		$\tilde{x}_1$	$\bar{x}_2$	x_3			RHS
z	1	3	5	0	0	1	-26
x_4	0	3	2	0	1	1	2
x_3	0	2	-1	1	0	-1	2

$\Rightarrow$ x_1 enters the basis.

x_2 will not enter the basis as it is at its upper bound.

28/1

$$\min \; -3x_1 - 2x_2 - 2x_3 - 4x_4$$

$$
\begin{aligned}
\text{st.} \quad x_1 + x_2 + 2x_3 + x_4 &\leq 10 \\
x_1 + 2x_2 &\leq 8 \\
x_2 &\leq 5 \\
x_3 + 3x_4 &\leq 6 \\
x_3 &\leq 4
\end{aligned}
$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Decomposition Algorithm

Breakdown the problem into sub problems.

$$\min \quad -3x_1 - 2x_2 - 2x_3 - 4x_4$$

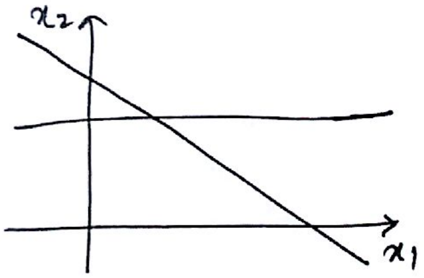
$$\text{st.} \quad x_1 + x_2 + 2x_3 + x_4 \leq 10$$

$$\underset{\sim}{x} \in \mathcal{X}$$

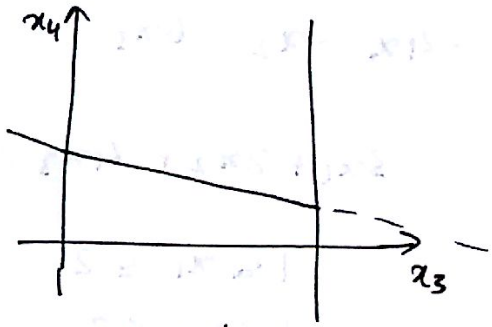
$$\text{where} \quad \mathcal{X} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{X}_1 \text{ and } \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in \mathcal{X}_2 \right\}$$

$$\mathcal{X}_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{array}{r} x_1 + 2x_2 \leq 8 \\ x_2 \leq 3 \end{array} \right\}$$

$$\mathcal{X}_2 = \left\{ \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} : \begin{array}{r} x_3 + 3x_4 \leq 6 \\ x_3 \leq 4 \end{array} \right\}$$



$\mathcal{X}_1$



$\mathcal{X}_2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \beta_{11}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \beta_{12}\begin{pmatrix} 8 \\ 0 \end{pmatrix} + \beta_{13}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \beta_{14}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \beta_{21}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \beta_{22}\begin{pmatrix} 6 \\ 0 \end{pmatrix} + \beta_{23}\begin{pmatrix} 4 \\ 2/3 \end{pmatrix} + \beta_{24}\begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\beta_{11} + \beta_{12} + \beta_{13} + \beta_{14} = 1 \qquad \beta_{1i}\text{'s} \geq 0 \quad \forall i$$

$$\beta_{21} + \beta_{22} + \beta_{23} + \beta_{24} = 1 \qquad \beta_{2j}\text{'s} \geq 0 \quad \forall j$$

Obj. fn : $-3(8\beta_{12} + 2\beta_{13}) - 2(3\beta_{13} + \beta_{14})$

$-2(4\beta_{23} + 6\beta_{22}) - 4(\frac{2}{3}\beta_{23} + 2\beta_{24})$

st. $8\beta_{12} + 2\beta_{13} + 3\beta_{13} + \beta_{14} + 2(4\beta_{23} +$

$+ (\frac{2}{3}\beta_{23} + 2\beta_{24}) \leq 10$

$\beta_{11} + \beta_{12} + \beta_{13} + \beta_{14} = 1$ $\qquad \beta_{1i} \geq 0$

$\beta_{21} + \beta_{22} + \beta_{23} + \beta_{24} = 1$ $\qquad \beta_{2j} \geq 0$

Dantzig - Wolfe Decomposition

min $-4x_1 - x_2 - 6x_3$

st. $3x_1 + 2x_2 + 4x_3 = 17$

$1 \leq x_1 \leq 2$

$1 \leq x_2 \leq 2$

$1 \leq x_3 \leq 2$

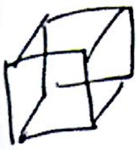
$\equiv$

min $-4x_1 - x_2 - 6x_3$

st. $3x_1 + 2x_2 + 4x_3 = 17$.

