7)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$x_1(0), x_2(0)$ & $x_3(0)$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Phi(t) = e^{At}$$

$$= \left(I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^k t^k}{k!} + \cdots\right) \quad \begin{bmatrix} 8 & 12 & 6 \\ 0 & 8 & 12 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} t + \begin{bmatrix} 4 & 4 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} \frac{t^2}{2!}$$

$$+ \begin{bmatrix} 8 & 12 & 6 \\ 0 & 8 & 12 \\ 0 & 0 & 8 \end{bmatrix} \frac{t^3}{3!} + \cdots$$

$$= \quad 1 + 2t + 2t^2 + \frac{\cancel{8}4}{3}t^3 + \cdots \qquad t + 2t^2 + 2t^3 \qquad t^2/2 + t^3$$

$$\begin{bmatrix} 1 + 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} + \cdots & t + \frac{4t^2}{2!} + \frac{12t^3}{3!} \quad t(1+2t+ & \frac{t^2}{2!} + \frac{6t^3}{3!} \\ 0 & 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} + \cdots & t + \frac{4t^2}{2!} + \frac{12t^3}{3!} \\ 0 & 0 & 1 + 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} \end{bmatrix}$$

$$\begin{bmatrix} e^{2t} & te^{2t} & \frac{t^2}{2} e^{2t} \\ 0 & e^{2t}-1 & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

$x_h(t) = \phi(t)x(0)$

$$\boxed{\begin{aligned} x_1(t) &= x_1(0)\, e^{2t} + x_2(0)\, te^{2t} + x_3(0) \frac{t^2}{2} e^{2t} \\ x_2(t) &= \cancel{x_2(0)}\; x_2(0)(e^{2t}-1) + x_3(0)\, te^{2t} \\ x_3(t) &= e^{2t} x_3(0) \end{aligned}}$$

6. $A = \operatorname{diag}\{a_{11}, a_{22}, \dots, a_{mm}\}$ $e^{A} = \operatorname{diag}\{e^{a_{11}}, e^{a_{22}}, \dots, e^{a_{mm}}\}$

(a) $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 16 & 12 \\ 0 & 4 & 0 \\ 0 & 9 & 1 \end{bmatrix}$

$2+2+12$

$\phi(t) = e^{At}$

$= I + At + \frac{A^2t^2}{2!} + \dots + \frac{A^k t^k}{k!} + \dots$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} t + \begin{bmatrix} 4 & 16 & 2 \\ 0 & 4 & 0 \\ 0 & 9 & 1 \end{bmatrix} t^2/2!$

$\begin{bmatrix} 8 & 72 & 28 \\ 0 & 8 & 0 \\ 0 & 21 & 1 \end{bmatrix} t^3/3!$

$4+31+32$

$4+3$

$\begin{bmatrix} 4 & 16 & 12 \\ 0 & 4 & 0 \\ 0 & 9 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & \cancel{38}\ 72 & 28 \\ 0 & 8 & 0 \\ 0 & 21 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1+2t+\frac{4t^2}{2!}+\frac{8t^3}{3!}+\cdots & t+\frac{\cancel{6}^{8}t^2}{2!}+\frac{\cancel{24}^{12}t^3}{3!} & 4t+\frac{2t^2}{2!}+\frac{28t^3}{3!}+ \\ 0 & e^{2t} & 0 \\ 0 & 3t+\frac{9t^2}{2!}+\frac{21t^3}{3!}+\cdots & e^t \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & t+8t^2+12t^3 & 4t+\frac{2t^2}{2!}+\frac{28t^3}{3!}+\cdots \\ 0 & e^{2t} & 0 \\ 0 & 3t\left[1+\frac{3t}{2!}+\frac{7t^2}{3!}+\cdots\right] & e^t \end{bmatrix}$$

(2)

B is an $n\times n$ matrix.

$$e^{Q^{-1}BQ} = Q^{-1}e^{B}Q$$

$Q^{-1}PQ$ → Diagonal matrix ⇒ $Q^{-1}BQ$ → diagonal matrix with eigen values of B

$$\Rightarrow \begin{bmatrix} e^{d_1t} & 0 & \cdots & 0 \\ 0 & e^{d_2t} & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & e^{d_nt} \end{bmatrix} = e^{Dt}$$

$$e^{At} = \begin{matrix} e^{a_1t} & \cdots & 0 \\ 0 & e^{a_2t}\cdots & 0 \end{matrix}$$

(c) $e^A e^B = e^{(A+B)}$ iff $AB = BA$

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & \cdots & 0 \\ & & \vdots \\ 0 & & \end{bmatrix}$$

