=1}^{m} \frac{\pi_j \times (\text{i만 투자})}{(\text{j번째 평균수익률})} \rightarrow \frac{\text{i의 평균}}{\text{포폴 전체 평균}}$$

Original prob : minimize $-\sum_{j=1}^{m} \pi_j \log(p_j^T x)$

If x is feasible $\sum \nabla f(x)^T (z-x) \geq 0$ for all z

$z \succeq 0, \ \mathbf{1}^T z = 1$

$x > 0 \rightarrow \nabla f =$ 모든 방향에 대해 같은 상수 λ

$x = 0 \rightarrow \nabla f \geq \lambda$

$$\nabla f(x)_i = -\sum_{j=1}^{m} \pi_j \frac{p_{ij}}{p_j^T x}$$

$$\nabla f(x)^T x = -\sum_{j=1}^{m} \pi_j \frac{p_j^T x}{p_j^T x} = -1 \Rightarrow \lambda = 1$$

Then, optimal conditions are,

$\mathbf{1}^T x = 1, \ x \succeq 0$ and for each

g

(b)

$$L(x, \nu, \lambda, \nu_0) = -\sum_{j=1}^{m} \pi_j \log y_j + \underline{(-\nu^T P^T - \lambda^T + \nu_0 1^T)} x + \nu^T y - \nu_0$$

if $\neq 0 \rightarrow$ unbounded below

if $= 0 \rightarrow$ bounded

minimize over y_j

$\Rightarrow$ maximize

$$g(\nu, \lambda, \nu_0) = 1 - \nu_0 + \sum_{j=1}^{m} \pi_j \log\left(\frac{\nu_j}{\pi_j}\right)$$

$$, \nu_0 1 - P\nu = \lambda^T \geq 0$$

$\rightarrow$ maximize $\quad 1 - \nu_0 + \sum_{j=1}^{m} \pi_j \log(\tilde{\nu}_j \nu_0 / \pi_j)$

$$\nu_0 P \tilde{\nu} \leq \nu_0 1$$

$\rightarrow$ maximize $\quad 1 - \nu_0 + \log \nu_0 + \sum \pi_j \log(\tilde{\nu}_j / \pi_j)$

s.t $\quad P\tilde{\nu} \leq 1$

$\nu_0 = 1$ is optimal

$\rightarrow$ max $\quad \sum_{1}^{m} \pi_j \log(\tilde{\nu}_j / \pi_j)$

S.t $\quad P\tilde{\nu} \leq 1$

④ : log-optimal investment strategy (4.60)

with CVXPY

1. x^*

Compare with

1) uniform $x = \left(\frac{1}{n}\right)\mathbf{1}$, pure investment in each

$x = e_i \quad i = 1 \cdots n$

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log-optimal이 좋다.

⑤ Maximizing house profit in gamble and imputed probabilities

(a)

House profit = benefit − cost

$$= \underbrace{\sum_{i=1}^{n} x_i p_i}_{=p^T x} - \sum_{i:j\in S_i} x_i$$

worst-case profit : $p^T x - \max_{j=1\cdots m} \sum_{j\in S_i} x_i$

$$\rightarrow \begin{cases} \text{Maximize} & p^T x - \max_{j=1\cdots m} a_j^T x \\ \text{s.t} & 0 \preceq x \preceq q \end{cases}$$

$$A_{ji} = \begin{cases} 1 & j \in S_i \\ 0 & \text{otherwise} \end{cases}$$

$\left(\max_j \cdots \right)$ is non-convex

↳ piece wise

$$\Rightarrow \begin{cases} \max_{x,t} & p^T x - t \\ \text{s.t} & \sum_{i:j\in S_i} x_i \le t \\ & 0 \le x_i \le q_i \end{cases} \Rightarrow \begin{cases} \max_{x,t} & p^T x - t \\ \text{s.t} & Ax \preceq t\mathbf{1} \\ & 0 \preceq x \preceq q \end{cases}$$

(b) LP for worst case

$$\begin{cases} \text{maximize} & p^T x - t \\ \text{s.t} & t\mathbf{1} \succeq Ax \\ & 0 \preceq x \preceq q \end{cases}$$

참가자 i에 x_i개 팜. j : 결과

$L(x, t, \lambda_1, \lambda_2, \lambda_3)$

$= p^T x - t + \lambda_1^T (Ax - t\mathbf{1}) - \lambda_2^T x + \lambda_3^T (x - q)$

$= (p^T + \lambda_1^T A - \lambda_2^T + \lambda_3^T) x - t + \lambda_1^T t\mathbf{1} - \lambda_3^T q$

LP just

$$\begin{cases} \max & \underline{p^T x - \pi^T A x} \\ \text{s.t} & 0 \preceq x \preceq q \end{cases}$$

→ 이 문제는 분리가능

$p_i - (A^T \pi)_i > 0 \rightarrow x_i = q_i$

$\quad$ 〃 $\quad < 0 \rightarrow x_i = 0$

$\quad\quad = 0 \rightarrow 0 \le x \le q$

Dual Problem for worst case

↳ Maximize $-\lambda_3^T q$

s.t $\quad \underline{p^T + \lambda_1^T A - \lambda_2^T + \lambda_3^T = 0}$,

$\quad\quad \underline{\mathbf{1}^T \lambda_1 = 1}$, $\quad \underline{\lambda_1}, \lambda_2, \lambda_3 \succeq 0$

* KKT 조건

$p - A^T \pi = \lambda_3^* - \lambda_2^*$

works as probability $\Rightarrow \lambda_1 = \pi$

만약 π가 λ_1라면

두 KKT 조건이 동일!

< Complementary slackness >

① : $\lambda_2^* x^* = 0$, $\lambda_3^* (x^* - q) = 0$

② $\lambda_2 x = 0$, $\lambda_3 (x - q) = 0$