Integer Linear Programming

Reference

Ragsdale: Chapter 6

Winston and Albright: Chapter 6

Learning Objectives

- Formulate business scenarios using binary integer variables.
  - Project selection
  - Capital budgeting
  - Fixed charge problems
  - Quantity discounts
  - Contract award problems
  - Set covering- location problems
  - Cutting stock problems

Introduction

- When one or more variables in an LP problem must assume an integer value we have an Integer Linear Programming (ILP) problem.
- Integer variables also allow us to build more accurate models for a number of common business problems.

Integer Linear Programming: ILP

Maximize/Minimize $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

$$\text{s.t.} \quad a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \left\{ \begin{matrix} \leq \\ \geq \\ = \end{matrix} \right\} b_i, \quad i = 1,\ldots,m$$

$$0 \leq x_j \leq u_j, \quad j = 1,\ldots,n$$

x_j integer for some or all $j = 1,\ldots,n$

Integrality Conditions- Blue Ridge Revisited

MAX:	$350X_1 + 300X_2$	} profit
S.T.:	$1X_1 + 1X_2 <= 200$	} pumps
	$9X_1 + 6X_2 <= 1566$	} labor
	$12X_1 + 16X_2 <= 2880$	} tubing
	$X_1, X_2 >= 0$	} nonnegativity
	X_1, X_2 must be integers	} integrality

Integrality conditions are easy to state but make the problem much more difficult (and sometimes impossible) to solve.

BINARY INTEGER PROGRAMMING

Binary Integer Programming

- ILP is not simply a matter of adding integer constraints to decision variables, such as the numbers of workers.
- Many inherently *nonlinear* problems can be transformed into linear models with the use of binary variables.
- The clever use of binary variables allows us to solve many interesting and difficult problems that LP algorithms are incapable of solving.

Binary Variables

- Binary variables are integer variables that can assume only two values: 0 or 1.
- These variables can be useful in a number of practical modeling situations….
- Perhaps the simplest binary IP model is the following capital budgeting example, which illustrates the “go–no go” nature of many IP models.

Making “yes-or-no” type decisions

– Invest in a project?
– Build a factory?
– Manufacture a product?
– Assign a person to a task?

A Capital Budgeting Problem: CRT Technologies

- In a capital budgeting problem, a decision maker is presented with several projects or investment alternatives and must determine which projects or investments to choose.
- The projects or investments typically require different amount of resources and generate cash flows to the company, which are converted to a net present value(NPV).
- Problem: Determine which set of projects or investments to select to achieve the maximum possible NPV.

A Capital Budgeting Problem: CRT Technologies

Project	Expected NPV (in $000s)	Capital (in $000s) Required in Year 1	Year 2	Year 3	Year 4	Year 5
1	$141	$75	$25	$20	$15	$10
2	$187	$90	$35	$0	$0	$30
3	$121	$60	$15	$15	$15	$15
4	$83	$30	$20	$10	$5	$5
5	$265	$100	$25	$20	$20	$20
6	$127	$50	$20	$10	$30	$40

- The company has $250,000 available to invest in new projects. It has budgeted $75,000 for continued support for these projects in year 2 and $50,000 per year for years 3, 4, and 5.
- Unused funds in any year cannot be carried over.

Defining the Decision Variables

$$X_i = \begin{cases} 1, \text{ if project } i \text{ is selected} \\ 0, \text{ otherwise} \end{cases} \quad i = 1,2,\ldots,6$$

Defining the Objective Function

Maximize the total NPV of selected projects.

$$\text{MAX: } 141X_1 + 187X_2 + 121X_3 + 83X_4 + 265X_5 + 127X_6$$

Defining the Constraints

- Capital Constraints

  $75X_1 + 90X_2 + 60X_3 + 30X_4 + 100X_5 + 50X_6 <= 250$ } year 1

  $25X_1 + 35X_2 + 15X_3 + 20X_4 + 25X_5 + 20X_6 <= 75$ } year 2

  $20X_1 + 0X_2 + 15X_3 + 10X_4 + 20X_5 + 10X_6 <= 50$ } year 3

  $15X_1 + 0X_2 + 15X_3 + 5X_4 + 20X_5 + 30X_6 <= 50$ } year 4

  $10X_1 + 30X_2 + 15X_3 + 5X_4 + 20X_5 + 40X_6 <= 50$ } year 5

- Binary Constraints

  All X_i must be binary

Binary Variables & Logical Conditions

- Binary variables are also useful in modeling a number of logical conditions.
  - Of projects 1, 3 & 6, no more than one may be selected