$\underset{\sim}{x} \in X$ $\qquad \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$X = \left\{ \underset{\sim}{x} \in \mathbb{R}^3 ; \begin{array}{l} 1 \leq x_1 \leq 2 \\ 1 \leq x_2 \leq 2 \\ 1 \leq x_3 \leq 2 \end{array} \right\}$



$$\begin{pmatrix}2\\2\\2\end{pmatrix}\begin{pmatrix}1\\1\\2\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix}\cdots\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \sum_{j=1}^{8} \lambda_j x^j$$

$$\min \ \underset{\sim}{c}^T \underset{\sim}{x} = \sum_{j=1}^{8} \lambda_j \underset{\sim}{c}^T \underset{\sim}{x}^j$$

st. $A\underset{\sim}{x} = \underset{\sim}{b}$

$$\sum_{j=1}^{8} \lambda_j A x^j = \underset{\sim}{b}$$

$$\sum_{j=1}^{8} \lambda_j = 1 \quad , \lambda_j \geq 0.$$

$$\begin{array}{cccccccc} x^1 & x^2 & x^3 & x^4 & x^5 & x^6 & x^7 & x^8 \\ \begin{pmatrix}2\\2\\2\end{pmatrix} & \begin{pmatrix}1\\1\\2\end{pmatrix} & \begin{pmatrix}2\\1\\2\end{pmatrix} & \begin{pmatrix}1\\2\\2\end{pmatrix} & \begin{pmatrix}1\\2\\1\end{pmatrix} & \begin{pmatrix}2\\1\\1\end{pmatrix} & \begin{pmatrix}2\\2\\1\end{pmatrix} & \begin{pmatrix}1\\1\\1\end{pmatrix} \end{array}$$

$$\underset{\sim}{c}^T \underset{\sim}{x} = \sum_{j=1}^{8} \lambda_j \underset{\sim}{c}^T \underset{\sim}{x}^j$$

st. $\lambda_1 (3 \ 2 \ 4) \begin{pmatrix}2\\2\\2\end{pmatrix} + \lambda_2 (3 \ 2 \ 4) \begin{pmatrix}1\\1\\2\end{pmatrix}$

$+ \cdots$

$\Rightarrow \quad 18\lambda_1 + 13\lambda_2 + \cdots = 17$

$\lambda_1 + \lambda_2 + \cdots + \lambda_8 = 1$

$\lambda_i \geq 0 \quad \forall i$

$c^T x$ $\lambda_j c^T x^j$

	-1	-4	-21
λ_1	$1/5$	$-13/5$	$4/5$
λ_2	$-1/5$	$18/5$	$1/5$

B^{-1} $\quad\quad$ $\bar{b}$

$$B = \begin{array}{cc} \lambda_1 & \lambda_2 \\ \end{array}$$
$$\begin{bmatrix} 18 & 13 \\ 1 & 1 \end{bmatrix}$$

$z_k - \hat{c}_k$

29/1

DECOMPOSITION ALGORITHM

$\min \ c^T x$

st $\quad Ax = b$

$\quad\quad x \in X$

Case (i)

X is a bounded set.

$x^1, x^2, \dots, x^t \rightarrow$ Extreme points of X.

