Graph theory: Connectivity

Introduction to Network Science

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- Paths and distances
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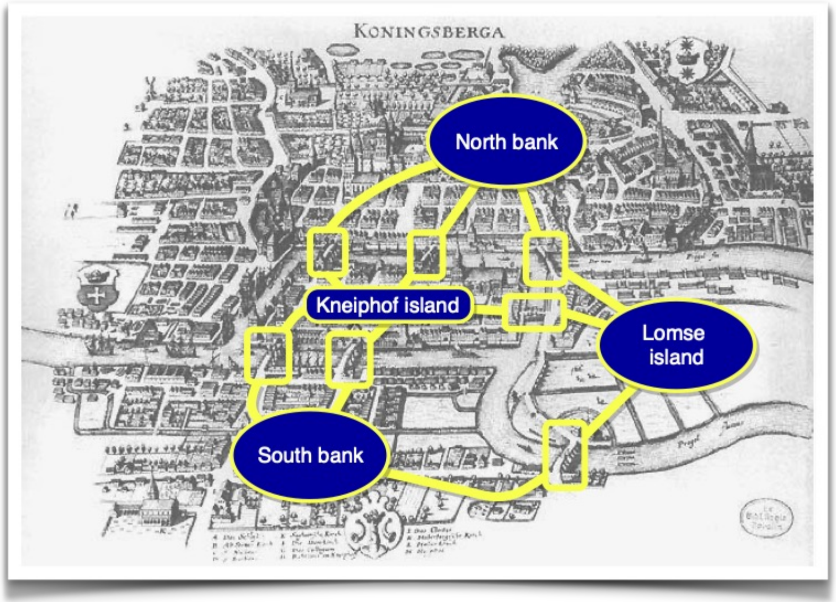
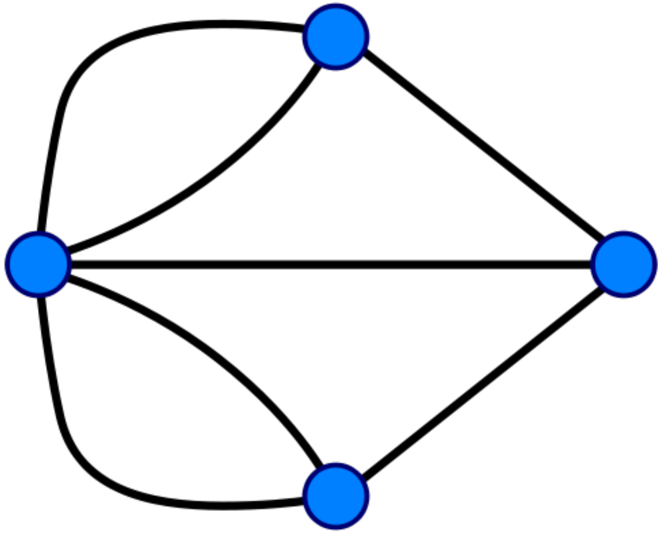
Paths and distances

Euler circa 1736: Koningsberg bridges



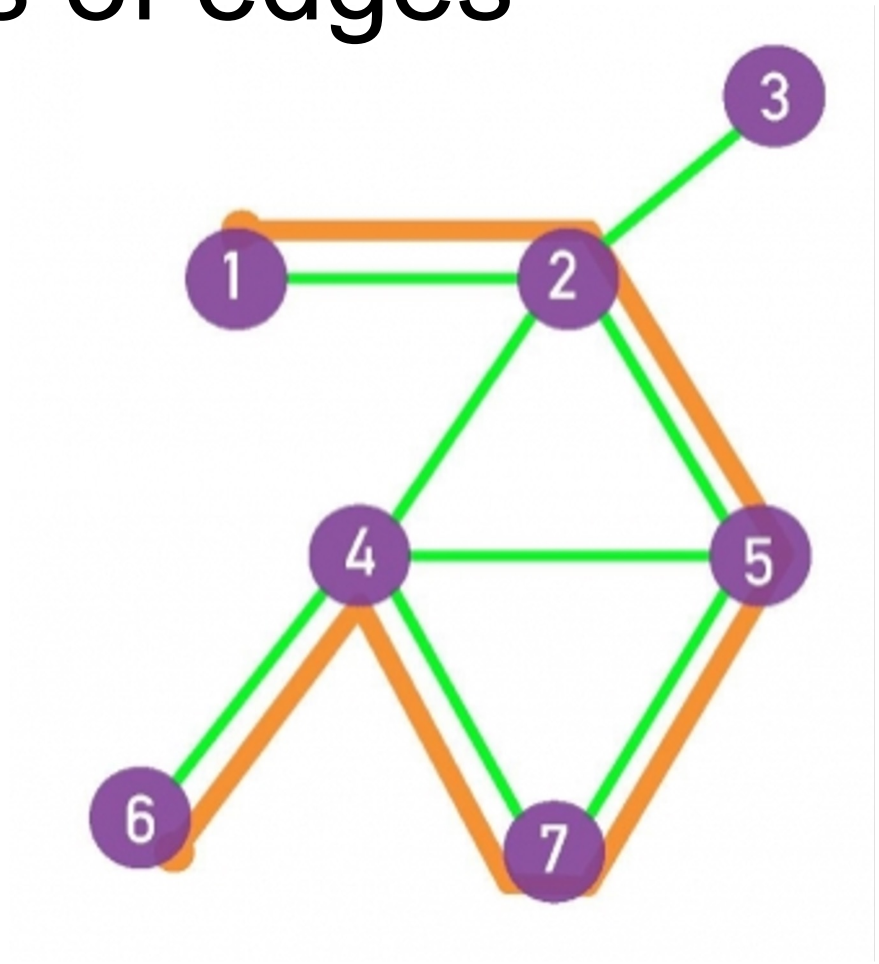


Q: Can you cross all 7 bridges just once each?