    $X_1 + X_3 + X_6 <= 1$
  - Of projects 1, 3 & 6, exactly one must be selected

    $X_1 + X_3 + X_6 = 1$
  - Project 4 cannot be selected unless project 5 is also selected

    $X_4 – X_5 <= 0$

Value of			
X4	X5	Meaning	Feasible?
0	0	Do not select either project	Yes
1	1	Select both projects	Yes
0	1	Select 5, but not 4	Yes
1	0	Select 4, but not 5	No

The Fixed-Charge Problem

- In many situations, a cost is incurred if an activity is undertaken at any positive level. This cost is independent of the level of the activity and is known as a fixed cost (or fixed charge).
- Fixed costs- costs regardless of what decision is made.

The Fixed-Charge Problem

- Many decisions result in a fixed or lump-sum cost being incurred:
  - The cost to lease, rent, or purchase a piece of equipment or a vehicle that will be required if a particular action is taken.
  - The setup cost required to prepare a machine or to produce a different type of product.
  - The cost to construct a new production line that will be required if a particular decision is made.
  - The cost of hiring additional personnel that will be required if a particular decision is made.

Example Fixed-Charge Problem: Remington Manufacturing

- Remington manufacturing is planning its next production cycle. The company can produce three products, each of which must undergo machining, grinding and assemble operations.
- Management of Remington wants to determine the most profitable mix of products to produce.

Example Fixed-Charge Problem: Remington Manufacturing

	Hours Required By:			
Operation	Prod. 1	Prod. 2	Prod. 3	Hours Available
Machining	2	3	6	600
Grinding	6	3	4	300
Assembly	5	6	2	400
Unit Profit	\$48	\$55	\$50	
Setup Cost	\$1000	\$800	\$900	

Defining the Decision Variables

X_i = the amount of product i to be produced, i = 1, 2, 3

Each of the X_i variables has a corresponding binary variable

$$Y_i = \begin{cases} 1, \text{if } X_i > 0 \\ 0, \text{if } X_i = 0 \end{cases} \quad i = 1, 2, 3$$

Defining the Objective Function

Maximize total profit.

MAX: $48X_1 + 55X_2 + 50X_3 - 1000Y_1 - 800Y_2 - 900Y_3$

Defining the Constraints

- Resource Constraints

  $2X_1 + 3X_2 + 6X_3 <= 600$ } machining

  $6X_1 + 3X_2 + 4X_3 <= 300$ } grinding

  $5X_1 + 6X_2 + 2X_3 <= 400$ } assembly

- Nonnegativity conditions

  $X_i >= 0$, $i = 1, 2, ..., 6$

- Binary Constraints

  All Y_i must be binary

- Is there a missing link?

Linking Constraints

- We must ensure that the required relationship between X_i and Y_i is enforced.
- In particular, the value of Y_i variables can be determined from the X_i variables.
- We need constraints to establish this link between the value of the Y_i variables and the X_i variables.

Defining the Constraints *(cont'd)*

- Linking Constraints (with "Big M")

  $X_1 <= M_1Y_1$ or $X_1 - M_1Y_1 <= 0$

  $X_2 <= M_2Y_2$ or $X_2 - M_2Y_2 <= 0$

  $X_3 <= M_3Y_3$ or $X_3 - M_3Y_3 <= 0$

- If $X_i > 0$ these constraints force the associated Y_i to equal 1.
- If $X_i = 0$ these constraints allow Y_i to equal 0 or 1, but the objective will cause Solver to choose 0.
- Note that M_i imposes an upper bounds on X_i.
- It helps to find reasonable values for the M_i.

Finding Reasonable Values for M_1

- Consider the resource constraints

  $2X_1 + 3X_2 + 6X_3 <= 600$ } machining

  $6X_1 + 3X_2 + 4X_3 <= 300$ } grinding

  $5X_1 + 6X_2 + 2X_3 <= 400$ } assembly

- What is the maximum value X_1 can assume?

  Let $X_2 = X_3 = 0$

  $X_1 = \text{MIN}(600/2, 300/6, 400/5)$

  $= \text{MIN}(300, 50, 80)$

  $= 50$

- Maximum values for X_2 & X_3 can be found similarly.

Summary of the Model

MAX: $48X_1 + 55X_2 + 50X_3 - 1000Y_1 - 800Y_2 - 900Y_3$

S.T.:

$2X_1 + 3X_2 + 6X_3 <= 600$ } machining

$6X_1 + 3X_2 + 4X_3 <= 300$ } grinding

$5X_1 + 6X_2 + 2X_3 <= 400$ } assembly

$$\left.\begin{array}{l} X_1 - 50Y_1 <= 0 \\ X_2 - 67Y_2 <= 0 \\ X_3 - 75Y_3 <= 0 \end{array}\right\} \text{linking constraints}$$

All Y_i must be binary

$X_i >= 0, i = 1, 2, 3$

Minimum Order Size Restrictions

- Many investment, production, and distribution problems have minimum purchase amounts or minimum production lot size requirements that must be met.
- Suppose Remington doesn’t want to manufacture any units of product 3 unless it produces at least 40 units...

The Constraints

- Consider,

$$X_3 <= M_3Y_3$$

$$X_3 >= 40\ Y_3$$

- The first constraint is a linking constraint
- According to second constraint, if Y_3 equals 1, then X_3 must be greater than or equal to 40.
- On the other hand, if X_3 equals 0, Y_3 must also equal 0 in order to satisfy both constraints.

Quantity Discounts

- We have so far assumed that the profit or cost coefficients in the objective function were constant.
- Recall the revised blue ridge hot tubs example

- MAX: $350X_1 + 300X_2$ } profit

  S.T.: $1X_1 + 1X_2 <= 200$ } pumps

  $9X_1 + 6X_2 <= 1566$ } labor

  $12X_1 + 16X_2 <= 2880$ } tubing

  $X_1, X_2 >= 0$ } nonnegativity

  X_1, X_2 must be integers } integrality

- AS production of these products increases, quantity discounts might be obtained on component parts that would cause the profit margin on these items to increase.

- Assume…

  If Blue Ridge Hot Tubs produces more than 75 Aqua-Spas, it obtains discounts that increase the unit profit to $375.

  If it produces more than 50 Hydro-Luxes, the profit increases to $325.

Formulation- Decision Variables

- X_{11} = Number of Aqua-Spas produced at $350 per unit
- X_{12} = Number of Aqua-Spas produced at $375 per unit
- X_{21} = Number of Aqua-Spas produced at $300 per unit
- X_{22} = Number of Aqua-Spas produced at $325 per unit

Formulation- With Objective

- MAX: $350X_{11} + 375X_{12} + 300X_{21} + 325X_{22}$
- S.T.: $1X_{11} + 1X_{12} + 1X_{21} + 1X_{22} <= 200$ } pumps
- $9X_{11} + 9X_{12} + 6X_{21} + 6X_{22} <= 1566$ } labor
- $12X_{11} + 12X_{12} + 16X_{21} + 16X_{22} <= 2880$ } tubing
- $X_{ij} >= 0$

  X_{ij} must be integers, Y_i must be binary
- Is the formulation complete?

The Missing Constraints

- Ensure the model does not allow any units of X_{12} to be produced unless we have produced 75 units of X_{11} .
  - $X_{12} <= M_{12} Y_1$
  - $X_{11} >= 75 Y_1$
- Ensure the model does not allow any units of X_{22} to be produced unless we have produced 50 units of X_{21} .
  - $X_{22} <= M_{22} Y_2$
  - $X_{21} >= 50 Y_2$

Quantity Discount Model

MAX: $350X_{11} + 375X_{12} + 300X_{21} + 325X_{22}$

S.T.: $1X_{11} + 1X_{12} + 1X_{21} + 1X_{22} <= 200$ } pumps

$9X_{11} + 9X_{12} + 6X_{21} + 6X_{22} <= 1566$ } labor

$12X_{11} + 12X_{12} + 16X_{21} + 16X_{22} <= 2880$ } tubing

$X_{12} <= M_{12}Y_1$

$X_{11} >= 75Y_1$

$X_{22} <= M_{22}Y_2$

$X_{21} >= 50Y_2$

$X_{ij} >= 0$

X_{ij} must be integers, Y_i must be binary

A Contract Award Problem

- B&G Construction has 4 building projects and can purchase cement from 3 companies for the following costs:

	Cost per Delivered Ton of Cement				Max.
	Project 1	Project 2	Project 3	Project 4	Supply
Co. 1	\$120	\$115	\$130	\$125	525
Co. 2	\$100	\$150	\$110	\$105	450
Co. 3	\$140	\$95	\$145	\$165	550
Needs (tons)	450	275	300	350	

A Contract Award Problem

- Side constraints:

  Co. 1 will not supply orders of less than 150 tons for any project

  Co. 2 can supply more than 200 tons to no more than one of the projects

  Co. 3 will accept only orders that total 200, 400, or 550 tons

A Contract Award Problem

- B & G can contract with more than one supplier to meet the cement requirements for a given project.
- The problem is to determine what amounts to purchase from each supplier to meet the demands for each project at the least total cost.

Defining the Decision Variables

X_{ij} = tons of cement purchased
from company i for project j

Defining the Objective Function

Minimize total cost

$$\begin{aligned} \text{MIN:}\quad & 120X_{11} + 115X_{12} + 130X_{13} + 125X_{14} \\ & + 100X_{21} + 150X_{22} + 110X_{23} + 105X_{24} \\ & + 140X_{31} + 95X_{32} + 145X_{33} + 165X_{34} \end{aligned}$$

Defining the Constraints

- Supply Constraints

$X_{11} + X_{12} + X_{13} + X_{14} <= 525$ } company 1

$X_{21} + X_{22} + X_{23} + X_{24} <= 450$ } company 2

$X_{31} + X_{32} + X_{33} + X_{34} <= 550$ } company 3

- Demand Constraints

$X_{11} + X_{21} + X_{31} = 450$ } project 1

$X_{12} + X_{22} + X_{32} = 275$ } project 2

$X_{13} + X_{23} + X_{33} = 300$ } project 3

$X_{14} + X_{24} + X_{34} = 350$ } project 4

Defining the Constraints-I

- Company 1 Side Constraints- Minimum Order Size restriction

$X_{11} <= 525Y_{11}$
$X_{12} <= 525Y_{12}$
$X_{13} <= 525Y_{13}$
$X_{14} <= 525Y_{14}$

Linking constraints to ensure if X_{11}, X_{12}, X_{13} or X_{14}, is greater than 0, then its associated binary variable must equal 1.

$X_{11} >= 150Y_{11}$
$X_{12} >= 150Y_{12}$
$X_{13} >= 150Y_{13}$
$X_{14} >= 150Y_{14}$

These constraints ensure that if X_{11}, X_{12}, X_{13} or X_{14}, is greater than 0, then it must be at least 150.

Y_{ij} binary

Defining the Constraints-II

- Company 2 Side Constraints- it can supply more than 200 tons to no more than one of the projects.

$X_{21} <= 200 + 250Y_{21}$

$X_{22} <= 200 + 250Y_{22}$

$X_{23} <= 200 + 250Y_{23}$

$X_{24} <= 200 + 250Y_{24}$

$Y_{21} + Y_{22} + Y_{23} + Y_{24} <= 1$

Y_{ij} binary

Defining the Constraints-III

- Company 3 Side Constraints

$$X_{31} + X_{32} + X_{33} + X_{34} = 200Y_{31} + 400Y_{32} + 550Y_{33}$$

$$Y_{31} + Y_{32} + Y_{33} <= 1$$

Set Covering Models

- Many companies have geographically dispersed customers that they must service in some way.
- To do this, they create service center facilities at selected locations and then assign each customer to one of the service centers.
- Various costs are incurred, including:
  - Fixed costs of locating service centers in particular locations;
  - Operating costs, depending on the service centers' locations; and
  - Transportation costs, depending on the distances between customers and their assigned service centers.

Set Covering Models

- We first examine a particular type of location model called a ***set-covering model***.
- In a set-covering model, each member of a given set (set 1) must be "covered" by an acceptable member of another set (set 2).
- The usual objective in a set-covering problem is to minimize the number of members in set 2 that are needed to cover all the members in set 1.
- Set-covering models have been applied to areas as diverse as site selection, airline crew scheduling, truck dispatching, political redistricting, and capital investment

Applications

- Site Selection
- Crew Scheduling

Selection of Sites for Emergency Services: The Caliente City Problem

- Caliente City is growing rapidly and spreading well beyond its original borders
- They still have only one fire station, located in the congested center of town
- The result has been long delays in fire trucks reaching the outer part of the city

Goal: Develop a plan for locating multiple fire stations throughout the city

New Policy: Response Time ≤ 10 minutes

Response Time and Cost Data for Caliente City

		Fire Station in Tract							
		1	2	3	4	5	6	7	8
Response times (minutes) for a fire in tract	1	2	8	18	9	23	22	16	28
	2	9	3	10	12	16	14	21	25
	3	17	8	4	20	21	8	22	17
	4	10	13	19	2	18	21	6	12
	5	21	12	16	13	5	11	9	12
	6	25	15	7	21	15	3	14	8
	7	14	22	18	7	13	15	2	9
	8	30	24	15	14	17	9	8	3
Cost of Station ($thousands)		350	250	450	300	50	400	300	200

Algebraic Formulation of Caliente City Problem

Let $x_j = 1$ if tract j is selected to receive a fire station; 0 otherwise ($j = 1, 2, \ldots, 8$)

Minimize $C = 350x_1 + 250x_2 + 450x_3 + 300x_4 + 50x_5 + 400x_6 + 300x_7 + 200x_8$

subject to

Tract 1: $x_1 + x_2 + x_4 \geq 1$
Tract 2: $x_1 + x_2 + x_3 \geq 1$
Tract 3: $x_2 + x_3 + x_6 \geq 1$
Tract 4: $x_1 + x_4 + x_7 \geq 1$
Tract 5: $x_5 + x_7 \geq 1$
Tract 6: $x_3 + x_6 + x_8 \geq 1$
Tract 7: $x_4 + x_7 + x_8 \geq 1$
Tract 8: $x_6 + x_7 + x_8 \geq 1$

and x_j are binary (for $j = 1, 2, \ldots, 8$).

Southwestern Airways Crew Scheduling

- Southwestern Airways needs to assign crews to cover all its upcoming flights.
- We will focus on assigning 3 crews based in San Francisco (SFO) to 11 flights.

Question: How should the 3 crews be assigned 3 sequences of flights so that every one of the 11 flights is covered?

Data for the Southwestern Airways Problem

	Feasible Sequence of Flights											
Flights	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1. SFO–LAX	1			1			1			1		
2. SFO–DEN		1			1			1			1	
3. SFO–SEA			1			1			1			1
4. LAX–ORD				2			2		3	2		3
5. LAX–SFO	2					3				5	5	
6. ORD–DEN				3	3				4			
7. ORD–SEA							3	3		3	3	4
8. DEN–SFO		2		4	4				5			
9. DEN–ORD					2			2			2	
10. SEA–SFO			2				4	4				5
11. SEA–LAX						2			2	4	4	2
Cost, $1,000s	2	3	4	6	7	5	7	8	9	9	8	9

Algebraic Formulation

Let $x_j = 1$ if flight sequence j is assigned to a crew; 0 otherwise. ($j = 1, 2, \ldots, 12$).

Minimize Cost = $2x_1 + 3x_2 + 4x_3 + 6x_4 + 7x_5 + 5x_6 + 7x_7 + 8x_8 + 9x_9 + 9x_{10} + 8x_{11} + 9x_{12}$

(in $thousands)

subject to

Flight 1 covered: $x_1 + x_4 + x_7 + x_{10} \geq 1$

Flight 2 covered: $x_2 + x_5 + x_8 + x_{11} \geq 1$

: :

Flight 11 covered: $x_6 + x_9 + x_{10} + x_{11} + x_{12} \geq 1$

Three Crews: $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} \leq 3$

and

x_j are binary ($j = 1, 2, \ldots, 12$).

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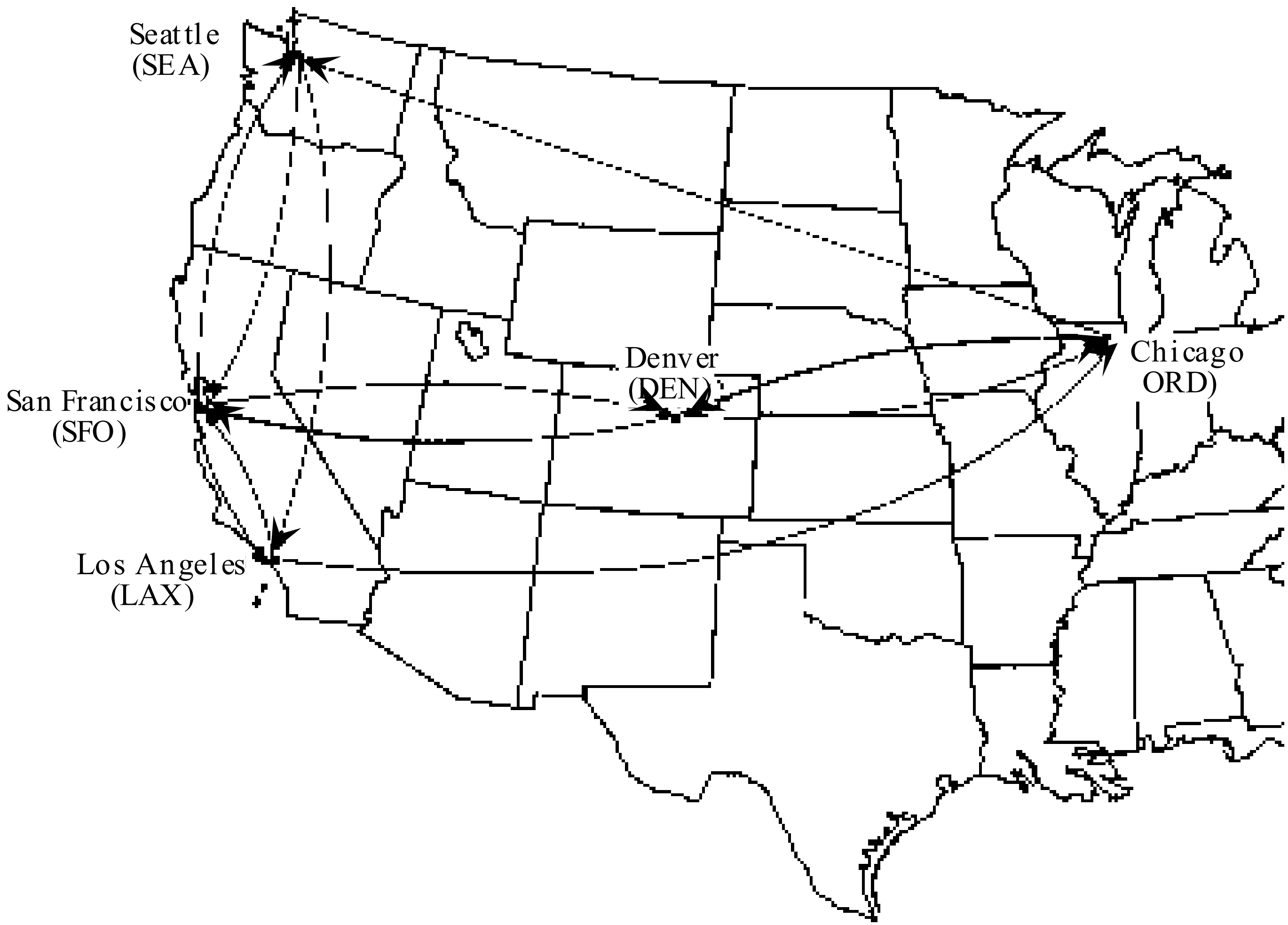
Southwestern Airways Flights



Data for the Southwestern Airways Problem

	Feasible Sequence of Flights											
Flights	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1. SFO–LAX	1			1			1			1		
2. SFO–DEN		1			1			1			1	
3. SFO–SEA			1			1			1			1
4. LAX–ORD				2			2		3	2		3
5. LAX–SFO	2					3				5	5	
6. ORD–DEN				3	3				4			
7. ORD–SEA							3	3		3	3	4
8. DEN–SFO		2		4	4				5			
9. DEN–ORD					2			2			2	
10. SEA–SFO			2				4	4				5
11. SEA–LAX						2			2	4	4	2
Cost, $1,000s	2	3	4	6	7	5	7	8	9	9	8	9

Algebraic Formulation

Let $x_j = 1$ if flight sequence j is assigned to a crew; 0 otherwise. ($j = 1, 2, \ldots, 12$).

Minimize Cost $= 2x_1 + 3x_2 + 4x_3 + 6x_4 + 7x_5 + 5x_6 + 7x_7 + 8x_8 + 9x_9 + 9x_{10} + 8x_{11} + 9x_{12}$

(in $thousands)

subject to

Flight 1 covered: $x_1 + x_4 + x_7 + x_{10} \geq 1$

Flight 2 covered: $x_2 + x_5 + x_8 + x_{11} \geq 1$

: :

Flight 11 covered: $x_6 + x_9 + x_{10} + x_{11} + x_{12} \geq 1$

Three Crews: $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} \leq 3$

and

x_j are binary ($j = 1, 2, \ldots, 12$).