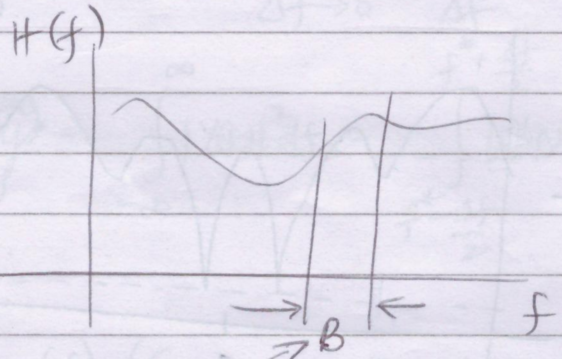
$$= e^{-j2\pi f\tau_1}\sum_{k=1}^{m}\alpha_k e^{-j2\pi f(\tau_k-\tau_1)}$$

$$= \underbrace{e^{-j2\pi f\tau_1}}_{\text{corresponds to pure delay}}\Big[\alpha_1 + \sum_{k=2}^{m}\alpha_k \underbrace{e^{-j2\pi f(\tau_k-\tau_1)}}_{\text{Period } \frac{1}{\tau_k-\tau_1}}\Big]$$

Smallest period $\frac{1}{\tau_m-\tau_1}$.

Define $\tau_d = \tau_m - \tau_1$ as the <u>delay spread</u>.



Consider freq. band of width B

Suppose B is significantly smaller than $\frac{1}{\tau_d}$

H(f) is approximately flat in that band

Define $B_C = \frac{1}{\tau_d}$ as the <u>coherence bandwidth</u>.

<u>Numerical</u> example:

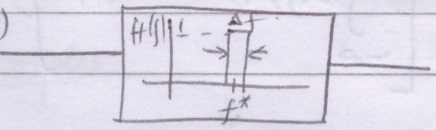
$$h(t) = \delta(t-1) + 0.5\delta(t-3.5) - 0.5\delta(t-1\ldots$$

(Let units be μs)

Delay spread $= 3.5 - 1 = \underline{2.5\,\mu s}$.

$$B_C = \frac{1}{2.5} = 0.4\,MHz = 400\,kHz.$$

Suppose we want to know the energy content of $u(t)$ in a small band of frequencies around frequency f^*.

$u(t)$  $y(t) \longrightarrow$ Measure the energy of $y(t)$

$$\int_{-\infty}^{\infty} |y(t)|^2 dt$$

$E_u(f^*)$ = Energy spectral density at $f^* = \dfrac{\text{Energy of } y(t)}{\Delta f}$ as $\Delta f \to 0$.

(or) $\lim\limits_{\Delta f \to 0} \dfrac{\text{Energy of } y(t)}{\Delta f}$.

$$\text{Energy of } y(t) = \int_{-\infty}^{\infty} |Y(f)|^2 df = \int_{f^*-\frac{\Delta f}{2}}^{f^*+\frac{\Delta f}{2}} |U(f)|^2 df \approx |U(f)|^2 \cdot \Delta f$$

$$E_u(f) \text{ (Energy spectral density)} = |U(f)|^2 .$$

Lecture 9 : 1 Feb 2016

Note that $|U(f)|^2$ can be written as $U(f)\,U^*(f)$.

Inverse Fourier transform of $U(f)\,U^*(f) = \,?$

Inverse Fourier transform of $U(f)$ $*$ Inverse Fourier transform of $U^*(f)$

$$|U(f)|^2 \longleftrightarrow u(t) * u^*(-t)$$

$$= \int_{-\infty}^{\infty} u(t)\, u^*(t-\tau)\, dt \triangleq R_u(\tau).$$

$$\mathcal{F}^{-1}\left[U^*(f)\right] = \int_{-\infty}^{\infty} U^*(f)\, e^{j2\pi f t}\, df$$

$$= \left[\int_{-\infty}^{\infty} U(f)\, e^{-j2\pi f t}\, df\right]^*$$

$$= \left[u(-t)\right]^* = u^*(-t).$$

$$x(\tau) * y(\tau) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$\begin{pmatrix} x(t) = u(t) \\ y(t) = u^*(-t) \end{pmatrix} \quad = \int_{-\infty}^{\infty} u(t)\, u^*(t-\tau)\, dt$$

---

* $R_u(\tau)$ is called the autocorrelation function of signal $u(t)$. We are using τ to denote the time variable here (convention for autocorrelation).

* Inner product of $u(t)$ with $u(t-\tau)$ for different τ's

<u>Bandwidth</u>

A measure of ~~the~~ band of frequencies that are "occupied" by a signal.

Different definitions possible.

(1) "Size" of the band over which $U(f)$ is non-zero.

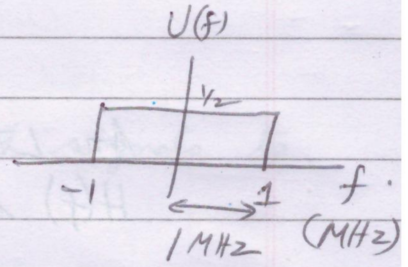
(2) "Size" of the band over which $|U(f)|^2$ is within some fraction of its peak value (e.g. $\frac{1}{2}$ ~~(3-dB bandwidth)~~)

(3) "Size" of smallest band over which a specified fraction of ~~bandwidth~~ signal energy is contained (called energy-containment bandwidth).

Examples

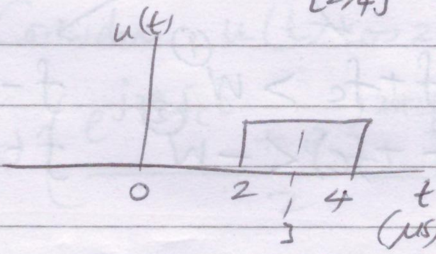
(1) $u(t) = \mathrm{sinc}(2t)$.
with μs as the time unit.

$$U(f) = \frac{1}{2} I_{[-1,1]}(f)$$



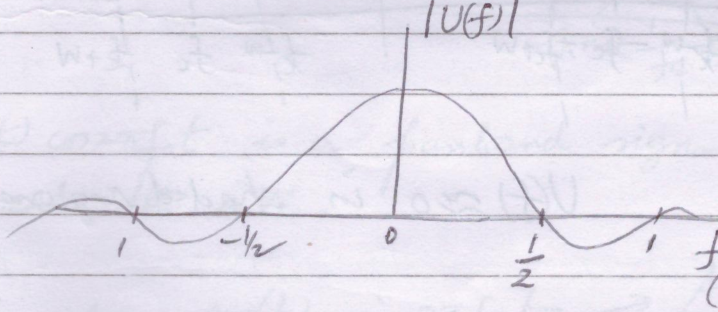
Bandwidth $\overset{W}{=}$ 1 MHz.
(We will consider only +ve frequencies when we talk about bandwidth of a real signal).

(2) $u(t) = I_{[2,4]}(t)$ [unit of t is μs]



$\longleftrightarrow U(f) = 2\,\mathrm{sinc}(2f)e^{-j}$

Strict definition (1) $\rightarrow$ Bandwidth ∞.



$\longrightarrow$ 99% energy containment bandwidth $\approx$ 5.1 MH
W

* For complex signals, we will define bandwidth as the size of the freq. band occupied over both +ve & −ve frequen

Baseband and passband signals

* A signal $u(t)$ is said to be baseband if the signal energy is concentrated in a band around DC ($f=0$), and

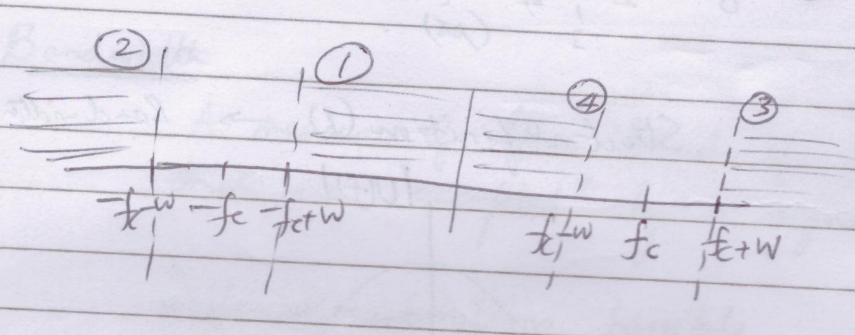
$$U(f) \approx 0 \quad \text{for } |f| > W \quad \text{for some } W>0.$$

An LTI system is a baseband system if $H(f)$ is baseband.