$e^{A+B} = a_{11}+b_{11} \quad 0 \cdots \quad 0$

$$e^A = \begin{bmatrix} e^{a_{11}} & 0 & \cdots & 0 \\ 0 & e^{a_{22}} & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & e^{a_{nn}} \end{bmatrix} \quad e^B = \begin{bmatrix} e^{b_{11}} & 0 & \cdots & 0 \\ 0 & e^{b_{22}} & \cdots & \\ \vdots & & & \\ 0 & \cdots & & e^{b_{nn}} \end{bmatrix}$$

$$e^A \cdot e^B = \begin{bmatrix} e^{a_{11}} \cdot e^{b_{11}} & 0 & \cdots & 0 \\ 0 & e^{a_{22}} \cdot e^{b_{22}} & \cdots & 0 \\ \vdots & & & e^{a_{nn}} \cdot e^{b_{nn}} \\ 0 & \cdots & & \end{bmatrix}$$

ifdl

Q5.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

transformed to

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda+6 & -1 & 0 \\ 11 & \lambda & -1 \\ 6 & 0 & \lambda \end{vmatrix} = 0$$

$\lambda+6(\lambda^2) + \lambda(11\lambda+6) = 0$

$(\lambda+6)(\lambda^2) + 1(11\lambda+6)$

$\lambda^2 + 6\lambda + 11\lambda^2 + 6\lambda = 0$

$\lambda^3 + 6\lambda^2 + 11\lambda^2 + 6\lambda = 0$

$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$

$\cancel{\lambda^2 + 17\lambda + 6 = 0}$

$\lambda(\lambda^2 + 17\lambda + 6) = 0$

$\lambda = -1, -2, -3$

$12\lambda^2 + 12\lambda = 0$

$289 - 24$
255

$12\lambda(10\lambda + 1) = 0$

$\lambda = 0, \lambda = -1$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$P\dot{z} = APz + Bu$

$x = Pz$

$\dot{z} = P^{-1}APz + P^{-1}Bu$

$x = Pz$

$\dot{x} = Ax + Bu$

$P\dot{z} = APz + Bu$

$\dot{z} = \underbrace{P^{-1}APz}_{①} + P^{-1}Bu$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}$$

$$P^{-1}B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$B = P\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = B =$

$P.C.C^{-1} = B.C$

$P. =$

$a_{11} = 3$

$a_{21} = 6$

$a_{31} = 2$

$AP = PB$

$-6a_{11} + -11a_{12} - 6a_{13} = 0$

a_{11}

5. $x = Pz$

$\dot{x} = P\dot{z}$

$P\dot{z} = APz + Bu$

$\dot{z} = P^{-1}APz + P^{-1}Bu$

$y = Cx + Du$

$y = CPz + Du$

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

$a_{11} = 2$

$a_{21} = 6$

$a_{31} = 2$

$AP = PQ$.

$$\begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

$$\begin{bmatrix} -6a_{11}+a_{21} & -6a_{12}+a_{22} & -6a_{13}+a_{23} \\ -11a_{11}+a_{31} & -11a_{12}+a_{32} & -11a_{13}+a_{33} \\ -6a_{11} & -6a_{12} & -6a_{13} \end{bmatrix} = \begin{bmatrix} a_{12} & a_{13} & -6a_{11}-11a_{12}-6a_{13} \\ a_{22} & a_{23} & -6a_{21}-11a_{22}-6a_{23} \\ a_{32} & a_{33} & -6a_{31}-11a_{32}-6a_{33} \end{bmatrix}$$

$-6a_{11} + a_{21} = a_{12}$

$a_{32} = -12$, $\quad -12 + 6 = a_{12}$

$a_{12} = -6$

$a_{22} = -22 + 2$

$= -20$

$36 - 20 = a_{13} = 16$

$a_{23} - 96 = -12 + 66 - 96$

$= -42$

$a_{33} = 36$

$-11 \times 16 + 36 = -36 - 11 \times -20 - 6a_{23}$

$-176 + 36 = -36 + 220 - 6a_{23}$

$\dfrac{-176 + 72 - 220}{6} = -a_{23}$

$a_{23} = 54$

$$P = \begin{bmatrix} 2 & -6 & 16 \\ 6 & -20 & -42 \\ 2 & -12 & 54 \end{bmatrix}$$

$x = Pz.$

$$y = \begin{bmatrix} 2 & -6 & 16 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}.$$

4.

$$A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

$$\text{Jordan}(A) = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$ using matlab.

1.

$\ddot{\theta} + g/L \sin\theta + k/M \dot{\theta} = 0$

$x_1 = \theta$

$x_2 = \dot{x}_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{L} \sin x_1 - k/M\, x_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -g/L \cos x_1 & -k/M \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ linearization

equip

$$\frac{dx_1}{dt} = 0 \Rightarrow x_2 = 0$$

$$\frac{dx_2}{dt} = 0 \Rightarrow \quad -\frac{g}{L}\sin x_1 - \frac{k}{M}x_2 = 0$$

$$\Rightarrow x_1 = \pm n\pi$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & -k/M \end{bmatrix}_{(2n\pi, 0)} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ +g/L & -k/M \end{bmatrix}_{((2n+1)\pi, 0)}$$

$$y = \theta = x_1$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2.

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^n + b_1 u^{n-1} + \cdots + b_{n-1}\dot{u} + b_n u$$

$$y_1 = x - \beta_0 u$$

$$y_2 = \dot{x} - \beta_0\dot{u} - \beta_1 u = \dot{y}_1 - \beta_1 u$$

$$y_3 = \ddot{x} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u = \dot{y}_2 - \beta_2 u$$

$$\vdots$$

$$\beta_0 = b_0$$

$$\beta_1 = b_1 - a_1\beta_0$$

$$\beta_2 = b_2 - a_2\beta_0 - a_2\beta_0$$

$$\vdots$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} u$$

$$x = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + u$$

$$s^3 X(s) + 3s^2 X(s) + 3s X(s) + X(s) = 3s^3 U(s) + 2s^2 U(s) + 4s U(s) + U(s)$$

$$X(s)\,(s^3 + 3s^2 + 3s + 1) = U(s)\,(3s^3 + 2s^2 + 4s + 1)$$

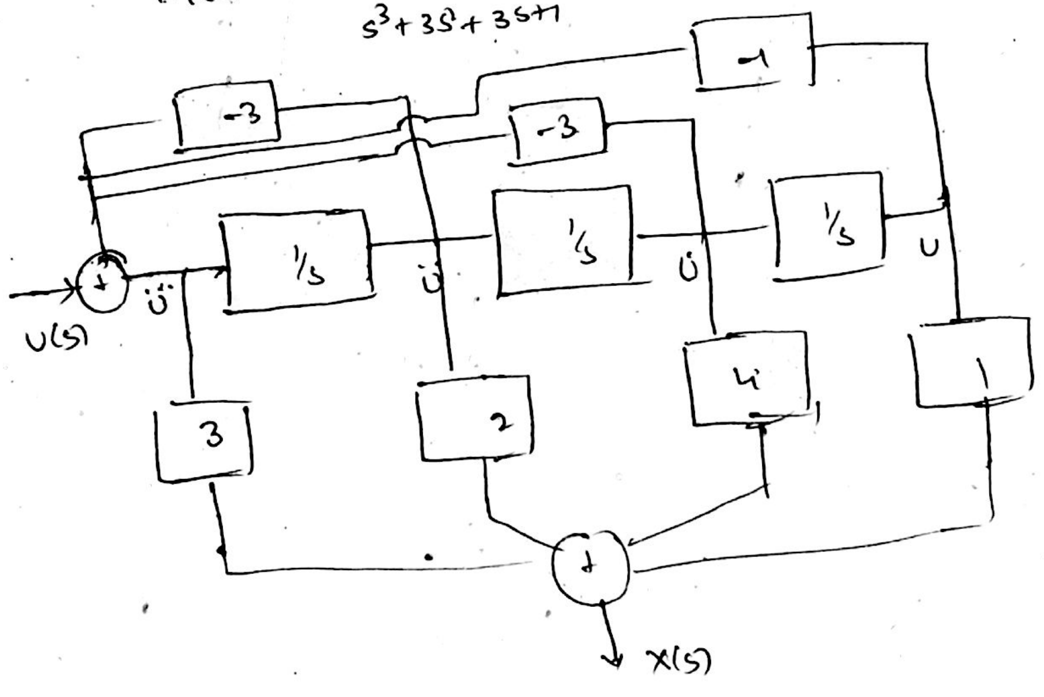
$$= \frac{3s^3 + 2s^2 + 4s + 1}{s^3 + 3s^2 + 3s + 1}$$

$$Z(s) = H_1(s)\,U(s).$$

$$X(s) = H_2(s)\,Z(s)$$

$$\frac{X(s)}{U(s)} = \frac{cs^2 + ds + e}{s^2 + as + b}$$

$$H_1(s) = \frac{1}{s^3 + 3s^2 + 3s + 1}$$



3.

$$T(s) = \frac{Y(s)}{U(s)} = \frac{s^2+2s+10}{s^3+4s^2+6s+10} = \frac{\frac{1}{s}}{1+10\left(\frac{1+\frac{4}{s}}{\frac{1+4+6}{12}}\right.}$$

$$T(s) = \frac{as^2+bs+c}{ds^3+es^2+fs+g}$$

$Z(s) = H_1(s)\,X(s)$ | $Y(s) = H_2(s)\,Z(s)$

$$= \frac{1}{s^3+4s^2+6s+10}\,X(s)$$