Systems & Control: Foundations & Applications

Daniel Liberzon

Switching in Systems and Control

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Contents

Preface

Many systems encountered in practice involve a coupling between continuous dynamics and discrete events. Systems in which these two kinds of dynamics coexist and interact are usually called *hybrid.* For example, the following phenomena give rise to hybrid behavior: a valve or a power switch opening and closing; a thermostat turning the heat on and off; biological cells growing and dividing; a server switching between buffers in a queueing network; aircraft entering, crossing, and leaving an air traffic control region; dynamics of a car changing abruptly due to wheels locking and unlocking on ice. Hybrid systems constitute a relatively new and very active area of current research. They present interesting theoretical challenges and are important in many real-world problems. Due to its inherently interdisciplinary nature, the field has attracted the attention of people with diverse backgrounds, primarily computer scientists, applied mathematicians, and engineers.

Researchers with a background and interest in continuous-time systems and control theory are concerned primarily with properties of the continuous dynamics, such as Lyapunov stability. A detailed investigation of the discrete behavior, on the other hand, is usually not a goal in itself. In fact, rather than dealing with specifics of the discrete dynamics, it is often useful to describe and analyze a more general category of systems which is known to contain a particular model of interest. This is accomplished by considering continuous-time systems with discrete switching events from a certain class. Such systems are called *switched systems* and can be viewed as higher-level abstractions of hybrid systems, although they are of interest in their own right.

The present book is not really a book on hybrid systems, but rather a book on switched systems written from a control-theoretic perspective. In particular, the reader will not find a formal definition of a hybrid system here. Such a definition is not necessary for the purposes of this book, the emphasis of which is on formulating and solving stability analysis and control design problems and not on studying general models of hybrid systems. The main goal of the book is to bridge the gap between classical mathematical control theory and the interdisciplinary field of hybrid systems, the former being the point of departure. More specifically, system-theoretic tools are used to analyze and synthesize systems that display quite nontrivial switching behavior and thus fall outside the scope of traditional control theory.

This book is based on lecture notes for an advanced graduate course on hybrid systems and control, which I taught at the University of Illinois at Urbana-Champaign in 2001–2002. The level at which the book is written is somewhere between a graduate textbook and a research monograph. All of the material can be covered in a semester course, although the instructor will probably need to skip some details in the treatment of more advanced topics and assign them as supplementary reading. The book can also serve as an introduction to the main research issues and results on switched systems and switching control for researchers working in various areas of control theory, as well as a reference source for experts in the field of hybrid systems and control.

It is assumed that the reader is familiar with basic linear systems theory. Some results on existence and uniqueness of solutions to differential equations, Lyapunov stability of nonlinear systems, nonlinear stabilization, and mathematical background are reviewed in suitable chapters and in the appendices. This material is covered in a somewhat informal style, to allow the reader to get to the main developments quickly. The level of rigor builds up as the reader reaches more advanced topics. My goal was to make the presentation accessible yet mathematically precise.

The main body of the book consists of three parts. The first part introduces the reader to the class of systems studied in the book. The second part is devoted to stability theory for switched systems; it deals with single and multiple Lyapunov function analysis methods, Lie-algebraic stability criteria, stability under limited-rate switching, and switched systems with various types of useful special structure. The third part is devoted to switching control design; it describes several wide classes of continuous-time control systems for which the logic-based switching paradigm emerges naturally as a control design tool, and presents switching control algorithms for several specific problems such as stabilization of nonholonomic systems, control with limited information, and switching adaptive control of uncertain systems. At the moment there is no general theory of switching control or a standard set of topics to discuss, and the choice of material in this part is based largely on my personal preferences. It is hoped, however, that the book will contribute to creating a commonly accepted body of material to be covered in courses on this subject.

Typically, results are first developed for linear systems and then extended to nonlinear systems. Complete proofs of most of the results are provided, other proofs are given in sketched form. A few exercises are scattered throughout the text. Since the book focuses on theoretical developments, students interested in applications will need to study other sources. In the course that I taught at the University of Illinois, the students were required to do final projects in which they could apply the theory developed in class to practical problems.

I would like to call special attention to the Notes and References section at the end of the book. It complements the main text by providing many additional comments and pointers to a large body of literature, from research articles on which this book is based to a variety of related topics not covered here. The reader should remember to consult this section often, as references in the main text are kept to a minimum. The literature on the subject is growing so rapidly, however, that the bibliography supplied here will quickly go out of date.

I am indebted to many people who influenced my thinking and offered valuable advice on the material of this book. I would especially like to thank my former advisors: Andrei Agrachev, Roger Brockett, and Steve Morse. This book would not have been possible without the research contributions of João Hespanha. It has also benefited greatly from my interactions with Eduardo Sontag. I am grateful to my colleagues at the University of Illinois for creating a very stimulating environment, and particularly to Tamer Başar who encouraged me to teach a course on hybrid systems and to publish this book. I am thankful for the numerous corrections and comments that I received from students while teaching the course. The support of the National Science Foundation and the DARPA/AFOSR MURI Program is gratefully acknowledged.

Champaign, IL

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Switching in Systems and Control

Part I

Introduction

1

Basic Concepts

1.1 Classes of hybrid and switched systems

Dynamical systems that are described by an interaction between continuous and discrete dynamics are usually called *hybrid systems*. Continuous dynamics may be represented by a continuous-time control system, such as a linear system $\dot{x} = Ax + Bu$ with state $x \in \mathbb{R}^n$ and control input $u \in \mathbb{R}^m$. As an example of discrete dynamics, one can consider a finite-state automaton, with state q taking values in some finite set $\mathcal{Q}$, where transitions between different discrete states are triggered by suitable values of an input variable v. When the input u to the continuous dynamics is some function of the discrete state q and, similarly, the value of the input v to the discrete dynamics is determined by the value of the continuous state x, a hybrid system arises (see Figure 1).

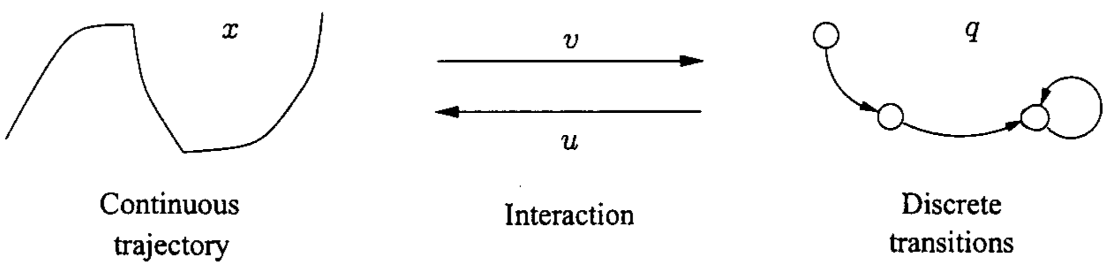

FIGURE 1. A hybrid system

Traditionally, control theory has focused either on continuous or on discrete behavior. However, many (if not most) of the dynamical systems encountered in practice are of hybrid nature. The following example is borrowed from [55].

Example 1.1 A very simple model that describes the motion of an automobile might take the form

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = f(a, q)$$

where x_1 is the position, x_2 is the velocity, $a \geq 0$ is the acceleration input, and $q \in \{1, 2, 3, 4, 5, -1, 0\}$ is the gear shift position. The function f should be negative and decreasing in a when $q = -1$, negative and independent of a when $q = 0$, and increasing in a, positive for sufficiently large a, and decreasing in q when $q > 0$. In this system, x_1 and x_2 are the continuous states and q is the discrete state. Clearly, the discrete transitions affect the continuous trajectory. In the case of an automatic transmission, the evolution of the continuous state x_2 is in turn used to determine the discrete transitions. In the case of a manual transmission, the discrete transitions are controlled by the driver. It is also natural to consider output variables that depend on both the continuous and the discrete states, such as the engine rotation rate (rpm) which is a function of x_2 and q. □

The field of hybrid systems has a strong interdisciplinary flavor, and different communities have developed different viewpoints. One approach, favored by researchers in computer science, is to concentrate on studying the discrete behavior of the system, while the continuous dynamics are assumed to take a relatively simple form. Basic issues in this context include well-posedness, simulation, and verification. Many researchers in systems and control theory, on the other hand, tend to regard hybrid systems as continuous systems with switching and place a greater emphasis on properties of the continuous state. The main issues then become stability analysis and control synthesis. It is the latter point of view that prevails in this book.

Thus we are interested in continuous-time systems with (isolated) discrete switching events. We refer to such systems as *switched systems*. A switched system may be obtained from a hybrid system by neglecting the details of the discrete behavior and instead considering all possible switching patterns from a certain class. This represents a significant departure from hybrid systems, especially at the analysis stage. In switching control design, specifics of the switching mechanism are of greater importance, although typically we will still characterize and exploit only essential properties of the discrete behavior. Having remarked for the purpose of motivation that switched systems can arise from hybrid systems, we henceforth choose switched systems as our focus of study and will generally make no explicit reference to the above connection.

Rather than give a universal formal definition of a switched system, we want to describe several specific categories of systems which will be our main objects of interest. Switching events in switched systems can be classified into

- *State-dependent* versus *time-dependent*;
- *Autonomous (uncontrolled)* versus *controlled*.

Of course, one can have combinations of several types of switching (cf. Example 5.2 in Chapter 5). We now briefly discuss all these possibilities.

1.1.1 State-dependent switching

Suppose that the continuous state space (e.g., $\mathbb{R}^n$) is partitioned into a finite or infinite number of *operating regions* by means of a family of *switching surfaces*, or *guards*. In each of these regions, a continuous-time dynamical system (described by differential equations, with or without controls) is given. Whenever the system trajectory hits a switching surface, the continuous state jumps instantaneously to a new value, specified by a *reset map*. In the simplest case, this is a map whose domain is the union of the switching surfaces and whose range is the entire state space, possibly excluding the switching surfaces (more general reset maps can also be considered, as explained below). In summary, the system is specified by

- The family of switching surfaces and the resulting operating regions;
- The family of continuous-time subsystems, one for each operating region;
- The reset map.

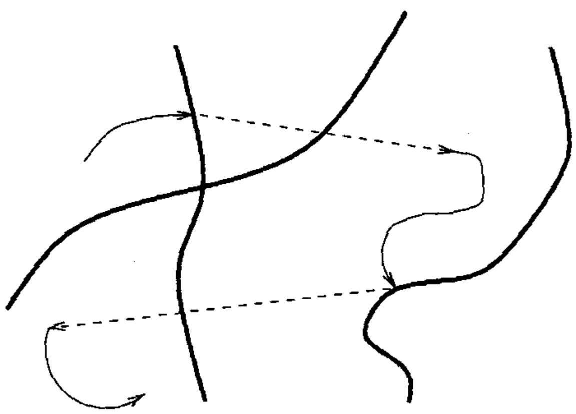

FIGURE 2. State-dependent switching

In Figure 2, the thick curves denote the switching surfaces, the thin curves with arrows denote the continuous portions of the trajectory, and the dashed lines symbolize the jumps. The instantaneous jumps of the continuous state are sometimes referred to as *impulse effects*. A special case is when such impulse effects are absent, i.e., the reset map is the identity. This means that the state trajectory is continuous everywhere, although it in general loses differentiability when it passes through a switching surface. In most of what follows, we restrict our attention to systems with no impulse effects. However, many of the results and techniques that we will discuss do generalize to systems with impulse effects. Another issue that we are ignoring for the moment is the possibility that some trajectories may "get stuck" on switching surfaces (cf. Section 1.2.3 below).

One may argue that the switched system model outlined above (state-dependent switching with no state jumps) is not really hybrid, because even though we can think of the set of operating regions as the discrete state space of the system, this is simply a discontinuous system whose description does not involve discrete dynamics. In other words, its evolution is uniquely determined by the continuous state. The system becomes truly hybrid if the discrete transitions explicitly depend on the value of the discrete state (i.e., the direction from which a switching surface is approached). More complicated state-dependent switching rules are also possible. For example, the operating regions may overlap, and a switching surface may be recognized by the system only in some discrete states. One paradigm that leads to this type of behavior is hysteresis switching, discussed later (see Section 1.2.4).

1.1.2 Time-dependent switching

Suppose that we are given a family f_p, $p \in \mathcal{P}$ of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, where $\mathcal{P}$ is some index set (typically, $\mathcal{P}$ is a subset of a finite-dimensional linear vector space). This gives rise to a family of systems

$$\dot{x} = f_p(x), \qquad p \in \mathcal{P} \tag{1.1}$$

evolving on $\mathbb{R}^n$. The functions f_p are assumed to be sufficiently regular (at least locally Lipschitz; see Section 1.2.1 below). The easiest case to think about is when all these systems are linear:

$$f_p(x) = A_p x, \qquad A_p \in \mathbb{R}^{n \times n},\ p \in \mathcal{P} \tag{1.2}$$

and the index set $\mathcal{P}$ is finite: $\mathcal{P} = \{1, 2, \dots, m\}$.

To define a switched system generated by the above family, we need the notion of a *switching signal*. This is a piecewise constant function $\sigma : [0, \infty) \to \mathcal{P}$. Such a function σ has a finite number of discontinuities—which we call the *switching times*—on every bounded time interval and takes a constant value on every interval between two consecutive switching times. The role of σ is to specify, at each time instant t, the index $\sigma(t) \in \mathcal{P}$ of the *active subsystem*, i.e., the system from the family (1.1) that is currently being followed. We assume for concreteness that σ is continuous from the right everywhere: $\sigma(t) = \lim_{\tau \to t^+} \sigma(\tau)$ for each $\tau \geq 0$. An example of such a switching signal for the case $\mathcal{P} = \{1, 2\}$ is depicted in Figure 3.

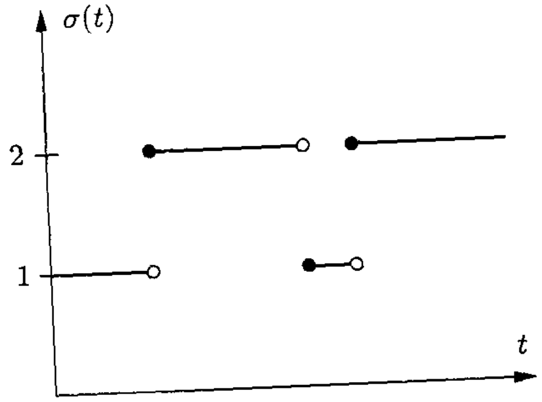

FIGURE 3. A switching signal

Thus a switched system with time-dependent switching can be described by the equation

$$\dot{x}(t) = f_{\sigma(t)}(x(t)).$$

A particular case is a *switched linear system*

$$\dot{x}(t) = A_{\sigma(t)} x(t)$$

which arises when all individual subsystems are linear, as in (1.2). To simplify the notation, we will often omit the time arguments and write

$$\dot{x} = f_\sigma(x) \tag{1.3}$$

and

$$\dot{x} = A_\sigma x \tag{1.4}$$

respectively.

Note that it is actually difficult to make a formal distinction between state-dependent and time-dependent switching. If the elements of the index set $\mathcal{P}$ from (1.1) are in 1-to-1 correspondence with the operating regions discussed in Section 1.1.1, and if the systems in these regions are those appearing in (1.1), then every possible trajectory of the system with state-dependent switching is also a solution of the system with time-dependent switching given by (1.3) for a suitably defined switching signal (but not vice versa). In view of this observation, the latter system can be regarded as a coarser model for the former, which can be used, for example, when the locations of the switching surfaces are unknown. This underscores the importance of developing analysis tools for switched systems like (1.3), which is the subject of Part II of the book.

1.1.3 Autonomous and controlled switching

By autonomous switching, we mean a situation where we have no direct control over the switching mechanism that triggers the discrete events. This category includes systems with state-dependent switching in which locations of the switching surfaces are predetermined, as well as systems with time-dependent switching in which the rule that defines the switching signal is unknown (or was ignored at the modeling stage). For example, abrupt changes in system dynamics may be caused by unpredictable environmental factors or component failures.

In contrast with the above, in many situations the switching is actually imposed by the designer in order to achieve a desired behavior of the system. In this case, we have direct control over the switching mechanism (which can be state-dependent or time-dependent) and may adjust it as the system evolves. For various reasons, it may be natural to apply discrete control actions, which leads to systems with controlled switching. Part III of this book is devoted entirely to problems of this kind. An important example, which provides motivation and can serve as a unifying framework for studying systems with controlled switching, is that of an *embedded system*, in which computer software interacts with physical devices (see Figure 4).

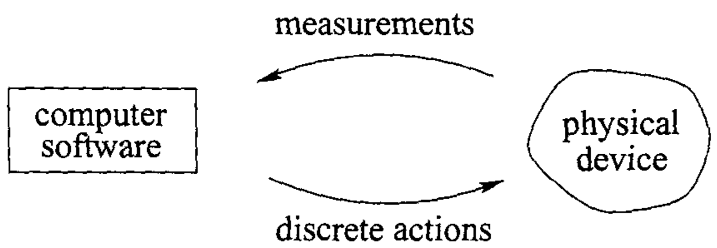

FIGURE 4. A computer-controlled system

It is not easy to draw a precise distinction between autonomous and controlled switching, or between state-dependent or time-dependent switching. In a given system, these different types of switching may coexist. For example, if the given process is prone to unpredictable environmental influences or component failures (autonomous switching), then it may be necessary to consider logic-based mechanisms for detecting such events and providing fault-correcting actions (controlled switching).

In the context of the automobile model discussed in Example 1.1, automatic transmission corresponds to autonomous state-dependent switching, whereas manual transmission corresponds to switching being controlled by the driver. In the latter case, state-dependent switching (shifting gears when reaching a certain value of the velocity or rpm) typically makes more sense than time-dependent switching. An exception is parallel parking, which may involve time-periodic switching patterns.

Switched systems with controlled time-dependent switching can be described in a language that is more standard in control theory. Assume that

$\mathcal{P}$ is a finite set, say, $\mathcal{P} = \{1, 2, \dots, m\}$. Then the switched system (1.3) can be recast as

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i \tag{1.5}$$

where the admissible controls are of the form $u_k = 1$, $u_i = 0$ for all $i \neq k$ (this corresponds to $\sigma = k$). In particular, the switched linear system (1.4) gives rise to the bilinear system

$$\dot{x} = \sum_{i=1}^{m} A_i x u_i.$$

1.2 Solutions of switched systems

This section touches upon a few delicate issues that arise in defining solutions of switched systems. In the subsequent chapters, these issues will mostly be avoided. We begin with some remarks on existence and uniqueness of solutions for systems described by ordinary differential equations.

1.2.1 Ordinary differential equations

Consider the system

$$\dot{x} = f(t, x), \qquad x \in \mathbb{R}^n. \tag{1.6}$$

We are looking for a solution $x(\cdot)$ of this system for given initial time t_0 and initial state $x(t_0) = x_0$. It is common to assume that the function f is continuous in t and locally Lipschitz in x uniformly over t. The second condition[1] means that for every pair (t_0, x_0) there exists a constant $L > 0$ such that the inequality

$$|f(t, x) - f(t, y)| \leq L|x - y| \tag{1.7}$$

holds for all (t, x) and (t, y) in some neighborhood of (t_0, x_0) in $[t_0, \infty) \times \mathbb{R}^n$. (Here and below, we denote by $|\cdot|$ the standard Euclidean norm on $\mathbb{R}^n$.) Under these assumptions, it is well known that the system (1.6) has a unique solution for every initial condition (t_0, x_0). This solution is defined on some maximal time interval $[t_0, T_{\max})$.

Example 1.2 To understand why the local Lipschitz condition is necessary, consider the scalar time-invariant system

$$\dot{x} = \sqrt{x}, \qquad x_0 = 0. \tag{1.8}$$

[1] In the time-invariant case, this reduces to the standard Lipschitz condition for a function $f : \mathbb{R}^n \to \mathbb{R}^n$.

The functions $x(t) \equiv 0$ and $x(t) = t^2/4$ both satisfy the differential equation (1.8) and the initial condition $x(0) = 0$. The uniqueness property fails here because the function $f(x) = \sqrt{x}$ is not locally Lipschitz at zero. This fact can be interpreted as follows: due to the rapid growth of $\sqrt{x}$ at zero, it is possible to "break away" from the zero equilibrium. Put differently, there exists a nonzero solution of (1.8) which, propagated backward in time, reaches the zero equilibrium in finite time. □

The maximal interval $[t_0, T_{\max})$ of existence of a solution may fail to be the entire semi-axis $[t_0, \infty)$. The next example illustrates that a solution may "escape to infinity in finite time." (This will not happen, however, if f is globally Lipschitz in x uniformly in t, i.e., if the Lipschitz condition (1.7) holds with some Lipschitz constant L for all $x, y \in \mathbb{R}^n$ and all $t \geq t_0$.)

Example 1.3 Consider the scalar time-invariant system

$$\dot{x} = x^2, \qquad x_0 > 0.$$

It is easy to verify that the (unique) solution satisfying $x(0) = x_0$ is given by the formula

$$x(t) = \frac{x_0}{1 - x_0 t}$$

and is only defined on the finite time interval $[0, 1/x_0)$. This is due to the rapid nonlinear growth at infinity of the function $f(x) = x^2$. □

Let us go back to the general situation described by the system (1.6). Since our view is toward systems with switching, the assumption that the function f is continuous in t is too restrictive. It turns out that for the existence and uniqueness result to hold, it is sufficient to demand that f be piecewise continuous in t. In this case one needs to work with a weaker concept of solution, namely, a continuous function $x(\cdot)$ that satisfies the corresponding integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau.$$

A function with these properties is piecewise differentiable and satisfies the differential equation (1.6) almost everywhere. Such functions are known as *absolutely continuous* and provide solutions of (1.6) *in the sense of Carathéodory*. Solutions of the switched system (1.3) will be interpreted in this way.

1.2.2 Zeno behavior

We now illustrate, with the help of the bouncing ball example, a peculiar type of behavior that can occur in switched systems.

Example 1.4 Consider a ball bouncing on the floor. Denote by h its height above the floor and by v its velocity (taking the positive velocity direction to be upwards). Normalizing the gravitational constant, we obtain the following equations of motion, valid between the impact times:

$$\begin{aligned}\dot{h} &= v \\ \dot{v} &= -1.\end{aligned} \tag{1.9}$$

At the time of an impact, i.e., when the ball hits the floor, its velocity changes according to the rule

$$v(t) = -rv(t^-) \tag{1.10}$$

where $v(t^-)$ is the ball's velocity right before the impact, $v(t)$ is the velocity right after the impact, and $r \in (0,1)$ is the restitution coefficient. This model can be viewed as a state-dependent switched system with impulse effects. Switching events (impacts) are triggered by the condition $h = 0$. They cause instantaneous jumps in the value of the velocity v which is one of the two continuous state variables. Since the continuous dynamics are always the same, all trajectories belong to the same operating region $\{(h,v) : h \geq 0,\ v \in \mathbb{R}\}$.

Integration of (1.9) gives

$$\begin{aligned}v(t) &= -(t-t_0) + v(t_0) \\ h(t) &= -\frac{(t-t_0)^2}{2} + v(t_0)(t-t_0) + h(t_0).\end{aligned} \tag{1.11}$$

Let the initial conditions be $t_0 = 0$, $h(0) = 0$, and $v(0) = 1$. By (1.11), until the first switching time we have

$$\begin{aligned}v(t) &= -t + 1 \\ h(t) &= -\frac{t^2}{2} + t.\end{aligned}$$

The first switch occurs at $t = 2$ since $h(2) = 0$. We have $v(2^-) = -1$, hence $v(2) = r$ in view of (1.10). Using (1.11) again with $t_0 = 2$, $h(2) = 0$, and $v(2) = r$, we obtain

$$\begin{aligned}v(t) &= -t + 2 + r \\ h(t) &= -\frac{(t-2)^2}{2} + (t-2)r.\end{aligned}$$

From this it is easy to deduce that the next switch occurs at $t = 2 + 2r$ and the velocity after this switch is $v(2+2r) = r^2$.

Continuing this analysis, one sees that the switching times form the sequence $2, 2+2r, 2+2r+2r^2, 2+2r+2r^2+2r^3, \ldots$ and that the corresponding velocities form the sequence r^2, r^3, r^4, and so on. The interesting conclusion is that the switching times have a finite *accumulation point*, which is the sum of the geometric series

$$\sum_{k=0}^{\infty} 2r^k = \frac{2}{1-r}.$$

At this time the switching events "pile up," i.e., the ball makes infinitely many bounces prior to this time! This is an example of the so-called *Zeno behavior* (see Figure 5).

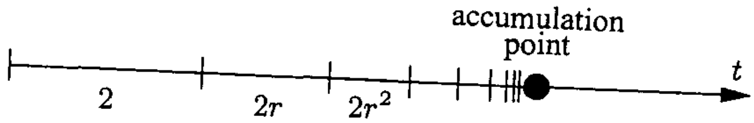

FIGURE 5. Zeno behavior

Since both $h(t)$ and $v(t)$ obtained by previous reasoning converge to zero as $t \to \frac{2}{1-r}$, it is natural to extend the solution beyond this time by setting

$$h(t), v(t) := 0, \qquad t \geq \frac{2}{1-r}.$$

Thus the ball stops bouncing, which is a reasonable outcome; of course, in reality this will happen after a finite number of jumps. □

In more complicated hybrid systems, the task of detecting possible Zeno trajectories and extending them beyond their accumulation points is far from trivial. This topic is beyond the scope of this book. In what follows, we either explicitly rule out Zeno behavior or show that it cannot occur.

1.2.3 Sliding modes

Consider a switched system with state-dependent switching, described by a single switching surface $\mathcal{S}$ and two subsystems $\dot{x} = f_i(x)$, $i = 1, 2$, one on each side of $\mathcal{S}$. Suppose that there are no impulse effects, so that the state does not jump at the switching events. In Section 1.1.1 we tacitly assumed that when the continuous trajectory hits $\mathcal{S}$, it crosses over to the other side. This will indeed be true if at the corresponding point $x \in \mathcal{S}$, both vectors $f_1(x)$ and $f_2(x)$ point in the same direction relative to $\mathcal{S}$, as in Figure 6(a); a solution in the sense of Carathéodory is then naturally obtained. However, consider the situation shown in Figure 6(b), where in the vicinity of $\mathcal{S}$ the vector fields f_1 and f_2 both point toward $\mathcal{S}$. In this case, we cannot describe the behavior of the system in the same way as before.

