Lecture 1 (11 Jan 16)

# COMMUNICATION SYSTEMS

Text book: V. Madhow, "Introduction to Communication Systems," Cambridge University Press, 2014.

An earlier version available in V. Madhow's website.

Topics: from Chapters 1 – 6.

Communication: the process of information transfer across space or time.

Eg: across space – Radio (AM, FM)
Television
Cell phones
Landline phones
⋮

across time – Storage – CDs, DVDs, Hard drives

Key: channel

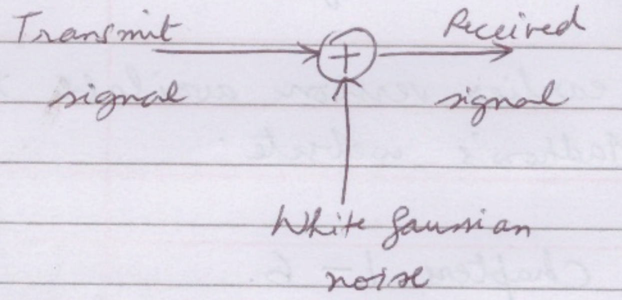
Transmit signal → Channel → Received signal

Typically, we assume the tx & rx signals as "analog".

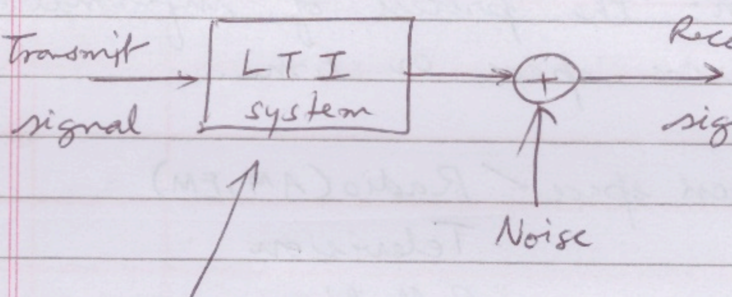
Need to answer the following:
- How should we model the channel?

the communication system?

* We will build from <u>simple</u> models.



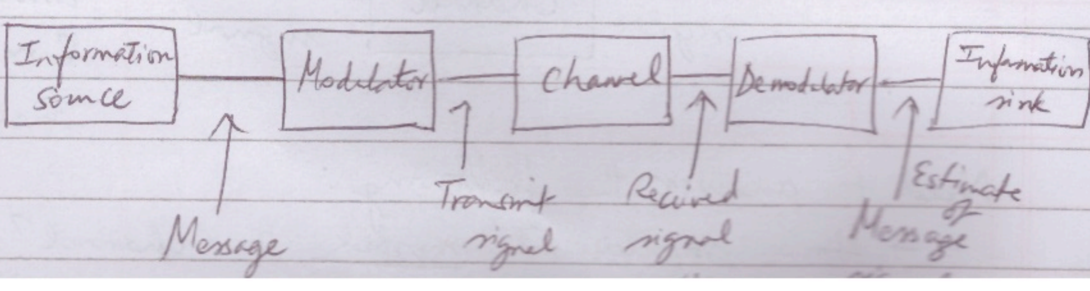

Additive white Gaussian noise (AWGN) model




Could be a simple ideal bandlimited filter
(or)
a more general LTI system.

Simple models provide the <u>insights</u>. Results can be generalized later to more realistic scenarios.

* Typical analog communication system.



"Analog" signals - continuous time continuous valued.
"Digital" signals - discrete time discrete valued.

Examples of analog sources : Audio, Speech, etc.

digital sources : files on a computer

Examples of analog communication systems:
Radio (AM, FM)
Broadcast television

Q: How should we design the modulator and demodulator?
What is the "performance" that we expect?

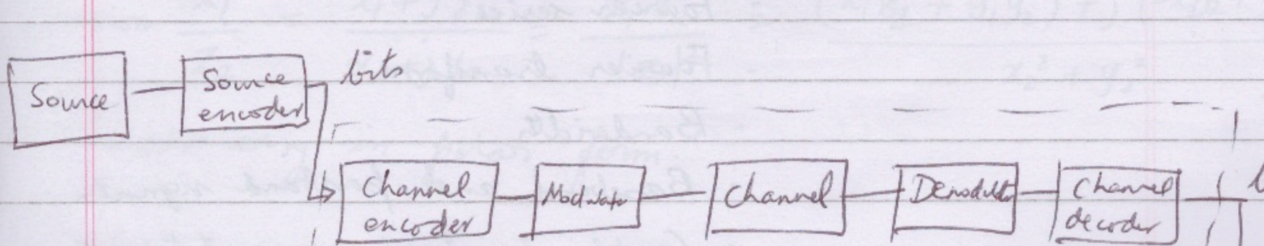
* Performance is harder to measure and
even harder to optimize for analog systems.

Eg: $\int (y(t) - x(t))^2 \, dt$ could be a possible measure.

* Performance of a digital system is easier to study
and optimal designs are possible.

- Probability of error
- Rate (bits/sec) for a given prob. of error.

* Typical digital communication system.



Is the separation into source and channel encoders optimal?
- Not always
- For AWGN channels under some assumptions
- But it helps a great deal in near-optimal design / intuitive understanding.

* We will learn about analog communications first. Even though most modern systems are digital, what we learn about analog communication is still useful. It does form a part of the overall digital communication system.

$\boxed{\text{Lecture 2 (12 Jan 16)}}$

* Signals and systems review with a focus on how to apply to communication systems.

  - Classify signals / systems based on the frequency band they occupy

  - Review of
    - complex numbers
    - common signals we will encounter
    - LTI systems
    - Fourier series
    - Fourier transforms
    - Bandwidth
    - Baseband and passband signals
    - Complex baseband representation of passband signals.

## Complex numbers:

Complex number $z = x + jy$, $x, y$ are real numbers
$j = \sqrt{-1}$.

$x = Re(z)$

$y = Im(z)$

In Polar form, $z = r e^{j\theta}$

$r = |z| = \sqrt{x^2 + y^2}$

$\theta = \angle z = \tan^{-1}\left(\frac{y}{x}\right)$.

$x = r\cos\theta$ ; $y = r\sin\theta$.

Complex conjugate of $z$ is $z^* = x - jy$.
$= r e^{-j\theta}$

Note that $Re(z) = \frac{z + z^*}{2}$, $Im(z) = \frac{z - z^*}{2j}$.

Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$.

Addition of complex numbers: Easy if complex no.s are represented in rectangular form.

Multiplication of complex numbers: Easy in polar form

ision $\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{z_1 z_2^*}{|z_2|^2} = \frac{(x_1 x_2 + y_1 y_2) + j(-x_1 y_2 + y_1 x_2)}{x_2^2 + y_2^2}$

also easy in polar form.

* Euler's formula can be used to quickly derive

eg: To get $\cos(\theta_1+\theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$,
consider $e^{j(\theta_1+\theta_2)}$

$$\cos(\theta_1+\theta_2) = Re\left[e^{j(\theta_1+\theta_2)}\right]$$
$$= Re\left[e^{j\theta_1} \cdot e^{j\theta_2}\right]$$
$$= Re\left[(\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2)\right]$$
$$= \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 .$$

$\boxed{\text{Signals}}$ (Some useful examples) & (notation)

Continuous time signals : $x(t)$, $s(t)$.
Discrete time signals : $x[n]$
$(x(nT_s + t_0) \triangleq x[n])$.