A: No. At most two nodes (start, end) may have odd degree

Paths: sequences of edges

- The destination of each edge is the origin of the next edge
  - In directed graphs, paths follow the direction of the edges
- The length of the path is the number of edges on it
  - Example: path in orange has length 5



Paths and diameter

We can use the shortest paths to characterize a network:

The **diameter** is the longest shortest-path length, or the maximum of the shortest path lengths across all pairs of nodes: $\ell_{max} = \max_{i,j} \ell_{ij}$

The **average path length** (APL) is the average of the shortest path lengths across all pairs of nodes

Undirected network:

$$\langle \ell \rangle = \frac{\sum_{i,j} \ell_{ij}}{\binom{N}{2}} = \frac{2\sum_{i,j} \ell_{ij}}{N(N-1)}$$

Directed network:

$$\langle \ell \rangle = \frac{\sum_{i,j} \ell_{ij}}{N(N-1)}$$

Paths and diameter

What if there is not a path between one or more pairs of nodes?

We can say APL and diameter are undefined (as NetworkX does)

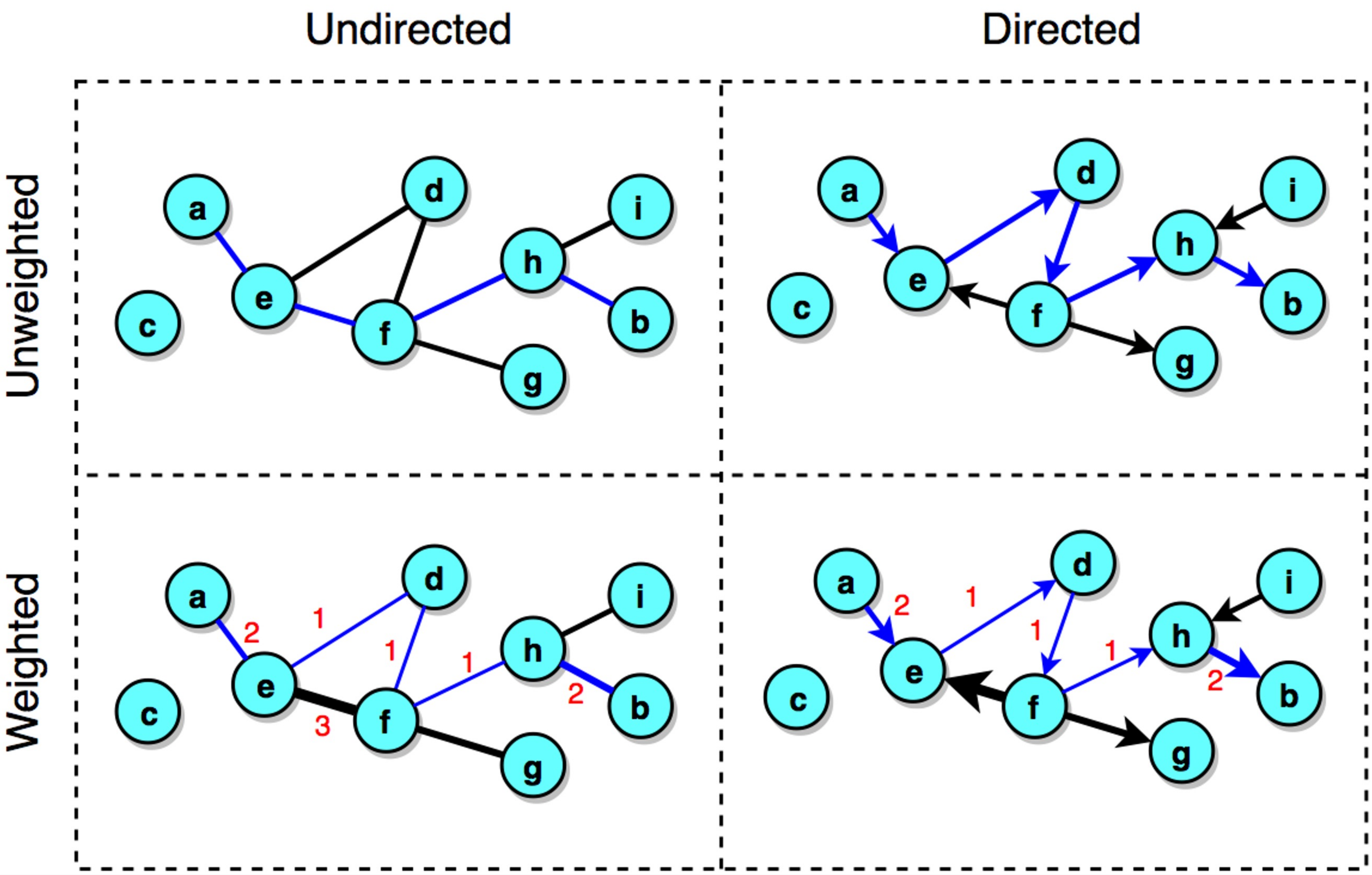
We can measure APL and diameter within the largest connected component (defined later)