A way to resolve the above difficulty is provided by a different concept of solution, introduced by Filippov to deal precisely with problems of this

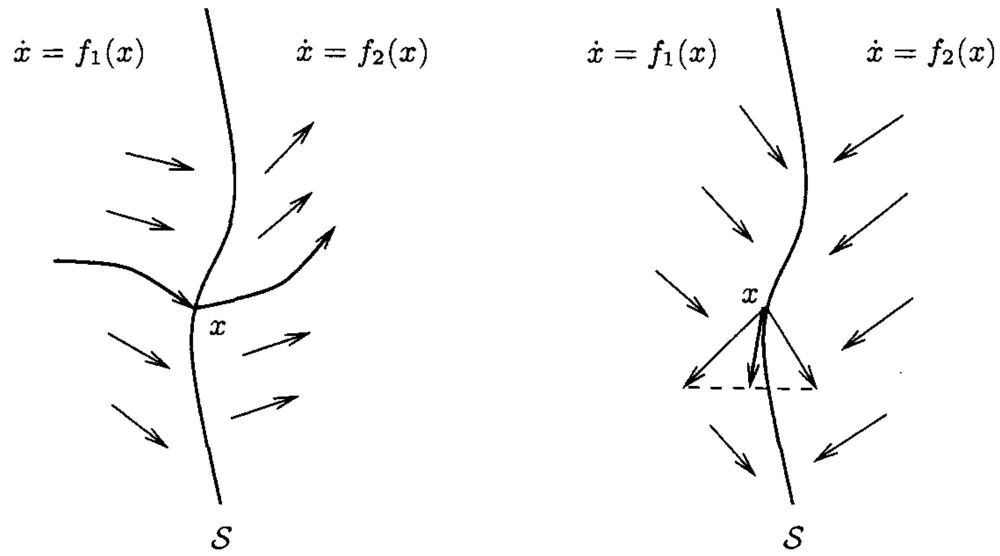

FIGURE 6. (a) Crossing a switching surface, (b) a sliding mode

kind. According to Filippov's definition, one enriches the set of admissible velocities for points $x \in \mathcal{S}$ by including all convex combinations of the vectors $f_1(x)$ and $f_2(x)$. Thus an absolutely continuous function $x(\cdot)$ is a solution of the switched system in the sense of Filippov if it satisfies the *differential inclusion*

$$\dot{x} \in F(x) \tag{1.12}$$

where F is a multi-valued function defined as follows. For $x \in \mathcal{S}$, we set

$$F(x) = \text{co}\{f_1(x), f_2(x)\} := \{\alpha f_1(x) + (1-\alpha)f_2(x) : \alpha \in [0,1]\}$$

while for $x \notin \mathcal{S}$, we simply set $F(x) = f_1(x)$ or $F(x) = f_2(x)$ depending on which side of S the point x lies on.

It is not hard to see what Filippov solutions look like in the situation shown in Figure 6(b). Once the trajectory hits the switching surface $\mathcal{S}$, it cannot leave it because the vector fields on both sides are pointing toward $\mathcal{S}$. Therefore, the only possible behavior for the solution is to slide on $\mathcal{S}$. We thus obtain what is known as a *sliding mode.* To describe the sliding motion precisely, note that there is a unique convex combination of $f_1(x)$ and $f_2(x)$ that is tangent to $\mathcal{S}$ at the point x. This convex combination determines the instantaneous velocity of the trajectory starting at x; see Figure 6(b). For every $x_0 \in \mathcal{S}$, the resulting solution $x(\cdot)$ is the only absolutely continuous function that satisfies the differential inclusion (1.12).

From the switched system viewpoint, a sliding mode can be interpreted as infinitely fast switching, or *chattering.* This phenomenon is often undesirable in mathematical models of real systems, because in practice it corresponds to very fast switching which causes excessive equipment wear. On the other hand, we see from the above discussion that a sliding mode yields a behavior that is significantly different from the behavior of each individual subsystem. For this reason, sliding modes are sometimes created on purpose to solve control problems that may be difficult or impossible to solve otherwise.

Example 1.5 Consider the following state-dependent switched linear system in the plane:

$$\dot{x} = \begin{cases} A_1 x & \text{if } x_2 \geq x_1 \\ A_2 x & \text{if } x_2 < x_1 \end{cases}$$

where

$$A_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} -1 & 0 \\ 0 & -\lambda \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2.$$

One can show that a sliding motion occurs in the first quadrant if $\lambda < 1$. For $\lambda > -1$ the corresponding trajectory approaches the origin along the switching line (a *stable sliding mode*) while for $\lambda < -1$ it goes away from the origin (an *unstable sliding mode*). □

Exercise 1.1 Prove this.

1.2.4 Hysteresis switching

We will often be interested in approximating a sliding mode behavior, while avoiding chattering and maintaining the property that two consecutive switching events are always separated by a time interval of positive length. Consider again the system shown in Figure 6(b). Construct two overlapping open regions Ω_1 and Ω_2 by offsetting the switching surface $\mathcal{S}$, as shown in Figure 7(a). In this figure, the original switching surface is shown by a dashed curve, the newly obtained switching surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ are shown by the two solid curves, the region Ω_1 is on the left, the region Ω_2 is on the right, and their intersection is the stripe between the new switching surfaces (excluding these surfaces themselves).

We want to follow the subsystem $\dot{x} = f_1(x)$ in the region Ω_1 and the subsystem $\dot{x} = f_2(x)$ in the region Ω_2. Thus switching events occur when the trajectory hits one of the switching surfaces $\mathcal{S}_1$, $\mathcal{S}_2$. This is formalized by introducing a discrete state σ, whose evolution is described as follows. Let $\sigma(0) = 1$ if $x(0) \in \Omega_1$ and $\sigma(0) = 2$ otherwise. For each $t > 0$, if $\sigma(t^-) = i \in \{1, 2\}$ and $x(t) \in \Omega_i$, keep $\sigma(t) = i$. On the other hand, if $\sigma(t^-) = 1$ but $x(t) \notin \Omega_1$, let $\sigma(t) = 2$. Similarly, if $\sigma(t^-) = 2$ but $x(t) \notin \Omega_2$, let $\sigma(t) = 1$. Repeating this procedure, we generate a piecewise constant signal σ which is continuous from the right everywhere. Since σ can change its value only after the continuous trajectory has passed through the intersection of Ω_1 and Ω_2, chattering is avoided. A typical solution trajectory is shown in Figure 7(b).

This standard idea, known as *hysteresis switching*, is very useful in control design (we will return to it in Chapter 6). The resulting closed-loop

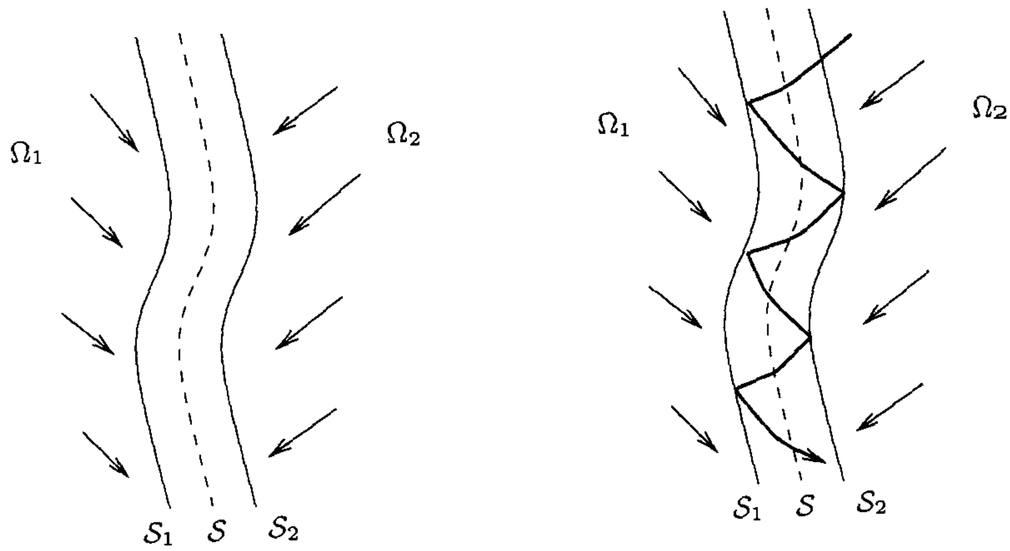

FIGURE 7. Hysteresis: (a) switching regions, (b) a typical trajectory

system is a hybrid system, σ being its discrete state. Unlike the system with state-dependent switching discussed in Section 1.1.1, this system is truly hybrid because its discrete part has "memory": the value of σ is not determined by the current value of x alone, but depends also on the previous value of σ. The instantaneous change in x is, in turn, dependent not only on the value of x but also on the value of σ.

Part II

Stability of Switched Systems

In this part we will be investigating stability issues for switched systems of the form (1.3). For the moment, we concern ourselves with asymptotic stability, although other forms of stability are also of interest. To understand what the basic questions are, consider the situation where $\mathcal{P} = \{1, 2\}$ and $x \in \mathbb{R}^2$, so that we are switching between two systems in the plane. First, suppose that the two individual subsystems are asymptotically stable, with trajectories as shown on the left in Figure 8 (the solid curve and the dotted curve). For different choices of the switching signal, the switched system might be asymptotically stable or unstable (these two possibilities are shown in Figure 8 on the right).

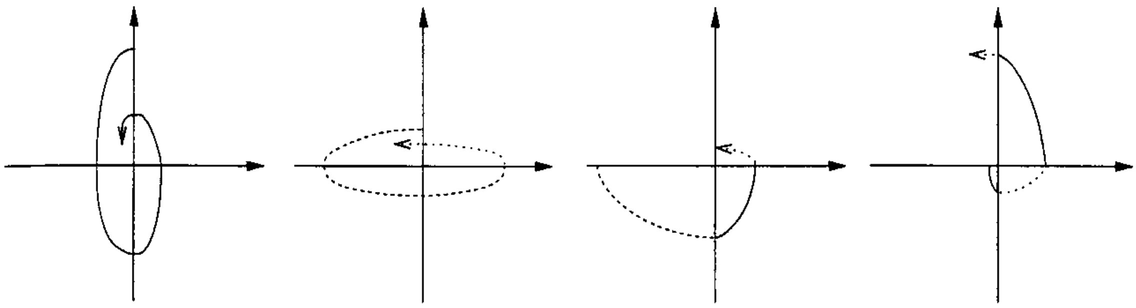

FIGURE 8. Switching between stable systems

Similarly, Figure 9 illustrates the case when both individual subsystems are unstable. Again, the switched system may be either asymptotically stable or unstable, depending on a particular switching signal.

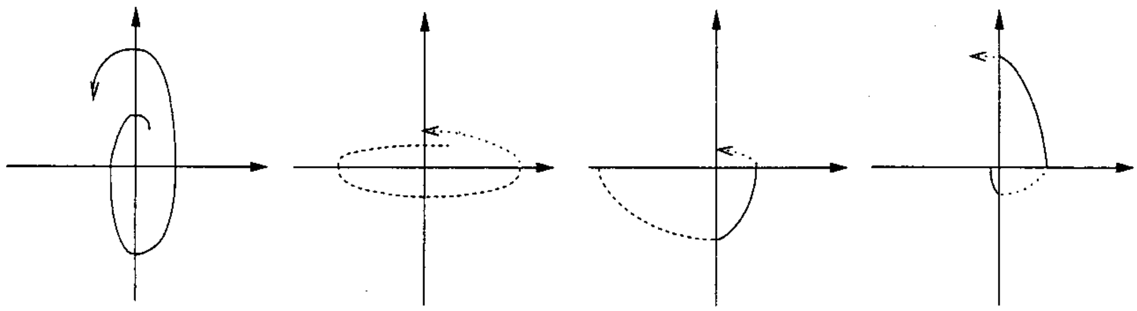

FIGURE 9. Switching between unstable systems

From these two examples, the following facts can be deduced:

- Unconstrained switching may destabilize a switched system even if all individual subsystems are stable.[2]

- It may be possible to stabilize a switched system by means of suitably constrained switching even if all individual subsystems are unstable.

---

[2]However, there are certain limitations to what types of instability are possible in this case. For example, it is easy to see that trajectories of such a switched system cannot escape to infinity in finite time.

Thus we will be studying the following two main problems:

1. Find conditions that guarantee asymptotic stability of a switched system for arbitrary switching signals.

2. If a switched system is not asymptotically stable for arbitrary switching, identify those switching signals for which it is asymptotically stable.

The first problem is relevant when the switching mechanism is either unknown or too complicated to be useful in the stability analysis. When studying the first problem, one is led to investigate possible sources of instability, which in turn provides insight into the more practical second problem.

In the context of the second problem, it is natural to distinguish between two situations. If some or all of the individual subsystems are asymptotically stable, then it is of interest to characterize, as completely as possible, the class of switching signals that preserve asymptotic stability (such switching signals clearly exist; for example, just let $\sigma(t) \equiv p$, where p is the index of some asymptotically stable subsystem). On the other hand, if all individual subsystems are unstable, then the task at hand is to construct at least one stabilizing switching signal, which may actually be quite difficult or even impossible.

The two problems described above are more rigorously formulated and studied in Chapters 2 and 3. In what follows, basic familiarity with Lyapunov's stability theory (for general nonlinear systems) is assumed. The reader is encouraged to consult Appendix A, which reviews necessary concepts and results.

2

Stability under Arbitrary Switching

2.1 Uniform stability and common Lyapunov functions

2.1.1 *Uniform stability concepts*

Given a family of systems (1.1), we want to study the following question: when is the switched system (1.3) asymptotically stable for every switching signal σ? We are assuming here that the individual subsystems have the origin as a common equilibrium point: $f_p(0) = 0$ for all $p \in \mathcal{P}$. Clearly, a necessary condition for (asymptotic) stability under arbitrary switching is that all of the individual subsystems are (asymptotically) stable. Indeed, if the pth system is unstable for some $p \in \mathcal{P}$, then the switched system is unstable for $\sigma(t) \equiv p$.

Therefore, throughout this chapter it will be assumed that all individual subsystems are asymptotically stable. Our earlier discussion shows that this condition is not sufficient for asymptotic stability under arbitrary switching. Thus one needs to determine what additional requirements on the systems from (1.1) must be imposed.

Recalling the equivalence between the switched system (1.3) for $\mathcal{P} = \{1, 2, \dots, m\}$ and the control system (1.5), we see that asymptotic stability of (1.3) for arbitrary switching corresponds to a lack of controllability of (1.5). Indeed, it means that for any admissible control input, the resulting solution trajectory must approach the origin.

Instead of just asymptotic stability for each particular switching signal, a somewhat stronger property is desirable, namely, asymptotic or exponential stability that is *uniform* over the set of all switching signals. The relevant stability concepts are the following appropriately modified versions of the standard stability concepts for time-invariant systems (see Appendix A for background and notation).

We will say that the switched system (1.3) is *uniformly asymptotically stable* if there exist a positive constant δ and a class $\mathcal{KL}$ function β such that for all switching signals σ the solutions of (1.3) with $|x(0)| \leq \delta$ satisfy the inequality

$$|x(t)| \leq \beta(|x(0)|, t) \qquad \forall\, t \geq 0. \tag{2.1}$$

If the function β takes the form $\beta(r, s) = cre^{-\lambda s}$ for some $c, \lambda > 0$, so that the above inequality takes the form

$$|x(t)| \leq c|x(0)|e^{-\lambda t} \qquad \forall\, t \geq 0 \tag{2.2}$$

then the system (1.3) is called *uniformly exponentially stable.* If the inequalities (2.1) and (2.2) are valid for all switching signals and all initial conditions, we obtain *global uniform asymptotic stability* (GUAS) and *global uniform exponential stability* (GUES), respectively.

Equivalent definitions can be given in terms of ε–δ properties of solutions. The term "uniform" is used here to describe uniformity with respect to switching signals. This is not to be confused with the more common usage which refers to uniformity with respect to the initial time for time-varying systems.

Exercise 2.1 Prove that for the switched linear system (1.4), GUAS implies GUES. Can you identify a larger class of switched systems for which the same statement holds?

2.1.2 Common Lyapunov functions

Lyapunov's stability theorem from Section A.3 has a direct extension which provides a basic tool for studying uniform stability of the switched system (1.3). This extension is obtained by requiring the existence of a single Lyapunov function whose derivative along solutions of all systems in the family (1.1) satisfies suitable inequalities. We are particularly interested in obtaining a Lyapunov condition for GUAS. To do this, we must take special care in formulating a counterpart of the inequality (A.5) which ensures a uniform rate of decay.

Given a positive definite continuously differentiable ($\mathcal{C}^1$) function $V : \mathbb{R}^n \to \mathbb{R}$, we will say that it is a *common Lyapunov function* for the family of systems (1.1) if there exists a positive definite continuous function $W : \mathbb{R}^n \to \mathbb{R}$ such that we have

$$\frac{\partial V}{\partial x} f_p(x) \leq -W(x) \qquad \forall\, x, \quad \forall\, p \in \mathcal{P}. \tag{2.3}$$

The following result will be used throughout this chapter.

Theorem 2.1 *If all systems in the family (1.1) share a radially unbounded common Lyapunov function, then the switched system (1.3) is GUAS.*

Theorem 2.1 is well known and can be derived in the same way as the standard Lyapunov stability theorem (cf. Section A.3). The main point is that the rate of decrease of V along solutions, given by (2.3), is not affected by switching, hence asymptotic stability is uniform with respect to σ.

Remark 2.1 In the special case when both V and W are quadratic (or, more generally, are bounded from above and below by monomials of the same degree in $|x|$), it is easy to show that the switched system is GUES. This situation will be discussed in detail later. □

If we replace the inequality (2.3) by the weaker condition

$$\frac{\partial V}{\partial x} f_p(x) < 0 \qquad \forall x \neq 0, \quad \forall p \in \mathcal{P} \tag{2.4}$$

then the result no longer holds. This can be seen from a close examination of the proof of Lyapunov's theorem and also from the following example.

Example 2.1 With reference to (1.1), let $f_p(x) = -px$ and $\mathcal{P} = (0, 1]$. This gives a family of systems, each of which is globally asymptotically stable and has $V(x) = x^2/2$ as a Lyapunov function. The resulting switched system

$$\dot{x} = -\sigma x$$

has the solutions

$$x(t) = e^{-\int_0^t \sigma(\tau) d\tau} x(0).$$

Thus we see that every switching signal $\sigma \in \mathcal{L}_1$ produces a trajectory that does not converge to zero. This happens because the rate of decay of V along the pth subsystem is given by

$$\frac{\partial V}{\partial x} f_p(x) = -px^2 \tag{2.5}$$

and this gets smaller for small values of p, so we do not have asymptotic stability if σ goes to zero too fast. Note that V is not a common Lyapunov function according to the above definition, since the right-hand sides in (2.5) for $0 < p \leq 1$ cannot all be upper-bounded by one negative definite function. □

The property expressed by (2.4) is sufficient for GUAS if $\mathcal{P}$ is a compact set and f_p depends continuously on p for each fixed x. (Under these conditions we can construct W by taking the maximum of the left-hand side of (2.4) over p, which is well defined.) This holds trivially if $\mathcal{P}$ is a finite set. For infinite $\mathcal{P}$, such compactness assumptions are usually reasonable and will be imposed in most of what follows.

Note that while we do not have asymptotic stability in the above example, stability in the sense of Lyapunov is always preserved under switching between one-dimensional stable systems. Interesting phenomena such as the one demonstrated by Figure 8 are only possible in dimensions 2 and higher.

Remark 2.2 If $\mathcal{P}$ is not a discrete set, it is also meaningful to consider the time-varying system described by (1.3) with a piecewise continuous (but not necessarily piecewise constant) signal σ. The existence of a common Lyapunov function implies global uniform asymptotic stability of this more general system; in fact, the same proof remains valid in this case. Although we will not mention it explicitly, many of the results presented below apply to such time-varying systems. □

The continuous differentiability assumption on V can sometimes be relaxed by requiring merely that V be continuous and decrease uniformly along solutions of each system in (1.1); this amounts to replacing the inequality (2.3) with its integral version. However, continuity of the gradient of V is important in the case of state-dependent switching with possible chattering, as demonstrated by the next example.

Example 2.2 Consider the system shown in Figure 10. Here, it is assumed that both systems $\dot{x} = f_i(x)$, $i = 1, 2$ share a common Lyapunov function V whose level set $\{x : V(x) = c\}$ is given by the heart-shaped curve, so that the gradient of V has a discontinuity on the switching surface $\mathcal{S}$. This results in a sliding motion along which V increases. □

In the context of the switched system (1.3), we explicitly rule out the possibility of undesirable behavior such as chattering or accumulation of switching events. When analyzing stability of (1.3), we restrict our attention to piecewise constant switching signals which are well defined for all $t \geq 0$.

2.1.3 A converse Lyapunov theorem

In the following sections we will be concerned with identifying classes of switched systems that are GUAS. The most common approach to this problem consists of searching for a common Lyapunov function shared by the individual subsystems. The question arises whether the existence of a common Lyapunov function is a more severe requirement than GUAS. A negative answer to this question—and a justification for the common Lyapunov function approach—follows from the converse Lyapunov theorem

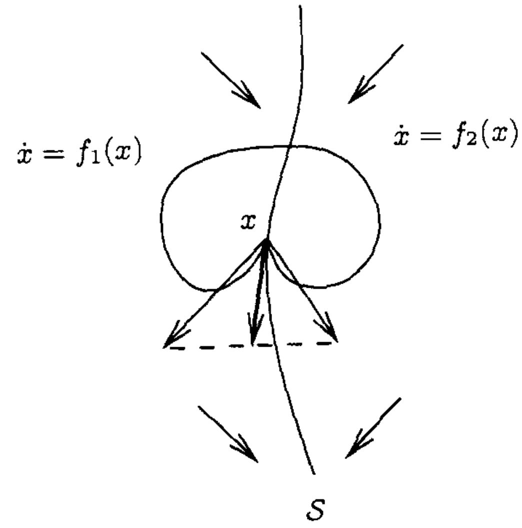

FIGURE 10. Illustration of Example 2.2

for switched systems, which says that the GUAS property of a switched system implies the existence of a common Lyapunov function. For such a converse Lyapunov theorem to hold, we need the family of systems (1.1) to satisfy suitable uniform (with respect to p) boundedness and regularity conditions. It is easy to see—and important to know—that these conditions automatically hold when the index set $\mathcal{P}$ is finite (recall that the functions f_p are always assumed to be locally Lipschitz in x).

Theorem 2.2 *Assume that the switched system (1.3) is GUAS, the set $\{f_p(x) : p \in \mathcal{P}\}$ is bounded for each x, and the function $(x,p) \mapsto f_p(x)$ is locally Lipschitz in x uniformly over p. Then all systems in the family (1.1) share a radially unbounded smooth common Lyapunov function.*

There is a useful result which we find convenient to state here as a corollary of Theorem 2.2. It says that if the switched system (1.3) is GUAS, then all "convex combinations" of the individual subsystems from the family (1.1) must be globally asymptotically stable. These convex combinations are defined by the vector fields

$$f_{p,q,\alpha}(x) := \alpha f_p(x) + (1-\alpha) f_q(x), \qquad p,q \in \mathcal{P}, \quad \alpha \in [0,1].$$

Corollary 2.3 *Under the assumptions of Theorem 2.2, for every $\alpha \in [0,1]$ and all $p,q \in \mathcal{P}$ the system*

$$\dot{x} = f_{p,q,\alpha}(x) \tag{2.6}$$

is globally asymptotically stable.

This can be proved by observing that a common Lyapunov function V provided by Theorem 2.2 decreases along solutions of (2.6). Indeed, from the inequality (2.3) we easily obtain

$$\frac{\partial V}{\partial x} f_{p,q,\alpha}(x) = \alpha \frac{\partial V}{\partial x} f_p(x) + (1-\alpha) \frac{\partial V}{\partial x} f_q(x) \le -W(x) \qquad \forall x. \tag{2.7}$$

A different justification of Corollary 2.3 comes from the fact that one can mimic the behavior of the convex combination (2.6) by means of fast switching between the subsystems $\dot{x} = f_p(x)$ and $\dot{x} = f_q(x)$, spending the correct proportion of time (α versus $1-\alpha$) on each one. This can be formalized with the help of the so-called *relaxation theorem* for differential inclusions, which in our context implies that the set of solutions of the switched system (1.3) is dense in the set of solutions of the "relaxed" switched system generated by the family of systems

$$\{\dot{x} = f_{p,q,\alpha}(x) : p, q \in \mathcal{P},\ \alpha \in [0,1]\}. \tag{2.8}$$

Therefore, if there exists a convex combination that is not asymptotically stable, then the switched system cannot be GUAS.

Remark 2.3 The formula (2.7) actually says more, namely, that V is a common Lyapunov function for the enlarged family of systems (2.8). By Theorem 2.1, the relaxed switched system generated by this family is also GUAS. □

A convex combination of two asymptotically stable vector fields is not necessarily asymptotically stable. As a simple example, consider the two matrices

$$A_1 := \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} -0.1 & 2 \\ -1 & -0.1 \end{pmatrix}.$$

These matrices are both Hurwitz, but their average $(A_1 + A_2)/2$ is not. Stability of all convex combinations often serves as an easily checkable necessary condition for GUAS. To see that this condition is not sufficient, consider the two matrices

$$A_1 := \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} -0.1 & -2 \\ 1 & -0.1 \end{pmatrix}.$$

It is easy to check that all convex combinations of these matrices are Hurwitz. Trajectories of the systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ look approximately the same as the first two plots in Figure 8 on page 19, and by switching it is possible to obtain unbounded trajectories such as the one shown on the last plot in that figure.

2.1.4 Switched linear systems

We now discuss how the above notions and results specialize to the switched linear system (1.4), in which all individual subsystems are linear. First, recall that for a linear time-invariant system $\dot{x} = Ax$, global exponential stability is equivalent to the seemingly weaker property of local attractivity (the latter means that all trajectories starting in some neighborhood of the origin converge to the origin). In fact, the different versions of asymptotic stability all amount to the property that A be a Hurwitz matrix—i.e., the eigenvalues of A lie in the open left half of the complex plane—and are characterized by the existence of a quadratic Lyapunov function

$$V(x) = x^T Px \tag{2.9}$$

where P is a positive definite symmetric matrix.

Now consider the switched linear system (1.4). Assume that $\{A_p : p \in \mathcal{P}\}$ is a *compact* (with respect to the usual topology in $\mathbb{R}^{n\times n}$) set of Hurwitz matrices. Similarly to the case of a linear system with no switching, the following is true.

Theorem 2.4 *The switched linear system* (1.4) *is GUES if and only if it is locally attractive for every switching signal.*

The equivalence between local attractivity and global exponential stability is not very surprising. A more interesting finding is that uniformity with respect to σ is automatically guaranteed: it cannot happen that all switching signals produce solutions decaying to zero but the rate of decay can be made arbitrarily small by varying the switching signal. (This is in fact true for switched nonlinear systems that are uniformly Lyapunov stable.) Moreover, we saw earlier that stability properties of the switched linear system do not change if we replace the set $\{A_p : p \in \mathcal{P}\}$ by its convex hull (see Remark 2.3).