(1) Sinusoid of frequency $f_0$

$$s(t) = A \cos(2\pi f_0 t + \theta)$$
where $A > 0$, $\theta \in [0, 2\pi]$.

$$s(t) = \underbrace{(A\cos\theta)}_{\triangleq A_c} \cos 2\pi f_0 t - \underbrace{(A\sin\theta)}_{\triangleq A_s} \sin(2\pi f_0 t)$$

In communication systems, $A, \theta$ are varied to convey "messages".

(2) Complex exponential at frequency $f_0$.

$$s(t) = A e^{j(2\pi f_0 t + \theta)} \quad [A > 0, \theta \in [0, 2\pi])$$
$$= \alpha e^{j 2\pi f_0 t} \quad \text{where } \alpha = A e^{j\theta}.$$

* Amplitude and phase are both captured by $\alpha$

$$Re\left(Ae^{j(2\pi f_0 t+\theta)}\right) = A\cos(2\pi f_0 t+\theta).$$

* $\{e^{j2\pi ft}\}_{f\in(-\infty,\infty)}$ forms a "basis" for a large class of signals that are of interest to us (will be discussed later - Fourier transform).

(3) <u>Unit impulse (or) delta function</u>

Remember the following properties

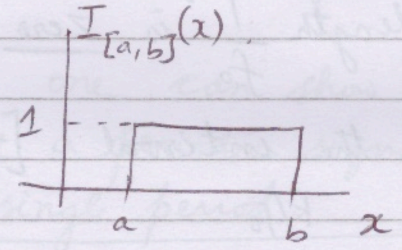
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$\int_{-\infty}^{\infty} s(t)\,\delta(t-t_0)\,dt = s(t_0).$$

(4) <u>Indicator function</u>

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise.} \end{cases}$$

e.g.

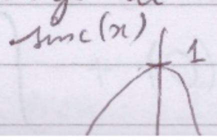



(5) <u>Sinc function</u>

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{(\pi x)}$$

At $x=0$, $\operatorname{sinc}(x) = \lim_{x\to 0} \frac{\sin \pi x}{\pi x} = 1$

zero crossings at $\pm 1, \pm 2, \ldots$



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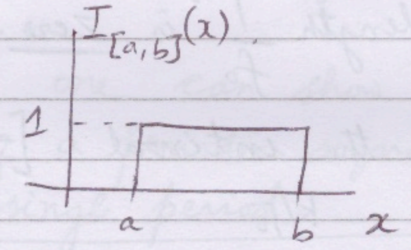


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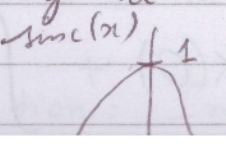


## Time average of a signal

For a function $g(t)$, define the time average as

$$\bar{g} = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g(t)\, dt .$$

(compute the average over a window $T_0$ and let $T_0$ become large).

Eg: $\bar{s}$ where $s(t) = A e^{j(2\pi f_0 t + \theta)}$.

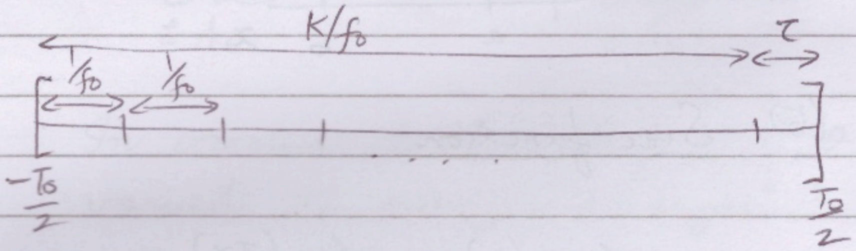
Claim: $\bar{s} = 0$.

Proof:

$s(t)$ is periodic with period $\frac{1}{f_0}$.

& The integral of $s(t)$ over any interval of length $\frac{1}{f_0}$ is zero.

Consider the interval $\left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$.




Express $T_0 = \frac{K}{f_0} + \tau$ where $K$ is an integer $> 0$ and $0 \le \tau \le \left(\frac{1}{f_0}\right)$

$$\int^{T_0/2} s(t)\, dt = K(0) + \int^{T_0/2} s(t)\, dt = \int^{T_0/2} s(t)\, dt .$$

Now, $-\int_{\frac{T_0}{2}-\tau}^{\frac{T_0}{2}} |s(t)|\,dt \leq \int_{\frac{T_0}{2}-\tau}^{\frac{T_0}{2}} s(t)\,dt \leq \int_{\frac{T_0}{2}-\tau}^{\frac{T_0}{2}} |s(t)|\,dt$

$$\int_{\frac{T_0}{2}-\tau}^{T_0/2} |s(t)|\,dt \leq \tau\left[\max_t |s(t)|\right] = \tau A \cdot \leq \frac{A}{f_0}$$

Therefore, we have

$$\frac{-A}{f_0} \leq \int_{\frac{T_0}{2}-\tau}^{\frac{T_0}{2}} s(t)\,dt \leq \frac{A}{f_0}$$

$$\Rightarrow \underbrace{-\lim_{T_0\to\infty} \frac{A}{f_0 T_0}}_{=0} \leq \overline{s} \leq \underbrace{\lim_{T_0\to\infty} \frac{A}{f_0 T_0}}_{=0}$$

$$\Rightarrow \overline{s} = 0.$$

Similarly, one can show that the time average of a periodic signal equals the average over a single period.

$\boxed{\text{Lecture 3 (14 Jan 16)}}$

Even though signals can be complicated function of time that live in an infinite-dimensional space, the mathematics for manipulating them is very similar to that for manipulating finite-dimensional vectors (sums are replaced by integrals).

Key definition: Inner product.

Just like we can think of vectors in a space like $\mathbb{R}^n$ (n-dimeasional real space) we can think of signals in a "signal space" as vectors. (particularly useful in digital communication)

Inner product for two $m \times 1$ complex vectors

$$\underline{s} = \begin{pmatrix} s[1] \\ \vdots \\ s[m] \end{pmatrix} \qquad \underline{r} = \begin{pmatrix} r[1] \\ \vdots \\ r[m] \end{pmatrix}.$$

Inner product $$\langle \underline{s}, \underline{r} \rangle = \sum_{i=1}^{m} s[i] r^*[i] = \underline{r}^H \underline{s}.$$

Inner product of two possibly complex signals

$$\langle s, r \rangle = \int_{-\infty}^{\infty} s(t) r^*(t) \, dt$$

The inner product obeys the following linearity properties.

$$\langle a_1 s_1 + a_2 s_2, r \rangle = a_1 \langle s_1, r \rangle + a_2 \langle s_2, r \rangle.$$

$$\langle s, a_1 r_1 + a_2 r_2 \rangle = a_1^* \langle s, r_1 \rangle + a_2^* \langle s, r_2 \rangle.$$

Energy and norm of a signal

* Energy $E_s$ of a signal $s$

$$= \int_{-\infty}^{\infty} |s(t)|^2 \, dt = \langle s, s \rangle = \|s\|^2$$

is the inner product of $s$ with itself.

* $E_s = 0 \implies s$ must be zero "almost everywhere" (e.g. cannot be non-zero over any interval however small the interval).