$x \in X$

$$x = \sum_{j=1}^{t} \lambda_j x^j$$

$$\sum_{j=1}^{t} \lambda_j = 1, \quad \lambda_j \geq 0 \quad \forall j$$

$$\min \sum_{j=1}^{t} \lambda_j (\underset{\sim}{c}^T \underset{\sim}{x}^j)$$

st.

$$\sum_{j=1}^{t} \lambda_j (A\underset{\sim}{x}^j) = b_i$$

$$\sum_{j=1}^{t} \lambda_j = 1$$

$$\lambda_j \geq 0 \quad , \; j = 1, 2, \ldots, t.$$

$$\min \quad -4x_1 - x_2 - 6x_3$$

st. $\quad 3x_1 + 2x_2 + 4x_3 = 17$

$$X = \{ \underset{\sim}{x} : \quad \begin{matrix} 1 \leq x_1 \leq 2 \\ 1 \leq x_2 \leq 2 \\ 1 \leq x_3 \leq 2 \end{matrix} \}$$

$$\Downarrow$$

$$\min \sum_{j=1}^{8} \lambda_j \underset{\sim}{c}^T \underset{\sim}{x}^j$$

st.

$$\sum_{j=1}^{8} \lambda_j A\underset{\sim}{x}^j = 1$$

$$\sum_{j=1}^{8} \lambda_j = 1$$

$$\lambda_j \geq 0.$$

$$\lambda = (\lambda_B, \lambda_N)$$

B , B^{-1} known.

$$\begin{array}{cc} d_k & d_l \end{array}$$

$$\begin{bmatrix} A\tilde{x}^k & A\tilde{x}^l \\ 1 & 1 \end{bmatrix} = B$$

$A = [3 \ 2 \ 4]$

$A\tilde{x}^1 = [3 \ 2 \ 4]\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$; $A\tilde{x}^2 = [3 \ 2 \ 4]\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$B = \begin{bmatrix} 18 & 13 \\ 1 & 1 \end{bmatrix}$; $B^{-1} = \begin{bmatrix} 1/5 & -13/5 \\ -1/5 & 18/5 \end{bmatrix}$

$\hat{c}_{d_i} = \tilde{c}^T \tilde{x}^i$

$\tilde{c}_B^T B^{-1}$	$\hat{c}_B \bar{b}$
B^{-1}	$\bar{b} = B^{-1}\binom{k}{1}$

	-1	-4	-21
d_1	$1/5$	$-13/5$	$+4/5$
d_2	$-1/5$	$18/5$	$1/5$

$$\bar{b} = \begin{bmatrix} 1/5 & -13/5 \\ -1/5 & 18/5 \end{bmatrix}\begin{bmatrix} 17 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 1/5 \end{bmatrix}$$

$$z_k - \hat{c}_k = \max_{1 \le j \le t} \{ z_j - \hat{c}_j \}$$

$$z_j = \underbrace{\hat{c}_B^T \; B^{-1}}_{\downarrow} \begin{bmatrix} A(\underset{\sim}{x}^j) \\ 1 \end{bmatrix}$$

$$(\underset{\sim}{w}^T \;\; \alpha) \begin{bmatrix} A\underset{\sim}{x}^j \\ 1 \end{bmatrix} - c^T \underset{\sim}{x}^j$$

$$\max_{1 \le j \le t} \{ \underset{\sim}{w}^T A \underset{\sim}{x}^j + \alpha - c^T \underset{\sim}{x}^j \}$$

$$z_k - \hat{c}_k = \max_{1 \le j \le t} \{ z_j - \hat{c}_j \}$$

$$= \max_{1 \le j \le t} \left[\underset{\sim}{w}^T A \underset{\sim}{x}^j + \alpha - \underset{\sim}{c}^T \underset{\sim}{x}^j \right]$$

SUB PROBLEM

$$\max \quad \underset{\sim}{w}^T A \underset{\sim}{x} + \alpha - \underset{\sim}{c}^T \underset{\sim}{x}$$

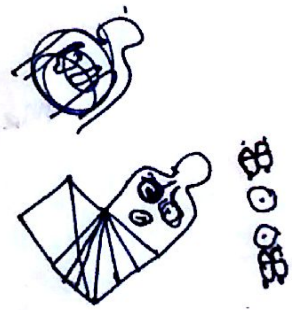
$$\text{s.t.} \quad \underset{\sim}{x} \in \mathcal{X}$$

$$\underset{\sim}{w}^T A = -1(3 \;\; 2 \;\; 4) = (-3 \;\; -2 \;\; -4)$$

$$\max \left[(-3 \;\; -2 \;\; -4) - (-4 \;\; -1 \;\; -6) \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \alpha$$

$$\underset{\sim}{x} \in \mathcal{X}$$

$$\boxed{\begin{array}{l} \max \quad x_1 - x_2 + 2x_3 + \alpha \\ \quad\quad \underset{\sim}{x} \in \mathcal{X} \end{array}}$$



$$\underset{\sim}{x}^3 = \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}$$

$$A\underset{\sim}{x}^3 = \begin{pmatrix} 3 & 2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} A\underset{\sim}{x}^3 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ 1 \end{pmatrix}$$