We can use a mathematical trick: $\langle \ell \rangle = \left(\frac{\sum_{i,j} \frac{1}{\ell_{ij}}}{\binom{N}{2}} \right)^{-1}$

Distance

- If two nodes i, j are in the same connected component:
  - the **distance** between i and j, denoted by d_{ij} is the **length of the shortest path** between them
- If they are not in the same connected component, the distance is by definition infinite (∞)

Blue = shortest path between nodes *a* and *b*



Source: Menczer, Fortunato, Davis: A First Course on Networks Science. Cambridge, 2020.

Diameter

- The **diameter** of a network is the maximum distance between two nodes on it, d_{max}
- The **effective diameter** (or effective-90% diameter) is a number d such that 90% of the pairs of nodes (i,j) are at a distance smaller than d
- The **average distance** is <d>, and is measured only for nodes that are in the same connected component

Connected components

Connectedness

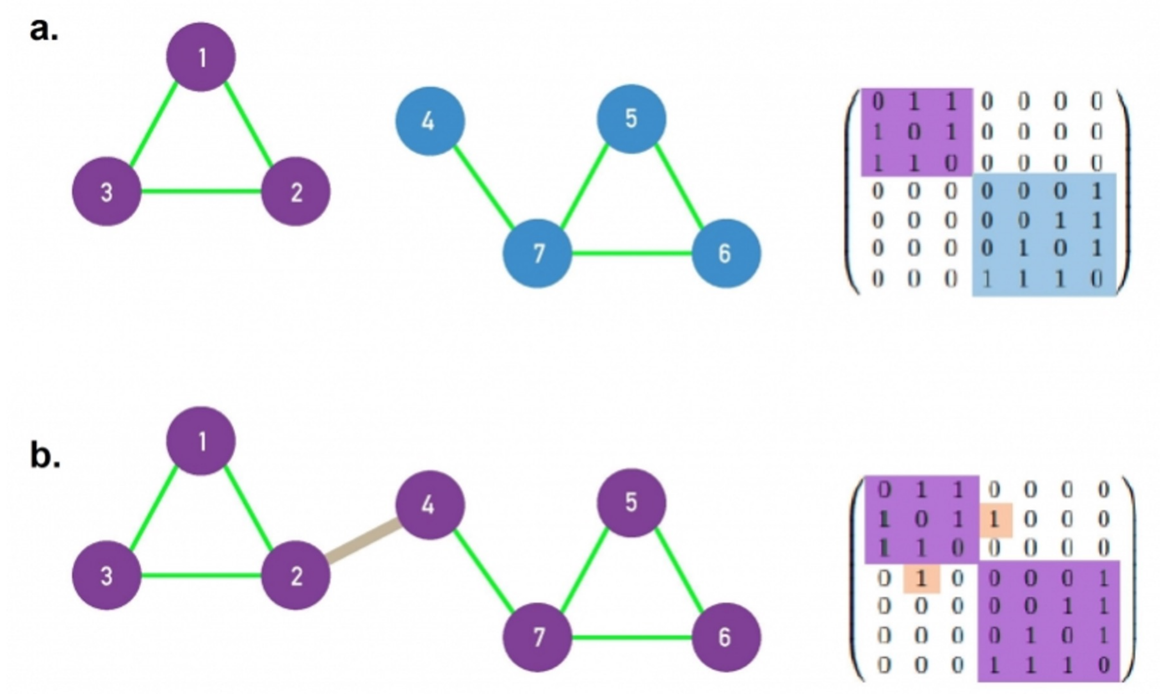
- If a path exists between two nodes i, j: those nodes are part of the same **connected component**
- A **connected graph** has only one connected component
- A **singleton** is a connected component with only one node

Connected graphs

A disconnected graph has an adjacency matrix that can be arranged in block diagonal form