For finite-dimensional vectors $\underline{s}$

$$E_s = \sum_i |s[i]|^2 = 0 \implies \underline{s} = \underline{0}.$$

For continuous-time signals, we will take $\|s\|^2 = 0$ to imply that $s = 0$ everywhere

Eg. (Energy computations)

(1) $s(t) = 2\, I_{[0,T)} + j\, I_{[T/2, 2T)}$

$$\text{i.e. } s(t) = \begin{cases} 2 & 0 \le t < \frac{T}{2} \\ 2+j & \frac{T}{2} \le t < T \\ j & T \le t < 2T \end{cases}$$

$$E_s = \int_0^{T/2} 2^2\, dt + \int_{T/2}^{T} |2+j|^2\, dt + \int_T^{2T} |j|^2\, dt$$

$$= 4\left(\frac{T}{2}\right) + 5\left(\frac{T}{2}\right) + T = \frac{11T}{2}.$$

(2) $s(t) = e^{-3|t| + j2\pi t}$

$$E_s = \int_{-\infty}^{\infty} \left| e^{-3|t| + j2\pi t} \right|^2 dt = \int_{-\infty}^{\infty} e^{-6|t|}\, dt = 2\int_0^{\infty} e^{-6t}\, dt$$

$$= \frac{1}{\cdot}.$$

## Power of a signal $s(t)$

is defined as the time average of its energy computed over a large time interval.

$$P_s = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |s(t)|^2 \, dt$$

$$= \overline{|s(t)|^2} \quad \text{(Average value of } |s(t)|^2\text{)}.$$

Note: Finite energy signals have zero power, i.e.,

$$E_s \text{ finite} \Rightarrow P_s = 0.$$

Eg (1) Suppose $s(t) = A e^{j(2\pi f_0 t + \theta)}$.

$$|s(t)|^2 = A^2$$

$$\Rightarrow P_s = A^2.$$

(2) Suppose $s(t) = A \cos(2\pi f_0 t + \theta)$.

$$\overline{s} = 0.$$

$$P_s = \overline{A^2 \cos^2(2\pi f_0 t + \theta)}$$

$$= \overline{\frac{A^2}{2} + \frac{A^2}{2}\cos(4\pi f_0 t + 2\theta)}$$

$$= \frac{A^2}{2}. \quad \left(\text{since } \overline{s_1 + s_2} = \overline{s_1} + \overline{s_2} \text{ \& } \overline{s_2} \text{ is } 0 \text{ in this case}\right)$$

# Systems

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

A system is specified once we characterize its input-output relationship, i.e, if we can determine output $y(t)$ for any possible input $x(t)$ in a given class of signals of interest.

## Linear systems

Let $x_1(t)$ and $x_2(t)$ be arbitrary input signals.
Let $y_1(t)$ and $y_2(t)$ be the corresponding system outputs, respectively.

A system is linear if for arbitrary scalars $a_1, a_2$ (complex scalars), the output corresponding to $a_1 x_1(t) + a_2 x_2(t)$ is $a_1 y_1(t) + a_2 y_2(t)$.

## Time-invariant systems

Let $x(t)$ be an arbitrary input signal.
Let $y(t)$ be the corresponding output signal.

A system is time-invariant if for each $t_0$ the output corresponding to $x(t-t_0)$ is $y(t-t_0)$

## Examples

Check if Linear, time-invariant

$y(t) = 2x(t-1) - jx(t-2)$

$y(t) = \int^{t+1} x(\tau)\,d\tau$

## Lecture 4 (18 Jan 2016)

A linear time-invariant (LTI) system has an input-output relationship that is easy to describe. If we know the response of an LTI system to the input $\delta(t)$, then we can compute the output for <u>any</u> input signal. The response to $\delta(t)$ is referred to as the system's <u>impulse response</u>.

– $$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

where $h(t)$ denotes the impulse response of the LTI system.

– $y(t)$ is nothing but the <u>convolution</u> of signals $x(t)$ and $h(t)$. Sometimes, we denote the above equ. as $y(t) = (x * h)(t)$

(or) $y = x * h$

(or) $y(t) = x(t) * h(t)$.

– Note that $$\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = -\int_{\infty}^{-\infty} x(t-\nu) h(\nu) d\nu$$

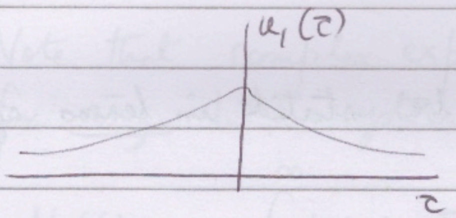
(let $t - \tau = \nu$)

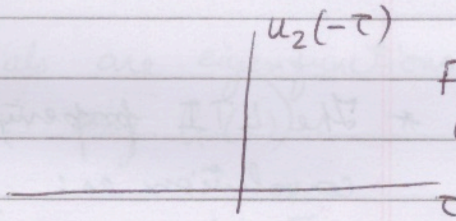
$$= \int_{-\infty}^{\infty} x(t-\tau) h(\nu) d\nu$$

Thus $x * h = h * x$.

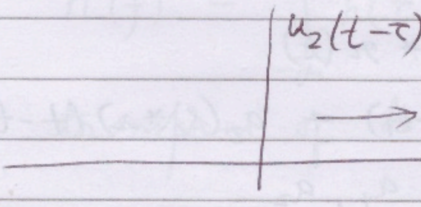
Computing the convolution:

$$v(t) = \int_{-\infty}^{\infty} u_1(\tau)\, u_2(t-\tau)\, d\tau$$

1. 



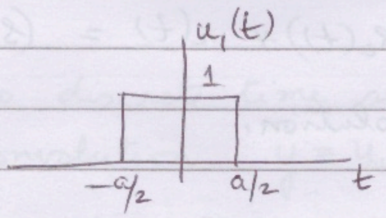
Flip $u_2(\cdot)$.

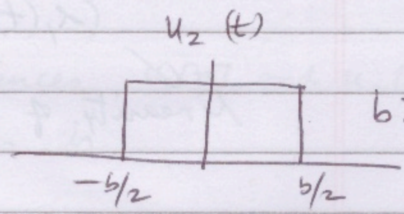
2. 

shift $u_2(-\tau)$ to the right by $t$.

3. Multiply $u_1(\tau)\, u_2(t-\tau)$ & integrate over $\tau$ to get $v(t)$

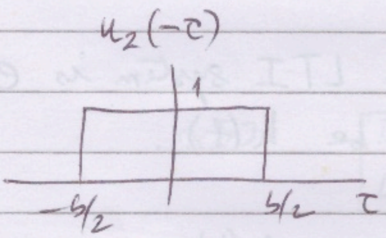
4. Repeat for each $t$.

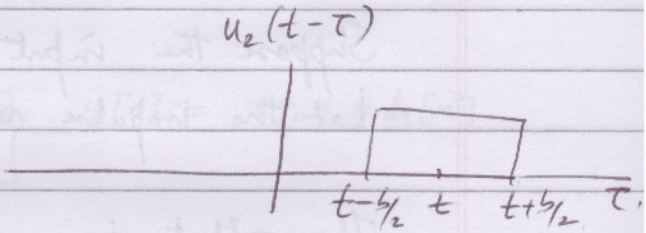
Example: 



$b > a$.