<u>$z_k - \hat{c}_k$</u>

$$y_k = B^{-1}\begin{pmatrix} 16 \\ 1 \end{pmatrix} = \begin{bmatrix} 1/5 & -13/5 \\ -1/5 & 18/5 \end{bmatrix} \begin{bmatrix} 16 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$$

	$-1/2$	-13	$-43/2$
λ_1	$1/2$	-8	$1/2$
λ_3	$-1/2$	9	$1/2$

<u>Sub Problem</u>

$$\text{Max} \left[\begin{pmatrix} -\frac{3}{2} & -1 & -2 \end{pmatrix} - \begin{pmatrix} -4 & -1 & -6 \end{pmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - 13$$

$$\underset{\sim}{x} \in \mathcal{X}$$

$$\text{Max} \quad \frac{5}{2}x_1 + 0x_2 + 4x_3 - 13$$

$$\underset{\sim}{x} \in \mathcal{X}.$$

$$\frac{1}{2}\begin{pmatrix} \cdot \\ 2 \\ 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \cdot \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ 3/2 \\ 2 \end{pmatrix}$$

4/2/16

Decomposition Method

$$\min \quad \underset{\sim}{c}^T \underset{\sim}{x}$$

$$\text{st.} \quad A\underset{\sim}{x} = \underset{\sim}{b}$$

$$\underset{\sim}{x} \in X$$

Case (i)

X is bounded set.

$\underset{\sim}{x}^1, \underset{\sim}{x}^2, \dots, \underset{\sim}{x}^t.$

$$\underset{\sim}{x} \in X \quad : \quad \sum_{j=1}^{t} \lambda_j \underset{\sim}{x}^j$$

$$\sum \lambda_j = 1, \quad \lambda_j \geq 0.$$

$$\min \quad \sum_{j=1}^{t} (c^T \underset{\sim}{x}^j)\lambda_j$$

$$\hat{\underset{\sim}{c}}_B B^{-1} = \underset{\sim}{w}\ \alpha$$

$$\text{st.} \quad \sum_{j=1}^{t} (A\underset{\sim}{x}^j)\lambda_j = \underset{\sim}{b} \longrightarrow \underset{\sim}{w}$$

$\underset{\sim}{w}\ \alpha$	$\hat{\underset{\sim}{c}}_B \bar{b}$
B^{-1}	$\bar{b}$

$$\sum_{j=1}^{t} \lambda_j = 1 \longrightarrow \alpha$$

$$\lambda_j \geq 0.$$

Sub problem

$$\max \quad (w^T A - c)x \quad + \alpha$$

$$\text{s.t.} \quad x \in X$$

Case (ii)

X is unbounded set. $\Leftarrow$

$$\min \quad c^T \left[\sum_{j=1}^{t} \lambda_j x^j + \dots \right.$$

$$\text{s.t.} \quad Ax \le b$$

$$x \in X$$

$x^1, x^2, \dots, x^t$ | $d^1, d^2, \dots, d_l$

$x \in X$: $\displaystyle\sum_{j=1}^{t} \lambda_j x^j + \sum_{j=1}^{l} \mu_j d_j$

$$\sum_{j=1}^{t} \lambda_j = 1 \quad ; \quad \lambda_j, \mu_j \ge 0.$$

$$\min \quad \sum_{j=1}^{t} (c^T x^j) \lambda_j + \sum_{j=1}^{l} (c^T d_j) \mu_j$$

s.t.

$$\sum_{j=1}^{t} (A x_j) \lambda_j + \sum_{j=1}^{l} (A d_j) \mu_j = b \rightarrow w$$

$$\sum_{j=1}^{t} \lambda_j = 1 \qquad \rightarrow \alpha$$

$$\lambda_j, \mu_j \ge 0.$$

Example

$$\text{Min} \quad -x_1 - 2x_2 - x_3$$

$$\begin{aligned} \text{st} \quad & x_1 + x_2 + x_3 \leq 12 \longrightarrow A = [1 \ 1 \ 1] \ ; \ b = 12 \\ & -x_1 + x_2 \leq 2 \\ & -x_1 + 2x_2 \leq 8 \\ & x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \Bigg] \ \underset{\sim}{x} \in \mathcal{X}$$

$$\mathcal{X} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{array}{r} -x_1 + x_2 \leq 2 \\ -x_1 + 2x_2 \leq 8 \\ x_3 \leq 3 \end{array} \right\}$$

$A = [1, 1, 1]$

$b = 12$

Non basic variables $\quad z_j - \hat{c}_j$

Non-basic x_j : $\quad w^T(A\underset{\sim}{x}^j) - (c^T\underset{\sim}{x}^j) + \alpha$

$$(w^T A - c)\underset{\sim}{x} + \alpha$$

Non basic d_j : $\quad (w \ \alpha)\begin{pmatrix} A\underset{\sim}{d}_j \\ 0 \end{pmatrix} - (c^T \underset{\sim}{d}_j)$

$$= w^T A \underset{\sim}{d}_j - \underset{\sim}{c}^T \underset{\sim}{d}_j = (w^T A - c)\underset{\sim}{d}_j$$

Subproblem

$$\max \quad (w^T A - c)x + \alpha$$

$$\text{s.t.} \quad x \in \mathcal{X}.$$

5/2/16

$$\min \quad -x_1 - 2x_2 - x_3$$

$$\text{st.} \quad x_1 + x_2 + x_3 \leq 12$$

$$-x_1 + x_2 \leq 2$$

$$-x_1 + 2x_2 \leq 8$$

$$x_3 \leq 3$$

$$x_1, x_2, x_3 \geq 0$$

$\rightarrow$

$$\min \quad c^T x$$

$$\text{st.} \quad Ax \leq b$$

$$x \in X$$

$$X = \left\{ x \in \mathbb{R}^3 \,\middle|\, \begin{array}{l} -x_1 \ldots \\ -x_1 \ldots \\ x \ldots \\ x_1, x_2, \ldots \end{array} \right.$$

$$Ax' = (1, 1, 1)\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$x' = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x = \sum_{j=1}^{t} \lambda_j x^j + \sum_{j=1}^{l} \mu_j d_j$$

MASTER PROBLEM

$$\min \quad \sum_{j=1}^{t} (c^T x^j) \lambda_j + \sum_{j=1}^{l} (c^T d_j \ldots$$

$$\text{st.} \quad \sum_{j=1}^{t} A(x^j)\lambda_j + \sum_{j=1}^{l} (Ad_j)\mu_j \leq$$

$$\sum_{j=1}^{t} \lambda_j = 1$$

$$\mu_j, \lambda_j \geq 0.$$

$$Ax' \leq b \longrightarrow \text{extreme point.}$$

$$\left[\begin{array}{c|c} I & Ax^j \\ \hline 0 & 1 \end{array}\right]$$

$$B = \begin{array}{c} s \\ \lambda_1 \end{array} \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \right]$$

Revised Simplex

$$\left[\begin{array}{c|c} w\alpha & \hat{c}_B \bar{b} \\ \hline B^{-1} & \bar{b} = B^{-1}\begin{pmatrix} \underset{\sim}{b} \\ 1 \end{pmatrix} \end{array} \right]$$

$$\hat{c}_B = (\hat{c}_s, \hat{c}_\lambda)$$

$$\hat{c}_B B^{-1} = (\underset{\sim}{w}, \alpha)$$

$$\begin{array}{c} \\ s \\ \lambda_1 \end{array} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ \hline 1 & 0 & 12 \\ 0 & 1 & 1 \end{array} \right]$$

$z_j - \hat{c}_j \leq 0 \ \forall$ non-basic u_j, d_j.

~~Sub problem~~

For non-basic d_j,

$$z_j - \hat{c}_j = \hat{c}_B B^{-1} \begin{pmatrix} A\underset{\sim}{x}^j \\ 1 \end{pmatrix} - c^T \underset{\sim}{x}^j$$

$$= (w^T A - c) \underset{\sim}{x}^j + \alpha$$

For basic u_j,

$$z_j - \hat{c}_j = \hat{c}_B B^{-1} \begin{pmatrix} A d_j \\ 0 \end{pmatrix} - \underset{\sim}{c}^T \underset{\sim}{d_j}$$

$$= (w^T A - c) \underset{\sim}{d_j}$$

Sub problem

$$\max \; (w^T A - c)\underset{\sim}{x} + \alpha$$

$$\text{st.} \quad x \in X .$$

Case (i) : Solution to subproblem is unbounded.

$$\exists k_0 : (w^T A - c)^T \underset{\sim}{d}_k > 0$$

$$\max \quad x_1 + 2x_2 + x_3$$

$$\text{st.} \quad \underset{\sim}{x} \in X .$$

	z	x_1	x_2	x_4	x_5	RHS
z	1	-1	-2	0	0	0
x_4	0	-1	1	1	0	2
x_5	0	-1	2	0	1	8

	z	x_1	x_2	x_4	x_5	RHS
z	1	0	0	-4	3	16
x_2	0	0	1	-1	1	6
x_1	0	1	0	-2	1	4

all y_k negative $\Rightarrow$ unbounded sol.

direction of unboundedness $= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$z_k - \hat{c}_k = 4$

$$\underset{\sim}{y_k} = B^{-1}\begin{pmatrix} A\underset{\sim}{d_k} \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{array}{c} \\ s \\ d_1 \end{array}\left[\begin{array}{cc|c} 0 & 0 & 0 \\ \hline 1 & 0 & 12 \\ 0 & 1 & 1 \end{array}\right] \begin{array}{c} 4 \\ 3 \\ 0 \end{array} \longrightarrow \begin{array}{c} \\ \mu_1 \\ d_1 \end{array}\left[\begin{array}{cc|c} -4/3 & 0 & -16/3 \\ \hline 1/3 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

$$\max \quad -\frac{1}{3}x_1 + \frac{2}{3}x_2 - \frac{1}{3}x_3$$

$$\text{st.} \quad \underset{\sim}{x} \in X.$$

$$\hat{c}_B = \left(\frac{2}{3}, \frac{-1}{3}\right)$$

	z	x_1	x_2	x_4	x_5	RHS
z	1	0	0	0	1/3	8/3
x_2	0	0	1	-1	1	6
x_1	0	1	0	-2	1	3

8|2|16

$$\min \quad \underset{\sim}{C}^T\underset{\sim}{x}$$

$$\text{st.} \quad A\underset{\sim}{x} = \underset{\sim}{b} \longrightarrow \underset{\sim}{w}$$

$$\underset{\sim}{x} \in X \longrightarrow \underset{\sim}{v}$$

dual variables.

$$X = \{\underset{\sim}{x} : D\underset{\sim}{x} \geq d ;\ \underset{\sim}{x} \geq 0\}$$

X is non-empty and bounded.

↑

Primal Problem.

Dual problem

$$\max \quad b^T w + d^T v$$
$$\text{st.} \quad \underset{\sim}{w}^T A + \underset{\sim}{v}^T D \le C^T$$
$$\underset{\sim}{w} \ge 0, \quad \underset{\sim}{v} \ge 0$$

$$\equiv$$

$$\max_{w \ge 0} \left\{ b^T w + \underbrace{\begin{array}{l} \max\ d^T v \\ \text{st.}\ v^T D \le C^T - \underset{\sim}{w}^T A \\ \underset{\sim}{v} \ge 0 \end{array}} \right\}$$

$\downarrow$ dual

$$\begin{array}{ll} \min & (C^T - w^T A)\underset{\sim}{x} \\ \text{st.} & D\underset{\sim}{x} \ge \underset{\sim}{d} \\ & \underset{\sim}{x} \ge 0 \end{array}$$

$$= \max_{w \ge 0} \left\{ b^T w + \min_{\underset{\sim}{x} \in X} (C^T - w^T A)\underset{\sim}{x} \right\}$$

$$= \max_{w \ge 0} \underbrace{\left\{ b^T w + \min_{j=1,\dots,t} (C^T - w^T A)\underset{\sim}{x}^j \right\}}_{z}$$

Let $\underset{\sim}{x}^1, \underset{\sim}{x}^2, \dots, \underset{\sim}{x}^t$ be extreme points of X.

Master Problem

$$\text{Max } z.$$

$$\text{st. } \quad z \le \underset{\sim}{b}^T \underset{\sim}{w} + (C^T - w^T A)\underset{\sim}{x}^j \quad , j=1,2,\dots,t$$

z, w unrestricted.

Example

$$\text{Min} \quad -2x_1 - x_2 - x_3 + x_4$$

$$\begin{aligned}
\text{st.}\ (w_1)\quad & x_1 + x_3 \le 2 \\
(w_2)\quad & x_1 + x_2 + 2x_4 \le 3
\end{aligned} \Bigg] \ A\underset{\sim}{x} \le \underset{\sim}{b}$$

$$\left.\begin{aligned}
(v_1)\quad & x_1 \le 2 \\
(v_2)\quad & x_1 + 2x_2 \le 5
\end{aligned}\right\} \begin{pmatrix}0\\0\end{pmatrix} \begin{pmatrix}2\\0\end{pmatrix} \begin{pmatrix}0\\5/2\end{pmatrix} \begin{pmatrix}2\\3/2\end{pmatrix}$$

$$\left.\begin{aligned}
(v_3)\quad & -x_3 + x_4 \le 2 \\
(v_4)\quad & 2x_3 + x_4 \le 6.
\end{aligned}\right\} \begin{pmatrix}0\\2\end{pmatrix} \begin{pmatrix}4/3\\10/3\end{pmatrix} \begin{pmatrix}3\\0\end{pmatrix} \begin{pmatrix}0\\0\end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \ge 0.$$

Dual

$$\text{max} \quad 2w_1 + 3w_2 + 2v_1 + 5v_2 + 2v_3 + 6v_4$$

$$\begin{aligned}
\text{st.}\quad & w_1 + w_2 + v_1 + v_2 \le -2 \\
& w_2 + 2v_2 \le -1 \\
& w_1 - v_3 + 2v_4 \le -1 \\
& 2w_1 + v_3 + v_4 \le 1 \\
& w_1, w_2, v_1, v_2, v_3, v_4 \le 0.
\end{aligned}$$

$$X = \left\{ \underset{\sim}{x} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} : \begin{matrix} x_1 \le 2 \\ x_1 + 2x_2 \le 5 \end{matrix} , \begin{matrix} -x_3 + x_4 \le 2 \\ 2x_3 + x_4 \le 6 \end{matrix} ; \ \underset{\sim}{x} \ge 0 \right\}$$

Max z

st. $z \leq 2w_1 + 3w_2 + (C^T - W^T A)x$.

$W^T A$

$(w_1 \; w_2) \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}$

$= (w_1 + w_2 , \; w_2 , \; w_1 , \; 2w_2)$

$C^T = (-2, -1, -1, 1)$

$C^T - W^T A = (-2 - w_1 - w_2 , \; -1 - w_2 , \; -1 - w_1 , \; 1 - 2w_2)$

<u>Master problem</u>

Max z

st. $z \leq 2w_1 + 3w_2 + (-2 - w_1 - w_2)x_1^j$
$+ (-1 - w_2)x_2^j$
$+ (-1 - w_1)x_3^j + (1 - 2w_2)x_4^j$

$w_1, w_2 \leq 0$.

; $x^j = \begin{pmatrix} x_1^j \\ x_2^j \\ x_3^j \\ x_4^j \end{pmatrix}$

For ~~bounda~~ extreme point $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$:

max z

st. $z \leq 2w_1 + 3w_2$

$w_1, w_2 \leq 0$.

Sub problem

$$\min \ \underset{\sim}{b}^T \underset{\sim}{\bar{w}} + (c^T - \bar{w}^T A)\, \underline{x}^0$$

11/2/16

Primal | **Dual**

$\min \ \underset{\sim}{c}^T \underset{\sim}{x}$ $\longrightarrow$ ~~Dual~~

st $\quad A\underset{\sim}{x} = \underset{\sim}{b} \longrightarrow \underset{\sim}{w}$

$\underset{\sim}{x} \in X = \left\{ \underset{\sim}{x} : \begin{array}{l} D\underset{\sim}{x} \geq \underset{\sim}{d} \\ \underset{\sim}{x} \geq 0 \end{array} \right\} \longrightarrow \underset{\sim}{v}$

$\max \ \underset{\sim}{w}^T \underset{\sim}{b} + \underset{\sim}{v}^T \underset{\sim}{d}$

st. $\quad \underset{\sim}{w}^T A + \underset{\sim}{v}^T D \leq \underset{\sim}{c}^T$

$\quad \underset{\sim}{w} \gtrless 0 \ , \ \underset{\sim}{v} \geq 0$

Assumption

X is non-empty and bounded.

$$\max_{w \gtrless 0} \left\{ \underset{\sim}{w}^T \underset{\sim}{b} + \begin{array}{l} \max \ \underset{\sim}{v}^T \underset{\sim}{d} \\ \text{st. } \ \underset{\sim}{v}^T D \leq \underset{\sim}{c}^T - \underset{\sim}{w}^T A \\ \quad \ \underset{\sim}{v} \geq 0 \end{array} \right\}$$

$$\max_{w \gtrless 0} \left\{ \underset{\sim}{w}^T b + \begin{array}{l} \min \ (c^T - w^T A)\underset{\sim}{x} \\ \text{st. } \ \underset{\sim}{x} \in X \end{array} \right\}$$

MASTER PROBLEM

Max z

st. $z \leq \underset{\sim}{w}^T \underset{\sim}{b} + (\underset{\sim}{c}^T - \underset{\sim}{w}^T A)\underset{\sim}{x}$

$j = 1, 2, \ldots, t$

$\underset{\sim}{w} \geq 0$; $(\bar{z}, \bar{w})$ solution in relaxed problem

$\theta(w) = \underset{\sim}{w}^T \underset{\sim}{b} + \min_{x \in X} (c^T - w^T A)\underset{\sim}{x}$

max $\theta(w)$: $w \geq 0$

SUB PROBLEM

$\underset{\sim}{\bar{w}}^T \underset{\sim}{b} + \min_{\underset{\sim}{x} \in X} (c^T - \bar{w}^T A)\underset{\sim}{x}$

---

no notes for 11/2/16

12/2/16

Primal	Dual
min $\underset{\sim}{c}^T \underset{\sim}{x}$	max $\underset{\sim}{w}^T \underset{\sim}{b}$
st $A\underset{\sim}{x} = \underset{\sim}{b}$	st $\underset{\sim}{w}^T A \leq \underset{\sim}{c}^T$
$\underset{\sim}{x} \geq 0$	$w \geq 0$

MASTER PROBLEM

Max z

st. $z \le \underset{\sim}{w}^T \underset{\sim}{b} + (\underset{\sim}{c}^T - \underset{\sim}{w}^T A)\underset{\sim}{x}$

$j = 1, 2, \dots, t$

$\underset{\sim}{w} \geq 0$; $(\bar{z}, \bar{w})$ solution in relaxed problem

$\theta(w) = \underset{\sim}{w}^T \underset{\sim}{b} + \min_{x \in X} (c^T - w^T A)\, \underset{\sim}{x}$

max $\theta(w)$: $w \geq 0$.

SUB PROBLEM

$\underset{\sim}{\bar{w}}^T \underset{\sim}{b} + \min_{\underset{\sim}{x} \in X} (c^T - \bar{w}^T A)\, \underset{\sim}{x}$

---

no notes for 11/2/16

12/2/16

Primal

min $\underset{\sim}{c}^T \underset{\sim}{x}$

st $A\underset{\sim}{x} = \underset{\sim}{b}$

$\underset{\sim}{x} \geq 0$

Dual

max $\underset{\sim}{w}^T \underset{\sim}{b}$

st $\underset{\sim}{w}^T A \le \underset{\sim}{c}^T$

$w \geq 0$