For switched linear systems, it is natural to consider *quadratic common Lyapunov functions*, i.e., functions of the form (2.9) such that for some positive definite symmetric matrix Q we have

$$A_p^T P + PA_p \leq -Q \qquad \forall p \in \mathcal{P}. \tag{2.10}$$

(The inequality $M \leq N$ or $M < N$ for two symmetric matrices M and N means that the matrix $M - N$ is nonpositive definite or negative definite, respectively.) In view of the compactness assumption made earlier, the inequality (2.10) is equivalent to the simpler one

$$A_p^T P + PA_p < 0 \qquad \forall p \in \mathcal{P} \tag{2.11}$$

(although in general they are different; cf. Example 2.1 in Section 2.1.2). One reason why quadratic common Lyapunov functions are attractive is that (2.11) is a system of *linear matrix inequalities* (LMIs) in P, and there are efficient methods for solving finite systems of such inequalities numerically. It is also known how to determine the infeasibility of (2.11): for

$\mathcal{P} = \{1, 2, \ldots, m\}$, a quadratic common Lyapunov function does not exist if and only if the equation

$$R_0 = \sum_{i=1}^{m} (A_i R_i + R_i A_i^T) \tag{2.12}$$

is satisfied by some nonnegative definite symmetric matrices R_0, R_1, $\ldots$, R_m which are not all zero.

A natural question to ask is whether it is sufficient to work with quadratic common Lyapunov functions. In other words, is it true that if the switched linear system (1.4) is GUES and thus all systems in the family

$$\dot{x} = A_p x, \qquad p \in \mathcal{P} \tag{2.13}$$

share a common Lyapunov function (by virtue of Theorem 2.2), then one can always find a common Lyapunov function that is quadratic? The example given in the next section shows that the answer to this question is negative. However, it is always possible to find a common Lyapunov function that is homogeneous of degree 2, and in particular, one that takes the piecewise quadratic form

$$V(x) = \max_{1 \le i \le k} (l_i^T x)^2,$$

where l_i, $i = 1, \ldots, k$ are constant vectors. Level sets of such a function are given by surfaces of polyhedra, orthogonal to these vectors.

2.1.5 *A counterexample*

The following counterexample, taken from [78], demonstrates that even for switched linear systems GUES does not imply the existence of a *quadratic* common Lyapunov function. Take $\mathcal{P} = \{1, 2\}$, and let the two matrices be

$$A_1 := \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}.$$

These matrices are both Hurwitz.

FACT 1. The systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ do not share a quadratic common Lyapunov function of the form (2.9).

Without loss of generality, we can look for a positive definite symmetric matrix P in the form

$$P = \begin{pmatrix} 1 & q \\ q & r \end{pmatrix}$$

which satisfies the inequality (2.11). We have

$$-A_1^T P - P A_1 = \begin{pmatrix} 2 - 2q & 2q + 1 - r \\ 2q + 1 - r & 2q + 2r \end{pmatrix}$$

and this is positive definite only if

$$q^2 + \frac{(r-3)^2}{8} < 1. \tag{2.14}$$

(Recall that a symmetric matrix is positive definite if and only if all its leading principal minors are positive.) Similarly,

$$-A_2^T P - PA_2 = \begin{pmatrix} 2 - \frac{q}{5} & 2q + 10 - \frac{r}{10} \\ 2q + 10 - \frac{r}{10} & 20q + 2r \end{pmatrix}$$

is positive definite only if

$$q^2 + \frac{(r-300)^2}{800} < 100. \tag{2.15}$$

It is straightforward to check that the ellipses whose interiors are given by the formulas (2.14) and (2.15) do not intersect (see Figure 11). Therefore, a quadratic common Lyapunov function does not exist.

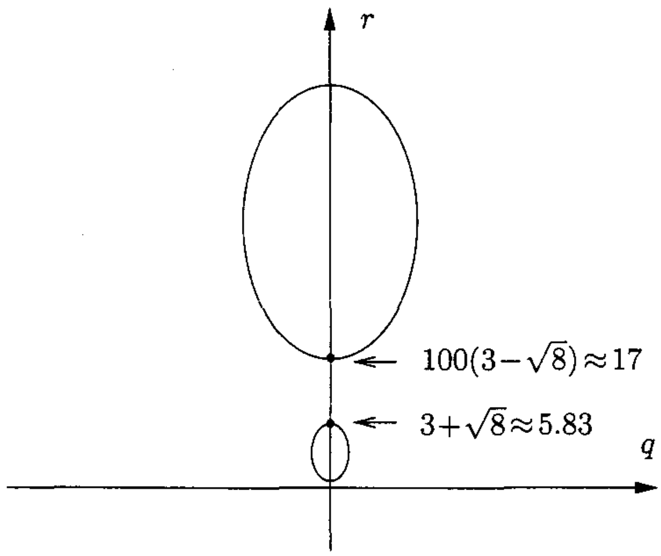

FIGURE 11. Ellipses in the counterexample

FACT 2. The switched linear system $\dot{x} = A_\sigma x$ is GUES.

This claim can be verified by analyzing the behavior of the system under the "worst-case switching," which is defined as follows. The vectors $A_1 x$ and $A_2 x$ are collinear on two lines going through the origin (the dashed lines in Figure 12). At all other points in $\mathbb{R}^2$, one of the two vectors points outwards relative to the other, i.e., it forms a smaller angle with the exiting radial direction. The worst-case switching strategy consists of following the vector field that points outwards, with switches occurring on the two lines. It turns out that this produces a trajectory converging to the origin, because the distance from the origin after one rotation decreases (see Figure 12). The

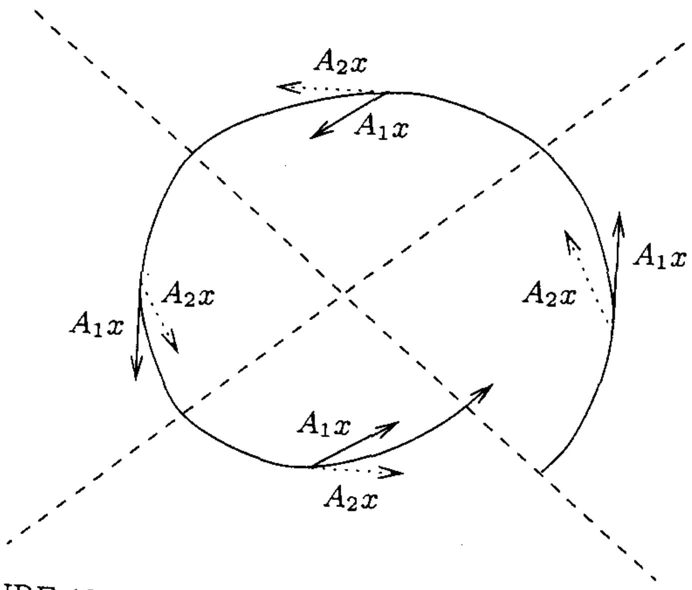

FIGURE 12. Worst-case switching in the counterexample

trajectories produced by all other switching signals also converge to the origin, and the worst-case trajectory described above provides a uniform lower bound on the rate of convergence. Thus the system is GUES.

2.2 Commutation relations and stability

The stability problem for switched systems can be studied from several different angles. In this section we explore a particular direction, namely, the role of commutation relations among the systems being switched.

2.2.1 Commuting systems

Linear systems

Consider the switched linear system (1.4), and assume for the moment that $\mathcal{P} = \{1, 2\}$ and that the matrices A_1 and A_2 commute: $A_1 A_2 = A_2 A_1$. We will often write the latter condition as $[A_1, A_2] = 0$, where the *commutator*, or *Lie bracket* $[\cdot, \cdot]$, is defined as

$$[A_1, A_2] := A_1 A_2 - A_2 A_1. \tag{2.16}$$

It is well known that in this case we have $e^{A_1}e^{A_2} = e^{A_2}e^{A_1}$, as can be seen from the definition of a matrix exponential via the series $e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \dots$, and more generally,

$$e^{A_1 t}e^{A_2 \tau} = e^{A_2 \tau}e^{A_1 t} \qquad \forall t, \tau > 0. \tag{2.17}$$

This means that the flows of the two individual subsystems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ commute.

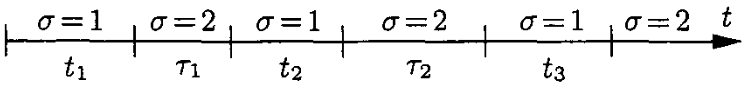

FIGURE 13. Switching between two systems

Now consider an arbitrary switching signal σ, and denote by t_i and τ_i the lengths of the time intervals on which σ equals 1 and 2, respectively (see Figure 13). The solution of the system produced by this switching signal is

$$x(t) = \cdots e^{A_2\tau_2}e^{A_1t_2}e^{A_2\tau_1}e^{A_1t_1}x(0)$$

which in view of (2.17) equals

$$x(t) = \cdots e^{A_2\tau_2}e^{A_2\tau_1}\cdots e^{A_1t_2}e^{A_1t_1}x(0). \tag{2.18}$$

Another fact that we need is

$$[A,B]=0 \quad \Rightarrow \quad e^Ae^B = e^{A+B}.$$

This is a consequence of the *Baker-Campbell-Hausdorff formula*

$$e^Ae^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]+[B,[A,B]])+\dots}.$$

Scalar multiples of the same matrix clearly commute with each other, hence we can rewrite (2.18) as

$$x(t) = e^{A_2(\tau_1+\tau_2+\dots)}e^{A_1(t_1+t_2+\dots)}x(0). \tag{2.19}$$

Since at least one of the series $t_1+t_2+\dots$ and $\tau_1+\tau_2+\dots$ converges to ∞ as $t\to\infty$, the corresponding matrix exponential converges to zero in view of stability of the matrices A_1 and A_2 (recall that asymptotic stability of individual subsystems is assumed throughout this chapter). We have thus proved that $x(t)\to 0$ for an arbitrary switching signal. Generalization to the case when $\mathcal{P}$ has more than two elements is straightforward. In fact, the following result holds.

Theorem 2.5 *If $\{A_p : p \in \mathcal{P}\}$ is a finite set of commuting Hurwitz matrices, then the corresponding switched linear system* (1.4) *is GUES.*

The above argument only shows global attractivity for every switching signal. To prove Theorem 2.5, one can invoke Theorem 2.4. There is also a more direct way to arrive at the result, which is based on constructing a common Lyapunov function. The following iterative procedure, taken from [222], can be used to obtain a quadratic common Lyapunov function for a finite family of commuting asymptotically stable linear systems.

Let $\{A_1, A_2, \dots, A_m\}$ be the given set of commuting Hurwitz matrices. Let P_1 be the unique positive definite symmetric solution of the Lyapunov equation

$$A_1^TP_1 + P_1A_1 = -I$$

(any other negative definite symmetric matrix could be used instead of $-I$ on the right-hand side). For $i = 1, \ldots, m$, let P_i be the unique positive definite symmetric solution of the Lyapunov equation

$$A_i^T P_i + P_i A_i = -P_{i-1}.$$

Then the function

$$V(x) = x^T P_m x \tag{2.20}$$

is a desired quadratic common Lyapunov function for the given family of linear systems.

To see why this is true, observe that the matrix P_m is given by the formula

$$P_m = \int_0^\infty e^{A_m^T t_m} \ldots \left(\int_0^\infty e^{A_1^T t_1} e^{A_1 t_1} dt_1 \right) \ldots e^{A_m t_m} dt_m$$

(see Example A.1 in Section A.3). Fix an arbitrary $i \in \{1, \ldots, m\}$. Since the matrix exponentials in the above expression commute, we can regroup them to obtain

$$P_m = \int_0^\infty e^{A_i^T t_i} \, Q_i \, e^{A_i t_i} \, dt_i \tag{2.21}$$

where Q_i is given by an expression involving $m-1$ integrals. This matrix Q_i can thus be obtained by applying $m-1$ steps of the above algorithm (all except the ith step), hence it is positive definite. Since (2.21) implies that $A_i^T P_m + P_m A_i = -Q_i$, we conclude V given by (2.20) is a Lyapunov function for the ith subsystem. It is also not hard to prove this directly by manipulating Lyapunov equations, as done in [222]. Incidentally, from the above formulas we also see that changing the order of the matrices $\{A_1, A_2, \ldots, A_m\}$ does not affect the resulting matrix P_m.

Nonlinear systems

To extend the above result to switched nonlinear systems, we first need the notion of a Lie bracket, or commutator, of two $\mathcal{C}^1$ vector fields. This is the vector field defined as follows:

$$[f_1, f_2](x) := \frac{\partial f_2(x)}{\partial x} f_1(x) - \frac{\partial f_1(x)}{\partial x} f_2(x).$$

For linear vector fields $f_1(x) = A_1 x$, $f_2(x) = A_2 x$ the right-hand side becomes $(A_2 A_1 - A_1 A_2)x$, which is consistent with the definition of the Lie bracket of two matrices (2.16) except for the difference in sign.

If the Lie bracket of two vector fields is identically zero, we will say that the two vector fields commute. The following result is a direct generalization of Theorem 2.5.

Theorem 2.6 *If $\{f_p : p \in \mathcal{P}\}$ is a finite set of commuting C^1 vector fields and the origin is a globally asymptotically stable equilibrium for all systems in the family* (1.1), *then the corresponding switched system* (1.3) *is GUAS.*

The proof of this result given in [190] establishes the GUAS property directly (in fact, commutativity of the flows is all that is needed, and the continuous differentiability assumption can be relaxed). It does not provide an explicit construction of a common Lyapunov function. Two alternative methods, discussed next, enable one to construct such a function. Unfortunately, they rely on the stronger assumption that the systems in the family (1.1) are *exponentially* stable, and provide a function that serves as a common Lyapunov function for this family only locally (in some neighborhood of the origin).

The first option is to employ Lyapunov's indirect method (described in Section A.5). To this end, consider the linearization matrices

$$A_p := \frac{\partial f_p}{\partial x}(0), \qquad p \in \mathcal{P}. \tag{2.22}$$

If the nonlinear vector fields commute, then the linearization matrices also commute.

Exercise 2.2 Prove this (assuming that $f_p \in C^1$ and $f_p(0) = 0$ for all $p \in \mathcal{P}$, and nothing else).

The converse does not necessarily hold, so commutativity of the linearization matrices is a weaker condition (which of course can be verified directly). The matrices A_p are Hurwitz if (and only if) the vector fields f_p are exponentially stable. Thus a quadratic common Lyapunov function for the linearized systems, constructed as explained earlier, serves as a local common Lyapunov function for the original finite family of nonlinear systems (1.1).

The second option is to use the iterative procedure described in [255]. This procedure, although not as constructive and practically useful as the previous one, parallels the procedure given earlier for commuting linear systems while working with the nonlinear vector fields directly. Let $\mathcal{P} = \{1, 2, \dots, m\}$ and suppose that the systems (1.1) are exponentially stable. For each $p \in \mathcal{P}$, denote by $\varphi_p(t, z)$ the solution of the system $\dot{x} = f_p(x)$ with initial condition $x(0) = z$. Define the functions

$$V_1(x) := \int_0^T |\varphi_1(\tau, x)|^2 d\tau$$

$$V_i(x) := \int_0^T V_{i-1}(\varphi_i(\tau, x)) d\tau, \qquad i = 2, \dots, m$$

where T is a sufficiently large positive constant. Then V_m is a local common Lyapunov function for the family (1.1). Moreover, if the functions $f_p, p \in \mathcal{P}$

are globally Lipschitz, then we obtain a global common Lyapunov function. For the case of linear systems $f_p(x) = A_p x, p \in \mathcal{P}$ we recover the algorithm described earlier upon setting $T = \infty$.

2.2.2 Nilpotent and solvable Lie algebras

Linear systems

Consider again the switched linear system (1.4). In view of the previous discussion, it is reasonable to conjecture that if the matrices A_p, $p \in \mathcal{P}$ do not commute, then stability of the switched system may still depend on the commutation relations between them. A useful object which reveals the nature of these commutation relations is the Lie algebra $\mathfrak{g} := \{A_p : p \in \mathcal{P}\}_{LA}$ generated by the matrices A_p, $p \in \mathcal{P}$, with respect to the standard Lie bracket (2.16). This is a linear vector space of dimension at most n^2, spanned by the given matrices and all their iterated Lie brackets. For an introduction to Lie algebras, see Appendix B. Note that the Lie bracket of two Hurwitz matrices is no longer Hurwitz, as can be seen from the formula $\mathrm{tr}[A, B] = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$.

Beyond the commuting case, the simplest relevant classes of Lie algebras are *nilpotent* and *solvable* ones. A Lie algebra is called nilpotent if all Lie brackets of sufficiently high order are zero. Solvable Lie algebras form a larger class of Lie algebras, in which all Lie brackets of sufficiently high order having a certain structure are zero. For precise definitions, see Section B.3.

The first nontrivial case is when we have $\mathcal{P} = \{1, 2\}$, $[A_1, A_2] \neq 0$, and $[A_1, [A_1, A_2]] = [A_2, [A_1, A_2]] = 0$. This means that $\mathfrak{g}$ is a nilpotent Lie algebra with order of nilpotency 2 and dimension 3 (as a basis we can choose $\{A_1, A_2, [A_1, A_2]\}$). Stability of the switched linear system corresponding to this situation—but in discrete time—was studied in [109]. The results obtained there for the discrete-time case can be easily adopted to continuous-time switched systems in which switching times are constrained to be integer multiples of a fixed positive number. In the spirit of the formula (2.19), in this case the solution of the switched system can be expressed as

$$x(t) = e^{A_2 \tau_1} e^{A_1 t_1} e^{A_2 \tau_2} e^{A_1 t_2} e^{A_2 \tau_3} x(0)$$

where at least one of the quantities $\tau_1, t_1, \tau_2, t_2, \tau_3$ converges to ∞ as $t \to \infty$. This expression is a consequence of the Baker-Campbell-Hausdorff formula. Similarly to the commuting case, it follows that the switched linear system is GUES provided that the matrices A_1 and A_2 are Hurwitz.

The following general result, whose proof is sketched below, includes the above example and also Theorem 2.5 as special cases. (A further generalization will be obtained in Section 2.2.3.)

Theorem 2.7 *If $\{A_p : p \in \mathcal{P}\}$ is a compact set of Hurwitz matrices and the Lie algebra $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ is solvable, then the switched linear system* (1.4) *is GUES.*

A standard example of a solvable Lie algebra is that generated by (non-strictly) upper-triangular matrices, i.e., matrices with zero entries everywhere below the main diagonal. Such a Lie algebra is solvable because when one computes Lie brackets, additional zeros are generated on and then above the main diagonal. We will exploit the fact that, up to a coordinate transformation, all solvable Lie algebras can be characterized in this way. This is a consequence of the classical Lie's theorem from the theory of Lie algebras (cf. Section B.3).

Proposition 2.8 (Lie) *If $\mathfrak{g}$ is a solvable Lie algebra, then there exists a (possibly complex) linear change of coordinates under which all matrices in $\mathfrak{g}$ are simultaneously transformed to the upper-triangular form.*

In view of this result we can assume, without loss of generality, that all matrices A_p, $p \in \mathcal{P}$ are upper-triangular. The following fact can now be used to finish the proof of Theorem 2.7.

Proposition 2.9 *If $\{A_p : p \in \mathcal{P}\}$ is a compact set of upper-triangular Hurwitz matrices, then the switched linear system* (1.4) *is GUES.*

To see why this proposition is true, suppose that $\mathcal{P} = \{1, 2\}$ and $x \in \mathbb{R}^2$. Let the two matrices be

$$A_1 := \begin{pmatrix} -a_1 & b_1 \\ 0 & -c_1 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} -a_2 & b_2 \\ 0 & -c_2 \end{pmatrix}. \tag{2.23}$$

Suppose for simplicity that their entries are real (the case of complex entries requires some care but the extension is not difficult). Since the eigenvalues of these matrices have negative real parts, we have $a_i, c_i > 0$, $i = 1, 2$. Now, consider the switched system $\dot{x} = A_\sigma x$. The second component of x satisfies the equation

$$\dot{x}_2 = -c_\sigma x_2.$$

Therefore, x_2 decays to zero exponentially fast, at the rate corresponding to $\min\{c_1, c_2\}$. The first component of x satisfies the equation

$$\dot{x}_1 = -a_\sigma x_1 + b_\sigma x_2.$$

This can be viewed as the exponentially stable system $\dot{x}_1 = -a_\sigma x_1$ perturbed by the exponentially decaying input $b_\sigma x_2$. Thus x_1 also converges to zero exponentially fast. It is not hard to extend this argument to systems of arbitrary dimension (by induction, proceeding from the bottom component of x upwards) and to infinite index sets $\mathcal{P}$ (by using the compactness assumption).

An alternative method of proving Fact 2, and thus completing the proof of Theorem 2.7, consists of constructing a common Lyapunov function for the family of linear systems (2.13). This construction leads to a cleaner proof and is of independent interest. It turns out that in the present case it is possible to find a quadratic common Lyapunov function of the form (2.9), with P a diagonal matrix. We illustrate this on the example of the two matrices (2.23). Let us look for P taking the form

$$P = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

where $d_1, d_2 > 0$. A straightforward calculation gives

$$-A_i^T P - PA_i = \begin{pmatrix} 2d_1 a_i & -d_1 b_i \\ -d_1 b_i & 2d_2 c_i \end{pmatrix}, \qquad i = 1, 2.$$

To ensure that this matrix is positive definite, we can first pick an arbitrary $d_1 > 0$, and then choose $d_2 > 0$ large enough to have

$$4d_2 d_1 a_i c_i - d_1^2 b_i^2 > 0, \qquad i = 1, 2.$$

Again, this construction can be extended to the general situation by using the compactness assumption and induction on the dimension of the system.

Exercise 2.3 Verify whether or not the switched linear systems in the plane generated by the following pairs of matrices are GUES: (a) $A_1 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$; (b) $A_1 = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$.

Nonlinear systems

Using Lyapunov's indirect method, we can obtain the following local version of Theorem 2.7 for switched nonlinear systems. Consider the family of nonlinear systems (1.1), assuming that each function f_p is C^1 and satisfies $f_p(0) = 0$. Consider also the corresponding family of linearization matrices A_p, $p \in \mathcal{P}$ defined by the formula (2.22).

Corollary 2.10 *Suppose that the linearization matrices A_p, $p \in \mathcal{P}$ are Hurwitz, $\mathcal{P}$ is a compact set, and $\frac{\partial f_p}{\partial x}(x)$ depends continuously on p for each x in some neighborhood of the origin. If the Lie algebra $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ is solvable, then the switched system (1.3) is locally uniformly exponentially stable.*

This is a relatively straightforward application of Lyapunov's indirect method (see Section A.5), although some additional technical assumptions are needed here because, unlike in Section 2.2.1, the set $\mathcal{P}$ is allowed to be infinite. The linearization matrices A_p, $p \in \mathcal{P}$ form a compact set because they are defined by the formula (2.22) and $\frac{\partial f_p}{\partial x}(x)$ is assumed to depend continuously on p. Moreover, since the matrices A_p, $p \in \mathcal{P}$ are Hurwitz and generate a solvable Lie algebra, the corresponding linear systems (2.13) share a quadratic common Lyapunov function (as we saw earlier). Then it is not hard to show that this function is also a common Lyapunov function for the original family (1.1) on a sufficiently small neighborhood of the origin.

More research is needed to understand how the structure of the Lie algebra generated by the original nonlinear vector fields f_p, $p \in \mathcal{P}$ is related to stability properties of the switched system (1.3). Taking higher-order terms into account, one may hope to obtain conditions that guarantee stability of switched nonlinear systems when the above linearization test fails. It is also of interest to investigate whether the equivalence between the switched system (1.3) for $\mathcal{P} = \{1, 2, \ldots, m\}$ and the control system (1.5) can lead to new insights. Note that while in the context of stability of (1.3) Lie-algebraic techniques seem to be a relatively new tool, they have been used for decades to study controllability properties of the system (1.5); see Section 4.2 for more on this topic.

2.2.3 More general Lie algebras

The material in this section[1] relies more heavily on the properties of Lie algebras discussed in Appendix B. As before, we study the switched linear system (1.4), where $\{A_p : p \in \mathcal{P}\}$ is a compact set of Hurwitz matrices. Consider a decomposition of the Lie algebra $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ into a semidirect sum $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$, where $\mathfrak{r}$ is a solvable ideal and $\mathfrak{s}$ is a subalgebra. For our purposes, the best choice is to let $\mathfrak{r}$ be the radical, in which case $\mathfrak{s}$ is semisimple and we have a Levi decomposition. If $\mathfrak{g}$ is not solvable, then $\mathfrak{s}$ is not zero.

The following result is a direct extension of Theorem 2.7. It states that the system (1.4) is still GUES if the subalgebra $\mathfrak{s}$ is compact (which amounts to saying that all matrices in $\mathfrak{s}$ are diagonalizable and have purely imaginary eigenvalues).

Theorem 2.11 *If $\{A_p : p \in \mathcal{P}\}$ is a compact set of Hurwitz matrices and the Lie algebra $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ is a semidirect sum of a solvable ideal and a compact subalgebra, then the switched linear system* (1.4) *is GUES.*

Sketch of proof. For an arbitrary $p \in \mathcal{P}$ we have $A_p = r_p + s_p$, where $r_p \in \mathfrak{r}$ and $s_p \in \mathfrak{s}$. Writing $e^{(r_p+s_p)t} = e^{s_p t}B_p(t)$, one can check that $B_p(t)$ satisfies $\dot{B}_p(t) = e^{-s_p t}r_p e^{s_p t}B_p(t)$. We have $e^{(r_p+s_p)t} \to 0$ as $t \to \infty$ because the matrix A_p is Hurwitz. Since $\mathfrak{s}$ is compact, there exists a constant $C > 0$ such that $|e^s x| \geq C|x|$ for all $s \in \mathfrak{s}$ and all x, hence we cannot have

[1] This section can be skipped without loss of continuity.

$e^{s_p t}x \to 0$ for any $x \neq 0$. Thus $B_p(t) \to 0$. This implies that r_p is a Hurwitz matrix.