Find $v(t) = (u_1 * u_2)(t)$.

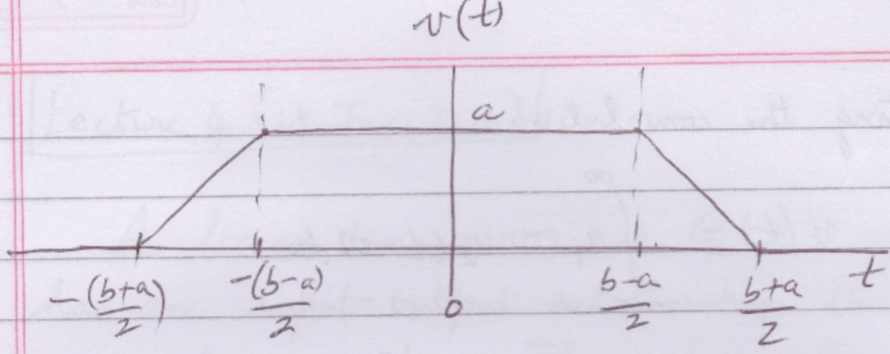





For $t + \frac{b}{2} < -\frac{a}{2}$ (or) $t - \frac{b}{2} > \frac{a}{2}$, $v(t) = 0$.

For $-\frac{a}{2} < t + b/2 < \frac{a}{2}$, $v(t)$ increases linearly with $t$.

For $\frac{a}{2} < t + \frac{b}{2}$ and $t - \frac{b}{2} < -\frac{a}{2}$, $v(t) = a$.




* The LTI property can be stated in terms of convolution as:

$$(a_1 s_1(t-t_1) + a_2 s_2(t-t_2)) * r(t)$$
$$= a_1 (s_1 * r)(t-t_1) + a_2 (s_2 * r)(t-t_2)$$

for any complex gains $a_1, a_2$ and any offsets $t_1$ and $t_2$.

Note that $\underbrace{s_1(t-t_1) * r(t)}_{\text{one of the signals that is convolved is delayed.}} = \underbrace{(s_1 * r)(t-t_1)}_{\text{output delayed}}$ $\Longleftrightarrow$

$$(s_1(t) + s_2(t)) * r(t) = (s_1 * r)(t) + (s_2 * r)(t)$$

linearity of convolution.

## Complex exponential through an LTI system

Suppose the input to an LTI system is $e^{j2\pi f_0 t}$. Let the impulse response be $h(t)$.

The output is $e^{j2\pi f_0 t} * h(t)$

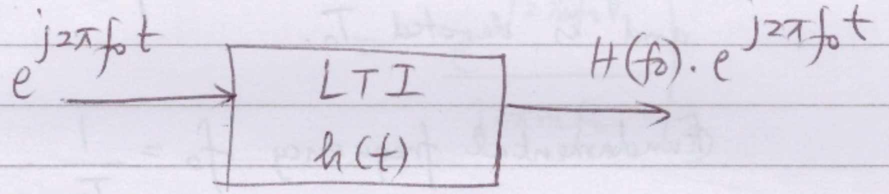
$$= \int_{-\infty}^{\infty} h(\tau) e^{j2\pi f_0 (t-\tau)} d\tau$$

$$= e^{j2\pi f_0 t} \left[ \int_{-\infty}^{\infty} h(\tau) e^{-j2\pi f_0 \tau} d\tau \right]$$

The output is also a complex exponential at the same frequency $f_0$. Such a function is called an <u>eigenfunction</u> of the LTI system. The corresponding <u>eigenvalue</u> is $H(f_0)$.

Note that complex exponentials are eigenfunctions for <u>any</u> LTI system (i.e. any $h(t)$).

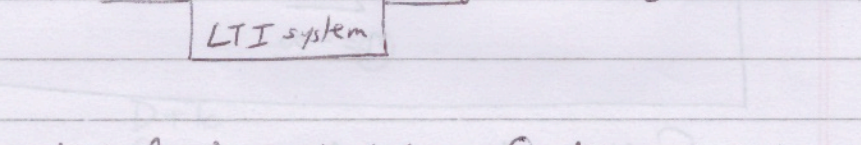
$$H(f) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j2\pi f\tau}\, d\tau$$ is the Fourier transform of $h(t)$.




## <u>Discrete-time convolution</u>

For two discrete-time sequences $u_1[n]$ and $u_2[n]$, their convolution $y = u_1 * u_2$ is

$$y[n] = \sum_{k=-\infty}^{\infty} u_1[k]\, u_2[n-k]$$




[ Reading material for self study: Sections 2.3.1 and 2.3.2.

- Discrete-time computation of continuous-time convolution
- Upsampling for digital modulation as an eg. of a multirate system ]

## Fourier Series:

* Periodic signals

$\left[\begin{array}{c} u(T) \text{ is periodic} \\ \text{with period } T \end{array}\right]$ if $u(t+T) = u(t) \ \forall t$

If $u(t)$ is periodic with period $T$, it is also periodic with period $nT$ where $n$ is any +ve integer.

The smallest interval for which $u(t)$ is periodic is called the <u>fundamental period</u>. and is devoted $T_0$.

Fundamental frequency $f_0 = \dfrac{1}{T_0}$.

If $u(t)$ is periodic with period $T$, then $T$ <u>must</u> be an integer multiple of $T_0$.

Lecture 5: 19 Jan 2016

Any periodic signal $u(t)$ with period $T_0$ (subject to some technical conditions) can be expressed as a linear combination of complex exponentials as

$$u(t) = \sum_{n=-\infty}^{\infty} u_n e^{j2\pi n f_0 t}$$

Sufficient conditions are available for existence of Fourier series: Dirichlet conditions

* $\int_{\text{over one period}} |\ | dt < \infty$

* Finite no. of discontinuities in one period

The series on the RHS converges to the function value at all pts. (i.e all $t$) where the function $u(t)$ is continuous.

Let $\psi_m(t) = e^{j2\pi m f_0 t}$, $m = 0, \pm 1, \pm 2, \ldots$

Observe that (for $m \neq 0$)

$$\int_D^{D+T_0} \psi_m(t)\,dt = \int_D^{D+T_0} e^{j2\pi m f_0 t}\,dt$$

$$= \left.\frac{e^{j2\pi m f_0 t}}{j2\pi m f_0}\right|_D^{D+T_0}$$

$$= \frac{e^{j2\pi m f_0 (D+T_0)} - e^{j2\pi m f_0 D}}{j2\pi m f_0}$$

$$= \frac{e^{j2\pi m f_0 D}\left(e^{j2\pi m f_0 T_0} - 1\right)}{j2\pi m f_0}$$

$$= \frac{e^{j2\pi m f_0 D}\left(e^{j2\pi m} - 1\right)}{j2\pi m f_0}$$

$$= 0.$$

Consider $\int_D^{D+T_0} \psi_n(t)\,\psi_k^*(t)\,dt$.