* A signal $u(t)$ is said to be passband if its energy is concentrated in a band away from DC, with

$$U(f) \approx 0 \quad \underbrace{|f \pm f_c| > W}_{} \quad \text{where } f_c > W > 0.$$

$$f + f_c > W \quad (1) \qquad\qquad f - f_c > W \quad (3)$$
$$+(f + f_c) < -W \quad (2) \qquad\qquad f - f_c < -W \quad (4)$$



$U(f) \approx 0$ in shaded regions.

* Baseband and passband systems/channels: Depending on whether $h(t)$ (the impulse response) is a baseband or passband signal.

In communication, we often have:

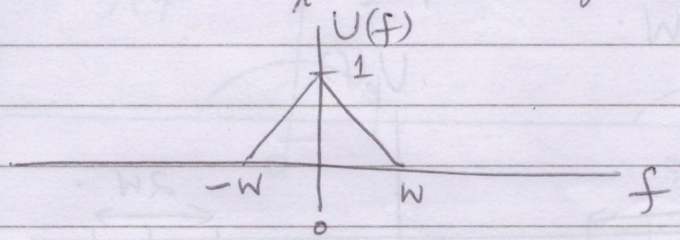
(1) baseband signals (eg. voice, music signals)

& (2) passband channels (eg. wireless, AM/FM radio, satellite, ...)

Some channels (eg. telephone wired) are also baseband channels.

+ How do we translate baseband signals to passband and viceversa?

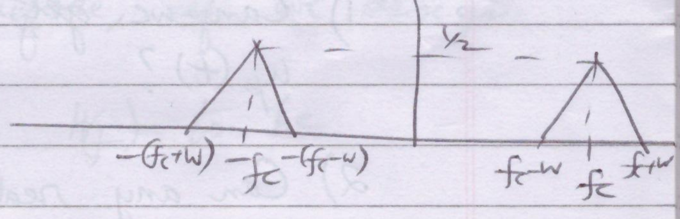
Let $u(t)$ be a real baseband signal of bandwidth W



real $u(t) \Rightarrow U(-f) = U^*(f)$

- Consider $u(t)\cos 2\pi f_c t$.

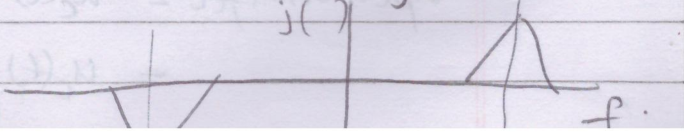
$$u(t)\left(\frac{e^{j2\pi f_c t}+e^{-j2\pi f_c t}}{2}\right) \longleftrightarrow \frac{U(f-f_c)+U(f+f_c)}{2}$$



$u(t)\cos 2\pi f_c t$ is a passband signal with bandwidth $2W$.

- Consider $v(t)\sin 2\pi f_c t \rightarrow$ Also a passband signal of bandwidth $2W$

$$v(t)\left(\frac{e^{j2\pi f_c t}-e^{-j2\pi f_c t}}{2j}\right) \Longleftrightarrow \frac{U(f-f_c)-U(f+f_c)}{2j}$$



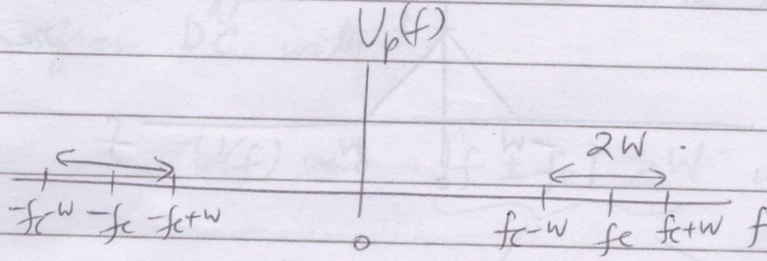
Lecture 10 : 2 Feb 2016

Now, observe that the sum (or difference) of $u(t)\cos 2\pi f_c t$ and $v(t)\sin 2\pi f_c t$ are also real passband signals. (for $f_c > W$)

Let $u_c(t)$ and $u_s(t)$ be baseband signals of bandwidth W. Then, for $f_c > W$,

$$\boxed{u_p(t) = u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t} \quad \text{— (A)}$$

is a real passband signal of bandwidth $2W$.



(A) is one way to obtain a real passband signal.

Questions:

1) Can we get $u_c(t)$ and $u_s(t)$ back from $u_p(t)$?

2) Can any real passband signal be represented as in (A)?

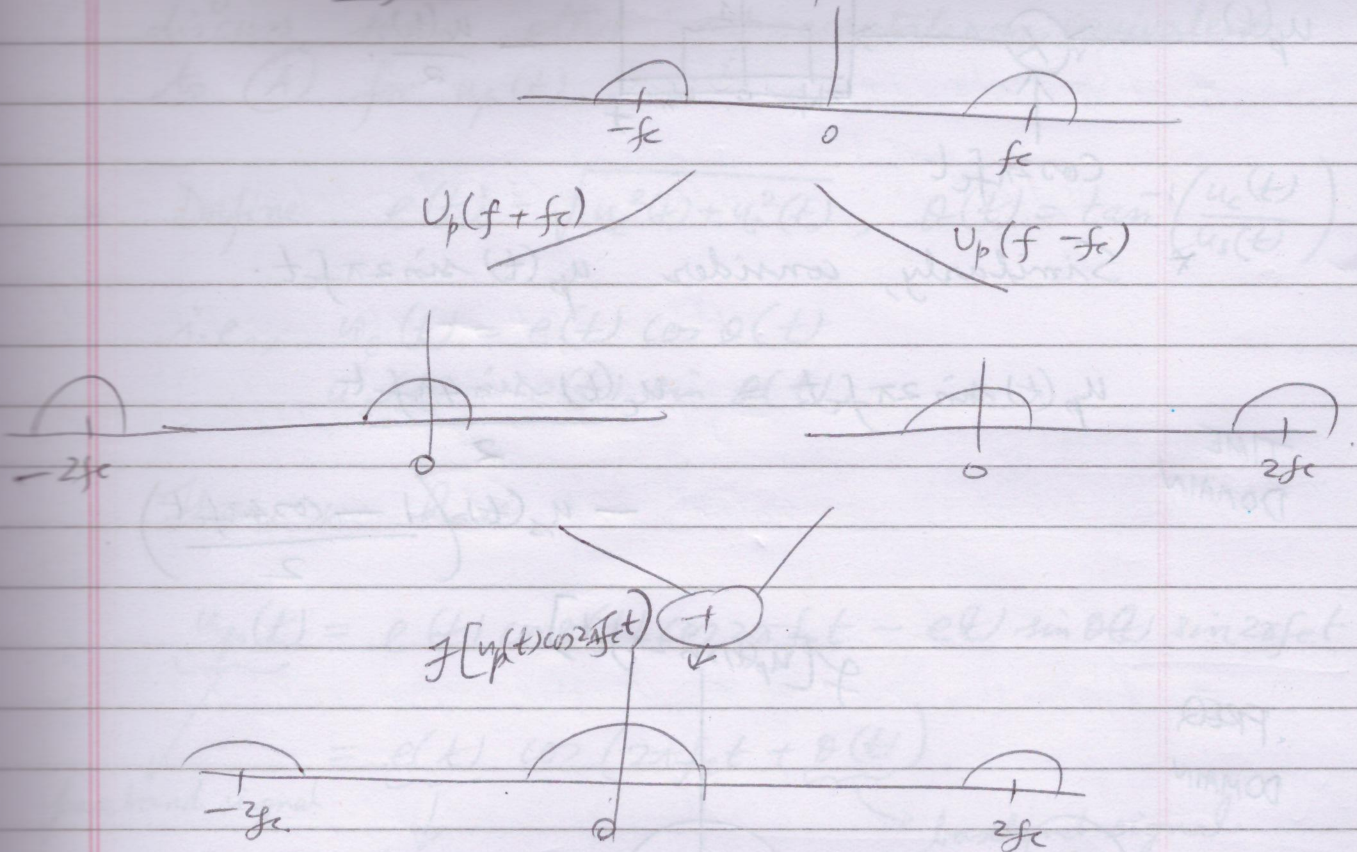
The answer to both questions is YES.