a. disconnected

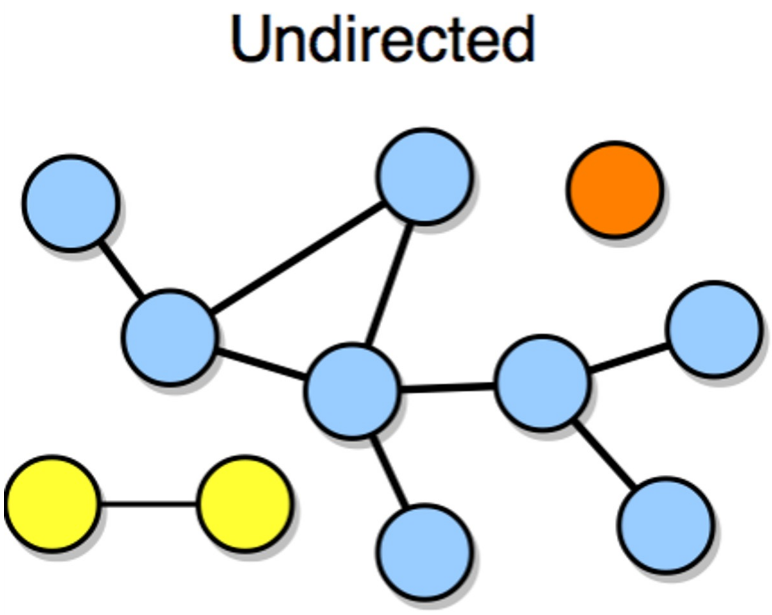
b. connected



Connectedness in directed graphs

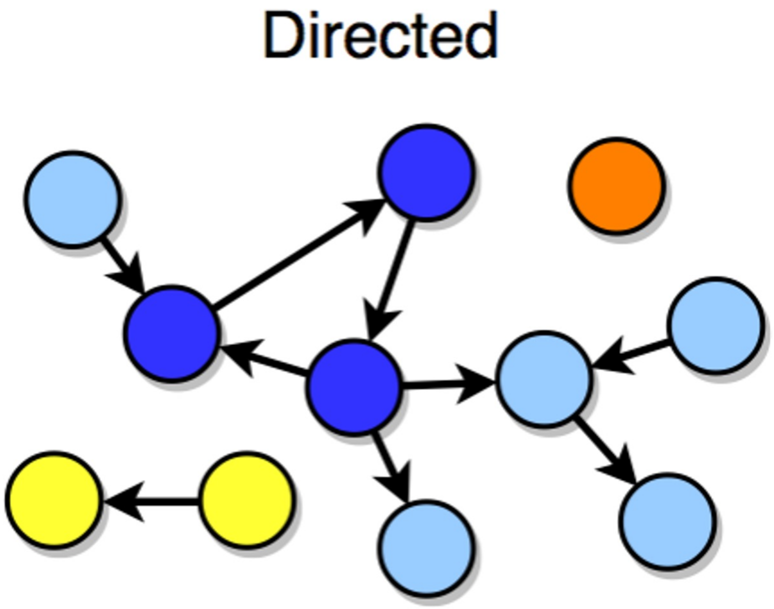
- A directed graph is **strongly connected** if it has only one connected component
- A directed graph is **weakly connected** if, when seen as undirected, has only one connected component

Connectedness example (undirected)



- Is not connected
- Has 3 connected components
- One of the connected components is a singleton

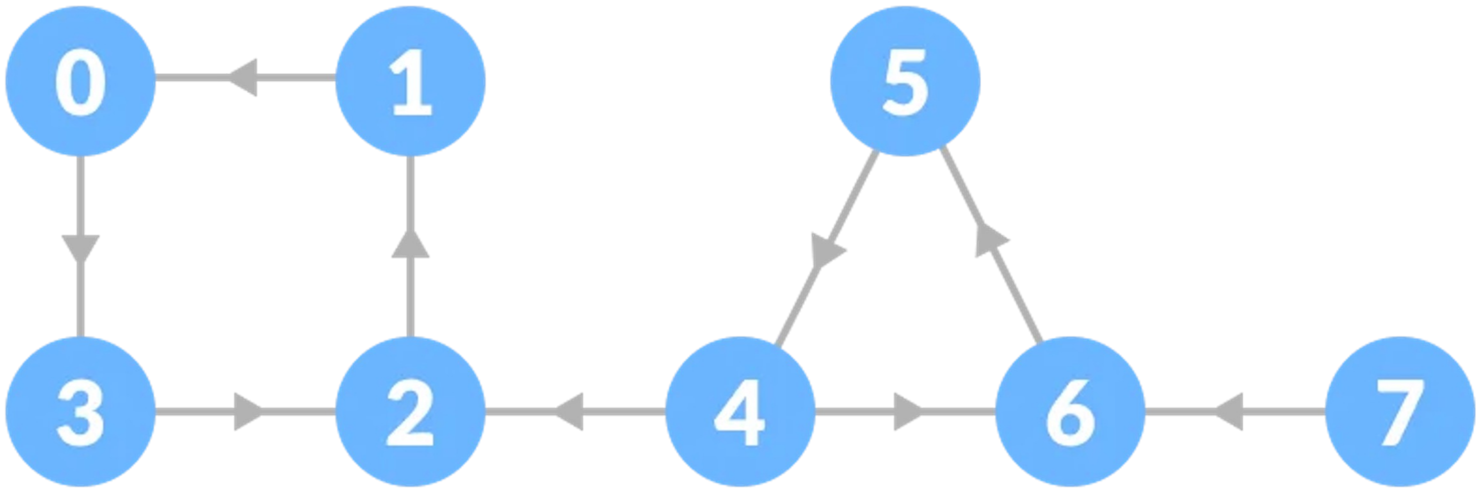
Connectedness example (directed)



- Is not strongly connected
- Is not weakly connected
- Has 3 connected components