(ORTHOGONALITY)

$$\int_D^{D+T_0} \psi_n(t)\,\psi_k^*(t)\,dt = \int_D^{D+T_0} e^{j2\pi(n-k)f_0 t}\,dt = \begin{cases} 0 & n \neq k \\ T_0 & n = k. \end{cases}$$

Therefore, we can get

$$\int_D^{D+T_0} u(t)\underbrace{e^{-j2\pi k f_0 t}}_{\psi_k^*(t)}\,dt = \int_D^{D+T_0} \left(\sum_{n=-\infty}^{\infty} u_n e^{j2\pi n f_0 t}\right)\psi_k^*(t)\,dt$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{\int_D^{D+T_0} u_n \psi_n(t)\psi_k^*(t)\,dt}_{= \begin{cases} u_k T_0 & n=k \\ 0 & n\neq k. \end{cases}}$$

$$= u_k T_0$$

$$\Rightarrow \boxed{u_k = \frac{1}{T_0}\int_D^{D+T_0} u(t)e^{-j2\pi k f_0 t}\,dt}$$

$\{u_k\}$ for a given $u(t)$ can be obtained using this relation.

$$u(t) = \sum_{n=-\infty}^{\infty} u_n \psi_n(t)$$

(Inner product).

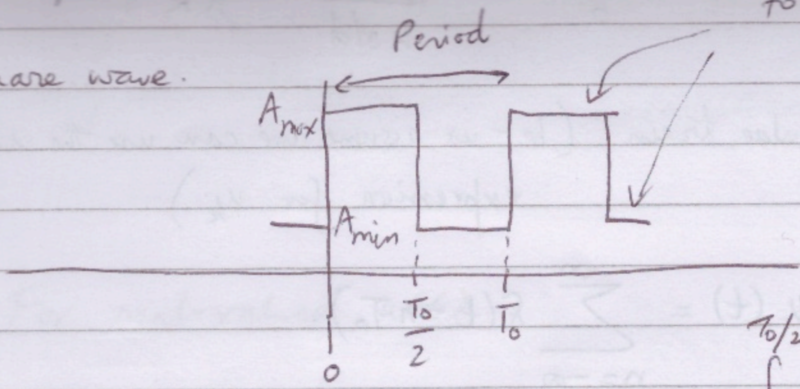
<u>Defn</u> $\langle u, v\rangle_{T_0} = \int_{T_0} u(t)v^*(t)\,dt$

$$\langle u, \psi_k\rangle_{T_0} = u_k \|\psi_k\|_{T_0}^2$$

$$u_k = \frac{\langle u, \psi_k\rangle_{T_0}}{\|\psi_k\|_{T_0}^2} = \frac{\langle u, \psi_k\rangle_{T_0}}{T_0}.$$

## Examples:

(i) Square wave.




The series will sum to $u(t)$ at all pts where $u(t)$ is continuous (in the flat portions here).

$$u_k = \int_{-T_0/2}^{T_0/2} u(t) e^{-j2\pi k f_0 t} dt$$

$$u_0 = \frac{A_{max} + A_{min}}{2}$$

$$u_k = \frac{1}{T_0}\int_{-T_0/2}^{0} A_{min} e^{-j\frac{2\pi k t}{T_0}} dt + \frac{1}{T_0}\int_{0}^{T_0/2} A_{max} e^{-j\frac{2\pi k t}{T_0}} dt$$

$$= \frac{A_{min}}{T_0} \left. \frac{e^{-j2\pi k t/T_0}}{-j2\pi k/T_0} \right|_{-T_0/2}^{0} + \frac{A_{max}}{T_0} \left. \frac{e^{-j2\pi k t/T_0}}{-j2\pi k/T_0} \right|_{0}^{T_0/2}$$

$$= \frac{A_{min}\left(1 - e^{+j\pi k}\right)}{-j2\pi k} + \frac{A_{max}\left(e^{-j\pi k} - 1\right)}{-j2\pi k}$$

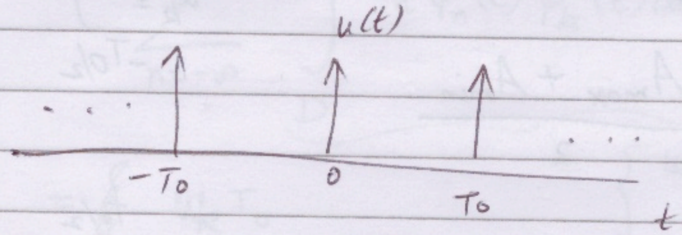
$$= \begin{cases} 0 & k \text{ even} \\ \dfrac{A_{max} - A_{min}}{j\pi k} & k \text{ odd} \end{cases}$$

(since $e^{j\pi k} = e^{-j\pi k} = 1$ for $k$ even
& $e^{-j\pi k} = e^{j\pi k} = -1$ for $k$ odd)

$$u(t) = \frac{A_{max} + A_{min}}{2} + \sum_{k \text{ odd}} \frac{2(A_{max} - A_{min})}{\pi k} \cancel{\sin} \left( \frac{2\pi k t}{T_0} \right).$$

(2) Impulse train (let us assume we can use the same expression for $u_k$).

$$u(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$




$$u_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} u(t) \, e^{-j2\pi f_0 k t} \, dt$$

$$= \frac{1}{T_0}.$$

<u>Properties</u> (Some, without proof).

(1) Let $u(t)$ & $v(t)$ be periodic with period $T_0$

Let $u(t) \longleftrightarrow \{u_k\}$

$v(t) \longleftrightarrow \{v_k\}$.

<u>Linearity</u> Then, for arbitrary complex no.s $\alpha$ and $\beta$,

$$\alpha u(t) + \beta v(t) \longleftrightarrow \{\alpha u_k + \beta v_k\}.$$

(2) $u(t-d) \longleftrightarrow \left\{ u_k e^{-j2\pi k f_0 d} \right\}$

delay in time domain

linear phase in freq. domain

(3) For real-valued $u(t)$

$$u_{-k} = u_k^*$$

$$\Rightarrow u(t) = u_0 + \sum_{k=1}^{\infty} u_k e^{j2\pi k f_0 t} + \sum_{k=1}^{\infty} u_{-k} e^{-j2\pi k f_0 t}$$

$$= u_0 + \sum_{k=1}^{\infty} 2|u_k| \cos(2\pi k f_0 t + \angle u_k).$$

(4) $x(t) = \frac{d}{dt} u(t) \longleftrightarrow x_k = j2\pi k f_0 u_k.$

$$\left(\text{Note: } x_0 = 0\right).$$

[ This property can be used to repeat example 1 (square wave) using the result of example 2 (impulse train)

$$x(t) = \frac{d}{dt} u(t) = (A_{max} - A_{min}) \left( \sum_k \delta(t - kT_0) - \sum_k \delta\left(t - kT_0 - \frac{T_0}{2}\right) \right)$$

$$\Rightarrow x_k = \frac{A_{max} - A_{min}}{T_0} - \frac{A_{max} - A_{min}}{T_0} \cdot e^{-j2\pi k f_0 \frac{T_0}{2}}$$

$$= \frac{A_{max} - A_{min}}{T_0} \left(1 - e^{-j\pi k}\right), \quad k \neq 0.$$

$$\Rightarrow u_k = \frac{x_k}{j2\pi f_0 k} = \frac{A_{max} - A_{min}}{-j \underbrace{2\pi k f_0 T_0}_{2\pi k}} \left(1 - e^{-j\pi k}\right), \quad k \neq 0$$ ]

(5) Parseval's identity

$$\boxed{\frac{1}{T_0}\int_{T_0} |u(t)|^2 dt = \sum_{k=-\infty}^{\infty} |u_k|^2 .}$$

Note: $\langle u, v\rangle_{T_0} = \int_{T_0} u(t)\, v^*(t)\, dt = T_0 \sum_{k=-\infty}^{\infty} u_k v_k^*$

Setting $u = v$, gives the previous expression
This is a more general relation.

