GOONWANTH
EE13B114

Assignment - II.

1).

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$A = [a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5]$$

$a_1 \; a_2 \rightarrow$ independent (obvious)

$a_1 - 2a_2 = a_3$.

$a_1 \; a_2 \; a_4 \rightarrow$ independent. (verified).

$a_1 + 3a_2 - 5a_4 = a_5$.

$\therefore$ Rank $(A) = 3$

fundamental theorem of linear algebra

rank (A) + nullity (A) = no. of colums.

$3 +$ nullity $= 5$

$\therefore$ nullity $(A) = 2$ //

4. $\dot{x} = Ax + Bu$

$y = Cx + Du$

$\dot{\omega} = -A^T\omega + C^T v$

$z = B^T\omega + D^T v$

$\mathcal{C}_1 = [B \quad AB \quad \cdots \quad A^{n-1}B]$

$\mathcal{O}_1 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

$\mathcal{C}_2 = [C^T \quad -A^T C^T \quad \cdots \quad (-A^T)^{n-1} C^T]$

$\mathcal{O}_2 = \begin{bmatrix} B^T \\ -B^T A^T \\ \vdots \\ +B^T (-A^T)^{n-1} \end{bmatrix}$.

If system one is fully controllable

$\Rightarrow \mathcal{C}_1 =$ full rank.

$$\mathcal{C}_1 \times \underbrace{\begin{bmatrix} 1 & 0 & 0 & \\ 0 & -1 & 0 & \\ \vdots & \vdots & 1 & \ddots & \cdots \\ 0 & 0 & 0 & \\ & & \vdots & \end{bmatrix}}_{P} = \text{full rank} \quad (\because \det P \neq 0).$$

$$\text{But } \underset{P^T \mathcal{C}_1^T}{(\mathcal{C}_1 P)^T} = \begin{bmatrix} 1 & 0 & 0 & \\ 0 & -1 & 0 & \\ 0 & \vdots & 1 & \ddots \\ \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix} = \text{full rank}.$$

$$= \begin{bmatrix} B^T \\ -B^T A^T \\ \vdots \\ B^T(-A^T)^{n-1} \end{bmatrix} = \mathcal{O}_2.$$

$\therefore$ If $\mathcal{C}_1$ = full rank $\Rightarrow \mathcal{O}_2$ = full rank

$\therefore$ system ② is observable.

Now Assume ~~that~~ that ① is observable

$$\Rightarrow \mathcal{O}_1 = \begin{bmatrix} C \\ | \\ CA^{n-1} \end{bmatrix} = \text{full Rank}$$

$$\Rightarrow \mathcal{O}_1^T = \text{full rank.} \quad (\because \text{square})$$

$$\mathcal{O}_1^T \times P = \begin{bmatrix} C^T & A^T C^T & \cdots & (A^{n-1})^T C^T \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & -1 & 0 & \\ \vdots & \vdots & +1 & \\ 0 & \vdots & 0 & \\ & & \vdots & \end{bmatrix}$$

$$= \begin{bmatrix} C^T & -A^T C^T & (A^T)^2 C^T & \cdots & (-A^T)^{n-1} C^T \end{bmatrix}$$

$$= \mathcal{C}_2 = \text{full rank} \quad (\det P \neq 0)$$

$\therefore$ If system ① is observable $\Rightarrow$ system ② is controllable.

converse can also be proved similarly

5)

$$\dot{x} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T u \quad ; \quad y = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} x$$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2 B \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -4 \\ 0 & 0 & 0 \\ 1 & 3 & 8 \end{bmatrix} \qquad \text{rank} = 2 < 3$$

$\therefore$ not controllable

$$T = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \end{bmatrix}$$

$$x = Tz$$

$$\Rightarrow \dot{z} = \left[\begin{array}{cc|c} 0 & -4 & 2 \\ 1 & 4 & -2 \\ \hline 0 & 0 & 1 \end{array}\right] z + \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right] u.$$

$$z = T^{-1} x = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$z_1 = 3x_1 + x_3$$

$$z_2 = -x_1$$

$$z_3 = x_2.$$

$\therefore$ x_2 is not controllable.

$$\Theta = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 2 \\ 4 & -7 & 4 \end{bmatrix}$$

$\operatorname{rank} \Theta = 2 < 3$

$\therefore$ not completely observable

$$T^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{z} = T^{-1} A T z + T^{-1} B u$$

$$\dot{z} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ \hline -2 & 3 & 0 \\ -5 & 3 & 2 \end{array}\right] z + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} u$$

$$y = \left[\begin{array}{cc|c} 1 & 0 & 0 \end{array}\right] z.$$

z_2 and z_3 are not observable.

$z_3 = x_3$

$\therefore x_3$ is not observable.

6. M [illegible] $]^T$

$$M = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$A \quad n \times n$

$C \quad 1 \times n$

$x \in \ker(M)$

$\Rightarrow Mx = 0$

$$\Rightarrow \begin{bmatrix} Cx \\ CAx \\ CA^2x \\ \vdots \\ CA^{n-1}x \end{bmatrix} = 0 \qquad — (1)$$

To prove $\quad Ax \in \ker(M)$

$$M \cdot Ax = \begin{bmatrix} CAx \\ CA^2x \\ \vdots \\ CA^n x \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ CA^n x \end{bmatrix} \qquad \text{from (1)}$$

from cayley hamilton theorem

$$A^n = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 - - - \quad \alpha_{n-1} A^{n-1}.$$

Linear comb^n

$\therefore$ A^n can be expressed as lin comb of $\{I\ A\ A^2 \ldots A^{n-1}\}$

$\because$ $Cx \quad CAx \quad \ldots \quad CA^{n-1}x = 0$

$\Rightarrow CA^n x = 0$

$$\therefore\ M\,Ax = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $Ax \in \text{Ker}(W)$.

7) $$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x \quad ; \quad y = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} x$$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & c_2 & c_3 \\ c_1 & 0 & -c_3 \\ c_1 & 0 & c_3 \end{bmatrix}$$

$\{c_1\ c_2\ c_3\} = \{1\ 1\ 0\} \rightarrow$ NOT Observable

$\{1\ 1\ 1\} \rightarrow$ OBSERVABLE

General condition rank $\mathcal{O}$ = 3

$\det \neq 0$

$\Rightarrow c_1 c_2 c_3 \neq 0.$

(iv) general case

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad ; \quad \Rightarrow A^2 = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

$$\mathcal{O} = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_1\lambda_1 & c_2\lambda_2 & c_3\lambda_3 \\ c_1\lambda_1^2 & c_2\lambda_2^2 & c_3\lambda_3^2 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}.$$

fully observable $\Rightarrow$ rank $= 3$ $\Rightarrow$ $\det \neq 0$

$\Rightarrow c_1 c_2 c_3 \neq 0$ & $(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) \neq 0$