The transition matrix of the switched linear system (1.4) takes the form

$$\Phi(t,0) = e^{(r_{p_k}+s_{p_k})t_k} \cdots e^{(r_{p_1}+s_{p_1})t_1} = e^{s_{p_k}t_k}B_{p_k}(t_k)\cdots e^{s_{p_1}t_1}B_{p_1}(t_1)$$

where $t_1,\ t_1+t_2,\ \dots,\ t_1+t_2+\dots+t_{k-1} < t$ are the switching times, $t_1+\dots+t_k = t$, and as before $\dot{B}_{p_i}(t) = e^{-s_{p_i}t}r_{p_i}e^{s_{p_i}t}B_{p_i}(t)$, $i = 1,\dots,k$. To simplify the notation, let $k = 2$. We can then write

$$\Phi(t,0) = e^{s_{p_2}t_2}e^{s_{p_1}t_1}\widetilde{B}_{p_2}(t_2)B_{p_1}(t_1)$$

where

$$\widetilde{B}_{p_2}(t) := e^{-s_{p_1}t_1}B_{p_2}(t)e^{s_{p_1}t_1}$$

and so

$$\dot{\widetilde{B}}_{p_2}(t) = e^{-s_{p_1}t_1}e^{-s_{p_2}t}r_{p_2}e^{s_{p_2}t}e^{s_{p_1}t_1}\widetilde{B}_{p_2}(t).$$

Therefore,

$$\Phi(t,0) = e^{s_{p_2}t_2}e^{s_{p_1}t_1}\cdot \overline{B}(t) \tag{2.24}$$

where $\overline{B}(t)$ is the transition matrix of a switched/time-varying system generated by matrices in the set $\bar{\mathfrak{r}} := \{e^{-s}r_pe^{s} : s \in \mathfrak{s},\ p \in \mathcal{P}\} \subset \mathfrak{r}$. The first term in the product on the right-hand side of (2.24) is bounded by compactness, while the second term converges to zero exponentially fast by Theorem 2.7 and Remark 2.2. □

Example 2.3 Suppose that the matrices A_p, $p \in \mathcal{P}$ take the form $A_p = -\lambda_p I + S_p$ where $\lambda_p > 0$ and $S_p^T = -S_p$ for all $p \in \mathcal{P}$. These are automatically Hurwitz matrices. If $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ contains the identity matrix, then the condition of Theorem 2.11 is satisfied with $\mathfrak{r} := \mathbb{R}I$ (scalar multiples of the identity matrix) and $\mathfrak{s} := \{S_p : p \in \mathcal{P}\}_{LA}$, which is compact. If $\mathfrak{g}$ does not contain the identity matrix, then $\mathfrak{g}$ is a proper subalgebra of $\mathbb{R}I \oplus \{S_p : p \in \mathcal{P}\}_{LA}$; it is not difficult to see that the result is still valid in this case. □

If the condition of Theorem 2.11 is satisfied, then the linear systems (2.13) share a quadratic common Lyapunov function. (The proof of this fact exploits the Haar measure on the Lie group corresponding to $\mathfrak{s}$ and is not as constructive as in the case when $\mathfrak{g}$ is solvable.) Considering the family of nonlinear systems (1.1) with $f_p(0) = 0$ for all $p \in \mathcal{P}$, together with the corresponding linearization matrices (2.22), we immediately obtain the following generalization of Corollary 2.10.

Corollary 2.12 *Suppose that the linearization matrices A_p, $p \in \mathcal{P}$ are Hurwitz, $\mathcal{P}$ is a compact set, and $\frac{\partial f_p}{\partial x}(x)$ depends continuously on p for each x in some neighborhood of the origin. If the Lie algebra $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ is a semidirect sum of a solvable ideal and a compact subalgebra, then the switched system (1.3) is locally uniformly exponentially stable.*

The result expressed by Theorem 2.11 is in some sense the strongest one that can be given on the Lie algebra level. To explain this more precisely, we need to introduce a possibly larger Lie algebra $\widehat{\mathfrak{g}}$ by adding to $\mathfrak{g}$ the scalar multiples of the identity matrix if necessary. In other words, define $\widehat{\mathfrak{g}} := \{I, A_p : p \in \mathcal{P}\}_{LA}$. The Levi decomposition of $\widehat{\mathfrak{g}}$ is given by $\widehat{\mathfrak{g}} = \widehat{\mathfrak{r}} \oplus \mathfrak{s}$ with $\widehat{\mathfrak{r}} \supset \mathfrak{r}$ (because the subspace $\mathbb{R}I$ belongs to the radical of $\widehat{\mathfrak{g}}$). Thus $\widehat{\mathfrak{g}}$ satisfies the hypothesis of Theorem 2.11 if and only if $\mathfrak{g}$ does.

It turns out that if $\widehat{\mathfrak{g}}$ cannot be decomposed as required by Theorem 2.11, then it can be generated by a family of Hurwitz matrices (which might in principle be different from A_p, $p \in \mathcal{P}$) with the property that the corresponding switched linear system is not GUES. On the other hand, there exists another set of Hurwitz generators for $\widehat{\mathfrak{g}}$ which does give rise to a GUES switched linear system. (In fact, both generator sets can always be chosen in such a way that they contain the same number of elements as the original set that was used to generate $\widehat{\mathfrak{g}}$.) Thus if the Lie algebra does not satisfy the hypothesis of Theorem 2.11, then this Lie algebra alone does not provide enough information to determine whether or not the original switched linear system is stable.

Theorem 2.13 *Suppose that a given matrix Lie algebra $\widehat{\mathfrak{g}}$ does not satisfy the hypothesis of Theorem 2.11. Then there exists a set of Hurwitz generators for $\widehat{\mathfrak{g}}$ such that the corresponding switched linear system is not GUES. There also exists another set of Hurwitz generators for $\widehat{\mathfrak{g}}$ such that the corresponding switched linear system is GUES.*

SKETCH OF PROOF. To prove the first statement of the theorem, in view of Corollary 2.3 it is sufficient to find two Hurwitz matrices $B_1, B_2 \in \widehat{\mathfrak{g}}$ that have an unstable convex combination. To do this, one uses the fact that $\mathfrak{s}$ is not compact and hence, as shown in Section B.5, contains a subalgebra isomorphic to $sl(2,\mathbb{R})$. We know from Section B.2 that this subalgebra contains three matrices of the form (B.2). It is not difficult to show that the desired B_1 and B_2 can be obtained by subtracting a sufficiently small positive multiple of the identity matrix from the last two matrices in that formula. To prove the second statement of the theorem, one subtracts a sufficiently large multiple of the identity from an arbitrary set of generators for $\widehat{\mathfrak{g}}$. $\square$

By virtue of this result, we have a complete characterization of all matrix Lie algebras $\widehat{\mathfrak{g}}$ that contain the identity matrix and have the property that every set of Hurwitz generators for $\widehat{\mathfrak{g}}$ gives rise to a GUES switched linear system. Namely, these are precisely the Lie algebras that admit a decomposition described in the statement of Theorem 2.11. The interesting—and rather surprising—discovery is that the above property depends only on the structure of $\widehat{\mathfrak{g}}$ as a Lie algebra, and not on the choice of a particular matrix representation of $\widehat{\mathfrak{g}}$.

For switched linear systems whose associated Lie algebras do not satisfy the condition of Theorem 2.11 but are low-dimensional, it is possible to reduce the investigation of stability to that of a switched linear system in $\mathbb{R}^2$. (This is useful in view of the available stability results for two-dimensional switched systems, to be discussed in Section 2.3.3.) For example, take $\mathcal{P} = \{1,2\}$, and define $\widetilde{A}_i := A_i - \frac{1}{n}\mathrm{tr}(A_i)I$, $i = 1,2$. Assume that all iterated Lie brackets of the matrices $\widetilde{A}_1$ and $\widetilde{A}_2$ are linear combinations of $\widetilde{A}_1$, $\widetilde{A}_2$, and $[\widetilde{A}_1, \widetilde{A}_2]$. This means that if we consider the Lie algebra $\mathfrak{g} = \{A_1, A_2\}_{LA}$ and add to it the identity matrix (if it is not already there), the resulting Lie algebra $\widehat{\mathfrak{g}}$ has dimension at most 4. In this case, the following algorithm can be used to verify that the switched linear system generated by A_1 and A_2 is GUES or, if this is not possible, to construct a two-dimensional switched linear system whose stability is equivalent to that of the original one.

Step 1. If $[\widetilde{A}_1, \widetilde{A}_2]$ is a linear combination of $\widetilde{A}_1$ and $\widetilde{A}_2$, stop: the system is GUES. Otherwise, write down the matrix of the Killing form for the Lie algebra $\widetilde{\mathfrak{g}} := \{\widetilde{A}_1, \widetilde{A}_2\}_{LA}$ relative to the basis given by $\widetilde{A}_1$, $\widetilde{A}_2$, and $[\widetilde{A}_1, \widetilde{A}_2]$. (This is a symmetric 3×3 matrix; see Section B.4 for the definition of the Killing form.)

Step 2. If this matrix is degenerate or negative definite, stop: the system is GUES. Otherwise, continue.

Step 3. Find three matrices h, e, and f in $\widetilde{\mathfrak{g}}$ with commutation relations $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$ (this is an $sl(2)$-triple which always exists in the present case). We can then write $\widetilde{A}_i = \beta_i e + \gamma_i f + \delta_i h$, where $\alpha_i, \beta_i, \gamma_i$ are constants, $i = 1,2$.

Step 4. Compute the largest eigenvalue of h. It will be an integer; denote it by k. Then the given system is GUES if and only if the switched linear system generated by the 2×2 matrices

$$\widehat{A}_1 := \begin{pmatrix} \frac{\mathrm{tr}(A_1)}{nk} - \delta_1 & -\beta_1 \\ -\gamma_1 & \frac{\mathrm{tr}(A_1)}{nk} + \delta_1 \end{pmatrix}, \quad \widehat{A}_2 := \begin{pmatrix} \frac{\mathrm{tr}(A_2)}{nk} - \delta_2 & -\beta_2 \\ -\gamma_2 & \frac{\mathrm{tr}(A_2)}{nk} + \delta_2 \end{pmatrix}$$

is GUES.

All steps in the above reduction procedure involve only standard matrix operations (addition, multiplication, and computation of eigenvalues and eigenvectors). The first two steps are justified by the fact that $\widehat{\mathfrak{g}}$ contains a noncompact semisimple subalgebra if and only if its dimension exactly equals 4 and the Killing form is nondegenerate and sign-indefinite on $\widetilde{\mathfrak{g}} = \widehat{\mathfrak{g}} \bmod \mathbb{R}I$ (see Section B.4). The justification for the last two steps comes from the classification of representations of $sl(2,\mathbb{R})$ given in Section B.2. If $\widehat{\mathfrak{g}}$ has dimension at most 4 but $\mathcal{P}$ has more than two elements, the above algorithm can be applied after finding an appropriate basis for $\widehat{\mathfrak{g}}$.

2.2.4 Discussion of Lie-algebraic stability criteria

Lie-algebraic stability criteria for switched systems are appealing because nontrivial mathematical tools are brought to bear on the problem and lead to interesting results. Another attractive feature of these conditions is that they are formulated in terms of the original data. Take, for example, Theorem 2.7. The proof of this theorem relies on the facts that the matrices in a solvable Lie algebra can be simultaneously triangularized and that switching between triangular matrices preserves stability. It is important to recognize, however, that it is a nontrivial matter to find a basis in which all matrices take the triangular form or even decide whether such a basis exists. To apply Theorem 2.7, no such basis needs to be found. Instead, one can check directly whether the Lie algebra generated by the given matrices is solvable.

In fact, classical results from the theory of Lie algebras can be employed to check the various stability conditions for switched linear systems presented above. Assume for simplicity that the set $\mathcal{P}$ is finite or a maximal linearly independent subset has been extracted from $\{A_p : p \in \mathcal{P}\}$. Then one can verify directly, using the definitions, whether or not the Lie algebra $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ is solvable (or nilpotent). To do this, one constructs a decreasing sequence of ideals by discarding lower-order Lie brackets at each step and checks whether the sequence of dimensions of these ideals strictly decreases to zero (see Section B.3). In specific examples involving a small number of matrices, it is usually not difficult to derive the relevant commutation relations between them. To do this systematically in more complicated situations, it is helpful to use a canonical basis known as a P. Hall basis.

Alternatively, one can use the Killing form, which is a canonical symmetric bilinear form defined on every Lie algebra (see Section B.3). Cartan's first criterion, stated in Section B.3, provides a necessary and sufficient condition for a Lie algebra to be solvable in terms of the Killing form. An explicit procedure for checking the hypothesis of Theorem 2.11 using the Killing form is described in Section B.4. In view of these remarks, Lie-algebraic tools yield stability conditions for switched systems which are both mathematically appealing and computationally efficient.

The main disadvantage of the Lie-algebraic stability criteria is their limited applicability. Clearly, they provide only sufficient and not necessary conditions for stability. (This can be seen from the second statement of Theorem 2.13 and also from the fact that they imply the existence of quadratic common Lyapunov functions—this property is interesting but, as we saw in Sections 2.1.4 and 2.1.5, does not hold for all GUES switched linear systems.)

Moreover, it turns out that even as sufficient conditions, the Lie-algebraic conditions are extremely nongeneric. To see why this is so, first note that the GUES property is robust with respect to sufficiently small perturbations of the matrices that define the individual subsystems. This follows via standard arguments from the converse Lyapunov theorem (Theorem 2.2). An especially transparent characterization of the indicated robustness property can be obtained in the case of linear systems sharing a quadratic common Lyapunov function, i.e., when there exist positive definite symmetric matrices P and Q satisfying the inequalities (2.10). Suppose that for every $p \in \mathcal{P}$, a perturbed matrix

$$\overline{A}_p := A_p + \Delta_p$$

is given. Let us derive an admissible bound on the perturbations Δ_p, $p \in \mathcal{P}$ such that the matrices $\overline{A}_p$, $p \in \mathcal{P}$ still share the same quadratic common Lyapunov function $x^T Px$. This is guaranteed if we have

$$|2x^T \Delta_p^T Px| < x^T Qx \qquad \forall p \in \mathcal{P}, \quad \forall x \neq 0. \tag{2.25}$$

We denote by $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ the smallest and the largest eigenvalue of a symmetric matrix, respectively. The right-hand side of (2.25) is lower-bounded by $\lambda_{\min}(Q)|x|^2$, while the left-hand side of (2.25) is upper-bounded by

$$2|\Delta_p x||Px| = 2\sqrt{x^T \Delta_p^T \Delta_p x} \cdot \sqrt{x^T P^2 x} \leq 2|x|^2 \sigma_{\max}(\Delta_p)\lambda_{\max}(P)$$

where $\sigma_{\max}(\Delta_p) := \sqrt{\lambda_{\max}(\Delta_p^T \Delta_p)}$ is the largest singular value of Δ_p. Therefore, a (conservative) admissible bound on Δ_p, $p \in \mathcal{P}$ is given by the formula

$$\sigma_{\max}(\Delta_p) < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}.$$

(The right-hand side is maximized when $Q = I$.)

On the other hand, the Lie-algebraic conditions of the type considered here do not possess the above robustness property. This follows from the fact, proved in Section B.6, that in an arbitrarily small neighborhood of any pair of $n \times n$ matrices there exists a pair of matrices that generate the entire Lie algebra $gl(n, \mathbb{R})$. In other words, the conditions given by Theorems 2.5, 2.7, and 2.11 are destroyed by arbitrarily small perturbations of the individual systems. To obtain more generic stability conditions, one needs to complement these results by a perturbation analysis.

2.3 Systems with special structure

The results discussed so far in this chapter apply to general switched systems. The questions related to stability of such systems are very difficult, and the findings discussed above certainly do not provide complete and satisfactory answers. On the other hand, specific structure of a given system can sometimes be utilized to obtain interesting results, even in the absence of a general theory. In this section we present a few results that are available for some special classes of switched systems.

2.3.1 Triangular systems

We already know (Proposition 2.9) that if $\{A_p : p \in \mathcal{P}\}$ is a compact set of Hurwitz matrices in the upper-triangular form, then the switched linear system (1.4) is GUES. In fact, under these hypotheses the linear systems in the family (2.13) share a quadratic common Lyapunov function. (The case of lower-triangular systems is completely analogous.) It is natural to ask to what extent this result is true for switched nonlinear systems.

Suppose that $\mathcal{P}$ is a compact set and that the family of systems (1.1) is such that for each $p \in \mathcal{P}$, the vector field f_p takes the upper-triangular form

$$f_p(x) = \begin{pmatrix} f_{p1}(x_1, x_2, \dots, x_n) \\ f_{p2}(x_2, \dots, x_n) \\ \vdots \\ f_{pn}(x_n) \end{pmatrix}. \tag{2.26}$$

If the linearization matrices (2.22) are Hurwitz and $\frac{\partial f_p}{\partial x}(x)$ depends continuously on p, then the linearized systems have a quadratic common Lyapunov function by virtue of the result mentioned above. It follows from Lyapunov's indirect method that in this case the original switched nonlinear system (1.3) is locally uniformly exponentially stable (cf. Corollary 2.10).

What about global stability results? One might be tempted to conjecture that under appropriate compactness assumptions, the switched system (1.3) is GUAS, provided that the individual subsystems (1.1) all share the origin as a globally asymptotically stable equilibrium. We now provide a counterexample showing that this is not true.

Example 2.4 Let $\mathcal{P} = \{1, 2\}$, and consider the vector fields

$$f_1(x) = \begin{pmatrix} -x_1 + 2\sin^2(x_1)x_1^2 x_2 \\ -x_2 \end{pmatrix}$$

and

$$f_2(x) = \begin{pmatrix} -x_1 + 2\cos^2(x_1)x_1^2 x_2 \\ -x_2 \end{pmatrix}$$

FACT 1. The systems $\dot{x} = f_1(x)$ and $\dot{x} = f_2(x)$ are globally asymptotically stable.

To see that the system $\dot{x} = f_1(x)$ is globally asymptotically stable, fix arbitrary initial values $x_1(0), x_2(0)$. We have $x_2(t) = x_2(0)e^{-t}$. As for x_1, note that $\sin(x_1)$ vanishes at the integer multiples of π. This implies that $|x_1(t)| \leq E$ for all $t \geq 0$, where E is the smallest integer multiple of π that is greater than or equal to $|x_1(0)|$. Since x_1 is bounded and x_2 converges to zero, it is easy to see that the linear term $-x_1$ eventually dominates and we have $x_1(t) \to 0$. We conclude that the system $\dot{x} = f_1(x)$ is globally attractive; its stability in the sense of Lyapunov can be shown by similar arguments. Global asymptotic stability of the system $\dot{x} = f_2(x)$ is established in the same way.

FACT 2. The switched system $\dot{x} = f_\sigma(x)$ is not GUAS.

If the switched system were GUAS, then Corollary 2.3 would guarantee global asymptotic stability of an arbitrary "convex combination"

$$\begin{aligned} \dot{x}_1 &= -x_1 + 2(\alpha \sin^2(x_1) + (1-\alpha)\cos^2(x_1))x_1^2 x_2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

of the two subsystems, where $0 \leq \alpha \leq 1$. In particular, for $\alpha = 1/2$ we arrive at the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 x_2 \\ \dot{x}_2 &= -x_2. \end{aligned} \tag{2.27}$$

We will now show that this system is not globally asymptotically stable; in fact, it even has solutions that are not defined globally in time. Recall that solutions of the equation $\dot{x} = x^2$ escape to infinity in finite time (see Example 1.3). In view of this, it is not hard to see that for sufficiently large initial conditions, the x_1-component of the solution of the system (2.27) escapes to infinity before x_2 becomes small enough to slow it down. The system (2.27) was actually discussed in [161, p. 8] in the context of adaptive control. Its solutions are given by the formulas

$$\begin{aligned} x_1(t) &= \frac{2x_1(0)}{x_1(0)x_2(0)e^{-t} + (2 - x_1(0)x_2(0))e^t} \\ x_2(t) &= x_2(0)e^{-t}. \end{aligned}$$

We see that solutions with $x_1(0)x_2(0) \geq 2$ are unbounded and, in particular, solutions with $x_1(0)x_2(0) > 2$ have a finite escape time. This proves that the switched system $\dot{x} = f_\sigma(x)$ is not GUAS. □

Thus in the case of switching among globally asymptotically stable nonlinear systems, the triangular structure alone is not sufficient for GUAS. One way to guarantee GUAS is to require that along solutions of the individual subsystems (1.1), each component of the state vector stay small if the subsequent components are small. The right notion in this regard turns out to be input-to-state stability (ISS), reviewed in Section A.6.

Theorem 2.14 *Assume that $\mathcal{P}$ is a compact set, f_p is continuous in p for each x, and the systems* (1.1) *are globally asymptotically stable and take the triangular form* (2.26). *If for each $i = 1, \ldots, n-1$ and each $p \in \mathcal{P}$ the system*

$$\dot{x}_i = f_{pi}(x_i, x_{i+1}, \ldots, x_n)$$

is ISS with respect to the input $u = (x_{i+1}, \ldots, x_n)^T$, then the switched system (1.3) *is GUAS.*

The first two hypotheses are trivially satisfied if $\mathcal{P}$ is a finite set. The theorem can be proved by starting with the bottom component of the state vector x and proceeding upwards, using ISS-Lyapunov functions (this is in the same spirit as the argument we used earlier to prove Proposition 2.9). For asymptotically stable linear systems, the ISS assumption is automatically satisfied, which explains why we did not need it in Section 2.2.2. Under certain additional conditions, it is possible to extend Theorem 2.14 to block-triangular systems.

If the triangular subsystems are asymptotically stable, $\mathcal{P}$ is a compact set, and f_p depends continuously on p for each x, then the ISS hypotheses of Theorem 2.14 are automatically satisfied in a sufficiently small neighborhood of the origin. Indeed, asymptotic stability of $\dot{x} = f_p(x)$ guarantees that the system $\dot{x}_i = f_{pi}(x_i, 0)$ is asymptotically stable[2] for each i, and we know from Section A.6 that this implies local ISS of $\dot{x}_i = f_{pi}(x_i, u)$. Thus the triangular switched system in question is always locally uniformly asymptotically stable, even if the linearization test mentioned earlier fails.

2.3.2 Feedback systems

Switched systems often arise from the feedback connection of different controllers with the same process (switching control mechanisms of this nature will be studied in Part III). Such feedback switched systems therefore assume particular interest in control theory. The fact that the process is fixed imposes some structure on the closed-loop systems, which sometimes facilitates the stability analysis. Additional flexibility is gained if input-output properties of the process and the controllers are specified but one has some freedom in choosing state-space realizations. We now briefly discuss several stability results for switched systems of this kind.

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[2]This is because all trajectories of the latter system are projections of trajectories of the former, for $x_{i+1}(0) = \cdots = x_n(0)$, onto the x_i-axis.

Passivity, positive realness, and absolute stability

Consider the control system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned}$$

with $x \in \mathbb{R}^n$ and $u, y \in \mathbb{R}^m$. By (strict) *passivity* we mean the property of this system characterized by the existence of a C^1 positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ (called a *storage function*) and a positive definite function $W : \mathbb{R}^n \to \mathbb{R}$ such that we have

$$\frac{\partial V}{\partial x} f(x, u) \leq -W(x) + u^T h(x). \tag{2.28}$$

(It is usually assumed that the storage function is merely nonnegative definite, but in the case of strict passivity its positive definiteness is automatic.) Passive systems frequently arise in a variety of applications, for example, in models of electrical circuits and mechanical devices.

Suppose that the inequality (2.28) holds. It is easy to see that for every $K \geq 0$, the closed-loop system obtained by setting $u = -Ky$ is asymptotically stable, with Lyapunov function V whose derivative along solutions satisfies

$$\dot{V}(x) \leq -W(x) - y^T K y.$$

In other words, V is a common Lyapunov function for the family of closed-loop systems corresponding to all nonpositive definite feedback gain matrices. It follows that the switched system generated by this family is uniformly asymptotically stable (GUAS if V is radially unbounded). Clearly, the function V also serves as a common Lyapunov function for all nonlinear feedback systems obtained by setting $u = -\varphi(y)$, where φ satisfies $y^T \varphi(y) \geq 0$ for all y. In the single-input, single-output (SISO) case, this reduces to the sector condition

$$0 \leq y\varphi(y) \qquad \forall y. \tag{2.29}$$

For linear systems, there is a very useful frequency-domain condition for passivity in terms of the concept of positive realness which we now define. We limit our discussion to SISO systems, although similar results hold for general systems. A proper rational function $g : \mathbb{C} \to \mathbb{C}$ is called *positive real* if $g(s) \in \mathbb{R}$ when $s \in \mathbb{R}$ and $\operatorname{Re} g(s) \geq 0$ when $\operatorname{Re} s \geq 0$, and *strictly positive real* if $g(s - \varepsilon)$ is positive real for some $\varepsilon > 0$. A positive real function has all its poles in the closed left half-plane; if all poles are in the open left half-plane, then it is enough to check the inequality $\operatorname{Re} g(s) \geq 0$ along the imaginary axis.

Every linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= c^T x \end{aligned}$$

with a Hurwitz matrix A and a strictly positive real transfer function

$$g(s) = c^T(sI - A)^{-1}b$$

is strictly passive. This follows from the famous Kalman-Yakubovich-Popov lemma, which guarantees the existence of a positive definite symmetric matrix P satisfying

$$\begin{aligned} A^T P + PA &\le -Q < 0 \\ Pb &= c. \end{aligned}$$

Letting $V(x) := \frac{1}{2}x^T P x$, we obtain

$$\frac{\partial V}{\partial x}(Ax + Bu) = \frac{1}{2}x^T(A^T P + PA)x + x^T P b u \le -x^T Q x + u^T y.$$

We conclude that if the open-loop transfer function is strictly positive real, then the closed-loop systems for all nonpositive feedback gains ($u = -ky$, $k \ge 0$) share a quadratic common Lyapunov function. (For systems of dimension $n \le 2$ the converse is also true: the existence of such a quadratic common Lyapunov function implies that the open-loop transfer function is strictly positive real.) We conclude that the corresponding switched linear system is GUES. Again, the above result immediately extends to nonlinear feedback systems

$$\dot{x} = Ax - b\varphi(c^T x) \tag{2.30}$$

with φ satisfying the inequality (2.29).

If the open-loop transfer function g is not strictly positive real but the function

$$\frac{1 + k_2 g}{1 + k_1 g} \tag{2.31}$$

is strictly positive real for some $k_2 > k_1 \ge 0$, where k_1 is a stabilizing gain, then a quadratic common Lyapunov function exists for the family of systems (2.30) under the following more restrictive sector condition on φ:

$$k_1 y^2 \le y\varphi(y) \le k_2 y^2 \qquad \forall y. \tag{2.32}$$

This result is usually referred to as the *circle criterion*, because the strict positive real property of the function (2.31) implies that the Nyquist locus of g lies outside the disk centered at the real axis which intersects the real axis at the points $(-1/k_1, 0)$ and $(-1/k_2, 0)$. For $k_1 = 0$ this disk becomes the half-plane $\{s : \operatorname{Re} s \le -1/k_2\}$.

The problem of determining stability of the system (2.30) for all nonlinearities φ lying in some given sector such as (2.29) or (2.32) is the well-known *absolute stability* problem. In the investigation of this problem, conditions that lead to the existence of a *quadratic* Lyapunov function are in general too restrictive. Less conservative frequency-domain conditions for absolute stability are provided by *Popov's criterion*. One version of this criterion can be stated as follows: if g has one pole at zero and the rest in the open left half-plane and the function $(1+\alpha s)g(s)$ is positive real for some $\alpha \geq 0$, then the system (2.30) is globally asymptotically stable for every function φ that satisfies the sector condition $0 < y\varphi(y)$ for all $y \neq 0$. Alternatively, if $(1+\alpha s)g(s)$ is strictly positive real for some $\alpha \geq 0$, then the weaker sector condition (2.29) is sufficient. When Popov's criterion applies, there exists a Lyapunov function for the closed-loop system in the form of a quadratic term plus an integral of the nonlinearity. Since this Lyapunov function depends explicitly on φ, a common Lyapunov function in general no longer exists; in other words, switching between different negative feedback gains or sector nonlinearities may cause instability.