$\boxed{\text{Lecture 6: 21 Jan 2016}}$

Fourier transform:

For an aperiodic, finite-energy waveform $u(t)$, the Fourier transform $U(f)$ is defined as

$$U(f) = \int_{-\infty}^{\infty} u(t)\, e^{-j2\pi f t} dt \qquad -\infty < f < \infty.$$

The inverse Fourier transform is given by

$$u(t) = \int_{-\infty}^{\infty} U(f)\, e^{j2\pi f t} df \qquad -\infty < t < \infty.$$

Any finite-energy signal can be written as a linear combination of a continuum of complex exponentials.

Notation: $u(t) \longleftrightarrow U(f)$

$U(f) = \mathcal{F}(u(t)).$

Example:

$$u(t) = I_{[-\frac{T}{2}, \frac{T}{2}]}(t).$$

$$U(f) = \int_{-T/2}^{T/2} e^{-j2\pi f t}\, dt$$

$$= \left. \frac{e^{-j2\pi f t}}{-j2\pi f} \right|_{-T/2}^{T/2}$$

$$= \frac{e^{-j\cancel{2}\pi f T} - e^{j\pi f T}}{-j2\pi f} = \frac{\sin(\pi f T)}{\pi f}$$

$$= T\ \mathrm{sinc}(fT).$$

$$I_{[-\frac{T}{2}, \frac{T}{2}]}(t) \longleftrightarrow T\ \mathrm{sinc}(fT).$$

Rectangular pulse      Sinc function.

Relation to Fourier series:

- Can think of the Fourier series as the period $T_0 \to \infty$.

Application to infinite-energy signals (especially delta functions)

- We will use these signals to ease modeling even though they are not realizable.

* Suppose

$u(t) = \delta(t)$

$$U(f) = \int_{-\infty}^{\infty} u(t) e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt = e^{-j2\pi f(0)} = 1.$$

$$\boxed{\delta(t) \longleftrightarrow 1} \triangleq I_{(-\infty,\infty)}(f).$$

* Suppose $U(f) = \delta(f - f_0)$.

$$u(t) = \int_{-\infty}^{\infty} \delta(f - f_0) e^{j2\pi f t} df = e^{j2\pi f_0 t}$$

$$\boxed{e^{j2\pi f_0 t} \longleftrightarrow \delta(f - f_0)}$$

* Fourier series in terms of Fourier transform

$$\underset{\substack{\text{Periodic} \\ T_0}}{u(t)} = \sum_{n=-\infty}^{\infty} u_n e^{j2\pi n f_0 t}.$$

$\downarrow$ Take Fourier transform

$$U(f) = \sum_{n=-\infty}^{\infty} \underbrace{u_n}_{\substack{\text{Coefficients} \\ \text{given by Fourier series.}}} \underbrace{\delta(f - n f_0)}_{\text{impulses at } nf_0.}$$

## Fourier transform properties:

Let $u(t) \Leftrightarrow U(f)$, $v(t) \Leftrightarrow V(f)$.

(1) Linearity

For arbitrary complex numbers $\alpha$ and $\beta$

$$\alpha u(t) + \beta v(t) \Longleftrightarrow \alpha U(f) + \beta V(f).$$

(2) Duality

If $u(t) \Longleftrightarrow U(f)$,

then $U(t) \Longleftrightarrow u(-f)$.

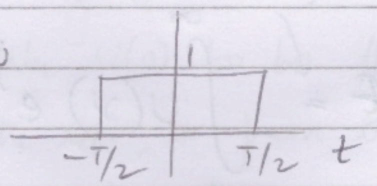
Proof: $$\mathcal{F}[U(t)] = \int_{-\infty}^{\infty} U(t) e^{-j2\pi f t} dt$$

We know $$\int_{-\infty}^{\infty} u(t) e^{-j2\pi f t} dt = U(f)$$

& $$\underbrace{\int_{-\infty}^{\infty} U(f) e^{j2\pi f t} df = u(t)}$$

$$\mathcal{F}[U(t)] = \int_{-\infty}^{\infty} U(t) e^{-j2\pi f t} dt = u(-f).$$

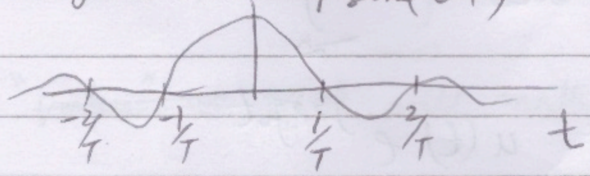
We know



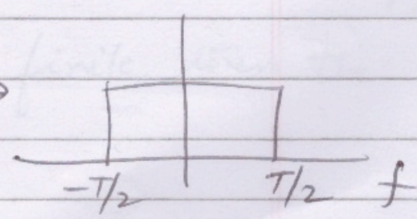

$\Longleftrightarrow T \,\text{sinc}(fT)$

Using duality

$T \,\text{sinc}(tT)$




$\Longleftrightarrow$



3) $$\underbrace{u(t-t_0)}_{\text{Time delay}} \longleftrightarrow U(f)\,\underbrace{e^{-j2\pi f t_0}}_{\text{Linear phase term}}$$

Proof : $$\mathcal{F}[u(t-t_0)] = \int_{-\infty}^{\infty} u(t-t_0)\, e^{-j2\pi f t}\, dt$$

$$[\text{Let } \tau = t - t_0] \quad = \int_{-\infty}^{\infty} u(\tau)\, e^{-j2\pi f(\tau+t_0)}\, d\tau$$

$$= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} u(\tau)\, e^{-j2\pi f \tau}\, d\tau$$

$$= e^{-j2\pi f t_0}\, U(f).$$

4) $$\underbrace{U(f-f_0)}_{\substack{\text{Frequency shift} \\ \text{by } f_0}} \longleftrightarrow \underbrace{u(t)\, e^{j2\pi f_0 t}}_{\substack{\text{modulation by complex} \\ \text{exponential} \\ \text{at freq } f_0.}}$$

Proof : Method (1) Use duality and time shift properties.

Method (2)
We know $\int_{-\infty}^{\infty} U(f)\, e^{j2\pi f t}\, df = u(t)$.

(Let $\nu = f - f_0$).

$$\int_{-\infty}^{\infty} U(f-f_0)\, e^{j2\pi f t}\, df = \int_{-\infty}^{\infty} U(\nu)\, e^{j2\pi(\nu+f_0)t}\, d\nu$$

$$= e^{j2\pi f_0 t} \int_{-\infty}^{\infty} U(\nu)\, e^{j2\pi \nu t}\, d\nu$$

$$= u(t)\, e^{j2\pi f_0 t}.$$

(5) If $u(t)$ is real, then $U(f) = U^*(-f)$.

Proof : (Exercise).

