4/2/16

Primal | **Dual**

$$\min \quad \underset{\sim}{c}^T \underset{\sim}{x} \longrightarrow \cancel{\text{Dual}}$$

$$\text{st} \quad A\underset{\sim}{x} = \underset{\sim}{b} \longrightarrow \underset{\sim}{w}$$

$$\underset{\sim}{x} \in X = \left\{ \underset{\sim}{x} : \begin{array}{l} D\underset{\sim}{x} \geq \underset{\sim}{d} \\ \underset{\sim}{x} \geq 0 \end{array} \right\} \longrightarrow \underset{\sim}{v}$$

$$\max \quad \underset{\sim}{w}^T \underset{\sim}{b} + \underset{\sim}{v}^T \underset{\sim}{d}$$

$$\text{st.} \quad \underset{\sim}{w}^T A + \underset{\sim}{v}^T D \leq \underset{\sim}{c}^T$$

$$\underset{\sim}{w} \gtrless 0, \; \underset{\sim}{v} \geq 0$$

Assumption

X is non-empty and bounded.

$$\underset{w \gtrless 0}{\text{Max}} \left\{ \underset{\sim}{w}^T \underset{\sim}{b} + \begin{array}{ll} \text{Max} & \underset{\sim}{v}^T \underset{\sim}{d} \\ \text{st.} & \underset{\sim}{v}^T D \leq \underset{\sim}{c}^T - \underset{\sim}{w}^T A \\ & \underset{\sim}{v} \geq 0 \end{array} \right\}$$

$$\underset{w \gtrless 0}{\max} \left\{ \underset{\sim}{w}^T b + \begin{array}{ll} \min & (c^T - w^T A)\underset{\sim}{x} \\ \text{st.} & \underset{\sim}{x} \in X \end{array} \right\}$$

MASTER PROBLEM

Max z

st. $z \leq \underset{\sim}{w}^T \underset{\sim}{b} + (\underset{\sim}{c}^T - \underset{\sim}{w}^T A)\underset{\sim}{x}$

$j = 1, 2, \ldots, t$

$\underset{\sim}{w} \geq 0$; $(\bar{z}, \bar{w})$ solution in relaxed problem

$\theta(w) = \underset{\sim}{w}^T \underset{\sim}{b} + \min_{x \in X} (c^T - w^T A)\underset{\sim}{x}$

max $\theta(w)$: $w \geq 0$

SUB PROBLEM

$\bar{w}^T \underset{\sim}{b} + \min_{\underset{\sim}{x} \in X} (c^T - \bar{w}^T A)\underset{\sim}{x}$

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no notes for 11/2/16

12/2/16

Primal	Dual
min $\underset{\sim}{c}^T \underset{\sim}{x}$	max $\underset{\sim}{w}^T \underset{\sim}{b}$
st $A\underset{\sim}{x} = \underset{\sim}{b}$	st $\underset{\sim}{w}^T A \leq \underset{\sim}{c}^T$
$\underset{\sim}{x} \geq 0$	$w \geq 0$

$$\min \quad 3x_1 + 4x_2 + 6x_3 + 7x_4 + x_5$$

$$\text{st} \quad 2x_1 - x_2 + x_3 + 6x_4 - 5x_5 - x_6 = 6$$

$$x_1 + x_2 + 2x_3 + x_4 + 2x_5 - x_7 = 3$$

$$x_1, x_2 \dots, x_7 \geq 0.$$

Dual

$$\text{Max} \quad 6w_1 + 3w_2$$

$$\text{st.} \quad 2w_1 + w_2 \leq 3$$

$$-w_1 + w_2 \leq 4$$

$$w_1 + 2w_2 \leq 6$$

$$6w_1 + w_2 \leq 7$$

$$-5w_1 + 2w_2 \leq 1$$

$$-w_1 \leq 0$$

$$-w_2 \leq 0.$$

$$w_1, w_2 \geq 0.$$

$\underset{\sim}{w} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $Q = \{6, 7\}$

$\boxed{\text{PRIMAL - DUAL ALGORITHM}}$

Start with a dual feasible solution.

$$Q = \{ j : w^T a_j - c_j = 0 \}$$

* Find initial basic feasible solution, add artificial vars. to create I.

Phase I : Restricted Primal

$$\min \quad x_8 + x_9$$

$$\text{st.} \quad -x_6 + x_8 = 6$$

$$-x_7 + x_9 = 3$$

$$x_6, x_7, x_8, x_9 \geq 0.$$

$$\rightarrow \begin{pmatrix} x_6 \\ x_7 \\ x_8 \\ x_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 3 \end{pmatrix}$$

$$\Rightarrow x_0 = 9 > 0$$

$$\left[\begin{aligned} &\min \ I\underset{\sim}{x_a} + \sum_{j \in Q} 0 x_j \\ &\text{st.} \quad \sum_{j \in Q} a_j x_j + I\underset{\sim}{x_a} = \underset{\sim}{b} \\ &\qquad x_j, \underset{\sim}{x_a} \geq 0 \end{aligned} \right]$$

Let x_0 is optimal solution to RP.

(i) $x_0 = 0 \Rightarrow$ optimal solution.

(ii) $x_0 > 0 \Rightarrow$ Modify the dual solution.

Dual of RP

$$\text{Max} \quad \underset{\sim}{v}^T \underset{\sim}{b}$$

$$\text{st.} \quad v^T a_j \leq 0 \quad j \in Q$$

$$v^T \leq 1$$

$$v \geq 0.$$

max $6v_1 + 3v_2$

st. $-v_1 \le 0$

$-v_2 \le 0$

$v_1 \le 1$

$v_2 \le 1$

$v_1, v_2 \ge 0$

$v^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Construct a new dual

$$\underset{\sim}{w}' = \underset{\sim}{w} + \theta \underset{\sim}{v}^*$$

$$w' a_j - c_j = (w + \theta v^*) a_j - c_j$$

$$= w a_j - c_j + \theta v^* a_j$$

- $\forall j \in Q$,

$$= 0 + \theta(-ve)$$

$$\Rightarrow w' a_j - c_j \le 0$$

- $\forall j \notin Q$,

$$= (w a_j - c_j) + \theta v^* a_j$$

$v^* a_j \le 0 \longrightarrow$ Dual unbounded $\Rightarrow$ Primal infeasible

$v^* a_j > 0 \longrightarrow \theta = \min_j \left\{ -\frac{(w a_j - c_j)}{v^* a_j} : v^* a_j > 0 \right\}$

$$w = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad v^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$j \in Q$

$$v^{*T} q_j = (1 \quad 1) \begin{array}{ccccc} q_1 & q_2 & q_3 & q_4 & q_5 \end{array} \begin{pmatrix} 2 & -1 & 1 & 6 & -5 \\ 1 & 1 & 2 & 1 & 2 \end{pmatrix}$$

$$= (3 \quad 0 \quad 3 \quad 7 \quad -3)$$

$$\theta = \min \left\{ +\frac{3}{3}, \frac{6}{3}, \frac{7}{7} \right\}$$

$$w' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$Q = \{1, 4\}$

<u>Phase I</u> (RP)

$$\begin{aligned} \min \quad & x_8 + x_9 \\ \text{st.} \quad & 2x_1 + 6x_4 + x_8 = 6 \\ & x_1 + x_4 + x_9 = 3 \\ & x_1, x_4, x_8, x_9 \geq 0. \end{aligned}$$

Check that optimal to this RP is

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_4 \\ x_8 \\ x_9 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad x_8 + x_9 = 0$

$\Rightarrow$ primal ~~is~~ solution is optimal.

primal optimal solution has all xs 0 except ~~for~~ x_1.

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INTEGER PROGRAMMING

* variables constrained to take integer values.

$$\min \ \underset{\sim}{c}^T \underset{\sim}{x}$$
$$\text{s.t.} \quad A\underset{\sim}{x} \leq \underset{\sim}{b}$$
$$\underset{\sim}{x} \geq 0$$
$$\underset{\sim}{x} = \text{integer}$$

$\Rightarrow$ Integer linear programming (ILP)

$$\left.\begin{aligned} \min \quad & \underset{\sim}{c}^T \underset{\sim}{x} + \underset{\sim}{d}^T \underset{\sim}{y} \\ \text{s.t.} \quad & A\underset{\sim}{x} + G\underset{\sim}{y} \leq \underset{\sim}{b} \\ & \underset{\sim}{x} \geq 0 \\ & \underset{\sim}{y} = \text{integer} \end{aligned}\right\} \text{Mixed integer linear programming (MILP).}$$

... integer values = {0,1}

$x_3 \quad 0 \mid \quad 2 \quad -1 \quad 1 \quad \quad 1$

$\Rightarrow$ x_1 enters the basis.

x_2 will not enter the basis as it is at its upper bound.

28/1

min $-3x_1 - 2x_2 - 2x_3 - 4x_4$

st.

$$
\begin{aligned}
x_1 + x_2 + 2x_3 + x_4 &\leq 10 \\
x_1 + 2x_2 &\leq 8 \\
x_2 &\leq 3 \\
x_3 + 3x_4 &\leq 6 \\
x_3 &\leq 4
\end{aligned}
$$

$x_1, x_2, x_3, x_4 \geq 0$