Q1) Consider $u_p(t)\cos 2\pi f_c t$

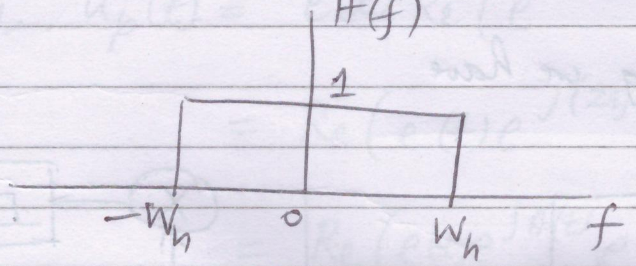
Time-domain

$$u_p(t)\cos 2\pi f_c t = u_c(t)\cos^2 2\pi f_c t - u_s(t)\sin 2\pi f_c t\cos 2\pi f_c t$$

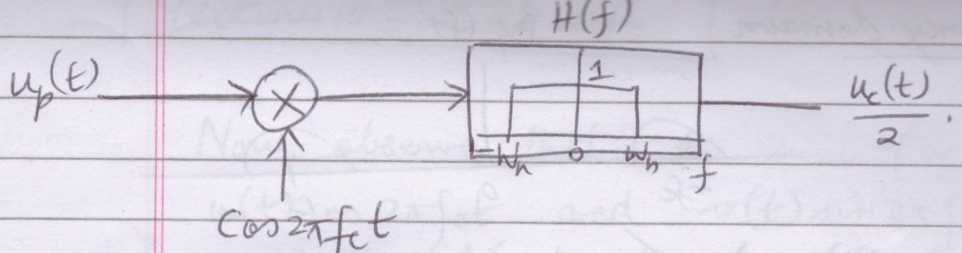
Frequency domain



Comparing with the time domain expression, we can easily see that the component near 0 corresponds to $\frac{u_c(t)}{2}$. The components around $2f_c$ & $-2f_c$ correspond to $\frac{u_c(t)\cos 4\pi f_c t}{2}$ & $\frac{-u_s(t)\sin 4\pi f_c t}{2}$. Therefore, if we design an LTI system with $H(f)$ to be



where $W < W_h < 2f_c - W$, we can separate the $u_c(t)$ term from the other terms.

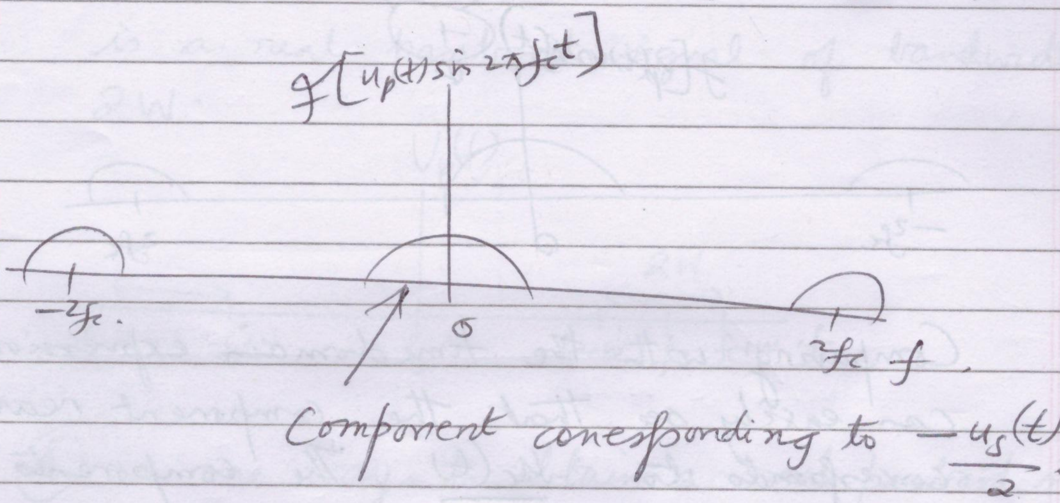


* Similarly, consider $u_p(t)\sin 2\pi f_c t$.

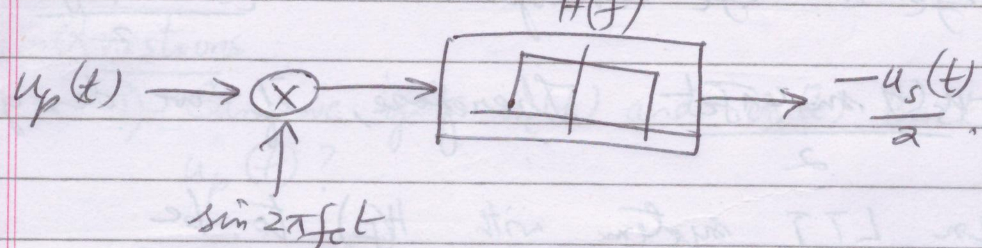
TIME DOMAIN

$$u_p(t)\sin 2\pi f_c t = \frac{u_c(t)\sin 4\pi f_c t}{2} - u_s(t)\left(\frac{1-\cos 4\pi f_c t}{2}\right).$$

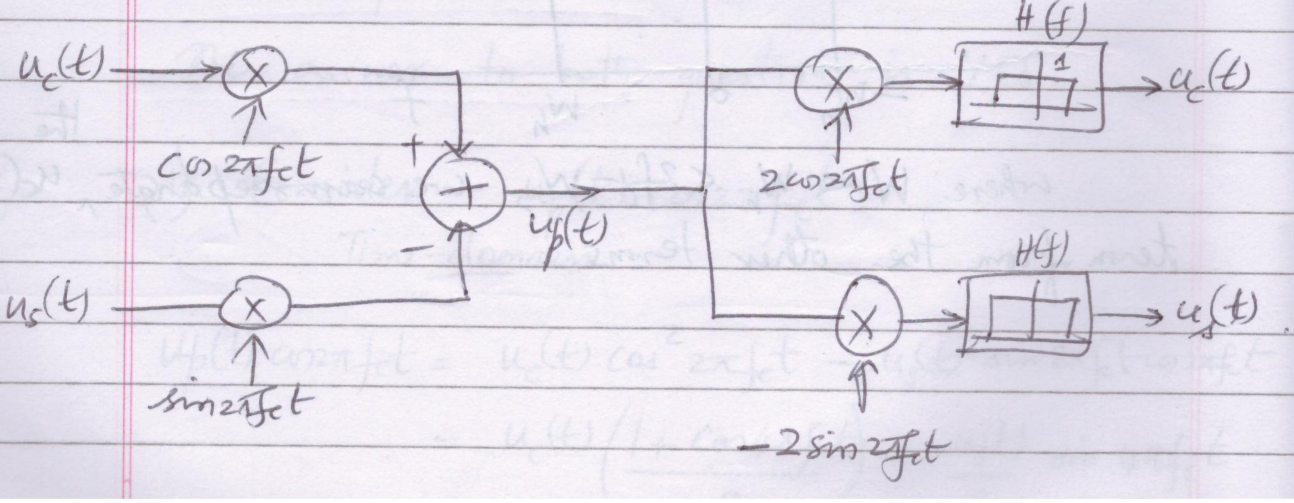
FREQ DOMAIN



$\Rightarrow$



* Summarizing, we have



* Before we get to Question 2, let us discuss some other representations equivalent to (A) for $u_p(t)$.