Small-gain theorem

Consider the output feedback switched linear system

$$\dot{x} = (A + BK_\sigma C)x. \tag{2.33}$$

Assume that A is a Hurwitz matrix and that $\|K_p\| \leq 1$ for all $p \in \mathcal{P}$, where $\|\cdot\|$ denotes the matrix norm induced by the Euclidean norm on $\mathbb{R}^n$. Then the classical small-gain theorem implies that (2.33) is GUES if

$$\|C(sI - A)^{-1}B\|_\infty < 1 \tag{2.34}$$

where $\|\cdot\|_\infty$ denotes the standard $\mathcal{H}_\infty$ norm of a transfer matrix, defined as $\|G\|_\infty := \max_{\mathrm{Re}\, s=0} \sigma_{\max}(G(s))$. This norm characterizes the $\mathcal{L}_2$ gain of the open-loop system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx. \end{aligned} \tag{2.35}$$

The condition (2.34) is satisfied if and only if there exists a solution $P > 0$ of the algebraic Riccati inequality

$$A^T P + PA + PBB^T P + C^T C < 0.$$

Under the present assumptions, this inequality actually provides a necessary and sufficient condition for the linear systems

$$\dot{x} = (A + BK_p C)x, \qquad p \in \mathcal{P} \tag{2.36}$$

to share a quadratic common Lyapunov function $V(x) = x^T Px$. A simple square completion argument demonstrates that the derivative of this Lyapunov function along solutions of the system (2.35) satisfies $\dot{V} \leq -|y|^2 + |u|^2 - \varepsilon|x|^2$ for some $\varepsilon > 0$. From this we see that V also serves as a common

Lyapunov function for the family of nonlinear feedback systems that result from setting $u = \varphi(y)$ with

$$|\varphi(y)| \leq |y| \qquad \forall y. \tag{2.37}$$

Since V is quadratic and the bound on its decay rate is also quadratic, it follows from Remark 2.1 that the switched system is still GUES.

Note that the inequality (2.37) is equivalent to the sector condition (2.32) with $k_1 = -1$ and $k_2 = 1$. The circle criterion can be applied in such situations too, except that now the Nyquist locus must lie *inside* an appropriate disk. This observation points to a unified framework for small-gain and passivity conditions.

Exercise 2.4 Investigate stability of the system $\ddot{x}+\dot{x} = u$ under nonlinear feedbacks of the form $u = -\varphi(x)$ by checking which of the above results (passivity criterion, circle criterion, Popov's criterion, small-gain theorem) can be applied. Support your findings by Lyapunov analysis.

If a given switched linear system is not in the form (2.33), it may be possible to construct an auxiliary switched linear system whose stability can be checked with the help of the above result and implies stability of the original system. As an example, consider the switched linear system (1.4) with $\mathcal{P} = \{1, 2\}$. It can be recast, for instance, as

$$\dot{x} = \frac{1}{2}(A_1 + A_2)x + \sigma\frac{1}{2}(A_1 - A_2)x$$

where σ takes values in the set $\{-1, 1\}$. It follows that this system is GUES if the inequality (2.34) is satisfied for $A = \frac{1}{2}(A_1 + A_2)$, $B = \frac{1}{2}I$, and $C = A_1 - A_2$. A similar trick can be used if one wants to apply the passivity criterion. Rewriting the same switched linear system as, say,

$$\dot{x} = A_1x - \sigma(A_1 - A_2)x$$

with σ now taking values in the set $\{0, 1\}$, we see that it is GUES if the open-loop system (2.35) with $A = A_1$, $B = I$, and $C = A_1 - A_2$ is strictly passive. Of course, the above choices of the auxiliary system parameters are quite arbitrary.

Coordinate changes and realizations

When each of the individual subsystems to be switched is specified up to a coordinate transformation (as is the case, for example, when one can choose realizations for given transfer matrices), a question that arises is whether it is possible to pick coordinates for each subsystem to ensure that the resulting switched system is stable. For linear systems, the answer is positive, as we now show.

Suppose that we are given a set $\{A_p : p \in \mathcal{P}\}$ of Hurwitz matrices. For each $p \in \mathcal{P}$, there exists a positive definite symmetric matrix P_p that satisfies

$$A_p^T P_p + P_p A_p = -I. \tag{2.38}$$

Consider the similarity transformations

$$A_p \mapsto \widetilde{A}_p := P_p^{1/2} A_p P_p^{-1/2}, \qquad p \in \mathcal{P}$$

(here $P_p^{1/2}$ is a square root of P_p and $P_p^{-1/2}$ is its inverse). These transformations correspond to linear coordinate changes applied to the linear systems (2.13). Multiplying (2.38) on both sides by $P_p^{-1/2}$, we obtain

$$\widetilde{A}_p^T + \widetilde{A}_p = -P_p^{-1} < 0.$$

This means that the linear systems $\dot{x} = \widetilde{A}_p x$, $p \in \mathcal{P}$ share the quadratic common Lyapunov function $V(x) = x^T x$, hence the switched linear system $\dot{x} = \widetilde{A}_\sigma x$ is GUES. The above result implies that given a family of stable transfer matrices (with an upper bound on their McMillan degree), we can always find suitable state-space realizations such that the linear systems describing their internal dynamics have a quadratic common Lyapunov function.

Now consider the situation where a given process $\mathbb{P}$ and a family of stabilizing controllers $\{\mathbb{C}_q : q \in \mathcal{Q}\}$ are specified by their transfer matrices. (When we say that a controller is stabilizing, we mean that the poles of the closed-loop transfer matrix that results from the feedback connection of this controller with the process have negative real parts.) Assume for simplicity that the index set $\mathcal{Q}$ is finite. Interestingly, it is always possible to find realizations for the process and the controllers, the latter of fixed dimension, with the property that the closed-loop systems share a quadratic common Lyapunov function and consequently the feedback switched system is GUES.

Theorem 2.15 *Given a strictly proper transfer matrix of the process and a finite family of transfer matrices of stabilizing controllers, there exist realizations of the process and the controllers such that the corresponding closed-loop systems share a quadratic common Lyapunov function.*

The proof of this result given in [131] relies on the previous observation regarding coordinate transformations and on the well-known Youla parameterization of stabilizing controllers. An interpretation of the above result is that stability properties of switched linear systems are not determined by the input-output behavior of the individual subsystems and can be gained (or lost) as a result of changing state-space realizations.

2.3.3 Two-dimensional systems

For switched homogeneous systems—and in particular for switched linear systems—defined by (the convex hull of) a finite family of systems in two dimensions, necessary and sufficient conditions for GUES have been known since [93]. We do not include the precise conditions here. Although somewhat complicated to state, they are formulated in terms of the original data and are relatively straightforward to check, at least for switched linear systems where they reduce to a set of algebraic and integral inequalities.

For two-dimensional switched linear systems with $\mathcal{P} = \{1, 2\}$, there are simple sufficient as well as necessary and sufficient conditions for the existence of a quadratic common Lyapunov function. (Recall that the existence of such a function is a stronger property than GUES.) We know that if the linear systems

$$\dot{x} = A_1 x, \qquad \dot{x} = A_2 x, \qquad x \in \mathbb{R}^2 \tag{2.39}$$

share a quadratic common Lyapunov function, then the convex combinations $\alpha A_1 + (1 - \alpha)A_2$, $\alpha \in [0, 1]$ must be Hurwitz. On the other hand, if all convex combinations of A_1 and A_2 have negative real eigenvalues, then a quadratic common Lyapunov function exists. Interestingly, taking into account the convex combinations of the given matrices and their inverses, one arrives at a necessary and sufficient condition for the existence of a quadratic common Lyapunov function.

Proposition 2.16 *The linear systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ share a quadratic common Lyapunov function if and only if all pairwise convex combinations of the matrices A_1, A_2, A_1^{-1}, and A_2^{-1} are Hurwitz.*

This result is usually stated in a slightly different form, namely, in terms of stability of the convex combinations of A_1 and A_2 and the convex combinations of A_1 and A_2^{-1}. The two statements are equivalent because all convex combinations of a Hurwitz matrix and its inverse are Hurwitz. To see why this is true, note that

$$A^T P + PA = -Q$$

implies

$$(A^{-1})^T P + PA^{-1} = -(A^{-1})^T Q A^{-1}$$

for every Hurwitz matrix A, so that $V(x) = x^T P x$ is a quadratic common Lyapunov function for the systems $\dot{x} = Ax$ and $\dot{x} = A^{-1}x$ and hence for all their convex combinations.

Being necessary and sufficient, all of the conditions mentioned above are robust with respect to sufficiently small perturbations of the systems being switched (this is not difficult to check directly). Thus they are quite useful for analyzing stability of two-dimensional switched systems. Since their proofs rely heavily on planar geometry, there is probably little hope of extending these results to higher dimensions.

3
Stability under Constrained Switching

3.1 Multiple Lyapunov functions

We begin this chapter by describing a useful tool for proving stability of switched systems, which relies on *multiple Lyapunov functions*, usually one or more for each of the individual subsystems being switched. To fix ideas, consider the switched system (1.3) with $\mathcal{P} = \{1, 2\}$. Suppose that both systems $\dot{x} = f_1(x)$ and $\dot{x} = f_2(x)$ are (globally) asymptotically stable, and let V_1 and V_2 be their respective (radially unbounded) Lyapunov functions. We are interested in the situation where a common Lyapunov function for the two systems is not known or does not exist. In this case, one can try to investigate stability of the switched system using V_1 and V_2.

In the absence of a common Lyapunov function, stability properties of the switched system in general depend on the switching signal σ. Let t_i, $i = 1, 2, \ldots$ be the switching times. If it so happens that the values of V_1 and V_2 coincide at each switching time, i.e., $V_{\sigma(t_{i-1})}(t_i) = V_{\sigma(t_i)}(t_i)$ for all i, then V_σ is a continuous Lyapunov function for the switched system, and asymptotic stability follows. This situation is depicted in Figure 14(a).

In general, however, the function V_σ will be discontinuous. While each V_p decreases when the pth subsystem is active, it may increase when the pth subsystem is inactive. This behavior is illustrated in Figure 14(b). The basic idea that allows one to show asymptotic stability in this case is the following. Let us look at the values of V_p at the beginning of each interval

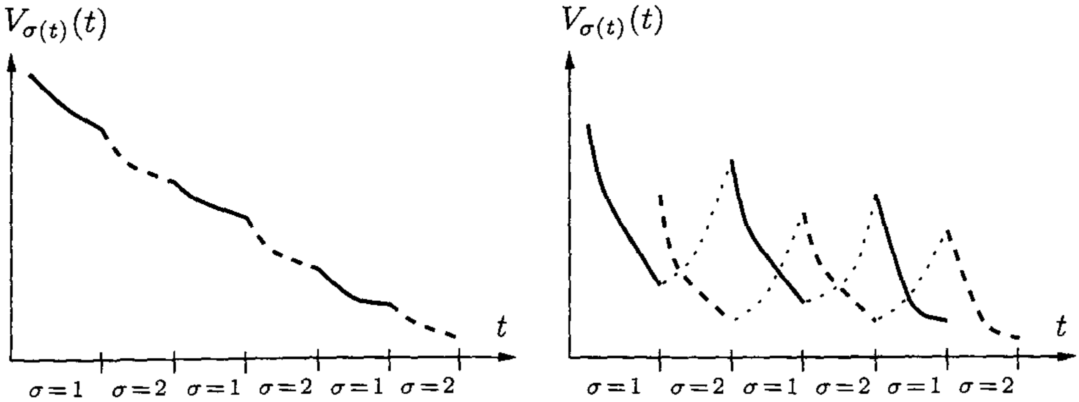

FIGURE 14. Two Lyapunov functions (solid graphs correspond to V_1, dashed graphs correspond to V_2): (a) continuous V_σ, (b) discontinuous V_σ

on which $\sigma = p$. For the switched system to be asymptotically stable, these values[1] must form a decreasing sequence for each p.

Theorem 3.1 *Let (1.1) be a finite family of globally asymptotically stable systems, and let V_p, $p \in \mathcal{P}$ be a family of corresponding radially unbounded Lyapunov functions. Suppose that there exists a family of positive definite continuous functions W_p, $p \in \mathcal{P}$ with the property that for every pair of switching times (t_i, t_j), $i < j$ such that $\sigma(t_i) = \sigma(t_j) = p \in \mathcal{P}$ and $\sigma(t_k) \neq p$ for $t_i < t_k < t_j$, we have*

$$V_p(x(t_j)) - V_p(x(t_i)) \leq -W_p(x(t_i)). \tag{3.1}$$

Then the switched system (1.3) is globally asymptotically stable.

PROOF. We first show stability of the origin in the sense of Lyapunov. Let m be the number of elements in $\mathcal{P}$. Without loss of generality, we assume that $\mathcal{P} = \{1, 2, \ldots, m\}$. Consider the ball around the origin of an arbitrary given radius $\varepsilon > 0$. Let $\mathcal{R}_m$ be a set of the form $\{x : V_m(x) \leq c_m\}$, $c_m > 0$, which is contained in this ball. For $i = m-1, \ldots, 1$, let $\mathcal{R}_i$ be a set of the form $\{x : V_i(x) \leq c_i\}$, $c_i > 0$, which is contained in the set $\mathcal{R}_{i+1}$. Denote by δ the radius of some ball around the origin which lies in the intersection of all nested sequences of sets constructed in this way for all possible permutations of $\{1, 2, \ldots, m\}$. Suppose that the initial condition satisfies $|x(0)| \leq \delta$. If the first k values of σ are distinct, where $k \leq m$, then by construction we have $|x(t_k)| \leq \varepsilon$. After that, the values of σ will start repeating, and the condition (3.1) guarantees that the state trajectory will always belong to one of the above sets. Figure 15 illustrates this argument for the case $m = 2$.

To show asymptotic stability, observe that due to the finiteness of $\mathcal{P}$ there exists an index $q \in \mathcal{P}$ that has associated with it an infinite sequence of

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[1] Alternatively, we could work with the values of V_p at the *end* of each interval on which $\sigma = p$.

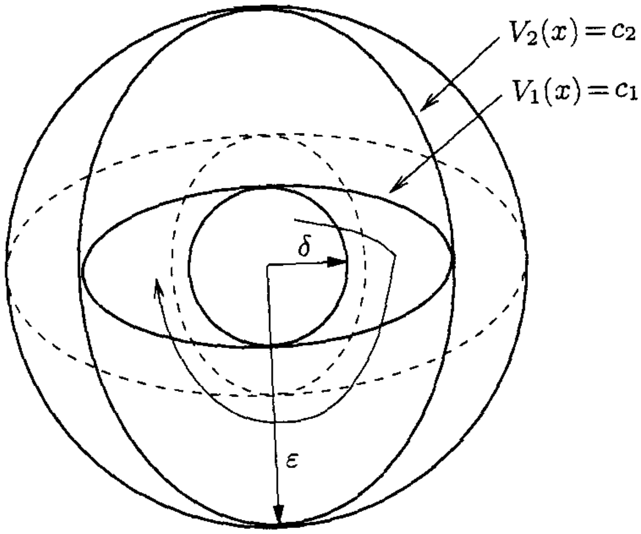

FIGURE 15. Proving Lyapunov stability in Theorem 3.1

switching times $t_{i_1}, t_{i_2}, \ldots$ such that $\sigma(t_{i_j}) = q$ (we are ruling out the trivial case when there are only finitely many switches). The sequence $V_q(x(t_{i_1}))$, $V_q(x(t_{i_2}))$, $\ldots$ is decreasing and positive and therefore has a limit $c \geq 0$. We have

$$\begin{aligned} 0 = c - c &= \lim_{j\to\infty} V_q(x(t_{i_{j+1}})) - \lim_{j\to\infty} V_q(x(t_{i_j})) \\ &= \lim_{j\to\infty} [V_q(x(t_{i_{j+1}})) - V_q(x(t_{i_j}))] \\ &\leq \lim_{j\to\infty} [-W_q(x(t_{i_j}))] \leq 0. \end{aligned}$$

Thus $W_q(x(t_{i_j})) \to 0$ as $j \to \infty$. We also know that W_q is positive definite. In view of radial unboundedness of V_p, $p \in \mathcal{P}$, an argument similar to the one used earlier to prove Lyapunov stability shows that $x(t)$ stays bounded. Therefore, $x(t_{i_j})$ must converge to zero as $j \to \infty$. It now follows from the Lyapunov stability property that $x(t) \to 0$ as $t \to \infty$. □

Remark 3.1 It is possible to obtain less conservative stability conditions involving multiple Lyapunov functions. In particular, one can relax the requirement that each V_p must decrease on the intervals on which the pth system is active, provided that the admissible growth of V_p on such intervals is bounded in a suitable way. Impulse effects can also be incorporated within the same framework. □

It is important to note that to apply Theorem 3.1, one must have some information about the solutions of the system. Namely, one needs to know the values of suitable Lyapunov functions at switching times, which in general requires the knowledge of the state at these times. This is to be contrasted with the classical Lyapunov stability results, which do not require the knowledge of solutions. (Of course, in both cases there remains the problem of finding candidate Lyapunov functions.) As we will see shortly, multiple Lyapunov function results such as Theorem 3.1 are useful when the class of admissible switching signals is constrained in a way that makes it possible to ensure the desired relationships between the values of Lyapunov functions at switching times.

3.2 Stability under slow switching

It is well known that a switched system is stable if all individual subsystems are stable and the switching is sufficiently slow, so as to allow the transient effects to dissipate after each switch. In this section we discuss how this property can be precisely formulated and justified using multiple Lyapunov function techniques.

3.2.1 Dwell time

The simplest way to specify slow switching is to introduce a number $\tau_d > 0$ and restrict the class of admissible switching signals to signals with the property that the switching times $t_1, t_2, \dots$ satisfy the inequality $t_{i+1} - t_i \geq \tau_d$ for all i. This number τ_d is usually called the *dwell time* (because σ "dwells" on each of its values for at least τ_d units of time).

It is a well-known fact that when all linear systems in the family (2.13) are asymptotically stable, the switched linear system (1.4) is asymptotically stable if the dwell time τ_d is sufficiently large. The required lower bound on τ_d can be explicitly calculated from the exponential decay bounds on the transition matrices of the individual subsystems.

Exercise 3.1 Consider a set of matrices $\{A_p : p \in \mathcal{P}\}$ with the property that for some positive constants c and λ_0 the inequality $\|e^{A_p t}\| \leq c e^{-\lambda_0 t}$ holds for all $t \geq 0$ and all $p \in \mathcal{P}$. Let the switched linear system (1.4) be defined by a switching signal σ with a dwell time τ_d. For an arbitrary number $\lambda \in (0, \lambda_0)$, derive a lower bound on τ_d that guarantees global exponential stability with stability margin λ (see Section A.1).

Under suitable assumptions, a sufficiently large dwell time also guarantees asymptotic stability of the switched system in the nonlinear case. Probably the best way to prove most general results of this kind is by using multiple Lyapunov functions. We now sketch the relevant argument.

Assume for simplicity that all systems in the family (1.1) are globally exponentially stable. Then for each $p \in \mathcal{P}$ there exists a Lyapunov function V_p which for some positive constants a_p, b_p, and c_p satisfies

$$a_p|x|^2 \leq V_p(x) \leq b_p|x|^2 \tag{3.2}$$

and

$$\frac{\partial V_p}{\partial x} f_p(x) \leq -c_p |x|^2. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$\frac{\partial V_p}{\partial x} f_p(x) \leq -2\lambda_p V_p(x), \qquad p \in \mathcal{P}$$

where

$$\lambda_p := \frac{c_p}{2b_p}, \qquad p \in \mathcal{P}.$$

This implies that

$$V_p(x(t_0 + \tau_d)) \leq e^{-2\lambda_p \tau_d} V_p(x(t_0))$$

provided that $\sigma(t) = p$ for $t \in [t_0, t_0 + \tau_d)$.

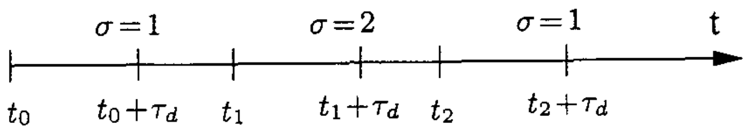

FIGURE 16. A dwell-time switching signal

To simplify the next calculation, let us consider the case when $\mathcal{P} = \{1, 2\}$ and σ takes the value 1 on $[t_0, t_1)$ and 2 on $[t_1, t_2)$, where $t_{i+1} - t_i \geq \tau_d$, $i = 0, 1$ (see Figure 16). From the above inequalities we have

$$V_2(t_1) \leq \frac{b_2}{a_1} V_1(t_1) \leq \frac{b_2}{a_1} e^{-2\lambda_1 \tau_d} V_1(t_0)$$

and furthermore

$$V_1(t_2) \leq \frac{b_1}{a_2} V_2(t_2) \leq \frac{b_1}{a_2} e^{-2\lambda_2 \tau_d} V_2(t_1) \leq \frac{b_1 b_2}{a_1 a_2} e^{-2(\lambda_1 + \lambda_2)\tau_d} V_1(t_0). \tag{3.4}$$

It is now straightforward to compute an explicit lower bound on τ_d which guarantees that the hypotheses of Theorem 3.1 are satisfied, implying that the switched system (1.3) is globally asymptotically stable. In fact, it is sufficient to ensure that

$$V_1(t_2) - V_1(t_0) \leq -\gamma |x(t_0)|^2$$

for some $\gamma > 0$. In view of (3.4), this will be true if we have

$$\left(\frac{b_1 b_2}{a_1 a_2} e^{-2(\lambda_1 + \lambda_2)\tau_d} - 1 \right) V_1(t_0) \leq -\gamma |x(t_0)|^2.$$

This will in turn hold, by virtue of (3.2), if

$$\left(\frac{b_1 b_2}{a_1 a_2} e^{-2(\lambda_1 + \lambda_2)\tau_d} - 1 \right) a_1 \leq -\gamma.$$

Since γ can be an arbitrary positive number, all we need to have is

$$\frac{b_1 b_2}{a_2} e^{-2(\lambda_1+\lambda_2)\tau_d} < a_1$$

which can be equivalently rewritten as

$$-2(\lambda_1 + \lambda_2)\tau_d < \log \frac{a_1 a_2}{b_1 b_2}$$

or finally as

$$\tau_d > \frac{1}{2(\lambda_1 + \lambda_2)} \log \frac{b_1 b_2}{a_1 a_2}. \tag{3.5}$$

This is a desired lower bound on the dwell time.

We do not discuss possible extensions and refinements here because a more general result will be presented below. Note, however, that the above reasoning would still be valid if the quadratic estimates in (3.2) and (3.3) were replaced by, say, quartic ones. In essence, all we used was the fact that there exists a positive constant μ such that

$$V_p(x) \le \mu V_q(x) \qquad \forall\, x \in \mathbb{R}^n, \quad \forall\, p, q \in \mathcal{P}. \tag{3.6}$$

If this inequality does not hold globally in the state space for any $\mu > 0$, then only local asymptotic stability can be established.

3.2.2 Average dwell time

In the context of controlled switching, specifying a dwell time may be too restrictive. If, after a switch occurs, there can be no more switches for the next τ_d units of time, then it is impossible to react to possible system failures during that time interval. When the purpose of switching is to choose the subsystem whose behavior is the best according to some performance criterion, as is often the case, there are no guarantees that the performance of the currently active subsystem will not deteriorate to an unacceptable level before the next switch is permitted (cf. Chapter 6). Thus it is of interest to relax the concept of dwell time, allowing the possibility of switching fast when necessary and then compensating for it by switching sufficiently slowly later.

The concept of average dwell time from [129] serves this purpose. Let us denote the number of discontinuities of a switching signal σ on an interval (t, T) by $N_\sigma(T, t)$. We say that σ has *average dwell time* τ_a if there exist two positive numbers N_0 and τ_a such that

$$N_\sigma(T, t) \le N_0 + \frac{T - t}{\tau_a} \qquad \forall\, T \ge t \ge 0. \tag{3.7}$$

For example, if $N_0 = 1$, then (3.7) implies that σ cannot switch twice on any interval of length smaller than τ_a. Switching signals with this property are exactly the switching signals with dwell time τ_a. Note also that $N_0 = 0$ corresponds to the case of no switching, since σ cannot switch at all on any interval of length smaller than τ_a. In general, if we discard the first N_0 switches (more precisely, the smallest integer greater than N_0), then the average time between consecutive switches is at least τ_a.

Besides being a natural extension of dwell time, the notion of average dwell time turns out to be very useful for analysis of the switching control algorithms to be studied in Chapter 6. Our present goal is to show that the property discussed earlier—namely, that asymptotic stability is preserved under switching with a sufficiently large dwell time—extends to switching signals with average dwell time. Although we cannot apply Theorem 3.1 directly to establish this result, the idea behind the proof is similar.

Exercise 3.2 Redo Exercise 3.1, this time working with the number of switches instead of assuming a fixed dwell time. Your answer should be of the form (3.7).

Theorem 3.2 *Consider the family of systems* (1.1). *Suppose that there exist C^1 functions $V_p : \mathbb{R}^n \to \mathbb{R}$, $p \in \mathcal{P}$, two class $\mathcal{K}_\infty$ functions α_1 and α_2, and a positive number λ_0 such that we have*

$$\alpha_1(|x|) \leq V_p(x) \leq \alpha_2(|x|) \qquad \forall x, \quad \forall p \in \mathcal{P} \tag{3.8}$$

and

$$\frac{\partial V_p}{\partial x} f_p(x) \leq -2\lambda_0 V_p(x) \qquad \forall x, \quad \forall p \in \mathcal{P}. \tag{3.9}$$

Suppose also that (3.6) *holds. Then the switched system* (1.3) *is globally asymptotically stable for every switching signal σ with average dwell time*[2]

$$\tau_a > \frac{\log \mu}{2\lambda_0} \tag{3.10}$$

(and N_0 arbitrary).