(6) $$\underbrace{x(t) = \frac{d}{dt} u(t)}_{\text{differentiation}} \Longleftrightarrow X(f) = j2\pi f \, U(f).$$

$\boxed{\text{Lecture 7: 28 Jan 2016}}$

Knowing the Fourier transform of $\frac{d}{dt}u(t)$, we can recover the Fourier transform of $u(t)$ for all $f$ except $f=0$.

$$U(f) = \frac{X(f)}{j2\pi f} \quad \text{for } f \neq 0.$$

$$U(f) = \int_{-\infty}^{\infty} u(t) e^{-j2\pi f t} dt \Rightarrow U(0) = \int_{-\infty}^{\infty} u(t)\, dt.$$

If $U(0)$ is finite, we can just take $U(f) = \frac{X(f)}{j2\pi f}$.

(and take $U(0)$ to be the limit of $U(f)$ as $f \to 0$ from the left, or from the right, or some number in between). The difference will have zero-energy. However, when $\int_{-\infty}^{\infty} u(t)\, dt$ is not finite, then the "~~value" at~~ $U(0)$ will matter.

One way to handle this:

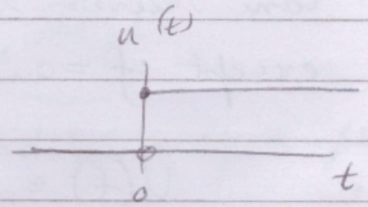
Suppose $\bar{u} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} u(t)\,dt$ is the average value of $u(t)$. (This may be finite even if $\int_{-\infty}^{\infty} u(t)\,dt$ is not, e.g., periodic signals)

Then, $\bar{u}(t) \triangleq \bar{u} \longleftrightarrow \bar{u}\,\delta(f)$. We can write

$$\boxed{U(f) = \frac{X(f)}{j2\pi f} + \bar{u}\,\delta(f)}.$$

Example:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$




Note: $\int_{-\infty}^{\infty} u(t)\,dt$ does not exist.

$$\bar{u} = \lim_{T \to \infty} \frac{1}{T}\left[\frac{T}{2}\right] = \frac{1}{2},$$

$$x(t) = \frac{d}{dt} u(t) = \delta(t) \longleftrightarrow 1.$$

$$U(f) = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f).$$

(7) Parseval's identity:

$$\langle u, v \rangle = \int_{-\infty}^{\infty} u(t)\, v^*(t)\,dt = \int_{-\infty}^{\infty} U(f)\, V^*(f)\,df$$

Setting $v = u$, we get

$$\int_{-\infty}^{\infty} |u(t)|^2 dt = \int_{-\infty}^{\infty} |U(f)|^2 df \triangleq \|u\|^2 .$$

Energy in time domain      Energy in freq. domain

Proof:

$$u(t) = \int_{-\infty}^{\infty} U(f) e^{j2\pi f t} df$$

$$v(t) = \int_{-\infty}^{\infty} V(f') e^{j2\pi f' t} df'$$

$$u(t) v^*(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(f) V(f') e^{j2\pi(f-f')t} df\, df'.$$

$$\int_{-\infty}^{\infty} u(t) v^*(t) dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(f) V(f') \left[\int_{-\infty}^{\infty} e^{j2\pi(f-f')t} dt\right] df\, df'$$

Assume interchange of integrals allowed.

$$\int_{-\infty}^{\infty} e^{j2\pi(f-f')t} dt = \delta(f-f')$$

$$\Rightarrow \int_{-\infty}^{\infty} u(t) v^*(t) dt = \int_{-\infty}^{\infty} U(f) V^*(f) df .$$

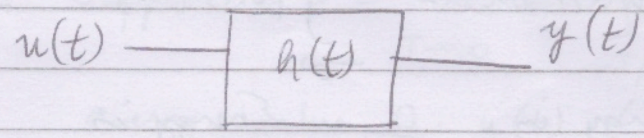
---

(8) Transfer function of an LTI system:

We know

$$e^{j2\pi f_0 t} \longrightarrow \boxed{\begin{matrix} LTI \\ h(t) \end{matrix}} \longrightarrow e^{j2\pi f_0 t} \underbrace{\left[\int_{-\infty}^{\infty} h(\tau) e^{-j2\pi f_0 \tau} d\tau\right]}_{H(f_0)}$$

Transfer function $H(f) \triangleq \mathcal{F}[h(t)]$




$$u(t) = \int_{-\infty}^{\infty} U(f) e^{j2\pi f t} df$$

$$y(t) = \int_{-\infty}^{\infty} \underbrace{Y(f) e^{j2\pi f t}}_{\text{should be } U(f) H(f) e^{j2\pi f t}} df$$

(Using linearly & $e^{j2\pi f t}$ as eig. fn.).

(or) $Y(f) = U(f) H(f)$.

Convolution in time domain $\equiv$ Multiplication in freq. domain.

$$\boxed{y(t) = (u * h)(t) \longleftrightarrow Y(f) = U(f) H(f).}$$

Proof:

$$y(t) = \int_{-\infty}^{\infty} u(\tau) h(t-\tau) d\tau$$

$$Y(f) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} u(\tau) h(t-\tau) d\tau \right] e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} u(\tau) \Big( \underbrace{\int_{-\infty}^{\infty} h(t-\tau) e^{-j2\pi f t} dt}_{H(f) e^{-j2\pi f \tau}} \Big) d\tau$$

$$= \int_{-\infty}^{\infty} u(\tau) H(f) e^{-j2\pi f \tau} d\tau = H(f) \int_{-\infty}^{\infty} u(\tau) e^{-j2\pi f \tau} d\tau$$

$$= H(f) U(f).$$

By duality, we also have

$$\boxed{y(t) = u(t)v(t) \longleftrightarrow Y(f) = (U * V)(f).}$$

Multiplication in time domain — Convolution in freq. domain.

## $\boxed{\text{Lecture 8 : 29 Jan 2016}}$

<u>LTI system response to real-valued sinusoidal signals</u>

$$u(t) = \cos(2\pi f_0 t + \theta) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ \text{real } h(t)\end{matrix}} \longrightarrow y(t)$$

$$y(t) = (u * h)(t)$$

$$u(t) = \frac{1}{2} e^{j(2\pi f_0 t + \theta)} + \frac{1}{2} e^{-j(2\pi f_0 t + \theta)}$$

$$y(t) = \frac{H(f_0)}{2} e^{j(2\pi f_0 t + \theta)} + \frac{H(-f_0)}{2} e^{-j(2\pi f_0 t + \theta)}$$

Since $h(t)$ is real, $H(-f_0) = H^*(f_0)$

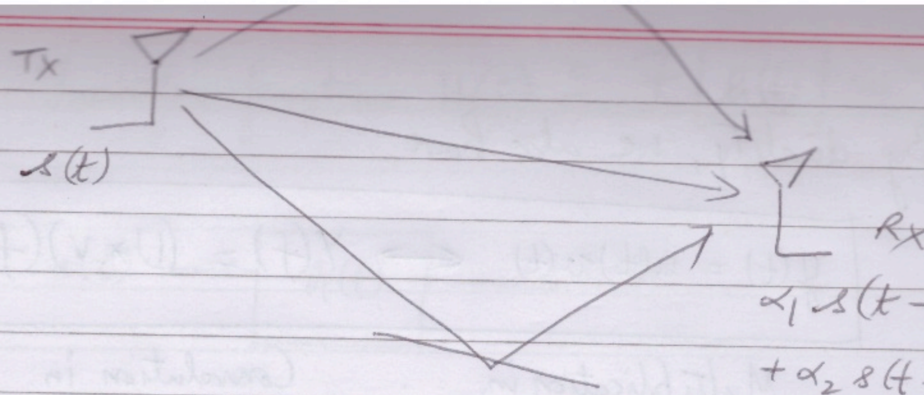
$$\Rightarrow y(t) = \operatorname{Re}\left(H(f_0) e^{j(2\pi f_0 t + \theta)}\right)$$

$$= \operatorname{Re}\left(|H(f_0)| e^{j(2\pi f_0 t + \theta + \angle H(f_0))}\right)$$

$$\boxed{y(t) = |H(f_0)| \cos(2\pi f_0 t + \theta + \angle H(f_0))}$$

<u>Example of a LTI system in communications</u>

– A multipath channel.