* Define $e(t) = \sqrt{u_c^2(t) + u_s^2(t)}$, $\theta(t) = \tan^{-1}\left(\frac{u_c(t)}{u_s(t)}\right)$

i.e., $u_c(t) = e(t)\cos\theta(t)$
$u_s(t) = e(t)\sin\theta(t)$.

Then, we have

$$\underbrace{u_p(t)}_{\text{passband signal}} = e(t)\cos\theta(t)\cos 2\pi f_c t - e(t)\sin\theta(t)\sin 2\pi f_c t$$

$$= \underbrace{e(t)}_{\substack{\text{baseband signal}\\ \text{(envelope)}}} \cos\left(2\pi f_c t + \underbrace{\theta(t)}_{\substack{\text{baseband signal}\\ \text{(phase)}}}\right)$$

$$\boxed{u_p(t) = e(t)\cos\left(2\pi f_c t + \theta(t)\right)} \text{ — (B)}$$

* Define $u(t) = u_c(t) + j\,u_s(t)$
$= e(t)e^{j\theta(t)}$.

Then, $u_p(t) = e(t)\,Re\left(e^{j(2\pi f_c t + \theta(t))}\right)$

$= Re\left(e(t)e^{j(2\pi f_c t + \theta(t))}\right)$

$= Re\left(e(t)e^{j\theta(t)}e^{j2\pi f_c t}\right)$

i.e. $\boxed{u_p(t) = Re\left(\underbrace{u(t)}_{\text{complex baseband signal.}}e^{j2\pi f_c t}\right)}$ — (C)

* Thus, we have three representations for $u_p(t)$

in (A), (B), and (C).

* Also note in (C) that

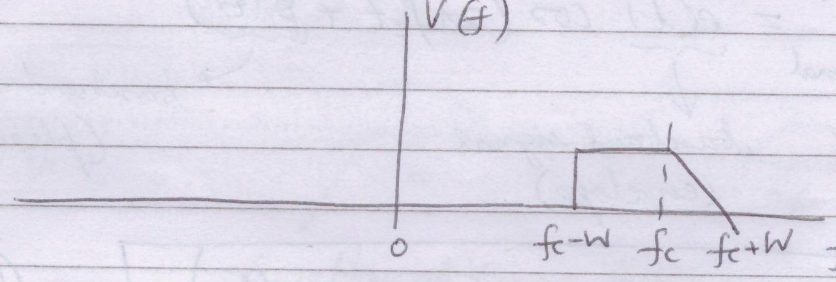
$$u(t)e^{j2\pi f_c t} = (u_c(t) + j\,u_s(t))(\cos 2\pi f_c t + j\sin 2\pi f_c t)$$

$$= u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t$$

$$+ j\,(u_s(t)\cos 2\pi f_c t + u_c(t)\sin 2\pi f_c t)$$

which is consistent with (A).

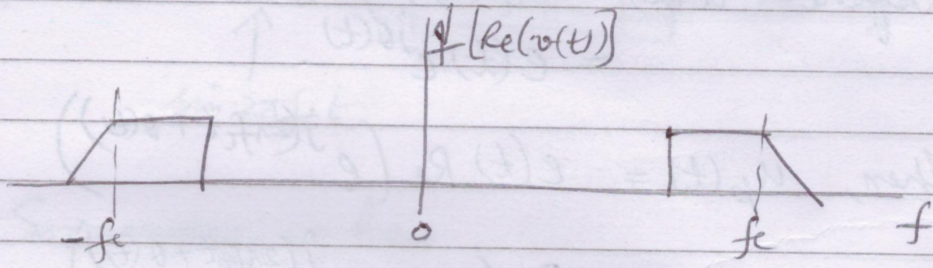
* Also if $v(t)$ is a complex signal with $V(f)$



$f_c > W$

then, $Re[v(t)]$ has spectrum (Fourier transform)

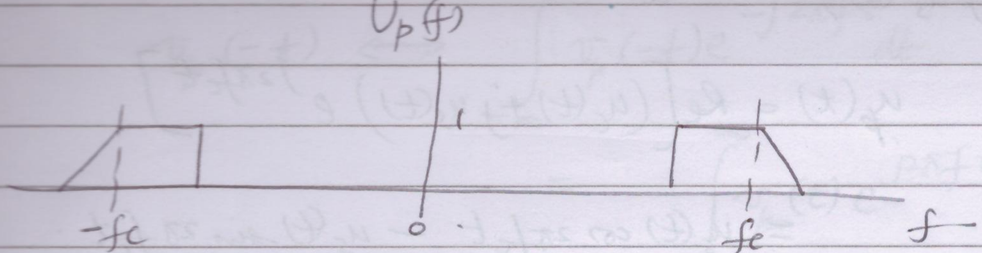
(I) $\rightarrow \mathcal{F}[Re(v(t))] = \mathcal{F}\left[\dfrac{v(t) + v^*(t)}{2}\right] = \dfrac{V(f) + V^*(-f)}{2}$



$\boxed{\text{Lecture 11: 4 Feb 2016}}$

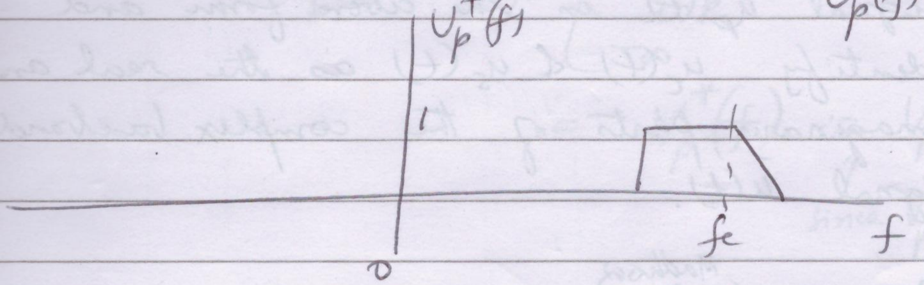
* Let us now interpret the representation in (C) in frequency domain and answer Q2.

Consider an arbitrary real passband signal $u_p(t)$.



Define $U_p^+(f)$ to be the following.

$$U_p^+(f) = \begin{cases} U_p(f) & f \geq 0 \\ 0 & f < 0. \end{cases}$$



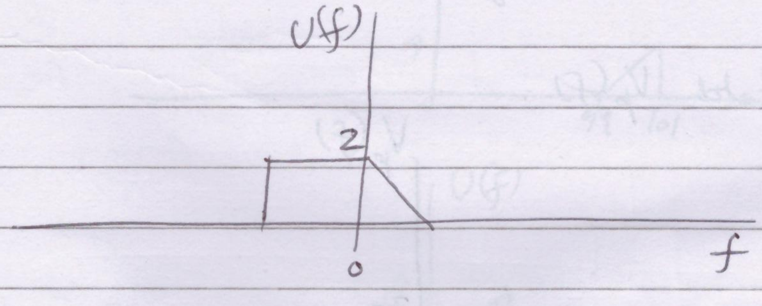
Define $C(f) = 2U_p^+(f) = \begin{cases} 2U_p(f) & f \geq 0 \\ 0 & f < 0 \end{cases}$.

From I in previous page (p46), we have

$$u_p(t) = Re[c(t)].$$

Now, define $u(t) = c(t)\, e^{-j2\pi f_c t}$

$$U(f) = C(f + f_c)$$



Therefore, we have $\underbrace{u_p(t)}_{\substack{\text{real} \\ \text{passband} \\ \text{signal}}} = Re\left(\underbrace{u(t)}_{\substack{\text{complex} \\ \text{baseband} \\ \text{signal}}} e^{j2\pi f_c t}\right)$.

$u(t)$ is a complex signal. Suppose we write $u(t) = u_c(t) + j u_s(t)$, we have

$$u_p(t) = Re\left[\left(u_c(t) + j\, u_s(t)\right) e^{j 2\pi f_c t}\right]$$

$$= u_c(t) \cos 2\pi f_c t - u_s(t) \sin 2\pi f_c t .$$

Thus, we can represent any real passband signal $u_p(t)$ in the above form and identify $u_c(t)$ & $u_s(t)$ as the real and imaginary parts of the complex baseband signal $u(t)$.

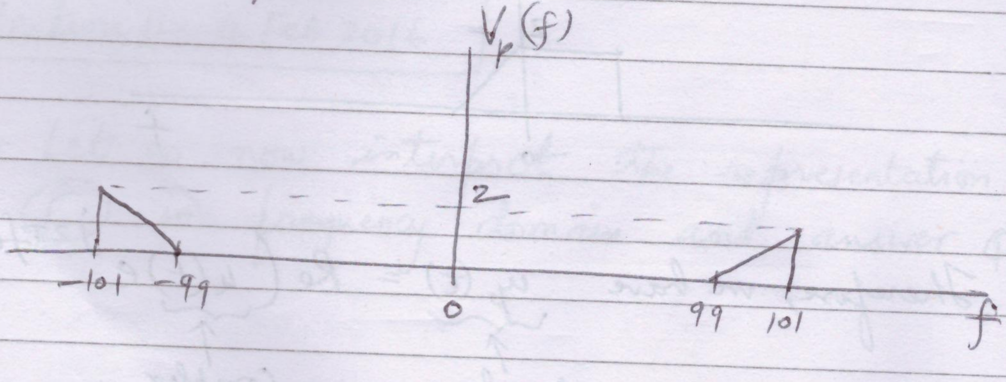
Example (2.8.4) Madhow

$v_p(t)$ is a real passband signal.
$V_p(f)$ for $f < 0$ is defined as

$$V_p(f) = \begin{cases} -(f+99) & -101 \le f \le -99 \\ 0 & f < -101 \\ 0 & -99 \le f \le 0 . \end{cases}$$

$V_p(f)$ for +ve frequencies can be obtained using symmetry conditions for Fourier transform of a real signal.

(a) Sketch $V_p(f)$.



(b) Is $v_p(t) = v_p(-t)$?

$$v_p(-t) \longleftrightarrow \int_{-\infty}^{\infty} v_p(-t) e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} v_p(\tau) e^{j2\pi f \tau} d\tau$$

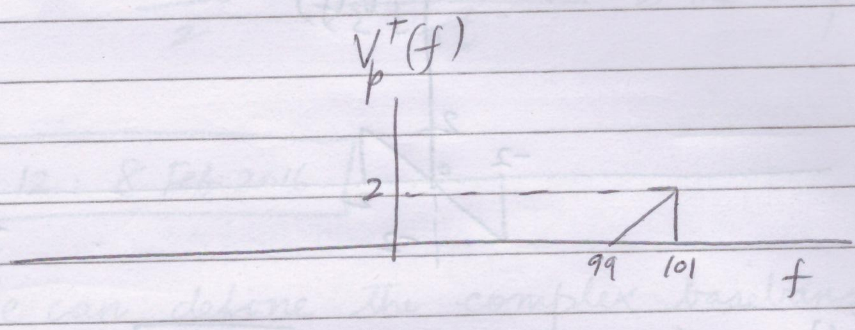
$$\left(\text{Since } v_p(t) \text{ is real}\right) = \left[\int_{-\infty}^{\infty} v_p(\tau) e^{-j2\pi f \tau} d\tau\right]^*$$

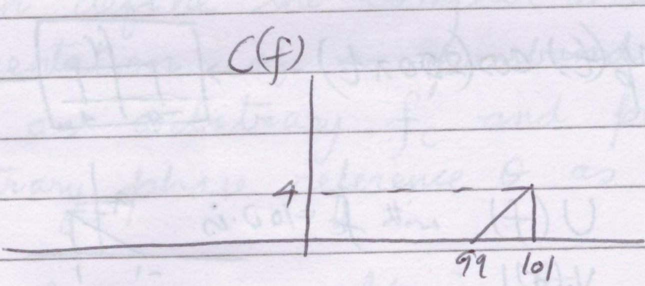
$$= V_p^*(f) \underset{\uparrow}{=} V_p(f)$$

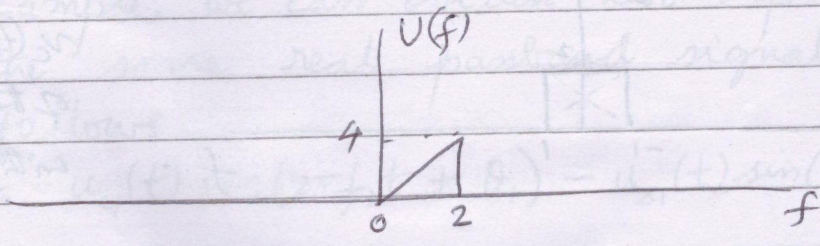
since $V_p(f)$ is real in this example.

$$\Rightarrow \quad v_p(t) = v_p(-t).$$

(c) Sketch $V_c(f)$, $V_s(f)$ for $f_c = 99$.

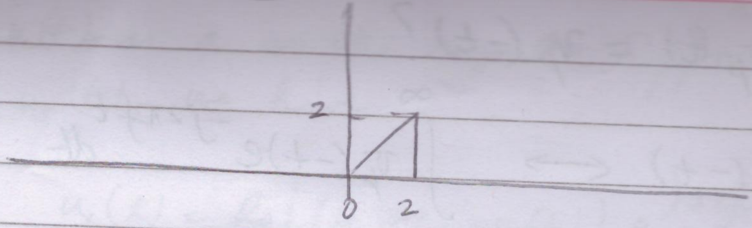




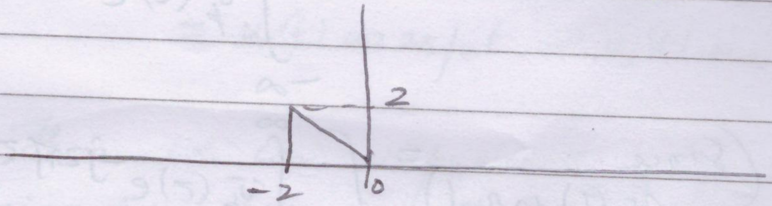


$$v_c(t) = Re[u(t)] = \frac{u(t) + u^*(t)}{2} \longleftrightarrow \frac{U(f) + U^*(-f)}{2}$$

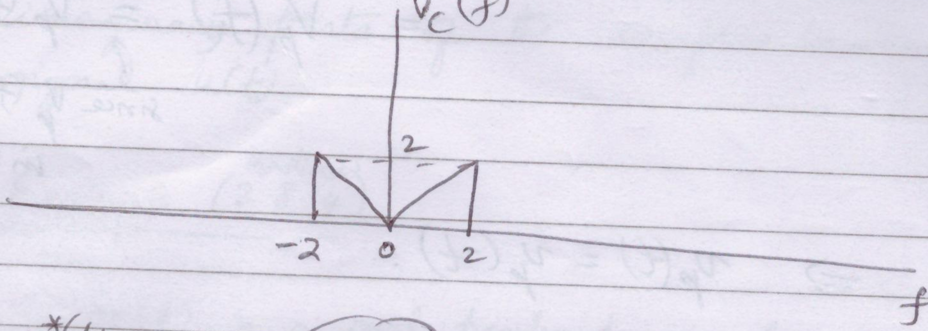
$V_c(f)$



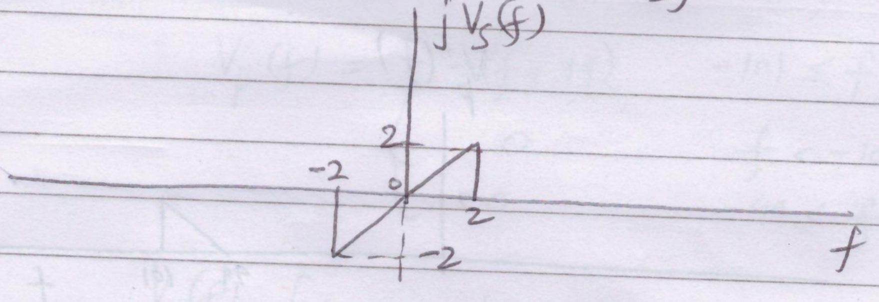
$+$

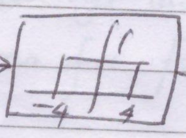


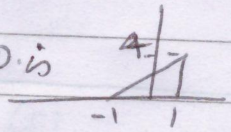
$=$

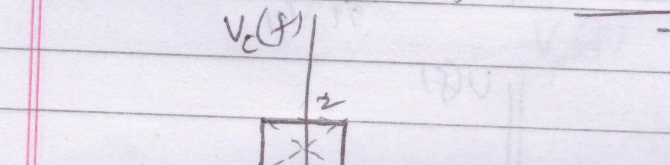


$v_s(t) = \dfrac{u(t) - u^*(t)}{2j}$ $\qquad V_s(f) = \dfrac{U(f) - U^*(-f)}{2j}$



(d) $v_p(t)\cos(200\pi t) \longrightarrow$  $\longrightarrow ? = \boxed{2\,\text{sinc}(2t)}$

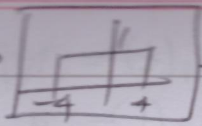
$U(f)$ with $f_c = 100$ is 



The output will be $\dfrac{v_c(t)}{2}$ where $v_c(t)$ is the real part of the complex envelope with $f_c = 100$.

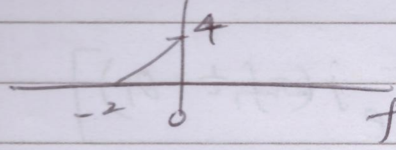
$\Rightarrow v_c(t) \longleftrightarrow 2\, I_{[-1,1]}(f)$

$\Rightarrow v_c(t) = 2\left[2\,\text{sinc}(2t)\right] = \underline{4\,\text{sinc}(2t)}.$

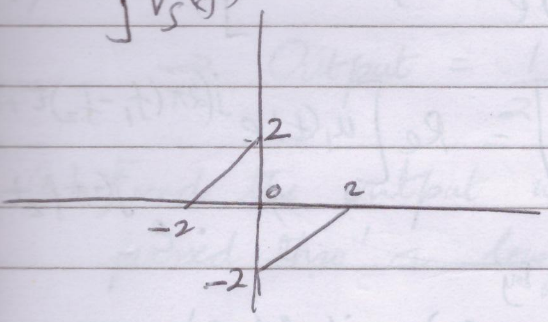
(e) $v_p(t)\sin(202\pi t) \longrightarrow$  $\longrightarrow$?

The output will be $\frac{-v_s(t)}{2}$ where $v_s(t)$ is the Imag. part of the complex envelope $u(t)$ with $f_c = 101$.

$U(f)$ with $f_c = 101$



$j V_s(f)$



$$V_s(f) \longleftrightarrow v_s(t) = \frac{2(\mathrm{sinc}(4t) - 1)}{\pi t}.$$

$$\Rightarrow \frac{-v_s(t)}{2} = \frac{1 - \mathrm{sinc}(4t)}{\pi t} \text{ is the output.}$$

Lecture 12: 8 Feb 2016

We can define the complex baseband representation of a passband signal (real) using an arbitrary f_c and possibly an arbitrary phase reference θ as well.

For example, we can obtain two representations for the same real passband signal $u_p(t)$ as follows.

$$u_p(t) = u_{c_1}(t)\cos(2\pi f_1 t + \theta_1) - u_{s_1}(t)\sin(2\pi f_1 t + \theta_1)$$

$$= u_{c_2}(t)\cos(2\pi f_2 t + \theta_2) - u_{s_2}(t)\sin(2\pi f_2 t + \theta_2).$$

Let $u_1(t) = u_{c1}(t) + j\, u_{s1}(t)$
$u_2(t) = u_{c2}(t) + j\, u_{s2}(t)$.

Now, we know

$$u_p(t) = Re\left[u_1(t)\, e^{j(2\pi f_1 t + \theta_1)} \right]$$
$$= Re\left[u_2(t)\, e^{j(2\pi f_2 t + \theta_2)} \right] \quad \text{— (1)}$$

$$Re\left[u_1(t)\, e^{j(2\pi f_1 t + \theta_1)} \right] = Re\left[u_1(t)\, e^{j(2\pi (f_1 - f_2)t + \theta_1 - \theta_2)} \cdot e^{j(2\pi f_2 t + \theta_2)} \right] \quad \text{— (2)}$$

multiply by

$e^{j(2\pi f_2 t + \theta_2)} \cdot e^{-j(2\pi f_2 t + \theta_2)}$

Thus, from (1) and (2), we can read

$$\boxed{u_2(t) = u_1(t)\, e^{j(2\pi (f_1 - f_2)t + \theta_1 - \theta_2)}} \quad \text{— (A)}$$

<u>Example</u>

$$u_p(t) = \underbrace{I_{[-1,1]}(t)}\, \cos(400\pi t).$$

Not exactly bandlimited but approximately bandlimited. We will still use the representation.

$f_c = 200$
$W \approx n(2)$
$n \approx 5$

(a) Find the output when $u_p(t) \cos(401\pi t)$ is passed through a lowpass filter.

Suppose $u_1(t) = u_{c1}(t) + j\, u_{s1}(t)$ is the complex envelope with respect to reference

$e^{j401\pi t}$, then the output is $\frac{u_{c,1}(t)}{2}$.

We know $u_p(t) = Re\left[I_{[-1,1]}(t)\, e^{j400\pi t}\right]$

$$= Re\left[\underbrace{I_{[-1,1]}(t)\, e^{-j\pi t}}\, e^{j401\pi t}\right].$$

$$\Rightarrow u_{c,1}(t) = I_{[-1,1]}(t) \cos \pi t$$

$$\Rightarrow \text{Output} = \frac{1}{2} I_{[-1,1]}(t) \cos \pi t.$$

(b) Find the output when $u_p(t) \sin(400\pi t - \pi/4)$ is passed thro' a lowpass filter.

$$u_p(t) = Re\left[I_{[-1,1]}(t)\, e^{j\pi/4} \cdot e^{j(400\pi t - \pi/4)}\right].$$

$$\text{Output} = \frac{-1}{2} I_{[-1,1]}(t) \sin \frac{\pi}{4}.$$

$$= -\frac{1}{2\sqrt{2}} I_{[-1,1]}(t).$$

* In a communication system, we have frequency and phase offsets between f_c & θ used at the transmitter and the f_c & θ used at the receiver. We can relate the recovered complex baseband signal with the transmitted complex baseband signal as in eqn. A in p.52.

Let the carrier frequency & phase of the receiver be f_c & 0. (i.e. we use as the ref the rx freq & phase).
Let the carrier freq. & phase of the transmitter be $f_c + \Delta f$ & θ.

$$u_p(t) = u_c(t)\cos(2\pi(f_c+\Delta f)t+\theta)$$
$$\qquad - u_s(t)\sin(2\pi(f_c+\Delta f)t+\theta).$$

$$= \mathrm{Re}\Big[\underbrace{u(t)}_{u_c(t)+j\,u_s(t).}\, e^{j(2\pi(f_c+\Delta f)t+\theta)}\Big]$$

$$= \mathrm{Re}\Big[\underbrace{u(t)\,e^{j2\pi\Delta f t}\,e^{j\theta}}_{=\tilde{u}(t)}\cdot e^{j2\pi f_c t}\Big]$$

$$= \mathrm{Re}\Big[\underbrace{\tilde{u}(t)}_{\tilde{u}_c(t)+j\,\tilde{u}_s(t)\text{ at the rx.}}\, e^{j2\pi f_c t}\Big].$$

$$\boxed{\tilde{u}(t) = u(t)\,e^{j2\pi\Delta f t}\,e^{j\theta}}$$

$$\tilde{u}_c + j\,\tilde{u}_s = (u_c + j u_s)\big(\cos(2\pi\Delta f t+\theta) + j\sin(2\pi\Delta f t$$

$$\boxed{\begin{aligned}\tilde{u}_c &= u_c\cos(2\pi\Delta f t+\theta) - u_s\sin(2\pi\Delta f t+\theta)\\ \tilde{u}_s &= u_c\sin(2\pi\Delta f t+\theta) + u_s\cos(2\pi\Delta f t+\theta)\end{aligned}}$$

Received: $\tilde{u}_c, \tilde{u}_s$. Transmitted: u_c, u_s.

<u>Discussion related to time-scaling:</u> (clarification Prob. Set 2)

Scaling property (Fourier transform):

If $u(t) \longleftrightarrow U(f)$,

then $u(at) \longleftrightarrow \dfrac{1}{|a|}U\!\left(\dfrac{f}{a}\right)$.

Proof: $\mathcal{F}[u(at)] = \int_{-\infty}^{\infty} u(at)\, e^{-j2\pi f t}\, dt$

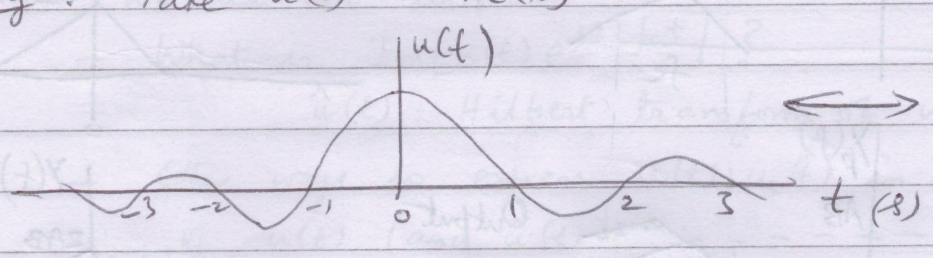
$\left[\begin{array}{l} at = \tau \\ a\,dt = d\tau \end{array}\right]$

$$= \begin{cases} \displaystyle\int_{-\infty}^{\infty} u(\tau)\, e^{-j2\pi f \frac{\tau}{a}} \frac{d\tau}{a} & (\text{for } a > 0) \\ \displaystyle\int_{\infty}^{-\infty} u(\tau)\, e^{-j2\pi f \frac{\tau}{a}} \frac{d\tau}{a} & (\text{for } a < 0) \end{cases}$$

$$= \frac{1}{|a|} \int_{-\infty}^{\infty} u(\tau)\, e^{-j2\pi\left(\frac{f}{a}\right)\tau}\, d\tau = \frac{1}{|a|} U\left(\frac{f}{a}\right).$$ ∎

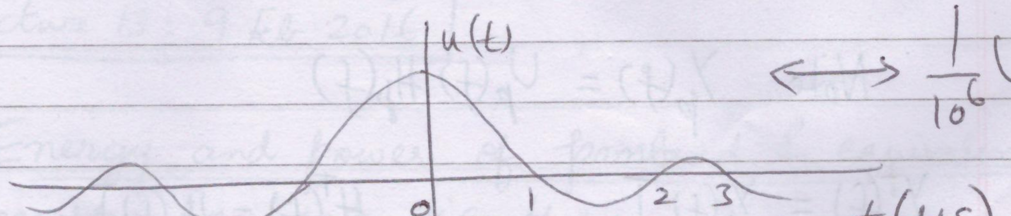
Suppose we scale the time axis by 10^6 to get μs, freq axis gets scaled to MHz scale. There is also a $\frac{1}{|a|}$ factor by which the Fourier transform is scaled.

Eg: Take $u(t) = \text{sinc}(t)$



$\longleftrightarrow U(f)$

Time scaling $\downarrow$ $\text{sinc}(10^6 t)$



$\longleftrightarrow \frac{1}{10^6} U\left(\frac{f}{10^6}\right)$

$\uparrow$

$\frac{f}{10^6}$ MHz

The Complex-baseband equivalent of passband filtering:

$$u_p(t) \longrightarrow \boxed{h_p(t)} \longrightarrow y_p(t) = (u_p * h_p)(t)$$

Real passband signal | Passband filter | Real Passband signal

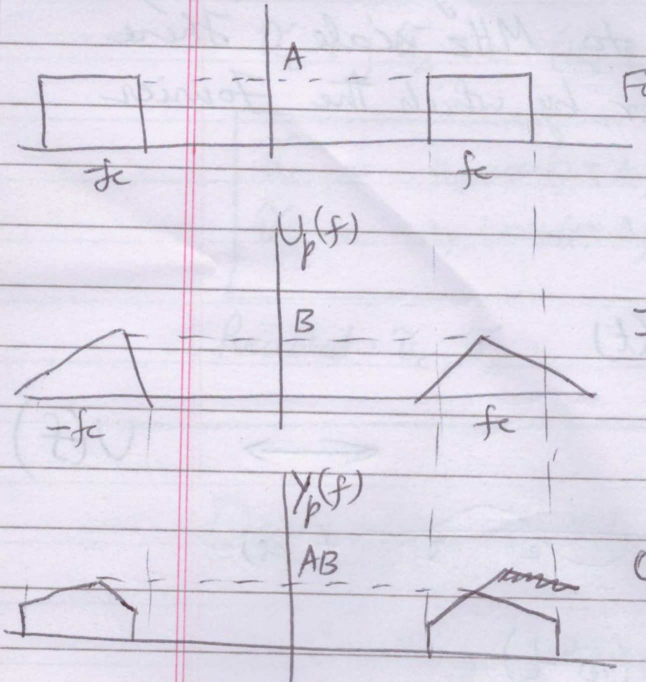
$u_p(t) \xrightarrow{\text{complex baseband equivalent}} u(t)$

$h_p(t) \xrightarrow{\text{complex baseband equiv}} h(t)$

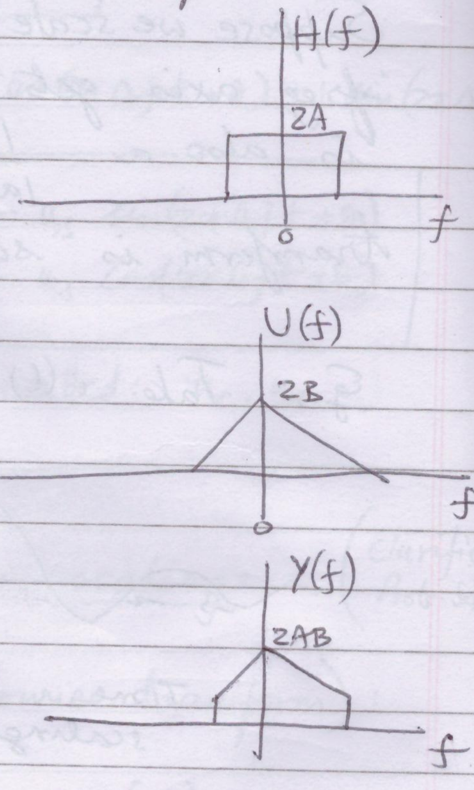
$y_p(t) \xrightarrow{\text{complex baseband equiv}} y(t)$.

$\boxed{\text{* How are } u(t),\ h(t), \text{ and } y(t) \text{ related?}}$

Passband



Complex Baseband



Note $Y_p(f) = U_p(f)\, H_p(f)$

$Y^+(f) = Y_p(f)\, I_{[f>0]}$

$H^+(f) = H_p(f)\, I_{[f>0]}$

$U^+(f) = U_p(f)\, I_{[f>0]}$

$$\Rightarrow Y^{+}(f) = U^{+}(f)\, H^{+}(f)$$

$$Y(f) = 2\, Y^{+}(f+f_c)$$

$$= 2\, U^{+}(f+f_c)\, H^{+}(f+f_c)$$

$$= \frac{1}{2}\left[2\, U(f)\; 2\, H(f)\right] = \frac{1}{2}\left[U(f)\, H(f)\right]$$

$$\Rightarrow \boxed{y(t) = \frac{1}{2}(u * h)(t).}$$

* As a consequence of the above result any desired passband filtering can be realized in complex baseband.

This requires 4 real baseband filters.

$$y_c = \frac{1}{2}(u_c * h_c - u_s * h_s)$$

$$y_s = \frac{1}{2}(u_s * h_c + u_c * h_s).$$

What we did not discuss?

- What is $Im[u(t)\, e^{j2\pi f_c t}]$?
  - $\hat{u}(t)$: Hilbert transform of $u(t)$.
- Other ways to express $u_c(t)$, $u_s(t)$ in terms of $u(t)$ (and $\hat{u}(t)$).

Lecture 13 : 9 Feb 2016

Energy and power of passband & equivalent complex baseband signals:

Real passband signal $\longleftrightarrow$ Complex baseband signal

$u_p(t)$ $\qquad\qquad$ $u_c(t) + j u_s(t)$.

$$\text{Energy of } u_p(t) = \|u_p\|^2 = \int_{-\infty}^{\infty} u_p^2(t)\,dt$$

$$= \int_{-\infty}^{\infty} \left(u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t\right)^2 dt$$

$$= \int_{-\infty}^{\infty} u_c^2(t)\cos^2 2\pi f_c t\,dt + \int_{-\infty}^{\infty} u_s^2(t)\sin^2 2\pi f_c t\,dt$$

$$- 2\int_{-\infty}^{\infty} \underbrace{u_c(t)\,u_s(t)}_{\substack{\text{has Fourier}\\ \text{transform}\\ U_c(f) * U_s(f)}}\ \underbrace{\cos 2\pi f_c t \sin 2\pi f_c t}_{\frac{\sin 4\pi f_c t}{2}}\,dt$$

Suppose $U_c(f)$ has bw W_1 $\quad (f_c > W_1)$

$U_s(f)$ has bw W_2 $\quad \underbrace{(f_c > W_2)}_{2f_c > W_1 + W_2}$

$U_c(f) * U_s(f)$ has bw $\leq W_1 + W_2 < 2f_c$

$$\Rightarrow U_c(f) * U_c(f)\Big|_{\substack{f = 2f_c\\ f=-2f_c}} = 0.$$

$$\Rightarrow \int_{-\infty}^{\infty} u_c(t)\,u_s(t)\,\frac{\sin 4\pi f_c t}{2}\,dt = 0.$$

$$\Rightarrow \|u_p\|^2 = \int_{-\infty}^{\infty} u_c^2(t)\left(\frac{1 + \cos 4\pi f_c t}{2}\right) dt$$

$$+ \int_{-\infty}^{\infty} u_s^2(t)\left(\frac{1 - \cos 4\pi f_c t}{2}\right) dt = \frac{1}{2}\left(\|u_c\|^2 + \|u_s\|^2\right)$$

$$\boxed{\|u_p\|^2 = \frac{1}{2}\left(\|u_c\|^2 + \|u_s\|^2\right) = \frac{1}{2}\left(\|u\|^2\right)}$$

Correlation between two signals

$$\langle u_p, v_p \rangle = \int_{-\infty}^{\infty} u_p(t) v_p(t)\, dt$$

$$= \int_{-\infty}^{\infty} \left(u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t\right) \left(v_c(t)\cos 2\pi f_c t - v_s(t)\sin 2\pi f_c t\right) dt$$

$$= \int_{-\infty}^{\infty} u_c(t) v_c(t) \left(\frac{1 + \cos 4\pi f_c t}{2}\right) dt + \int_{-\infty}^{\infty} u_s(t) v_s(t)\left(\frac{1 - \cos 4\pi f_c t}{2}\right) dt + \left(\underset{=0}{\text{crossterms}}\right)$$

$$= \frac{1}{2}\left[\langle u_c, v_c\rangle + \langle u_s, v_s\rangle\right]$$

Further note that

$$\langle u(t), v(t)\rangle = \int_{-\infty}^{\infty} u(t) v^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} \left(u_c(t) + j u_s(t)\right)\left(v_c(t) - j v_s(t)\right) dt$$

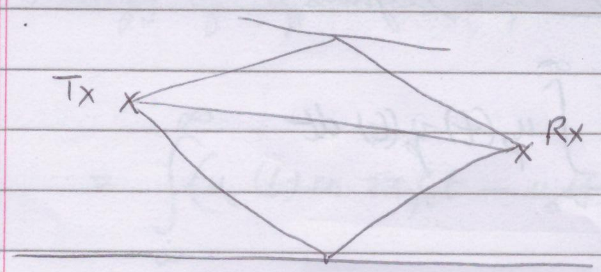
$$= \left(\langle u_c, v_c\rangle + \langle u_s, v_s\rangle\right) + j\left(\langle u_s, v_c\rangle - \langle u_c, v_s\rangle\right)$$

Thus

$$\boxed{\langle u_p, v_p\rangle = \frac{1}{2}\left[\langle u_c, v_c\rangle + \langle u_s, v_s\rangle\right] = \frac{1}{2}\,\mathrm{Re}\left[\langle u, v\rangle\right]}$$

Wireless channel modeling in complex baseband

Key feature of a wireless channel: Multipath



Addition of paths can be constructive or destructive. (Depends on relative locations of Tx & Rx and relative delays of paths)

Consider a passband signal at carrier frequency f_c of the form

$$u_p(t) = u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t$$
$$= e(t)\cos(2\pi f_c t + \theta(t))$$
$$u(t) = e(t)\, e^{j\theta(t)}.$$

For each path of path length r :

Attenuation factor $\frac{1}{r}$

Phase shift ϕ (due to reflections)

Delay $\tau(r) = \frac{r}{c}$, c = speed of light

Received signal (passband) for path i

$$\frac{A_i}{r_i}\, e(t-\tau_i)\cos(2\pi f_c(t-\tau_i) + \theta(t-\tau_i) + \phi_i)$$

Complex baseband received signal

$$\frac{A_i}{r}\, u(t-\tau_i)\, e^{-j(2\pi f_c \tau_i + \phi_i)}$$

Complex envelope of overall received signal

$$\sum_i \frac{A_i}{r_i}\, u(t-\tau_i)\, e^{-j(2\pi f_c \tau_i + \phi_i)}$$

Complex-baseband channel impulse response

$$h(t) = \sum_i \underbrace{\frac{A_i}{r_i} e^{-j(2\pi f_c \tau_i + \phi_i)}}_{\text{This was } \alpha_i \text{ earlier when we had } \sum_i \alpha_i \delta(t-\tau_i)} \delta(t-\tau_i).$$

$\mathcal{F}(\cdot)$

$$H(f) = \sum_i \frac{A_i}{r_i} e^{-j(2\pi f_c \tau_i + \phi_i)} e^{-j2\pi f \tau_i}.$$

Since we are modeling in complex baseband, $f=0$ corresponds to $f=f_c$ for the passband signal.

Channel delay spread $\tau_d = \tau_{max} - \tau_{min}$.
Coherence bandwidth $B_c = \frac{1}{\tau_d}$.

* A baseband signal of bandwidth W is said to be <u>narrowband</u> if $W\tau_d = \frac{W}{B_c} \ll 1$.
(i.e. $W \ll B_c$).

In this case we have the received signal to be approximately a scaled & delayed version of the transmit signal with the scaling

$$h \approx H(0) = \sum_i \frac{A_i}{r_i} e^{-j(2\pi f_c \tau_i + \phi_i)}.$$

<u>Summary</u>:

- Review of signals & systems, Fourier transforms
- Complex-baseband representation of passband signals
  - * Useful in modeling
  - * Useful in implementation & design.