Let us examine the hypotheses of this theorem. If all systems in the family (1.1) are globally asymptotically stable, then for each $p \in \mathcal{P}$ there exists a Lyapunov function V_p which for all x satisfies

$$\alpha_{1,p}(|x|) \leq V_p(x) \leq \alpha_{2,p}(|x|)$$

and

$$\frac{\partial V_p}{\partial x} f_p(x) \leq -W_p(x) \tag{3.11}$$

[2] Note that $\log \mu > 0$ because $\mu > 1$ in view of the interchangeability of p and q in (3.6).

where W_p is positive definite. It is known (although nontrivial to prove) that there is no loss of generality in taking $W_p(x) = 2\lambda_p V_p(x)$ for some $\lambda_p > 0$, modifying V_p if necessary. Moreover, if $\mathcal{P}$ is a finite set or if it is compact and appropriate continuity assumptions are made, then we may choose functions α_1, α_2 and a constant λ_0, independent of p, such that the inequalities (3.8) and (3.9) hold. Thus the only really restrictive assumption is (3.6). It does not hold, for example, if V_p is quadratic for one value of p and quartic for another. If the systems (1.1) are globally exponentially stable, then the functions V_p, $p \in \mathcal{P}$ can be taken to be quadratic with quadratic decay rates, so that all hypotheses are verified.

Proof of Theorem 3.2. Pick an arbitrary $T > 0$, let $t_0 := 0$, and denote the switching times on the interval $(0, T)$ by $t_1, \ldots, t_{N_\sigma(T,0)}$. Consider the function

$$W(t) := e^{2\lambda_0 t} V_{\sigma(t)}(x(t)).$$

This function is piecewise differentiable along solutions of (1.3). On each interval $[t_i, t_{i+1})$ we have

$$\dot{W} = 2\lambda_0 W + e^{2\lambda_0 t} \frac{\partial V_{\sigma(t_i)}}{\partial x} f_{\sigma(t_i)}(x)$$

and this is nonpositive by virtue of (3.9), i.e., W is nonincreasing between the switching times. This together with (3.6) implies that

$$\begin{aligned} W(t_{i+1}) &= e^{2\lambda_0 t_{i+1}} V_{\sigma(t_{i+1})}(x(t_{i+1})) \le \mu e^{2\lambda_0 t_{i+1}} V_{\sigma(t_i)}(x(t_{i+1})) \\ &= \mu W(t_{i+1}^-) \le \mu W(t_i). \end{aligned}$$

Iterating this inequality from $i = 0$ to $i = N_\sigma(T, 0) - 1$, we have

$$W(T^-) \le W(t_{N_\sigma(T,0)}) \le \mu^{N_\sigma(T,0)} W(0).$$

It then follows from the definition of W that

$$e^{2\lambda_0 T} V_{\sigma(T^-)}(x(T)) \le \mu^{N_\sigma(T,0)} V_{\sigma(0)}(x(0)). \tag{3.12}$$

Now suppose that σ has the average dwell time property expressed by the inequality (3.7). Then we can rewrite (3.12) as

$$\begin{aligned} V_{\sigma(T^-)}(x(T)) &\le e^{-2\lambda_0 T + (N_0 + \frac{T}{\tau_a}) \log \mu} V_{\sigma(0)}(x(0)) \\ &= e^{N_0 \log \mu} e^{(\frac{\log \mu}{\tau_a} - 2\lambda_0) T} V_{\sigma(0)}(x(0)). \end{aligned}$$

We conclude that if τ_a satisfies the bound (3.10), then $V_{\sigma(T^-)}(x(T))$ converges to zero exponentially as $T \to \infty$; namely, it is upper-bounded by $\mu^{N_0} e^{-2\lambda T} V_{\sigma(0)}(x(0))$ for some $\lambda \in (0, \lambda_0)$. Using (3.8), we have $|x(T)| \le \alpha_1^{-1}(\mu^{N_0} e^{-2\lambda T} \alpha_2(|x(0)|))$, which proves global asymptotic stability. □

Remark 3.2 Similarly to the way uniform stability over all switching signals was defined in Section 2.1.1, we can define uniform stability properties over switching signals from a certain class. Since the above argument gives the same conclusion for every switching signal with average dwell time satisfying the inequality (3.10), we see that under the assumptions of Theorem 3.2 the switched system (1.3) is GUAS in this sense over all such switching signals. (The switched system is GUES if the functions α_1 and α_2 are monomials of the same degree, e.g., quadratic; cf. Remark 2.1.) □

Remark 3.3 It is clear from the proof of Theorem 3.2 that exponential convergence of V_σ at the rate 2λ for an arbitrary $\lambda \in (0, \lambda_0)$ can be achieved by requiring that

$$\tau_a \geq \frac{\log \mu}{2(\lambda_0 - \lambda)}.$$

When the subsystems are linear, we can take the Lyapunov functions V_p, $p \in \mathcal{P}$ to be quadratic, and λ_0 corresponds to a common lower bound on stability margins of the individual subsystems. Thus the exponential decay rate λ for the switched linear system can be made arbitrarily close to the smallest one among the linear subsystems if the average dwell time is restricted to be sufficiently large. It is instructive to compare this with Exercise 3.2. □

The constant N_0 affects the overshoot bound for Lyapunov stability but otherwise does not change stability properties of the switched system. Also note that in the above stability proof we only used the bound on the number of switches on an interval starting at the initial time. The formula (3.7) provides a bound on the number of switches—and consequently a uniform decay bound for the state—on every interval, not necessarily of this form. For linear systems, this property guarantees that various induced norms of the switched system in the presence of inputs are finite (cf. Lemma 6.6 in Section 6.6).

3.3 Stability under state-dependent switching

In the previous section we studied stability of switched systems under time-dependent switching satisfying suitable constraints. Another example of constrained switching is state-dependent switching, where a switching event can occur only when the trajectory crosses a switching surface (see Section 1.1.1). In this case, stability analysis is often facilitated by the fact that properties of each individual subsystem are of concern only in the regions where this system is active, and the behavior of this system in other parts of the state space has no influence on the switched system.

Example 3.1 Consider the 2×2 matrices

$$A_1 := \begin{pmatrix} \gamma & -1 \\ 2 & \gamma \end{pmatrix}, \qquad A_2 := \begin{pmatrix} \gamma & -2 \\ 1 & \gamma \end{pmatrix} \tag{3.13}$$

where γ is a negative number sufficiently close to zero, so that the trajectories of the systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ look, at least qualitatively, as depicted on the first two plots in Figure 8 on page 19. Now, define a state-dependent switched linear system in the plane by

$$\dot{x} = \begin{cases} A_1 x & \text{if } x_1 x_2 \leq 0 \\ A_2 x & \text{if } x_1 x_2 > 0\,. \end{cases} \tag{3.14}$$

It is easy to check that the function $V(x) := x^T x$ satisfies $\dot{V} < 0$ along all nonzero solutions of this switched system, hence we have global asymptotic stability. The trajectories of (3.14) look approximately as shown on the third plot in Figure 8.

For the above argument to apply, the individual subsystems do not even need to be asymptotically stable. Again, this is because the Lyapunov function only needs to decrease along solutions of each subsystem in an appropriate region, and not necessarily everywhere. If we set $\gamma = 0$, then V still decreases along all nonzero solutions of the switched system (3.14). From a perturbation argument it is clear that if γ is a sufficiently small positive number, then (3.14) is still globally asymptotically stable, even though the individual subsystems are unstable. (For one idea about how to prove asymptotic stability directly in the latter case, see Remark 3.1 in Section 3.1).

It is important to note that V serves as a Lyapunov function only in suitable regions for each subsystem. In fact, there is no global common Lyapunov function for the two subsystems. Indeed, if one changes the switching rule to

$$\dot{x} = \begin{cases} A_1 x & \text{if } x_1 x_2 > 0 \\ A_2 x & \text{if } x_1 x_2 \leq 0 \end{cases}$$

then the resulting switched system is unstable (cf. the last plot in Figure 8). □

Observe that the state-dependent switching strategies considered in the above example can be converted to time-dependent ones, because the time needed for a linear time-invariant system to cross a quadrant can be explicitly calculated and is independent of the trajectory. This remark applies to most of the switched systems considered in the remainder of this chapter.

If a stability analysis based on a single Lyapunov function breaks down, one can use multiple Lyapunov functions. The following modification of the previous example illustrates this method.

Example 3.2 Let us use the same matrices (3.13), with γ negative but close to zero, to define a different state-dependent switched linear system, namely,

$$\dot{x} = \begin{cases} A_1 x & \text{if } x_1 \geq 0 \\ A_2 x & \text{if } x_1 < 0 \,. \end{cases}$$

We know that the linear systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ do not share a quadratic common Lyapunov function. Moreover, it is also impossible to find a single quadratic function that, as in Example 3.1, decreases along solutions of each subsystem in the corresponding region. Indeed, since each region is a half-plane, by symmetry this would give a quadratic common Lyapunov function.

However, consider the two positive definite symmetric matrices

$$P_1 := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_2 := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

The functions $V_1(x) := x^T P_1 x$ and $V_2(x) := x^T P_2 x$ are Lyapunov functions for the systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$, respectively. Moreover, on the switching surface $\{x : x_1 = 0\}$ their values match. Thus the function V_σ, where σ is the switching signal taking the value 1 for $x_1 \geq 0$ and 2 for $x_1 < 0$, is continuous along solutions of the switched system and behaves as in Figure 14(a). This proves global asymptotic stability. The level sets of the Lyapunov functions in the appropriate regions and a typical trajectory of the switched system are plotted in Figure 17. □

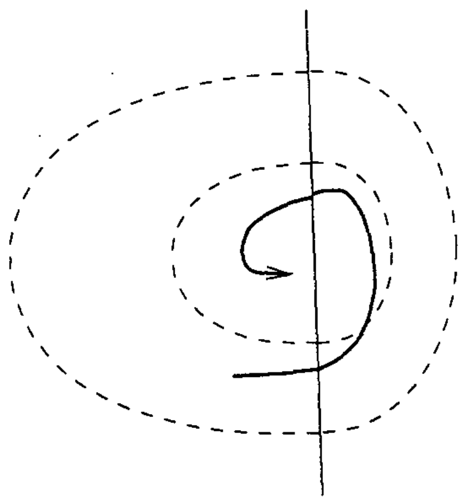

FIGURE 17. Illustrating stability in Example 3.2

Recall that Theorem 3.1 provides a less conservative condition for asymptotic stability, in the sense that multiple Lyapunov functions are allowed to behave like in Figure 14(b). If in Example 3.2 we multiplied the matrices P_1

and P_2 by two arbitrary positive numbers, the resulting Lyapunov functions would still satisfy the hypotheses of Theorem 3.1. In general, however, it is more difficult to use that theorem when the values of Lyapunov functions do not coincide on the switching surfaces.

As before, there is in general no need to associate with each subsystem a global Lyapunov function. It is enough to require that each function V_p decrease along solutions of the pth subsystem in the region Ω_p where this system is active (or may be active, if the index of the active subsystem is not uniquely determined by the value of x). This leads to relaxed stability conditions. For the case of a switched linear system and quadratic Lyapunov functions $V_p(x) = x^T P_p x$, $p \in \mathcal{P}$ these conditions can be brought to a computationally tractable form. This is achieved by means of the following well-known result, which will also be useful later.

Lemma 3.3 ("S-procedure") *Let T_0 and T_1 be two symmetric matrices. Consider the following two conditions:*

$$x^T T_0 x > 0 \text{ whenever } x^T T_1 x \geq 0 \text{ and } x \neq 0 \tag{3.15}$$

and

$$\exists \beta \geq 0 \text{ such that } T_0 - \beta T_1 > 0. \tag{3.16}$$

Condition (3.16) *always implies condition* (3.15). *If there is some x_0 such that $x_0^T T_1 x_0 > 0$, then* (3.15) *implies* (3.16).

Suppose that there exist symmetric matrices S_p, $p \in \mathcal{P}$ such that $\Omega_p \subset \{x : x^T S_p x \geq 0\}$ for all $p \in \mathcal{P}$. This means that each operating region Ω_p is embedded in a conic region. Then the S-procedure allows us to replace the condition

$$x^T (A_p^T P_p + P_p A_p) x < 0 \qquad \forall x \in \Omega_p \setminus \{0\}$$

by the linear matrix inequality

$$A_p^T P_p + P_p A_p + \beta_p S_p < 0, \qquad \beta_p \geq 0.$$

We also need to restrict the search for Lyapunov functions to ensure their continuity across the switching surfaces. If the boundary between Ω_p and Ω_q is of the form $\{x : f_{pq}^T x = 0\}$, $f_{pq} \in \mathbb{R}^n$, then we must have $P_p - P_q = f_{pq} t_{pq}^T + t_{pq} f_{pq}^T$ for some $t_{pq} \in \mathbb{R}^n$.

One can further reduce conservatism by considering several Lyapunov functions for each subsystem. In other words, one can introduce further partitioning of the regions Ω_p, $p \in \mathcal{P}$ and assign a function to each of the resulting regions. Stability conditions will be of the same form as before; we will simply have more regions, with groups of regions corresponding to the same dynamics. This provides greater flexibility in treating multiple subsystems and switching surfaces—especially in the presence of unstable subsystems—although the complexity of the required computations also grows.

3.4 Stabilization by state-dependent switching

In the previous section we discussed the problem of verifying stability of a given state-dependent switched linear system. In this section we study a related problem: given a family of linear systems, specify a state-dependent switching rule that makes the resulting switched linear system asymptotically stable. Of course, if at least one of the individual subsystems is asymptotically stable, this problem is trivial (just keep $\sigma(t) \equiv p$, where p is the index of this asymptotically stable subsystem). Therefore, for the rest of this chapter it will be understood that none of the individual subsystems is asymptotically stable.

3.4.1 Stable convex combinations

Suppose that $\mathcal{P} = \{1,2\}$ and that the individual subsystems are linear: $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$. As demonstrated in [303], one assumption that leads to an elegant construction of a stabilizing switching signal in this case is the following one:

ASSUMPTION 1. There exists an $\alpha \in (0,1)$ such that the convex combination

$$A := \alpha A_1 + (1-\alpha)A_2 \tag{3.17}$$

is Hurwitz. (The endpoints $\alpha = 0$ and $\alpha = 1$ are excluded because A_1 and A_2 are not Hurwitz.)

We know that in this case the switched system can be stabilized by fast switching designed so as to approximate the behavior of $\dot{x} = Ax$ (see Section 2.1.3). The procedure presented below allows one to avoid fast switching and is somewhat more systematic.

Under Assumption 1, there exist positive definite symmetric matrices P and Q which satisfy

$$A^T P + PA = -Q. \tag{3.18}$$

Using (3.17), we can rewrite (3.18) as

$$\alpha(A_1^T P + PA_1) + (1-\alpha)(A_2^T P + PA_2) = -Q$$

which is equivalent to

$$\alpha x^T(A_1^T P + PA_1)x + (1-\alpha)x^T(A_2^T P + PA_2)x = -x^T Q x < 0 \qquad \forall x \neq 0.$$

This implies that for every nonzero x we have either $x^T(A_1^T P + PA_1)x < 0$ or $x^T(A_2^T P + PA_2)x < 0$.

Let us define two regions

$$\Omega_i := \{x : x^T(A_i^T P + PA_i)x < 0\}, \qquad i = 1,2.$$

These are open conic regions ($x \in \Omega_i \;\Rightarrow\; \lambda x \in \Omega_i \; \forall \lambda \in \mathbb{R}$) which overlap and together cover $\mathbb{R}^n \setminus \{0\}$. It is now clear that we want to orchestrate the switching in such a way that the system $\dot{x} = A_i x$ is active in the region Ω_i, because this will make the function $V(x) := x^T P x$ decrease along solutions.

In implementing the above idea, we will pay special attention to two issues. First, we would like to have a positive lower bound on the rate of decrease of V. This will be achieved by means of modifying the original regions Ω_1, Ω_2. Second, we want to avoid chattering on the boundaries of the regions. This will be achieved with the help of hysteresis (cf. Section 1.2.4). We now describe the details of this construction.

Pick two new open conic regions Ω_i', $i = 1, 2$ such that each Ω_i' is a strict subset of Ω_i and we still have $\Omega_1' \cup \Omega_2' = \mathbb{R}^n \setminus \{0\}$. The understanding here is that each Ω_i' is obtained from Ω_i by a small amount of shrinking. Then the number

$$\varepsilon_i := - \max_{x \in \mathrm{cl}\, \Omega_i',\ |x|=1} x^T (A_i^T P + P A_i) x$$

is well defined and positive for each $i \in \{1, 2\}$, where "cl" denotes the closure of a set. Choosing a positive number $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$, we obtain

$$x^T (A_i^T P + P A_i) x < -\varepsilon |x|^2 \qquad \forall x \in \Omega_i', \quad i = 1, 2.$$

A hysteresis-based stabilizing switching strategy can be described as follows. Let $\sigma(0) = 1$ if $x(0) \in \Omega_1'$ and $\sigma(0) = 2$ otherwise. For each $t > 0$, if $\sigma(t^-) = i \in \{1, 2\}$ and $x(t) \in \Omega_i'$, keep $\sigma(t) = i$. On the other hand, if $\sigma(t^-) = 1$ but $x(t) \notin \Omega_1'$, let $\sigma(t) = 2$. Similarly, if $\sigma(t^-) = 2$ but $x(t) \notin \Omega_2'$, let $\sigma(t) = 1$. Thus σ changes its value when the trajectory leaves one of the regions, and the next switch can occur only when the trajectory leaves the other region after having traversed the intersection $\Omega_1' \cap \Omega_2'$. This situation is illustrated in Figure 18.

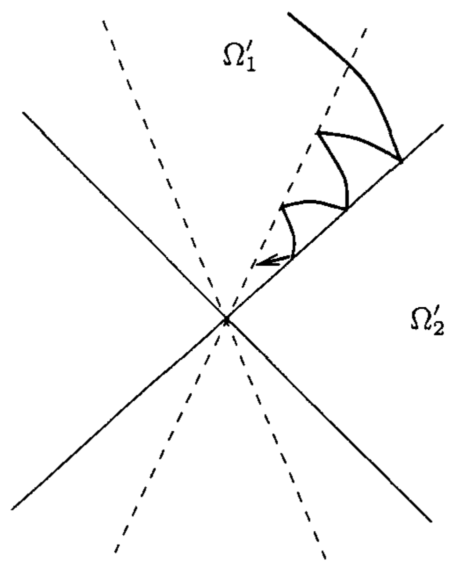

FIGURE 18. Conic regions and a possible trajectory (the boundary of Ω_1' is shown by solid lines and the boundary of Ω_2' is shown by dashed lines)

The above discussion implies that the derivative of V along the solutions of the resulting state-dependent switched linear system satisfies

$$\frac{d}{dt}x^T Px < -\varepsilon|x|^2 \qquad \forall\, x \neq 0. \tag{3.19}$$

This property is known as *quadratic stability* and is in general stronger than just global asymptotic stability, even for switched linear systems (see Example 3.3 below). We arrive at the following result.

Theorem 3.4 *If the matrices A_1 and A_2 have a Hurwitz convex combination, then there exists a state-dependent switching strategy that makes the switched linear system* (1.4) *with $\mathcal{P} = \{1, 2\}$ quadratically stable.*

Exercise 3.3 Construct an example of two unstable 2×2 matrices with a Hurwitz convex combination and implement the above procedure via computer simulation.

When the number of individual subsystems is greater than 2, one can try to single out from the corresponding set of matrices a pair that has a Hurwitz convex combination. If that fails, it might be possible to find a Hurwitz convex combination of three or more matrices from the given set, and then the above method for constructing a stabilizing switching signal can be applied with minor modifications.

A converse result

An interesting observation made in [91] is that Assumption 1 is not only sufficient but also necessary for quadratic stabilizability via state-dependent switching. This means that we cannot hope to achieve quadratic stability unless a given pair of matrices has a Hurwitz convex combination.

Theorem 3.5 *If there exists a state-dependent switching strategy that makes the switched linear system* (1.4) *with $\mathcal{P} = \{1, 2\}$ quadratically stable, then the matrices A_1 and A_2 have a Hurwitz convex combination.*

PROOF. Suppose that the switched linear system is quadratically stable, i.e., there exists a Lyapunov function $V(x) = x^T Px$ whose derivative along solutions of the switched system satisfies $\dot{V} < -\varepsilon|x|^2$ for some $\varepsilon > 0$. Since the switching is state-dependent, this implies that for every nonzero x we must have either $x^T(A_1^T P + PA_1)x < -\varepsilon|x|^2$ or $x^T(A_2^T P + PA_2)x < -\varepsilon|x|^2$. We can restate this as follows:

$$x^T(-A_1^T P - PA_1 - \varepsilon I)x > 0 \;\text{ whenever }\; x^T(A_2^T P + PA_2 + \varepsilon I)x \geq 0 \tag{3.20}$$

and

$$x^T(-A_2^T P - PA_2 - \varepsilon I)x > 0 \;\text{ whenever }\; x^T(A_1^T P + PA_1 + \varepsilon I)x \geq 0. \tag{3.21}$$

If $x^T(A_1^T P + PA_1 + \varepsilon I)x \le 0$ for all $x \neq 0$, then the matrix A_1 is Hurwitz and there is nothing to prove. Similarly, if $x^T(A_2^T P + PA_2 + \varepsilon I)x \le 0$ for all $x \neq 0$, then A_2 is Hurwitz. Discarding these trivial cases, we can apply the S-procedure (Lemma 3.3) to one of the last two conditions, say, to (3.20), and conclude that for some $\beta \ge 0$ we have

$$A_1^T P + PA_1 + \beta(A_2^T P + PA_2) < -(1+\beta)\varepsilon I$$

or, equivalently,

$$\frac{(A_1 + \beta A_2)^T}{1+\beta} P + P\frac{(A_1 + \beta A_2)}{1+\beta} < -\varepsilon I.$$

Therefore, the matrix $(A_1 + \beta A_2)/(1+\beta)$, which is a convex combination of A_1 and A_2, is Hurwitz, and so Assumption 1 is satisfied. □

We emphasize that the above result is limited to two linear subsystems, state-dependent switching signals, and quadratic stability.

3.4.2 Unstable convex combinations

A given pair of matrices may not possess a Hurwitz convex combination, in which case we know from Theorem 3.5 that quadratic stabilization is impossible. Also note that even if Assumption 1 is satisfied, in order to apply the procedure of Section 3.4.1 we need to identify a Hurwitz convex combination explicitly, which is a nontrivial task (in fact, this problem is known to be NP-hard). However, global asymptotic stabilization via state-dependent switching may still be possible even if no Hurwitz convex combination can be found.

Example 3.3 Consider the matrices

$$A_1 := \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}.$$

Define a two-dimensional state-dependent switched linear system by the rule

$$\dot{x} = \begin{cases} A_1 x & \text{if } x_1 x_2 \le 0 \\ A_2 x & \text{if } x_1 x_2 > 0\,. \end{cases}$$

This is the system of Example 3.1 with $\gamma = 0$. The trajectories of the individual subsystems and the switched system are shown in Figure 19.

It is not hard to see that this switched system is globally asymptotically stable. For example, the derivative of the function $V(x) := x^T x$ along solutions is negative away from the coordinate axes, i.e., we have $\dot{V} < 0$ when $x_1 x_2 \neq 0$. Moreover, the smallest invariant set contained in the union

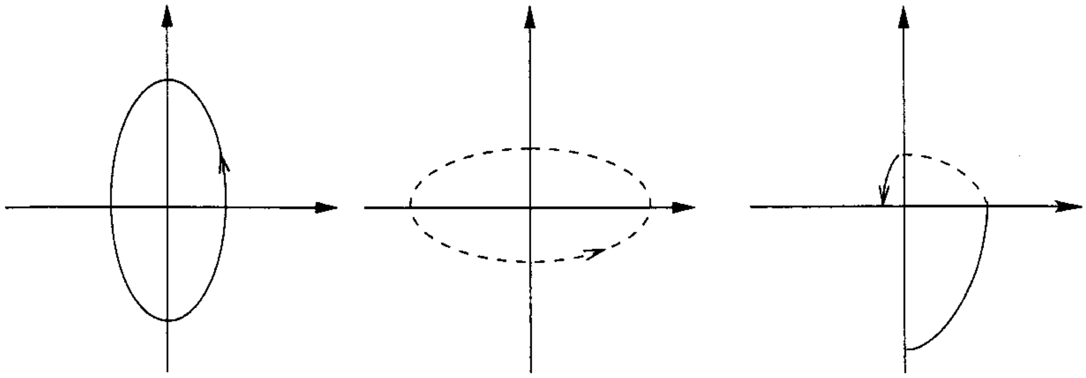

FIGURE 19. Switching between critically stable systems

of the coordinate axes is the origin, thus global asymptotic stability follows from LaSalle's invariance principle (Section A.4).

Since the derivative of V vanishes for some nonzero x, we do not have quadratic stability. More precisely, the inequality (3.19) cannot be satisfied with any $P = P^T > 0$ and $\varepsilon > 0$, and actually with any other choice of a state-dependent switching signal either. This follows from Theorem 3.5 because all convex combinations of the matrices A_1 and A_2 have purely imaginary eigenvalues.

The switching law used to asymptotically stabilize the switched system in this example is a special case of what is called a *conic switching law.* The switching occurs on the lines where the two vector fields are collinear, and the active subsystem is always the one whose vector field points inwards relative to the other. The system is globally asymptotically stable because the distance to the origin decreases after each rotation. No other switching signal would lead to better convergence.

It is interesting to draw a comparison with the system considered in Section 2.1.5. There, the opposite switching strategy was considered, whereby the active subsystem is always the one whose vector field points *outwards* relative to the other. If this does not destabilize the system, then no other switching signal will.

In the above example, both subsystems rotate in the same direction. It is possible to apply similar switching laws to planar subsystems rotating in opposite directions. This may produce different types of trajectories, such as the one shown in Figure 18. □

Multiple Lyapunov functions

Both in Section 3.4.1 and in Example 3.3, the stability analysis was carried out with the help of a single Lyapunov function. When this does not seem possible, in view of the results presented in Sections 3.1 and 3.3 one can try to find a stabilizing switching signal and prove stability by using multiple Lyapunov functions.

Suppose again that we are switching between two linear systems $\dot{x} = A_1x$ and $\dot{x} = A_2x$. Associate with the first system a function $V_1(x) = x^TP_1x$, with $P_1 = P_1^T > 0$, which decreases along its solutions in a nonempty conic region Ω_1. It can be shown that this is always possible unless A_1 is a nonnegative multiple of the identity matrix. Similarly, associate with the second system a function $V_2(x) = x^T P_2 x$, with $P_2 = P_2^T > 0$, which decreases along its solutions in a nonempty conic region Ω_2. If the union of the regions Ω_1 and Ω_2 covers $\mathbb{R}^n \setminus \{0\}$, then one can try to orchestrate the switching in such a way that the conditions of Theorem 3.1 are satisfied.

Using the ideas discussed in Section 3.3, one can derive algebraic conditions (in the form of bilinear matrix inequalities) under which such a stabilizing switching signal exists and can be constructed explicitly. Suppose that the following condition holds:

CONDITION 1. We have

$$x^T(A_1^T P_1 + P_1 A_1)x < 0 \quad \text{whenever} \quad x^T P_1 x \leq x^T P_2 x \quad \text{and} \quad x \neq 0$$

and

$$x^T(A_2^T P_2 + P_2 A_2)x < 0 \quad \text{whenever} \quad x^T P_1 x \geq x^T P_2 x \quad \text{and} \quad x \neq 0.$$

If this condition is satisfied, then a stabilizing switching signal can be defined by

$$\sigma(t) := \arg\min\{V_i(x(t)) : i = 1, 2\}.$$

Indeed, let us first suppose that no sliding motion occurs on the switching surface $\mathcal{S} := \{x : x^T P_1 x = x^T P_2 x\}$. Then the function V_σ is continuous and decreases along solutions of the switched system, which guarantees global asymptotic stability. The existence of a sliding mode, on the other hand, is easily seen to be characterized by the inequalities

$$x^T\big(A_1^T(P_1 - P_2) + (P_1 - P_2)A_1\big)x \geq 0$$

and

$$x^T\big(A_2^T(P_1 - P_2) + (P_1 - P_2)A_2\big)x \leq 0 \tag{3.22}$$

for $x \in \mathcal{S}$. If a sliding motion occurs on $\mathcal{S}$, then σ is not uniquely defined, so we let $\sigma = 1$ on $\mathcal{S}$ without loss of generality. Let us show that V_1 decreases along the corresponding Filippov solution. For every $\alpha \in (0, 1)$, we have

$$\begin{aligned}
&x^T\Big(\big(\alpha A_1 + (1-\alpha)A_2\big)^T P_1 + P_1\big(\alpha A_1 + (1-\alpha)A_2\big)\Big)x \\
&= \alpha x^T(A_1^T P_1 + P_1 A_1)x + (1-\alpha)x^T(A_2^T P_1 + P_1 A_2)x \\
&\leq \alpha x^T(A_1^T P_1 + P_1 A_1)x + (1-\alpha)x^T(A_2^T P_2 + P_2 A_2)x < 0
\end{aligned}$$

where the first inequality follows from (3.22) while the last one follows from Condition 1. Therefore, the switched system is still globally asymptotically stable. (Contrast this situation with the one described in Example 2.2.)

Condition 1 holds if the following condition is satisfied (by virtue of the S-procedure, the two conditions are equivalent, provided that there exist $x_1, x_2 \in \mathbb{R}^n$ such that $x_1^T(P_2 - P_1)x_1 > 0$ and $x_2^T(P_1 - P_2)x_2 > 0$):

CONDITION 2. There exist $\beta_1, \beta_2 \geq 0$ such that we have

$$-A_1^T P_1 - P_1 A_1 - \beta_1(P_2 - P_1) > 0 \tag{3.23}$$

and

$$-A_2^T P_2 - P_2 A_2 - \beta_2(P_1 - P_2) > 0. \tag{3.24}$$

The problem of finding a stabilizing switching signal can thus be reduced to finding two positive definite matrices P_1 and P_2 such that the above inequalities are satisfied. Similarly, if $\beta_1, \beta_2 \leq 0$, then a stabilizing switching signal can be defined by

$$\sigma(t) := \arg\max\{V_i(x(t)) : i = 1, 2\}.$$

A somewhat surprising difference, however, is that this alternative approach does not guarantee stability of a sliding mode, so sliding modes need to be ruled out.

Part III

Switching Control

This part of the book is devoted to *switching control*. This material is motivated primarily by problems of the following kind: given a process, typically described by a continuous-time control system, find a controller such that the closed-loop system displays a desired behavior. In some cases, this can be achieved by applying a continuous static or dynamic feedback control law. In other cases, a continuous feedback law that solves the problem may not exist. A possible alternative in such situations is to incorporate logic-based decisions into the control law and implement switching among a family of controllers. This yields a switched (or hybrid) closed-loop system, shown schematically in Figure 20.

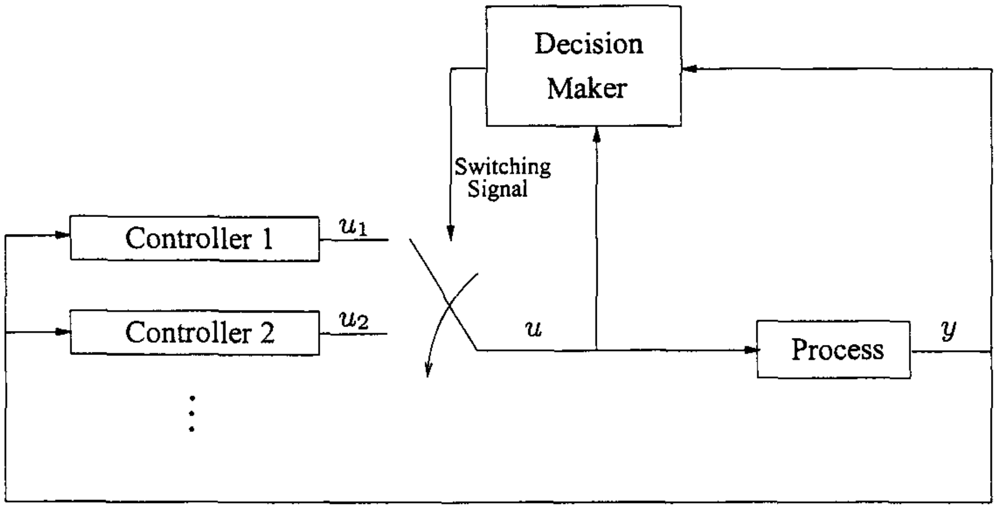

FIGURE 20. Switching control

We single out the following categories of control problems for which one might want—or need—to consider switching control (of course, combinations of two or more of these are also possible):

1. Due to the nature of the problem itself, continuous control is not suitable.

2. Due to sensor and/or actuator limitations, continuous control cannot be implemented.

3. The model of the system is highly uncertain, and a single continuous control law cannot be found.

There are actually several different scenarios that fit into the first category. If the given process is prone to unpredictable environmental influences or component failures, then it may be necessary to consider logic-based mechanisms for detecting such events and providing counteractions. If the desired system trajectory is composed of several pieces of significantly different types (e.g., aircraft maneuvers), then one might need to employ different controllers at different stages. The need for logic-based decisions also arises when the state space of the process contains obstacles. Perhaps more interestingly, there exist systems that are smooth and defined on spaces with no obstacles (e.g., $\mathbb{R}^n$) yet do not admit continuous feedback laws for tasks as basic as asymptotic stabilization. In other words, an obstruction to continuous stabilization may come from the mathematics of the system itself. A well-known class of such systems is given by nonholonomic control systems. Switching control techniques for some of these classes of systems will be discussed in Chapter 4.

The second of the categories mentioned above also encompasses several different classes of problems. The simplest example of an actuator limitation is when the control input is bounded, e.g., due to saturation. It is well known that optimal control of such systems involves switching (bang-bang) control. Control using output feedback, when the number of outputs is smaller than the number of states, can be viewed as control under sensor limitations. Typically, stabilization by a static output feedback is not possible, while implementing a dynamic output feedback may be undesirable. On the other hand, a simple switching control strategy can sometimes provide an effective solution to the output feedback stabilization problem. Perhaps more interestingly, a switched system naturally arises if the process dynamics are continuous-time but information is communicated only at discrete instants of time or over a finite-bandwidth channel, or if event-driven actuators are used. Thus we view switching control as a natural set of tools that can be applied to systems with sensor and actuator constraints. Specific problems in this area will be studied in Chapter 5.

The third category includes problems of controlling systems with large modeling uncertainty. As an alternative to traditional adaptive control, where controller selection is achieved by means of continuous tuning, it is possible to carry out the controller selection procedure with the help of logic-based switching among a family of control laws. This latter approach turns out to have some advantages over more conventional adaptive control algorithms, having to do with modularity of the design, simplicity of the analysis, and wider applicability. Switching control of uncertain systems is the subject of Chapter 6.

4

Systems Not Stabilizable by Continuous Feedback

4.1 Obstructions to continuous stabilization

Some systems cannot be globally asymptotically stabilized by smooth (or even continuous) feedback. This is not simply a lack of controllability. It might happen that, while every state can be steered to the origin by some control law, these control laws cannot be patched together in a continuous fashion to yield a globally defined stabilizing feedback. In this section we discuss how this situation can occur.

4.1.1 State-space obstacles

Consider a continuous-time system $\dot{x} = f(x)$ defined on some state space $\mathcal{X} \subset \mathbb{R}^n$. Assume that it has an asymptotically stable equilibrium, which we take to be the origin with no loss of generality. The region of attraction, which we denote by $\mathcal{D}$, must be a contractible set. This means that there exists a continuous mapping $H : [0,1] \times \mathcal{D} \to \mathcal{D}$ such that $H(0,x) = x$ and $H(1,x) \equiv 0$. The mapping H can be constructed in a natural way using the flow of the system.

One implication of the above result is that if the system is globally asymptotically stable, then its state space $\mathcal{X}$ must be contractible. Therefore, while there exist globally asymptotically stable systems on $\mathbb{R}^n$, no system on a circle can have a single globally asymptotically stable equilibrium.

Now, suppose that we are given a control system

$$\dot{x} = f(x,u), \qquad x \in \mathcal{X} \subset \mathbb{R}^n, \quad u \in \mathcal{U} \subset \mathbb{R}^m. \tag{4.1}$$

If a feedback law $u = k(x)$ is sufficiently regular (i.e., smooth or at least continuous) and renders the closed-loop system

$$\dot{x} = f(x, k(x)) \tag{4.2}$$

asymptotically stable, then the previous considerations apply to (4.2). In particular, global asymptotic stabilization is impossible if $\mathcal{X}$ is not a contractible space (for example, a circle).

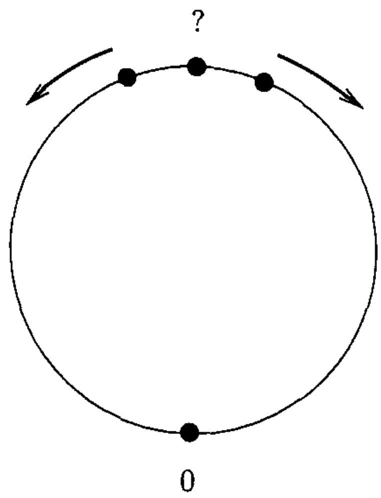

FIGURE 21. A system on a circle: global continuous stabilization is impossible

An intuitive explanation of this fact can be given with the help of Figure 21. Solutions with initial conditions on the right side need to move clockwise, whereas solutions on the left side need to move counterclockwise. Since there can be no equilibria other than the origin, there must be a point on the circle at which the feedback law is discontinuous. We can say that at that point a logical decision (whether to move left or right) is necessary.

Instead of a system evolving on a circle, we may want to consider a system evolving on the plane with a circular (or similar) obstacle. The same argument shows that such a system cannot be continuously globally asymptotically stabilized. Indeed, it is intuitively clear that there is no globally defined continuous feedback strategy for approaching a point behind a table.

When a continuous feedback law cannot solve the stabilization problem, what are the alternative ways to stabilize the system? One option is to use static discontinuous feedback control. In the context of the above example, this means picking a feedback function k that is continuous everywhere except at one point. At this discontinuity point, we simply make an arbitrary decision, say, to move to the right.

One of the shortcomings of this solution is that it requires precise information about the state and so the resulting closed-loop system is highly sensitive to measurement errors. Namely, in the presence of small random measurement noise it may happen that, near the discontinuity point, we misjudge which side of this point we are currently on and start moving toward it instead of away from it. If this happens often enough, the solution will oscillate around the discontinuity point and may never reach the origin.

Using a different logic to define the control law, it is possible to achieve robustness with respect to measurement errors. For example, if we sample and hold each value of the control for a long enough period of time, then we are guaranteed to move sufficiently far away from the discontinuity point, where small errors will no longer cause a motion in the wrong direction. After that we can go back to the usual feedback implementation in order to ensure convergence to the equilibrium. This initial discussion illustrates potential advantages of using logic-based switching control algorithms.

4.1.2 Brockett's condition

The examples discussed above illustrate possible obstructions to global asymptotic stabilization which arise due to certain topological properties of the state space. We now present an important result which shows that even local asymptotic stabilization by continuous feedback is impossible for some systems. Since a sufficiently small neighborhood of an equilibrium point has the same topological properties as $\mathbb{R}^n$, this means that such an obstruction to continuous stabilization has nothing to do with the properties of the state space and is instead embedded into the system equations.

Theorem 4.1 (Brockett) *Consider the control system* (4.1) *with* $\mathcal{X} = \mathbb{R}^n$ *and* $\mathcal{U} = \mathbb{R}^m$, *and suppose that there exists a continuous*[1] *feedback law* $u = k(x)$ *satisfying* $k(0) = 0$ *which makes the origin a (locally) asymptotically stable equilibrium of the closed-loop system* (4.2). *Then the image of every neighborhood of* $(0,0)$ *in* $\mathbb{R}^n \times \mathbb{R}^m$ *under the map*

$$(x, u) \mapsto f(x, u) \tag{4.3}$$

contains some neighborhood of zero in $\mathbb{R}^n$.

Sketch of proof. If k is asymptotically stabilizing, then by a converse Lyapunov theorem there exists a Lyapunov function V which decreases along nonzero solutions of (4.2), locally in some neighborhood of zero. Consider the set $\mathcal{R} := \{x : V(x) \leq c\}$, where c is a sufficiently small positive number. Then the vector field $f(x, k(x))$ points inside $\mathcal{R}$ everywhere on its boundary. By compactness of $\mathcal{R}$, it is easy to see that the vector field

[1] To ensure uniqueness of solutions, we in principle need to impose stronger regularity assumptions, but we ignore this issue here because the result is valid without such additional assumptions.

$f(x, k(x)) - \xi$ also points inside on the boundary of $\mathcal{R}$, where $\xi \in \mathbb{R}^n$ is chosen so that $|\xi|$ is sufficiently small.

We can now apply a standard fixed point argument to show that we must have $f(x, k(x)) = \xi$ for some x in $\mathcal{R}$. Namely, suppose that the vector field $f(x, k(x)) - \xi$ is nonzero on $\mathcal{R}$. Then for every $x \in \mathcal{R}$ we can draw a ray in the direction provided by this vector field until it hits the boundary of $\mathcal{R}$ (see Figure 22). This yields a continuous map from $\mathcal{R}$ to its boundary, and it is well known that such a map cannot exist.

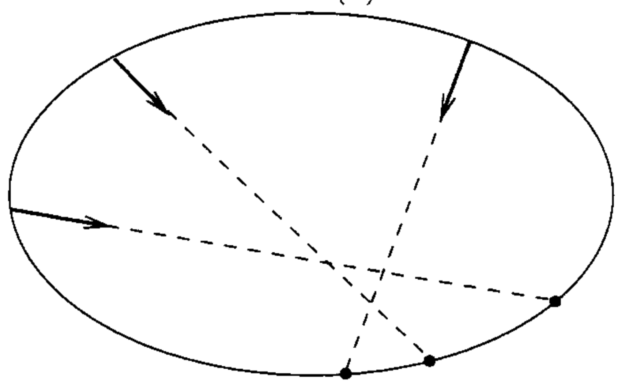

FIGURE 22. Illustrating the proof of Theorem 4.1

We have thus shown that the equation $f(x, k(x)) = \xi$ can be solved for x in a given sufficiently small neighborhood of zero in $\mathbb{R}^n$, provided that $|\xi|$ is sufficiently small. In other words, the image of every neighborhood of zero under the map

$$x \mapsto f(x, k(x)) \tag{4.4}$$

contains a neighborhood of zero. Moreover, if x is small, then $k(x)$ is small in view of continuity of k and the fact that $k(0) = 0$. It follows that the image under the map (4.3) of every neighborhood of $(0, 0)$ contains a neighborhood of zero. □

Theorem 4.1 provides a very useful necessary condition for asymptotic stabilizability by continuous feedback. Intuitively, it means that, starting near zero and applying small controls, we must be able to move in all directions. (The statement that a set of admissible velocity vectors does not contain a neighborhood of zero means that there are some directions in which we cannot move, even by a small amount.) Note that this condition is formulated in terms of the original open-loop system. If the map (4.3) satisfies the hypothesis of the theorem, then it is said to be *open* at zero.

Clearly, a system cannot be feedback stabilizable unless it is asymptotically open-loop controllable to the origin. It is important to keep in mind the difference between these two notions. The latter one says that, given an initial condition, we can find a control law that drives the state to the origin; the former property is stronger and means that there exists a feedback law that drives the state to the origin, regardless of the initial condition. In the next section we study an important class, as well as some specific examples, of controllable nonlinear systems which fail to satisfy Brockett's condition. Since Theorem 4.1 does not apply to switched systems, we will see that switching control provides an effective approach to the feedback stabilization problem for systems that are not stabilizable by continuous feedback.

Note that when the system (4.1) has a controllable (or at least stabilizable) linearization $\dot{x} = Ax + Bu$, it can be locally asymptotically stabilized (by linear feedback). However, controllability of a nonlinear system in general does not imply controllability of its linearization.

Exercise 4.1 Show directly that every controllable linear system satisfies Brockett's condition.

4.2 Nonholonomic systems

Consider the system

$$\dot{x} = \sum_{i=1}^{m} g_i(x)u_i = G(x)u, \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad G \in \mathbb{R}^{n \times m}. \tag{4.5}$$

Systems of this form are known as (driftless, kinematic) *nonholonomic control systems.* Nonholonomy means that the system is subject to constraints involving both the state x (position) and its derivative $\dot{x}$ (velocity). Namely, since there are fewer control variables than state variables, the velocity vector $\dot{x}$ at each x is constrained to lie in the proper subspace of $\mathbb{R}^n$ spanned by the vectors $g_i(x)$, $i = 1, \dots, m$. Under the assumptions that $\operatorname{rank} G(0) = m$ and $m < n$, the system (4.5) violates Brockett's condition, and we have the following corollary of Theorem 4.1.

Corollary 4.2 *The system* (4.5) *with* $\operatorname{rank} G(0) = m < n$ *cannot be asymptotically stabilized by a continuous feedback law.*

PROOF. Rearrange coordinates so that G takes the form

$$G(x) = \begin{pmatrix} G_1(x) \\ G_2(x) \end{pmatrix}$$

where $G_1(x)$ is an $m \times m$ matrix which is nonsingular in some neighborhood N of zero. Then the image of $N \times \mathbb{R}^m$ under the map

$$(x, u) \mapsto G(x)u = \begin{pmatrix} G_1(x)u \\ G_2(x)u \end{pmatrix}$$

does not contain vectors of the form $\binom{0}{a}$, where $a \in \mathbb{R}^{n-m} \neq 0$. Indeed, if $G_1(x)u = 0$, then we have $u = 0$ since G_1 is nonsingular, and this implies $G_2(x)u = 0$. □

In the singular case, i.e., when $G(x)$ drops rank at 0, the above result does not hold. Thus the full-rank assumption imposed on G is essential for making the class of systems (4.5) interesting in the present context. Nonholonomic systems satisfying this assumption are called *nonsingular.*

Even though infinitesimally the state of the system (4.5) can only move along linear combinations of the m available control directions, it is possible to generate motions in other directions by a suitable choice of controls. For example, consider the following (switching) control strategy. First, starting at some x_0, move along the vector field g_1 for ε units of time (by setting $u_1 = 1$ and $u_i = 0$ for all $i \neq 1$). Then move along the vector field g_2 for ε units of time. Next, move along $-g_1$ for ε units of time ($u_1 = -1$, $u_i = 0$ for $i \neq 1$), and finally along $-g_2$ for ε units of time. It is straightforward (although quite tedious) to check that for small ε the resulting motion is approximated, up to the second order in ε, by $\varepsilon^2[g_1, g_2](x_0)$ (see Section 2.2.1 for the definition of the Lie bracket of two nonlinear vector fields). This situation is depicted in Figure 23.

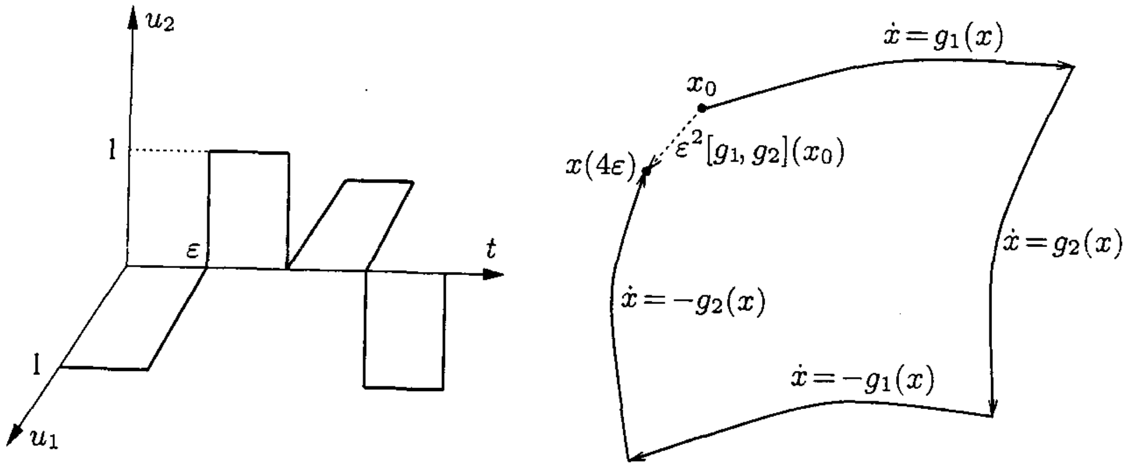

FIGURE 23. (a) Switching between two control directions, (b) the resulting approximate motion along the Lie bracket

The above example illustrates the general principle that by switching among the principal control directions, one can generate slower "secondary" motions in the directions of the corresponding Lie brackets. More complicated switching patterns give rise to motions in the directions of higher-order iterated Lie brackets. This explains the importance of the *controllability Lie algebra* $\{g_i : i = 1, \ldots, m\}_{LA}$ spanned by the control vector fields g_i (see Appendix B for background on Lie algebras). If this Lie algebra has rank n for all x, the system is said to satisfy the *Lie algebra rank condition* (LARC). In this case, it is well known that the system is completely controllable, in the sense that every state can be steered to every other state (Chow's theorem). The LARC guarantees that the motion of the system is not confined to any proper submanifold of $\mathbb{R}^n$ (Frobenius's theorem). In other words, the nonholonomic constraints are not integrable, i.e., cannot be expressed as constraints involving x only. Control systems of the form (4.5) satisfying the LARC are referred to as *completely nonholonomic.*

Nonsingular, completely nonholonomic control systems are of special interest to us. Indeed, we have shown that such systems cannot be stabilized by continuous feedback, even though they are controllable (and, in particular, asymptotically open-loop controllable to the origin). One way to overcome this difficulty is to employ switching control techniques.

As we already mentioned in Section 1.1.3, the system (4.5) becomes equivalent to the switched system (1.3) with $\mathcal{P} = \{1, 2, \dots, m\}$ and $f_i = g_i$ for all i if we restrict the admissible controls to be of the form $u_k = 1$, $u_i = 0$ for $i \neq k$ (this gives $\sigma = k$). In particular, the bilinear system

$$\dot{x} = \sum_{i=1}^{m} A_i x u_i$$

corresponds to the switched linear system (1.4). It is intuitively clear that asymptotic stability of the switched system (1.3) for arbitrary switching—the property studied in Chapter 2—corresponds to a lack of controllability for (4.5). Indeed, it implies that for every admissible control function, the resulting solution trajectory of (4.5) must approach the origin. As we have just seen, Lie algebras naturally arise in characterizing controllability of (4.5); this perhaps makes their relevance for stability analysis of (1.3), unveiled by the results discussed in Section 2.2, less surprising.

4.2.1 The unicycle and the nonholonomic integrator

As a simple example of a nonholonomic system, we consider the wheeled mobile robot of unicycle type shown in Figure 24. We henceforth refer to it informally as the unicycle.

The state variables are x_1, x_2, and θ, where x_1, x_2 are the coordinates of the point in the middle of the rear axle and θ denotes the angle that the vehicle makes with the x_1-axis (for convenience, we can assume that θ takes values in $\mathbb{R}$). The front wheel turns freely and balances the front end of the robot above the ground. When the same angular velocity is applied to both rear wheels, the robot moves straight forward. When the angular velocities applied to the rear wheels are different, the robot turns. The kinematics of the robot can be modeled by the equations

$$\begin{aligned} \dot{x}_1 &= u_1 \cos\theta \\ \dot{x}_2 &= u_1 \sin\theta \\ \dot{\theta} &= u_2 \end{aligned} \tag{4.6}$$

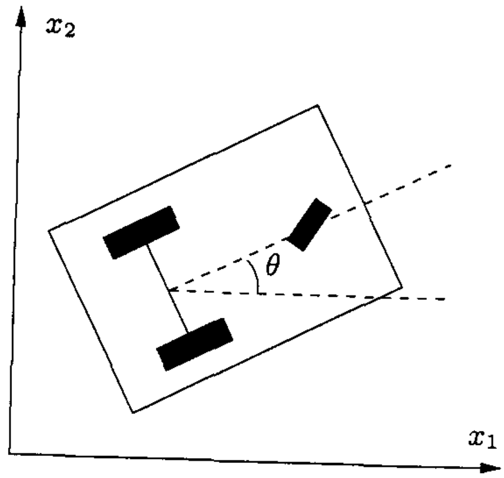

FIGURE 24. The unicycle

where u_1 and u_2 are the control inputs (the forward and the angular velocity, respectively). We assume that the no-slip condition is imposed on the wheels, so the robot cannot move sideways (this is precisely the nonholonomic constraint). Asymptotic stabilization of this system amounts to parking the unicycle at the origin and aligning it with the x_1-axis.

The system (4.6) is a nonsingular completely nonholonomic system. Indeed, it takes the form (4.5) with $n = 3$, $m = 2$,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \theta \end{pmatrix}, \qquad g_1(x) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \qquad g_2(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The vectors g_1 and g_2 are linearly independent for all x_1, x_2, θ. Moreover, we have

$$[g_1, g_2](x) = -\begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}.$$

Thus the vector field $[g_1, g_2]$ is orthogonal to both g_1 and g_2 everywhere, hence the LARC is satisfied.

Incidentally, controllability of the system (4.6) justifies the common strategy for parallel parking. After normalization, this strategy can be modeled by switching between the control laws $u_1 = -1, u_2 = -1$ (moving backward, the steering wheel turned all the way to the right), $u_1 = -1, u_2 = 1$ (moving backward, the wheel turned all the way to the left), $u_1 = 1, u_2 = -1$ (moving forward, the wheel turned to the right), $u_1 = 1, u_2 = 1$ (moving forward, the wheel turned to the left). As we explained earlier (see Figure 23), the resulting motion is approximately along the Lie bracket of the corresponding vector fields, which is easily seen to be the direction perpendicular to the straight motion (i.e., sideways). The factor ε^2 explains the frustration often associated with parallel parking.