$$\alpha_1 s(t-\tau_1) + \alpha_2 s(t-\tau_2) + \dots + \alpha_M s(t-\tau_m).$$

$$s(t) \longrightarrow \boxed{h(t)} \longrightarrow \sum_{k=1}^{m} \alpha_k s(t-\tau_k)$$

$$\boxed{h(t) = \sum_{k=1}^{m} \alpha_k \delta(t-\tau_k).}$$ Channel Model.

We will allow both $s(t)$, $\{\alpha_k\}$ to be complex. Reason will be clear later.

$$H(f) = \alpha_1 e^{-j2\pi f \tau_1} + \alpha_2 e^{-j2\pi f \tau_2} + \dots + \alpha_m e^{-j2\pi f \tau_m}$$

Linear combination of complex exponentials in the frequency domain. These complex exponentials can constructively or destructively add leading to significant fluctuations as a fn. of $f$ in $H(f)$.

* In the context of wireless communication, the variation of $H(f)$ with $f$ is termed <u>frequency-selective fading</u>.

W.L.O.G. assume $\tau_1 < \tau_2 < \dots < \tau_m$.

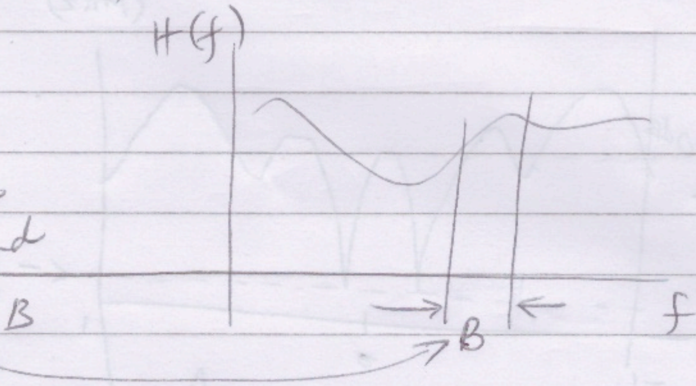
$$H(f) = \sum_{k=1}^{m} \alpha_k e^{-j2\pi f \tau_k}$$

$$= e^{-j2\pi f\tau_1} \sum_{k=1}^{m} \alpha_k e^{-j2\pi f(\tau_k-\tau_1)}$$

$$= \underbrace{e^{-j2\pi f\tau_1}}_{\text{corresponds to pure delay}} \Big[ \alpha_1 + \sum_{k=2}^{m} \alpha_k \underbrace{e^{-j2\pi f(\tau_k-\tau_1)}}_{\text{Period } \frac{1}{\tau_k-\tau_1}} \Big]$$

$$\underbrace{\text{Period } \frac{1}{\tau_k-\tau_1}}_{\text{Smallest period } \frac{1}{\tau_m-\tau_1}}$$

Define $\tau_d = \tau_m - \tau_1$ as the delay spread.




Consider freq. band of width B

Suppose B is significantly smaller than $\frac{1}{\tau_d}$

H(f) is approximately flat in that band.

Define $B_C = \frac{1}{\tau_d}$ as the coherence bandwidth.

Numerical example:

$$h(t) = \delta(t-1) + 0.5\,\delta(t-3.5) - 0.5\,\delta(t-1.5)$$

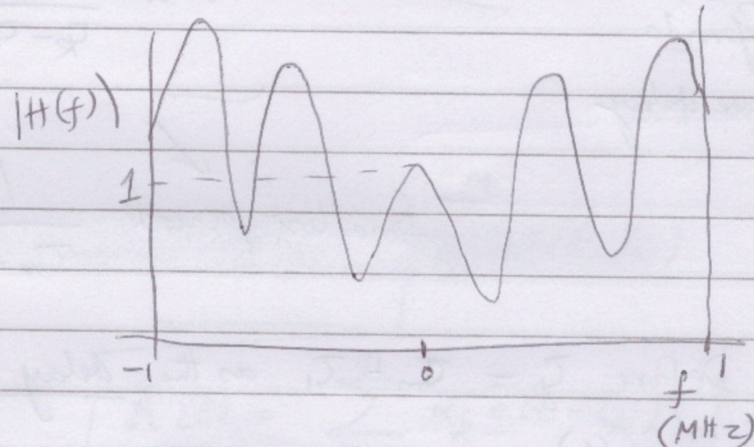
(Let units be $\mu s$)

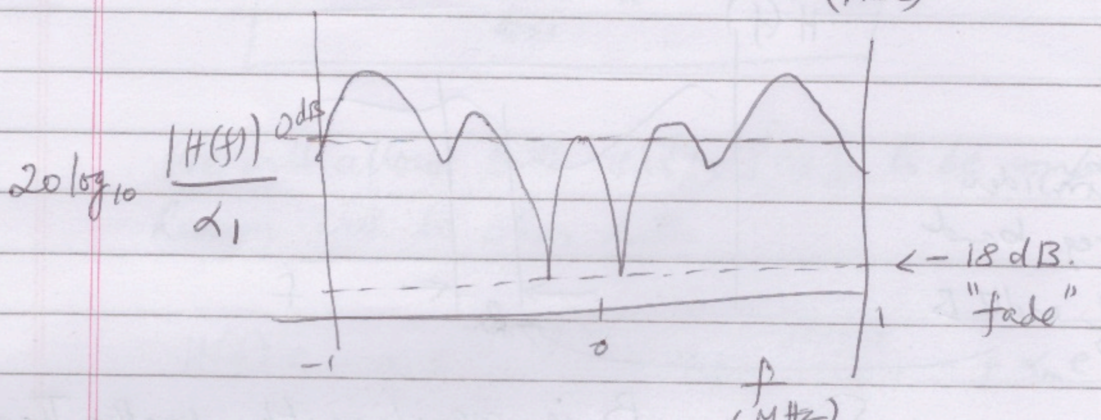
Delay spread $= 3.5 - 1 = \underline{2.5\,\mu s}$.

$$B_C = \frac{1}{2.5} = 0.4\,\text{MHz} = 400\,\text{kHz}.$$

Let us say we can approximate $H(f)$ to be constant over bands of width 40 kHz ($\frac{1}{10}$ of 400 kHz).

$$|H(f)| = \underbrace{1}_{\alpha_1} - 0.5e^{-j2\pi f} + 0.5e^{-j5\pi f}$$




→ Using a narrowband signal can result in a "deep fade" (if band coincides with channel fade).

Self-study Section 2.5.2
Numerical computation of Fourier transform using DFT.

## Energy Spectral Density & Bandwidth: