MS3530: Advanced Operations Research

Practice Assignment 3

Q1. A real estate development firm, Peterson and Johnson, is considering five possible development projects. The following table shows the estimated long-run profit (net present value) that each project would generate, as well as the amount of investment required to undertake the project, in units of millions of dollars.

	Development Project				
	1	**2**	**3**	**4**	**5**
Estimated profit	1	1.8	1.6	0.8	1.4
Capital required	6	12	10	4	8

The owners of the firm, Dave Peterson and Ron Johnson, have raised \$20 million of investment capital for these projects. Dave and Ron now want to select the combination of projects that will maximize their total estimated long-run profit (net present value) without investing more that \$20 million.

Formulate a BIP model for this problem.

Q2. The Research and Development Division of the Progressive Company has been developing four possible new product lines. Management must now make a decision as to which of these four products actually will be produced and at what levels. Therefore, an operations research study has been requested to find the most profitable product mix.

A substantial cost is associated with beginning the production of any product, as given in the first row of the following table. Management's objective is to find the product mix that maximizes the total profit (total net revenue minus start-up costs).

	Product			
	1	**2**	**3**	**4**
Start-up cost	\$50,000	\$40,000	\$70,000	\$60,000
Marginal revenue	\$ 70	\$ 60	\$ 90	\$ 80

Let the continuous decision variables x_1, x_2, x_3, and x_4 be the production levels of products 1, 2, 3, and 4, respectively. Management has imposed the following policy constraints on these variables:

1. No more than two of the products can be produced.
2. Either product 3 or 4 can be produced only if either product 1 or 2 is produced.
3. Either $5x_1 + 3x_2 + 6x_3 + 4x_4 \leq 6{,}000$ or
   $4x_1 + 6x_2 + 3x_3 + 5x_4 \leq 6{,}000$.

Introduce auxiliary binary variables to formulate a mixed BIP Model for this problem.

Q3 The Fly-Right Airplane Company builds small jet airplanes to sell to corporations for the use of their executives. To meet the needs of these executives, the company's customers sometimes or- der a custom design of the airplanes being purchased. When this occurs, a substantial start-up cost is incurred to initiate the production of these airplanes.

Fly-Right has recently received purchase requests from three customers with short deadlines. However, because the company's production facilities already are almost completely tied up filling previous orders, it will not be able to accept all three orders. There- fore, a decision now needs to be made on the number of airplanes the company will agree to produce (if any) for each of the three customers.

The relevant data are given in the next table. The first row gives the start-up cost required to initiate the production of the air- planes for each customer. Once production is under way, the marginal net revenue (which is the purchase price minus the marginal production cost) from each airplane produced is shown in the second row. The third row gives the percentage of the available production capacity that would be used for each airplane produced. The last row indicates the maximum number of airplanes requested by each customer (but less will be accepted).

	Customer		
	1	**2**	**3**
Start-up cost	\$3 million	\$2 million	0
Marginal net revenue	\$2 million	\$3 million	\$0.8 million
Capacity used per plane	20%	40%	20%
Maximum order	3 planes	2 planes	5 planes

Fly-Right now wants to determine how many airplanes to pro- duce for each customer (if any) to maximize the company's total profit (total net revenue minus start-up costs).

Formulate a model with both integer variables and binary variables for this problem.

Q4. Reconsider the Fly-Right Airplane Co. problem introduced in Q3. A more detailed analysis of the various cost and revenue factors now has revealed that the potential profit from producing airplanes for each customer cannot be expressed simply in terms of a start-up cost and a fixed marginal net revenue per air- plane produced. Instead, the profits are given by the following table.

Airplanes Produced	Profit from Customer		
	1	2	3
0	0	0	0
1	−\$1 million	\$1 million	\$1 million
2	\$2 million	\$5 million	\$3 million
3	\$4 million		\$5 million
4			\$6 million
5			\$7 million

Formulate a BIP model for this problem that includes constraints for mutually exclusive alternatives.

Q5. Suppose that a state sends R persons to the U.S. House of Representatives. There are D counties in the state ($D > R$), and the state legislature wants to group these counties into R distinct electoral districts, each of which sends a delegate to Congress. The total population of the state is P, and the legislature wants to form districts whose population approximates $p = P/R$. Suppose that the appropriate legislative committee studying the electoral districting problem generates a long list of N candidates to be districts ($N > R$). Each of these candidates contains contiguous counties and a total population p_j (j =1, 2, . . . , N) that is acceptably close to p.
Define $c_j = |p_j - p|$. Each county i (i = 1, 2. . . , D) is included in at least one candidate and typically will be included in a considerable number of candidates (in order to provide many feasible ways of selecting a set of R candidates that includes each county exactly once). Define

$$a_{ij} = \begin{cases} 1 & \text{If country } i \text{ is included in candidate} \\ 0 & \text{If not} \end{cases}$$

Given the values of the c_j and the a_{ij}, the objective is to select R of these N possible districts such that each county is contained in a single district and such that the largest of the associated c_j is as small as possible.

Formulate a BIP model for this problem.

Q6. Pawtucket University is planning to buy new copier machines for its library. Three members of its Operations Research Department are analyzing what to buy. They are considering two different models: Model A, a high-speed copier, and Model B, a lower-speed but less expensive copier. Model A can handle 20,000 copies a day, and costs $6,000. Model B can handle 10,000 copies a day, but costs only $4,000. They would like to have at least six copiers so that they can spread them throughout the library. They also would like to have at least one high-speed copier. Finally, the copiers need to be able to handle a capacity of at least 75,000 copies per day. The objective is to determine the mix of these two copiers that will handle all these requirements at minimum cost.

(a) Formulate an IP model for this problem.

(b) Use a graphical approach to solve this model.

Q7. A U.S. professor will be spending a short sabbatical leave at the University of Iceland. She wishes to bring all needed items with her on the airplane. After collecting the professional items she must have, she finds that airline regulations on space and weight for checked luggage will severely limit the clothes that she can take. (She plans to carry on a warm coat and then purchase a warm Icelandic sweater upon arriving in Iceland.) Clothes under consideration for checked luggage include 3 skirts, 3 slacks, 4 tops, and 3 dresses. The professor wants to maximize the number of outfits she will have in Iceland (including the special dress she will wear on the airplane). Each dress constitutes an outfit. Other outfits consist of a combination of a top and either a skirt or slacks. However, certain combinations are not fashionable and so will not qualify as an outfit.

In the following table, the combinations that will make an outfit are marked with an x.

		Top				
		1	**2**	**3**	**4**	**Icelandic Sweater**
	1	x	x			x
Skirt	2	x			x	
	3		x	x	x	x
	1	x		x		
Slacks	2	x	x		x	x
	3			x	x	x

The weight (in grams) and volume (in cubic centimeters) of each item are shown in the following table:

		Weight	**Volume**
Skirt	1	600	5,000
	2	450	3,500
	3	700	3,000
Slacks	1	600	3,500
	2	550	6,000
	3	500	4,000
Top	1	350	4,000
	2	300	3,500
	3	300	3,000
	4	450	5,000
Dress	1	600	6,000
	2	700	5,000
	3	800	4,000
Total allowed		4,000	32,000

Formulate a BIP model to choose which items of clothing to take. (Hint: After using binary decision variables to represent the individual items, you should introduce auxiliary binary variables to represent outfits involving combinations of items. Then use constraints and the objective function to ensure that these auxiliary variables have the correct values, given the values of the decision variables.)

Q8. Consider the following IP problem.

Maximize $Z = 5x_1 + x_2$,

subject to

$-x_1 + 2x_2 \leq 4$

$x_1 - x_2 \leq 1$

$4x_1 + x_2 \leq 12$

And

$x_1 \geq 0, \; x_2 \geq 0$

x_1, x_2 are integers.

(a) Solve this problem graphically.

(b) Solve the LP relaxation graphically. Round this solution to the *nearest* integer solution and check whether it is feasible. Then enumerate *all* the rounded solutions by rounding this solution for the LP relaxation in *all* possible ways (i.e., by rounding each noninteger value both up and down). For each rounded solution, check for feasibility and, if feasible, calculate Z. Are any of these feasible rounded solutions optimal for the IP problem?

Q9. Use the BIP branch-and-bound algorithm to solve the following problem interactively.

Maximize $Z = 2x_1 - x_2 + 5x_3 - 3x_4 + 4x_5$,

Subject to

$3x_1 - x_2 + x_3 + x_4 - 2x_5 \leq 6$

$x_1 - x_2 + 2x_3 - 4x_4 + 2x_5 \leq 0$

and

x_j is binary, for $j = 1, 2, \ldots, 5$

Q10. Consider the assignment problem with the following cost table:

		Task				
		1	**2**	**3**	**4**	**5**
	1	39	65	69	66	57
	2	64	84	24	92	22
Assignee	3	49	50	61	31	45
	4	48	45	55	23	50
	5	59	34	30	34	18

(a) Design a branch-and-bound algorithm for solving such assignment problems by specifying how the branching, bounding, and fathoming steps would be performed. (*Hint:* For the assignees not yet assigned for the current subproblem, form the relaxation by deleting the constraints that each of these as- signees must perform exactly one task.)

(b) Use this algorithm to solve this problem.

Q11. Consider the following IP problem.

Maximize $Z = 220x_1 + 80x_2,$

subject to

$$5x_1 + 2x_2 \leq 16$$

$$2x_1 - x_2 \leq 4$$

$$-x_1 + 2x_2 \leq 4$$

and

$$x_1 \geq 0, \quad x_2 \geq 0,$$

x_1, x_2 are integers.

a) Use the MIP branch-and-bound algorithm to solve this problem by hand. For each subproblem, solve its LP relaxation graphically.

Q12. Use the MIP branch-and-bound algorithm to solve the following MIP problem interactively.

Maximize $Z = 3x_1 + 4x_2 + 2x_3 + x_4 + 2x_5,$

subject to

$2x_1 - x_2 + x_3 + x_4 + x_5 \leq 3$

$-x_1 + 3x_2 + x_3 - x_4 - 2x_5 \leq 2$

$2x_1 + x_2 - x_3 + x_4 + 3x_5 \leq 1$

and

$x_j \geq 0,$ for j 1, 2, 3, 4, 5

x_j is binary, for j 1, 2, 3.

Q13. For each of the following constraints of pure BIP problems, identify which ones are made redundant by the binary constraints. Explain why each one is, or is not, redundant.

(a) $2x_1 + x_2 + 2x_3 \leq 5$

(b) $3x_1 - 4x_2 + 5x_3 \leq 5$

(c) $x_1 + x_2 + x_3 \geq 2$

(d) $3x_1 - x_2 - 2x_3 \geq -4$

Q14. Apply the procedure for tightening constraints to the following constraint for a pure BIP problem.

$3x_1 - 2x_2 + x_3 \leq 3$