Quick exercise

Find the strongly connected components

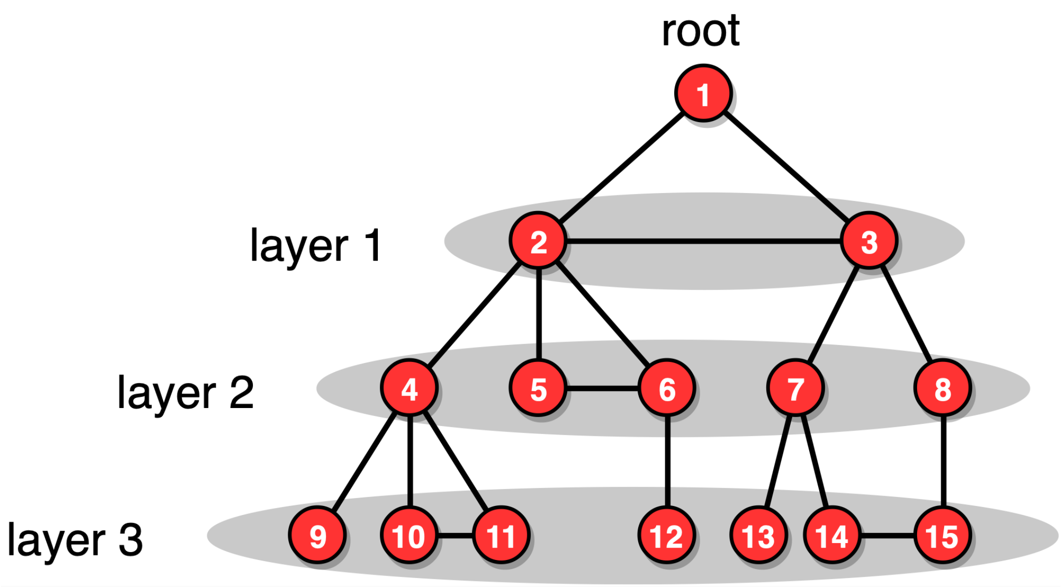


https://www.programiz.com/dsa/strongly-connected-components

Finding shortest paths

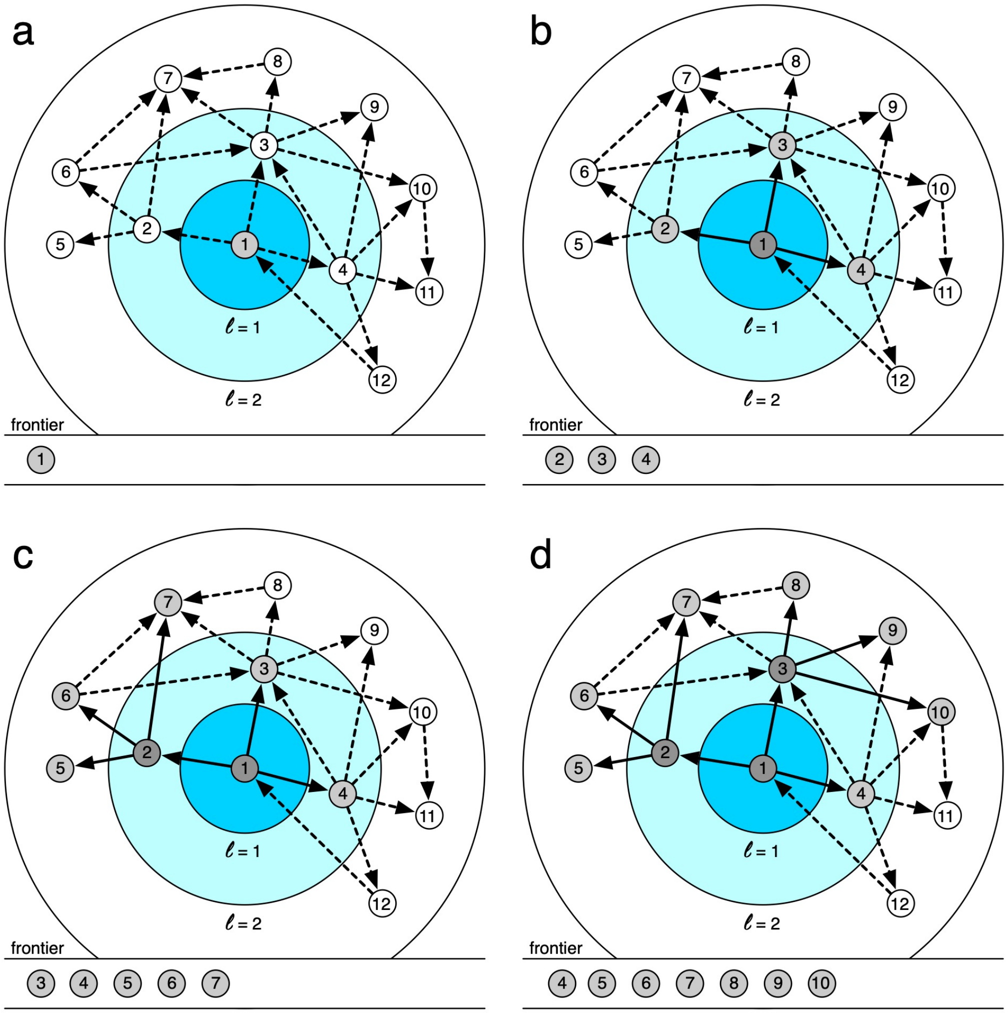
One algorithm used to find shortest paths is called **breadth-first search**

- Start from a source node (root)
- Visit the entire breadth of the network, within some distance from the source, before we move to a greater depth, farther away from the source
- Start from each node to find all-pairs-shortest-paths (slow: $O(N^2)$)



Breadth-first search (BFS)

- Each node has an attribute storing its **distance** ℓ from the source, initially ℓ = –1 except ℓ(source) = 0
- A queue (FIFO) holds the **frontier**, initially contains the source
- A directed **shortest path tree**, initially all the nodes and no links
- Iterate until the frontier is empty:
  - Remove next node *i* in frontier
  - For each successor *j* of *i* with ℓ(*j*) = –1:
    - Queue *j* into frontier
    - ℓ(*j*) = ℓ(*i*) + 1
    - Add link (*i* → *j*) to shortest-path tree



Distance in real networks

Image by Kmhkmh - CC BY 3.0
https://commons.wikimedia.org/w/index.php?curid=38087162

- How close or far are two nodes in a social network?
- Let us start by considering coauthorship networks, in which nodes are scholars and links represent two people having coauthored one or more publications
- Paul Erdos: Considered the father of graph theory together with Alfréd Rényi
- He collaborated with over 500 coauthors: a hub in the coauthorship network!

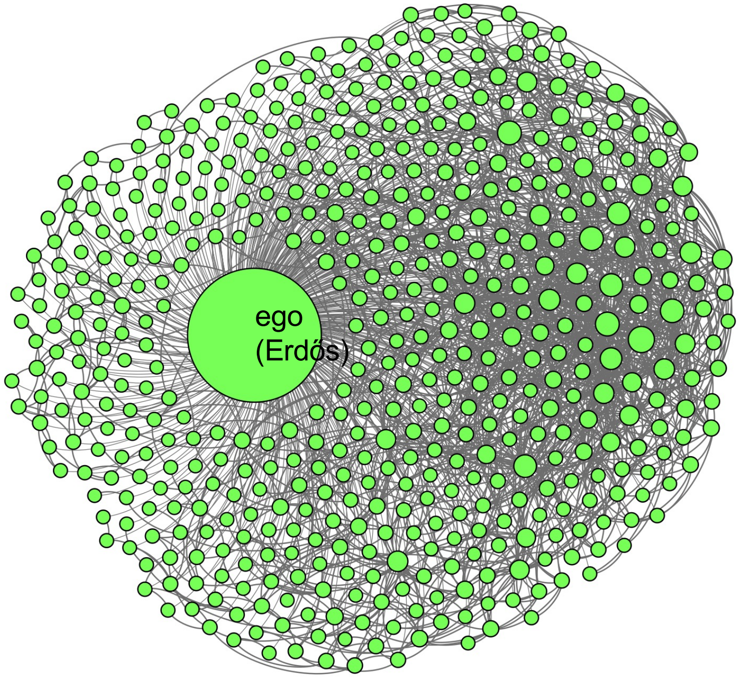
Erdős number

- An author's Erdős number is the length of the shortest path between them and Erdős in the coauthorship network
- Many mathematicians are proud to have a small Erdős number

Tool to compute one's Erdős number:

mathscinet.ams.org/mathscinet/collaborationDistance.html

Erdős ego coauthorhip network



Erdős numbers

Davis

BY EMILIO FERRARA, ONUR VAROL, CLAYTON DAVIS, FILIPPO MENCZER, AND ALESSANDRO FLAMMINI

The Rise of Social Bots

4

Menczer

Fortunato

Topical interests and the mitigation of search engine bias

S. Fortunato, A. Flammini, F. Menczer, and A. Vespignani

PNAS August 22, 2006 103 (34) 12684-12689; https://doi.org/10.1073/pnas.0605525103

Communicated by Elinor Ostrom, Indiana University, Bloomington, IN, July 1, 2006 (received for review March 2, 2006)

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Paul Erdős
(Image by Kmhkmh - CC BY 3.0 commons.wikimedia.org/w/index.php?curid=38087162)

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Journal of **Graph Theory**

Article | Full Access

Highly irregular graphs†

Yousef Alavi, Gary Chartrand, F. R. K. Chung, Paul Erdös, R. L. Graham, Ortrud R. Oellermann

1

Fan Chung (Image by Che Graham - CC BY 3.0 commons.wikimedia.org/w/index.php?curid=19475582)

The Workshop on Internet Topology (WIT) Report

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Alessandro Vespignani - Image CC BY-SA 2.0 commons.wikimedia.org/w/index.php?curid=60299947

Six Degrees of Kevin Bacon

Short paths are found among all authors, not just Erdős…

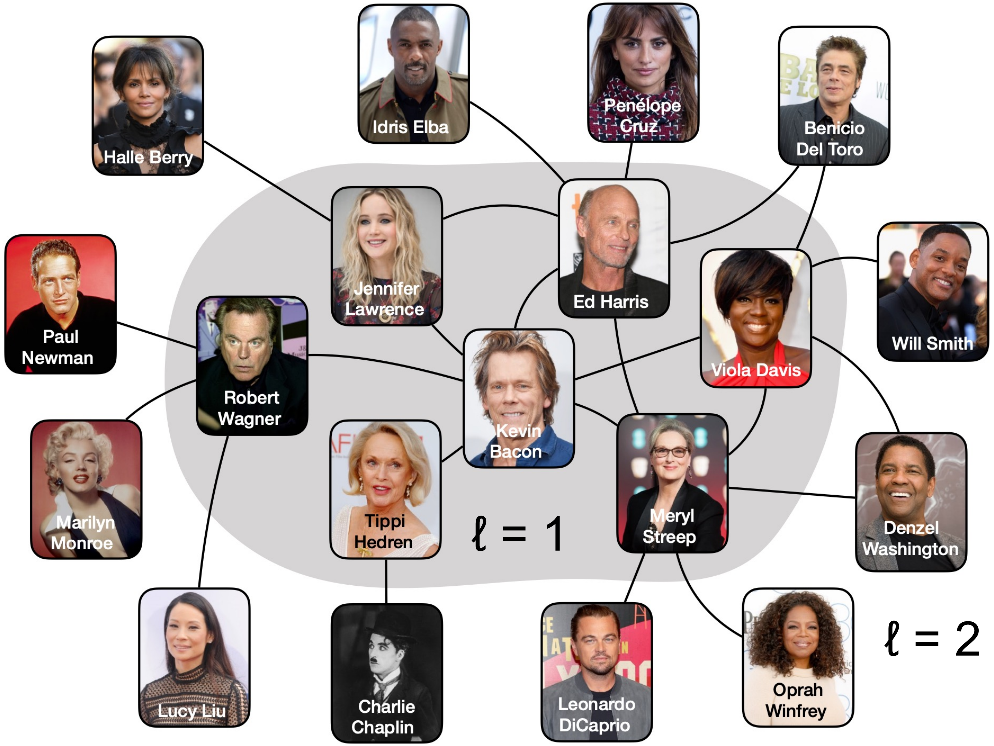
…And in all social networks, not just coauthorship

Consider the movie co-star network as a second example

Let's play the Oracle of Bacon game: oracleofbacon.org

Not just Kevin Bacon…

Can you find two stars separated by more that four links? Play the game and try!

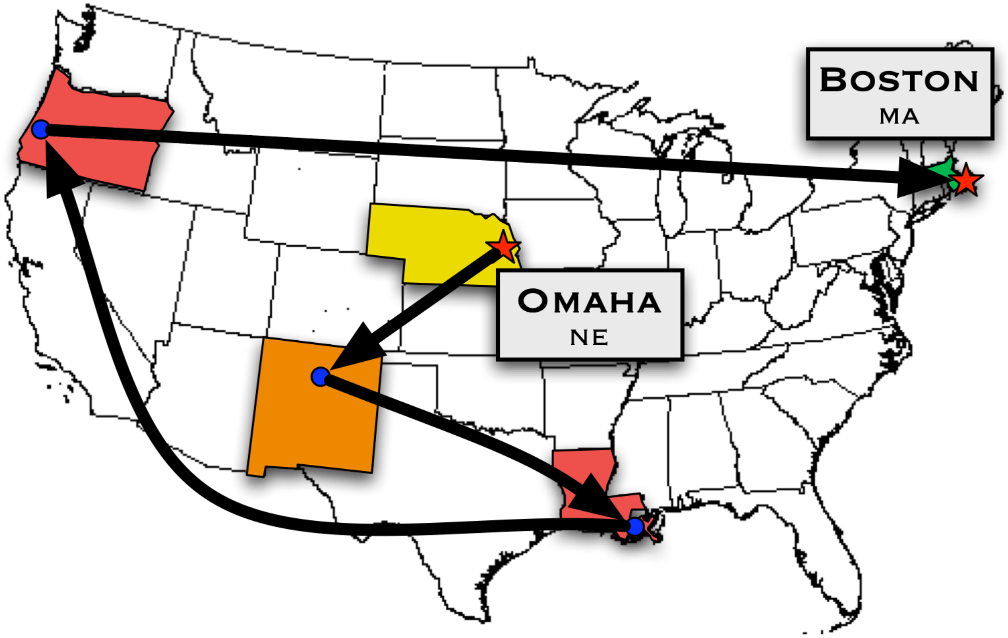


Small worlds

- What have we learned? Social networks tend to have very **short paths**
- **Six degrees of separation**: the idea that any two people are at most six steps away from each other in the social network
- First idea was in the short story “Chains” by Hungarian writer Frigyes Karinthy in 1929
- Psychologist **Stanley Milgram** provided first evidence in 1967 through a famous experiment to measure the **social distance** between any two people in the US
- John Guare coined the term “six degrees of separation” in a 1991 play (movie, too)

Milgram's experiment

- Instructions: send to personal acquaintance who is more likely to know target
- 160 letters to people in Omaha, NE and Wichita, KS
- 2 targets in Mass: the wife of a student in Sharon and a stockbroker in Boston
- 42 letters made it back (only 26%)
- Average: 6.5 steps (range: 3-12 steps)
- Much lower than most people expected!
- “Small world” effect is still surprising



More small world experiments

- Milgram's experiment was replicated in 2003 by Yahoo Research using email
  - 18 targets in 13 countries
  - 384 completed chains out of more than 24 thousand started
  - APL = 4 but when accounting for broken chains, estimated median PL of 5–7 steps
- Replicated by researchers at Facebook and University of Milan in 2011
  - 721 million active Facebook users
  - 69 billion friendships
  - APL = 4.74 steps: even shorter!


YAHOO! RESEARCH
SMALL WORLD EXPERIMENT

About the Experiment

The Small World Experiment is designed to test the hypothesis that anyone in the world can get a message to anyone else in just "six degrees of separation" by passing it from friend to friend. Sociologists have tried to prove (or disprove) this claim for decades, but it is still unresolved.

Now, using Facebook we finally have the technology to put the hypothesis to a proper scientific test. By participating in this experiment, you'll not only get to see how you're connected to people you might never otherwise encounter, you will also be helping to advance the science of social networks.

Become a Sender

We have already recruited a number of Target Persons from around the world.

Now we want you to try to reach them by becoming a Sender

Click on the Participate Button below, and you'll be shown your assigned target. Then you'll get to choose a friend to pass the message to. That person will then get the same instructions, and so on....

If everyone passes the messages along, your message will reach the target. How many steps will it take? There's only one way to find out.

Continue


The New York Times

Business Day

Technology

WORLD | U.S. | N.Y. / REGION | BUSINESS | TECHNOLOGY | SCIENCE | HEALTH | SPORTS | OPI

SAMSUNG The Next Big Thing Is Here GALAXY S4

Separating You and Me? 4.74 Degrees

By JOHN MARKOFF and SOMINI SENGUPTA

Published: November 21, 2011

The world is even smaller than you thought.

Enlarge This Image

Adding a new chapter to the research that cemented the phrase "six degrees of separation" into the language, scientists at Facebook and the University of Milan reported on Monday that the average number of acquaintances separating any two people in the world was not six but 4.74.

The original "six degrees" finding, published in 1967 by the psychologist Stanley Milgram,

RECOMMEND
TWITTER
LINKEDIN
SIGN IN TO E-MAIL
PRINT
REPRINTS
SHARE

Enough S… Now Playi…

Short paths

- What do we mean by "short paths"? When can we call a path "short"?
- It depends on the size of the network!
- Observe the relationship between APL and network size when considering networks (or subnetworks) of different sizes
- We say that the average path length is **short** when it **grows very slowly** with the size of the network, say, logarithmically:

$$\langle \ell \rangle \sim \log N$$

Small worlds

Small worlds Networks examples:

- Air transportation, the Internet, the Web, and Wikipedia.
  - Play Wikiracing games to convince yourself
  - Example: The Wiki Game (thewikigame.com)

Exceptions: grid-like networks

Table 2.1 Average path length and clustering coefficient of various network examples. The networks are the same as in Table 1.1, their numbers of nodes and links are listed as well. Link weights are ignored. The average path length is measured only on the giant component; for directed networks we consider directed paths in the giant strongly connected component. To measure the clustering coefficient in directed networks, we ignore link directions.

Network	Nodes (N)	Links (L)	Average path length ($\langle \ell \rangle$)	Clustering coefficient (C)
Facebook Northwestern Univ.	10,567	488,337	2.7	0.24
IMDB movies and stars	563,443	921,160	12.1	0
IMDB co-stars	252,999	1,015,187	6.8	0.67
Twitter US politics	18,470	48,365	5.6	0.03
Enron Email	87,273	321,918	3.6	0.12
Wikipedia math	15,220	194,103	3.9	0.31
Internet routers	190,914	607,610	7.0	0.16
US air transportation	546	2,781	3.2	0.49
World air transportation	3,179	18,617	4.0	0.49
Yeast protein interactions	1,870	2,277	6.8	0.07
C. elegans brain	297	2,345	4.0	0.29
Everglades ecological food web	69	916	2.2	0.55

Summary

Things to remember

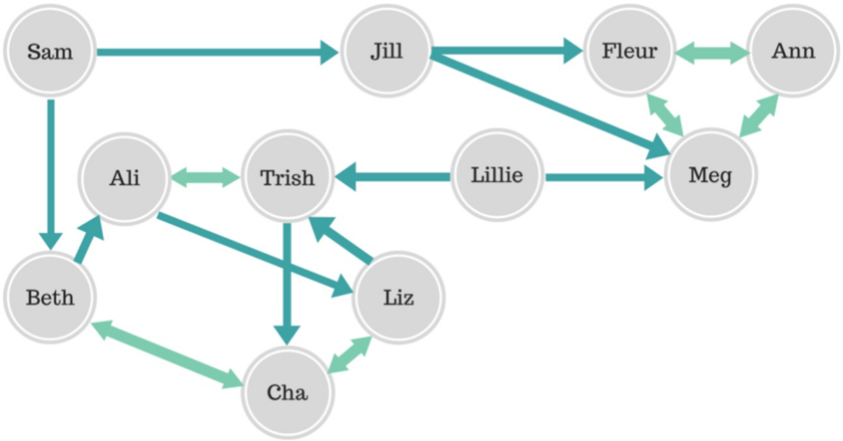
- Distance, diameter, effective diameter
  - In directed and undirected graphs
- Connected components
  - In directed and undirected graphs
  - d graphs
- Small words networks
  - Milgram’s experiment
  - Examples of MW networks

Sources

- A. L. Barabási (2016). Network Science - Chapter 01
- F. Menczer, S. Fortunato, C. A. Davis (2020). A First Course in Network Science – Chapter 02
- URLs cited in the footer of specific slides

Practice on your own

- Measure the sparsity of this graph $L/L_{\max}$
- (ignore direction of links)



Practice on your own (cont.)

- Compute the distance between two nodes
- Compute the diameter of a graph
- Identify connected components