We know from Corollary 4.2 that the parking problem for the unicycle cannot be solved by means of continuous feedback. It is actually not hard to show directly that the system (4.6) fails to satisfy Brockett's necessary condition for continuous stabilizability (expressed by Theorem 4.1). The map (4.3) in this case is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ \theta \\ u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \cos\theta \\ u_1 \sin\theta \\ u_2 \end{pmatrix}.$$

Pick a neighborhood in the x, u space where $|\theta| < \pi/2$. The image of such a neighborhood under the above map does not contain vectors of the form

$$\begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix}, \qquad a \neq 0.$$

Indeed, $u_1 \cos\theta = 0$ implies $u_1 = 0$ because $\cos\theta \neq 0$, hence we must also have $u_1 \sin\theta = 0$. Thus for small values of the angle θ we cannot move in all directions, which is an immediate consequence of the fact that the wheels are not allowed to slip.

We can intuitively understand why there does not exist a continuous stabilizing feedback. The argument is similar to the one we gave earlier for systems on a circle (see Section 4.1.1). Since the unicycle cannot move sideways, we need to decide which way to turn. If we start rotating clockwise from some initial configurations and counterclockwise from others, then the need for a logical decision will arise for a certain set of initial configurations (see Figure 25). Thus the nonholonomic constraint plays a role similar to that of a state-space obstacle.

Our next objective is to demonstrate how the system (4.6) can be asymptotically stabilized by a switching feedback control law. To this end, it is convenient to consider the following state and control coordinate transformation:

$$\begin{aligned} x &= x_1 \cos\theta + x_2 \sin\theta \\ y &= \theta \\ z &= 2(x_1 \sin\theta - x_2 \cos\theta) - \theta(x_1 \cos\theta + x_2 \sin\theta) \\ u &= u_1 - u_2(x_1 \sin\theta - x_2 \cos\theta) \\ v &= u_2. \end{aligned} \tag{4.7}$$

This transformation is well defined and preserves the origin, and in the new coordinates the system takes the form

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= xv - yu. \end{aligned} \tag{4.8}$$

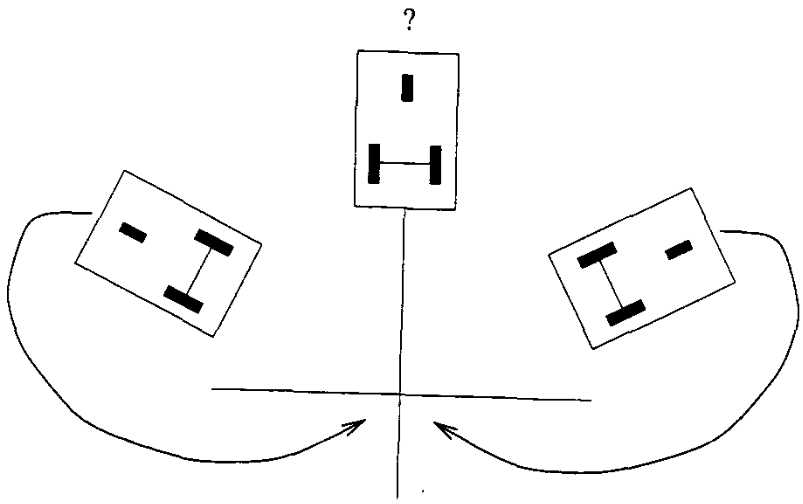

FIGURE 25. Parking the unicycle

The system (4.8) is known as Brockett's *nonholonomic integrator*. Given a feedback law that stabilizes this system, by reversing the above change of coordinates one obtains a stabilizing feedback law for the unicycle. It is therefore clear that a continuous feedback law that stabilizes the nonholonomic integrator does not exist. This can also be seen directly: the image of the map

$$\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \\ xv - yu \end{pmatrix}$$

does not contain vectors of the form

$$\begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}, \qquad a \neq 0.$$

Note that, unlike in the case of the unicycle, we did not even need to restrict the above map to a sufficiently small neighborhood of the origin.

The nonholonomic integrator is also controllable, and in fact its controllability has an interesting geometric interpretation. Suppose that we steer the system from the origin to some point $(x(t), y(t), z(t))^T$. Then from (4.8) and Green's theorem we have

$$z(t) = \int_0^t (x\dot{y} - y\dot{x})dt = \int_{\partial\mathcal{D}} xdy - ydx = 2\int_{\mathcal{D}} dxdy$$

where $\mathcal{D}$ is the area defined by the projection of the solution trajectory onto the xy-plane, completed by the straight line from $(x(t), y(t))$ to $(0, 0)$, and $\partial\mathcal{D}$ is its boundary (see Figure 26). Note that the integral along the line is zero. Thus the net change in z equals twice the signed area of $\mathcal{D}$. Since the subsystem of (4.8) that corresponds to the "base" coordinates x, y is obviously controllable, it is not difficult to see how to drive the system from the origin to a desired final state $(x(t), y(t), z(t))^T$. For example, we can first find a control law that generates a closed path in the xy-plane of area $z(t)/2$, and then apply a control law that induces the motion from the origin to the point $(x(t), y(t))^T$ along a straight line.

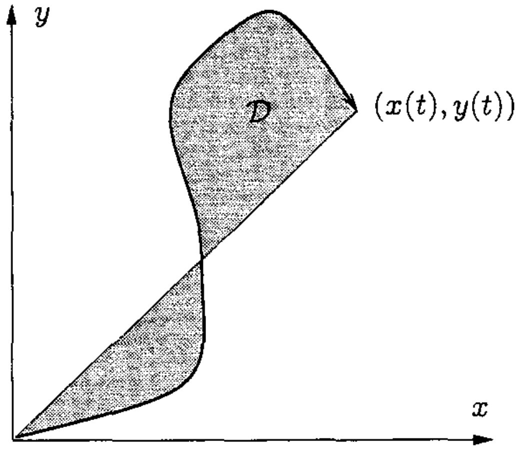

FIGURE 26. Illustrating controllability of the nonholonomic integrator

We now describe a simple example of a switching feedback control law that asymptotically stabilizes the nonholonomic integrator (4.8). Let us consider another state and control coordinate transformation, given by

$$\begin{aligned} x &= r\cos\psi \\ y &= r\sin\psi \\ \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}. \end{aligned}$$

Of course, the above transformation is only defined when $x^2 + y^2 = r^2 \neq 0$. We obtain the following equations in the new cylindrical coordinates:

$$\begin{aligned} \dot{r} &= \bar{u} \\ \dot{\psi} &= \bar{v}/r \\ \dot{z} &= r\bar{v}. \end{aligned}$$

The feedback law

$$\bar{u} = -r^2, \qquad \bar{v} = -z \tag{4.9}$$

yields the closed-loop system

$$\begin{aligned} \dot{r} &= -r^2 \\ \dot{z} &= -rz \\ \dot{\psi} &= -\frac{z}{r}. \end{aligned} \tag{4.10}$$

Its solutions satisfy the formulas

$$r(t) = \frac{1}{t + \frac{1}{r(0)}}$$

and

$$z(t) = e^{-\int_0^t r(\tau)d\tau} z(0)$$

from which it is not difficult to conclude that if $r(0) \neq 0$, then we have $r(t)$, $z(t) \to 0$ as $t \to \infty$, and moreover $r(t) \neq 0$ for all t. This implies that x, y, and z converge to zero (the exact behavior of the angle variable ψ is not important).

We now need to explain what to do if $r(0) = 0$. The simplest solution is to apply some control law that moves the state of (4.8) away from the z-axis (for example, $u = v = 1$) for a certain amount of time T, and then switch to the control law defined by (4.9). We formally describe this procedure by introducing a logical variable s, which is initially set to 0 if $r(0) = 0$ and to 1 otherwise. If $s(0) = 0$, then at the switching time T it is reset to 1. In fact, it is possible to achieve asymptotic stability in the Lyapunov sense, e.g., if we move away from the singularity line $r = 0$ with the speed proportional to $z(0)$, as in

$$u = z(0), \qquad v = z(0). \tag{4.11}$$

Figure 27 shows a computer-like diagram illustrating this switching logic, as well as a typical trajectory of the resulting switched system. A reset integrator is used to determine the switching time.

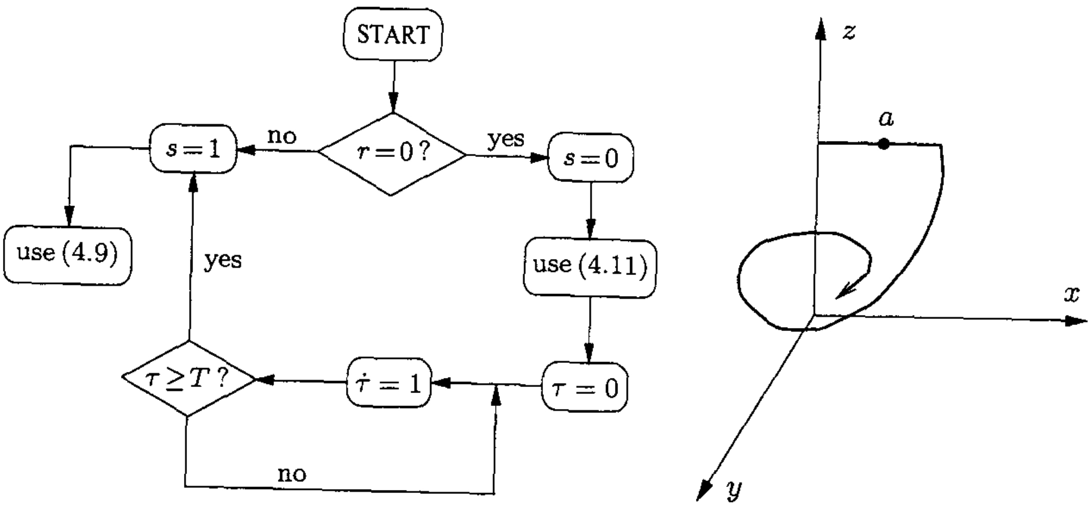

FIGURE 27. Stabilizing the nonholonomic integrator: (a) the switching logic, (b) a typical trajectory of the closed-loop system

Exercise 4.2 Implement the above switching stabilizing control law for the unicycle via computer simulation.

Note that the closed-loop system is a truly hybrid system, in the sense that its description necessarily involves discrete dynamics. Indeed, the value of the control is not completely determined by the value of the state, but also depends on the value of the discrete variable s. For example, at the point a shown in Figure 27(b), the control takes different values for $s = 0$ and $s = 1$. Thus the control law that we have just described is a *hybrid control law*, with two discrete states. To increase robustness with respect to measurement errors, we could replace the condition $r(0) = 0$ with $|r(0)| < \varepsilon$ for some $\varepsilon > 0$ (this condition would be checked only once, at $t = 0$). We will revisit the parking problem in a more general setting in Section 6.8.

4.3 Stabilizing an inverted pendulum

In this section we briefly discuss the popular benchmark problem of stabilizing an inverted pendulum on a cart in the upright position (see Figure 28). Simplified equations of motion are

$$\ddot{x} = u \tag{4.12}$$

$$J\ddot{\theta} = mgl\sin\theta - ml\cos\theta\, u \tag{4.13}$$

where x is the location of the cart, θ is the angle between the pendulum and the vertical axis, the control input u is the acceleration of the cart, l is the length of the rod, m is the mass of the pendulum (concentrated at its tip), and J is the moment of inertia with respect to the pivot point.

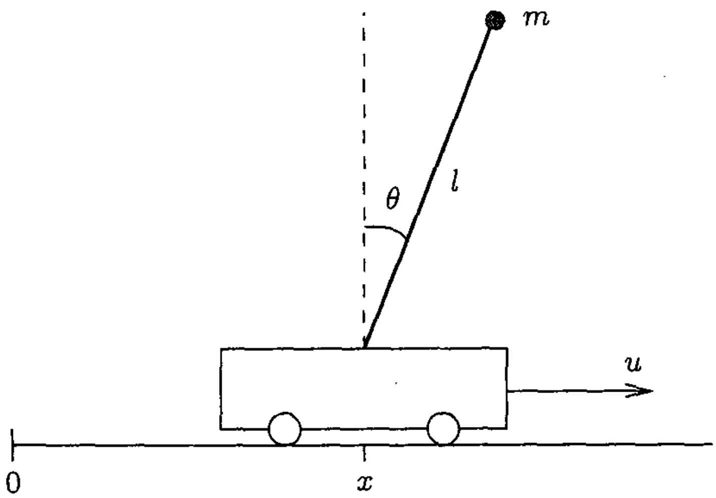

FIGURE 28. An inverted pendulum on a cart

If we are only concerned with the pendulum position and not the cart location, we can limit our attention to the subsystem (4.13). Once a feedback control law that stabilizes this subsystem is known, it is possible to obtain a feedback law that stabilizes the entire system (4.12)–(4.13).

The most natural state space for the system (4.13) is $\mathbb{S} \times \mathbb{R}$, where $\mathbb{S}$ denotes a circle. Since this space is not contractible, global continuous feedback stabilization is impossible (see Section 4.1.1). There is also a more direct way to see this. The point $(\theta, \dot{\theta})^T$ is an equilibrium of the closed-loop system

$$J\ddot{\theta} = mgl \sin\theta - ml\cos\theta\, k(\theta, \dot{\theta})$$

if and only if $\dot{\theta} = 0$ and

$$mgl \sin\theta - ml\cos\theta\, k(\theta, 0) = 0. \tag{4.14}$$

The left-hand side of (4.14) takes the positive value mgl at $\theta = \pi/2$ and the negative value $-mgl$ at $\theta = 3\pi/2$. Thus if the feedback law k is continuous, then there will necessarily be an undesired equilibrium for some $\theta \in (\pi/2, 3\pi/2)$. This means that the system (4.13) cannot be globally asymptotically stabilized by continuous feedback—in fact, even if it is viewed as a system evolving on $\mathbb{R}^2$.

Let us see how we can stabilize the inverted pendulum using switching control. A natural way to do this is to use "energy injection." The mechanical energy of the pendulum (kinetic plus potential) is given by

$$E := \frac{1}{2}J\dot{\theta}^2 + mgl(1 + \cos\theta)$$

(we have $E = 0$ in the downward equilibrium). The derivative of E along solutions of the system (4.13) is

$$\dot{E} = -ml\dot{\theta}\cos\theta\, u. \tag{4.15}$$

When the pendulum is in the upright position, we have $E = 2mgl$. It is not hard to see from the equation (4.15) how to control the energy to this desired value. Assume that the pendulum starts at rest away from the upright position, so that its energy is smaller than desired. Apply a constant control value u of large magnitude (the largest possible if the control saturates) whose sign is chosen so as to make E increase. Switch the sign of u whenever $\dot{\theta}$ or $\cos\theta$ become equal to zero (at this stage the control strategy is "bang-bang"; cf. Section 5.1). When E reaches the desired value, switch the control off.

As a result of the above control strategy, the pendulum will go through a series of swings, each one ending at a higher position than the previous one. The values of $\cos\theta$ at the times when the pendulum comes to rest can be calculated from the values of E. After the control is turned off, E will remain equal to $2mgl$, and so θ will converge asymptotically to zero as $\dot{\theta}$ approaches zero. If a sufficiently large control magnitude is admissible, then this can be accomplished with just one swing. In the presence of a saturation constraint, a larger number of swings is required in general.

The above switching control law is not very efficient near $\theta = 0$, because even for very small values of θ it can cause the pendulum to make a full rotation before reaching the upright position. It is therefore reasonable to combine this control law with another one, which stabilizes the system locally in a neighborhood of zero. For example, one can switch to a linear locally stabilizing feedback law $u = k_1\theta + k_2\dot{\theta}$ after the previously described swing-up strategy causes the system to enter the region of attraction for this linear feedback. Another example of a locally stabilizing smooth feedback law is

$$k(\theta, \dot{\theta}) = \frac{J}{ml\cos\theta}\Big(\frac{mgl}{J}\sin\theta + \tan\theta + \arctan\dot{\theta}\Big). \tag{4.16}$$

Its domain of attraction is $\{(\theta, \dot{\theta})^T : -\pi/2 < \theta < \pi/2\}$, so we can "catch" the pendulum with this feedback law after we bring it above the horizontal position by energy control. (The difficulty in obtaining a larger domain of attraction with continuous feedback stems from the following consequence of equation (4.15): when the pendulum starts at rest in the horizontal position, the energy cannot be instantaneously increased by any choice of control, so the only direction in which the pendulum can start moving is downwards.)

Similar stabilizing switching control strategies can be developed for pendulums with carts moving on a circle rather than on a straight line, multiple link pendulums, and pendulums actuated by rotating disks.

5
Systems with Sensor or Actuator Constraints

5.1 The bang-bang principle of time-optimal control

It is well known that switching control laws arise in time-optimal control of systems with controls taking values in bounded sets. For such systems, time-optimal control laws switch between boundary points of the admissible control set (hence the name "bang-bang controls").

We restrict our attention here to the linear time-invariant system

$$\dot{x} = Ax + Bu \tag{5.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^m$, and A and B are matrices of appropriate dimensions. Let us take the set of admissible controls to be the hypercube

$$\mathcal{U} := \{u \in \mathbb{R}^m : |u_i| \leq 1,\ i = 1, \dots, m\}.$$

Suppose that the control problem of interest is to steer the state x of the system (5.1) in the smallest possible time from an initial state x_0 to a given target state x_1.

To guarantee that a solution to this problem always exists, we assume that the state x_1 can be reached at *some* time using admissible controls. To ensure uniqueness of the solution, as we will see, we need to assume that (A, b_i) is a controllable pair for each i, where b_i, $i = 1, \dots, m$ are the columns of B. This means that the system (5.1) is controllable with respect to every component of the input vector u. Systems with this property are called *normal*.

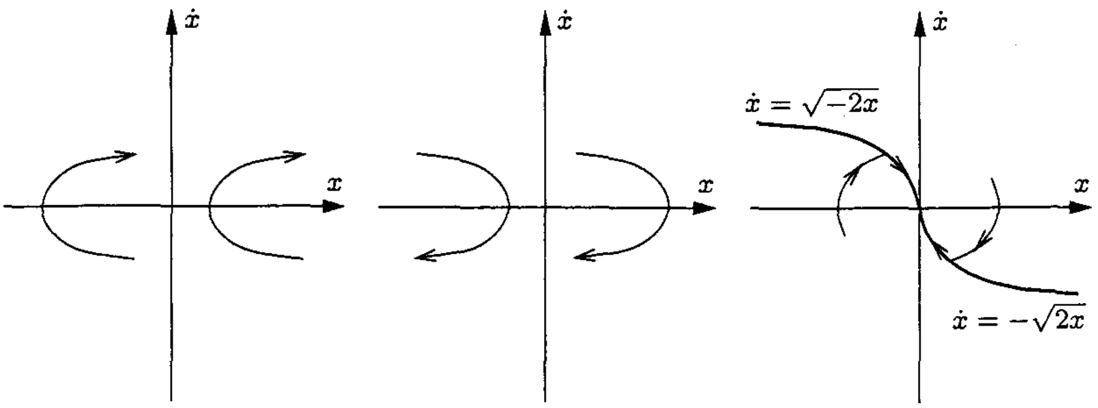

FIGURE 30. Bang-bang time-optimal control of the double integrator: (a) trajectories for $u = 1$, (b) trajectories for $u = -1$, (c) the switching curve

consider the time-optimal control problem which consists of bringing x to rest at the origin in minimal time, using controls that satisfy $|u| \leq 1$. Describe the optimal control laws and the resulting solution trajectories.

5.2 Hybrid output feedback

Suppose that we are given a linear time-invariant control system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{5.7}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and A, B, and C are matrices of suitable dimensions. Suppose that the system (5.7) is stabilizable and detectable (i.e., there exist matrices F and K such that the eigenvalues of $A + BF$ and the eigenvalues of $A + KC$ have negative real parts). Then, as is well known, there exists a linear dynamic output feedback law that asymptotically stabilizes the system (e.g., the standard Luenberger observer-based output feedback). The existence of a *static* stabilizing output feedback, on the other hand, is an exception rather than a rule.

In practice, a continuous dynamic feedback law might not be implementable, and a suitable discrete version of the controller is often desired. Stabilization by a hybrid output feedback controller that uses a countable number of discrete states can be achieved by means of a suitable discretization process. A logical question to ask next is whether it is possible to stabilize the system by using a hybrid output feedback with only a *finite* number of discrete states. Indeed, this is the only type of feedback that is feasible for implementation on a digital computer. A finite-state stabilizing hybrid feedback is unlikely to be obtained from a continuous one by discretization, and no systematic techniques for synthesizing desired feedbacks seem to be available for systems of dimension higher than 2. However, we will see in this section that sometimes a simple solution can be found.

One approach to the problem of stabilizing the linear system (5.7) via finite-state hybrid output feedback is prompted by the following observation. Suppose that we are given a collection of gain matrices $K_1, \dots, K_m$ of suitable dimensions. Setting $u = K_i y$ for some $i \in \{1, \dots, m\}$, we obtain the system

$$\dot{x} = (A + BK_iC)x.$$

Thus the stabilization problem for the original system (5.7) will be solved if we can orchestrate the switching between the systems in the above form in such a way as to achieve asymptotic stability. Defining

$$A_i := A + BK_iC, \qquad i \in \{1, \dots, m\} \tag{5.8}$$

we are led to the following question: using the measurements of the output $y = Cx$, can we find a switching signal σ such that the switched system $\dot{x} = A_\sigma x$ is asymptotically stable? The value of σ at a given time t might just depend on t and/or $y(t)$, or a more general hybrid feedback may be used. We are assuming, of course, that no static output feedback gain K yields a Hurwitz matrix $A+BKC$ (otherwise the problem would be trivial), so in particular none of the matrices A_i is Hurwitz.

Observe that the existence of a Hurwitz convex combination $\alpha A_i + (1-\alpha)A_j$ for some $i, j \in \{1, \dots, m\}$ and $\alpha \in (0,1)$ would imply that the system (5.7) can be stabilized by the linear static output feedback $u = Ky$ with $K := \alpha K_i + (1-\alpha)K_j$, contrary to the assumption that we just made. Therefore, the procedure described in Section 3.4.1 is not applicable here, which makes the problem at hand more challenging. In fact, Theorem 3.5 implies that a quadratically stabilizing switching signal does not exist. However, it might still be possible to construct an asymptotically stabilizing switching signal and even base a stability proof on a single Lyapunov function. The following example from [27] illustrates this point.

Example 5.2 Consider the forced harmonic oscillator with position measurements, given by the system (5.6) with $y = x$ or, more explicitly, by

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= x_1. \end{aligned} \tag{5.9}$$

Although this system is controllable and observable, it cannot be stabilized by (arbitrary nonlinear, even discontinuous) static output feedback. Indeed, applying a feedback law $u = k(x_1)$, we obtain the closed-loop system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + k(x_1). \end{aligned} \tag{5.10}$$

Consider the function

$$V(x_1, x_2) := \frac{1}{2}x_2^2 + \int_0^{x_1} (\xi - k(\xi))d\xi.$$

The derivative of V along solutions of (5.10) evaluates to

$$\dot{V} = -x_2x_1 + x_2k(x_1) + x_2x_1 - x_2k(x_1) = 0$$

which means that V remains constant. Since $V(0, x_2) = x_2^2/2$, we see for example that solutions with initial conditions on the x_2-axis cannot converge to the origin.

The obstruction to output feedback stabilization of the system (5.9) disappears if one allows switching controls. We now describe one possible stabilizing switching control strategy, which actually switches between just two linear static output gains. Letting $u = -y$, we obtain the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.11}$$

while $u = \frac{1}{2}y$ yields the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{5.12}$$

Define $V(x_1, x_2) := x_1^2 + x_2^2$. This function decreases along solutions of (5.11) when $x_1x_2 > 0$ and decreases along solutions of (5.12) when $x_1x_2 < 0$. Therefore, if the system (5.11) is active in the first and third quadrants, while the system (5.12) is active in the second and fourth quadrants, we have $\dot{V} < 0$ whenever $x_1x_2 \neq 0$, and the switched system is asymptotically stable by LaSalle's invariance principle. The trajectories of the individual subsystems (5.11) and (5.12) and a possible trajectory of the switched system are sketched in Figure 31. (This situation is quite similar to the one studied in Example 3.3.)

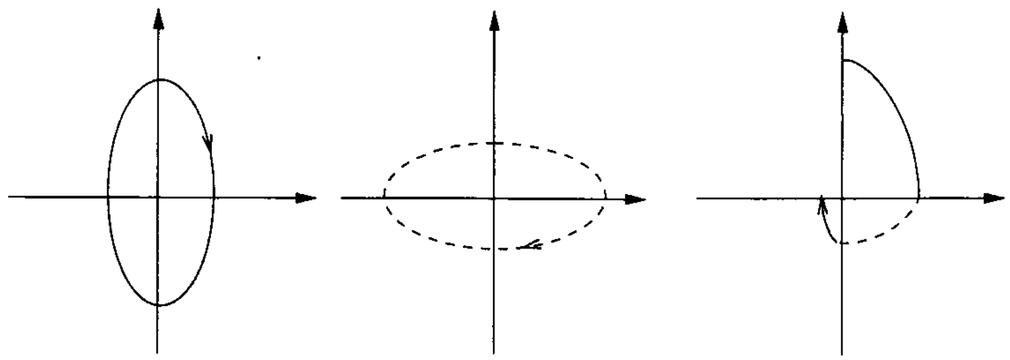

FIGURE 31. Stabilizing the harmonic oscillator

Let us examine more carefully what type of feedback law can induce this behavior. Although we can express the above switching strategy as $u = k(x)y$, where k is a function from $\mathbb{R}^2$ to $\{-1, \frac{1}{2}\}$, this is not implementable because x_2 is not available for measurement. Note, however, that both systems being switched are linear time-invariant, and so the time between a crossing of the x_1-axis and the next crossing of the x_2-axis can be explicitly calculated and is independent of the trajectory. This means that the above switching strategy can be implemented via hybrid feedback based just on the output measurements. We let T be the time needed for a trajectory of the system (5.11) to pass through the first or the third quadrant. The feedback control will be of the form $u = ky$, where the gain k will switch between the values -1 and $\frac{1}{2}$. When x_1 changes sign, we keep $k = -1$ for the next T units of time, after which we let $u = \frac{1}{2}$, wait for the next change in sign of x_1, and so on.

Interestingly, this switching control strategy employs a combination of time-dependent and state-dependent switching. Other possibilities of course exist. For example, we could use the measurements of x_1 to detect changes in sign of $x_2 = \dot{x}_1$. Alternatively, we could compute the time it takes for trajectories of the system (5.12) to traverse the second or the fourth quadrant and make the switching rule entirely time-dependent after the first change in sign of x_1 is detected (however, this strategy would be less robust with respect to modeling uncertainty or measurement errors).

The above control strategy can be illustrated by the computer-like diagram shown in Figure 32. An auxiliary variable r is introduced to detect a change in sign of x_1 (the left branch), and a reset integrator is employed to determine the transition time (the right branch). □

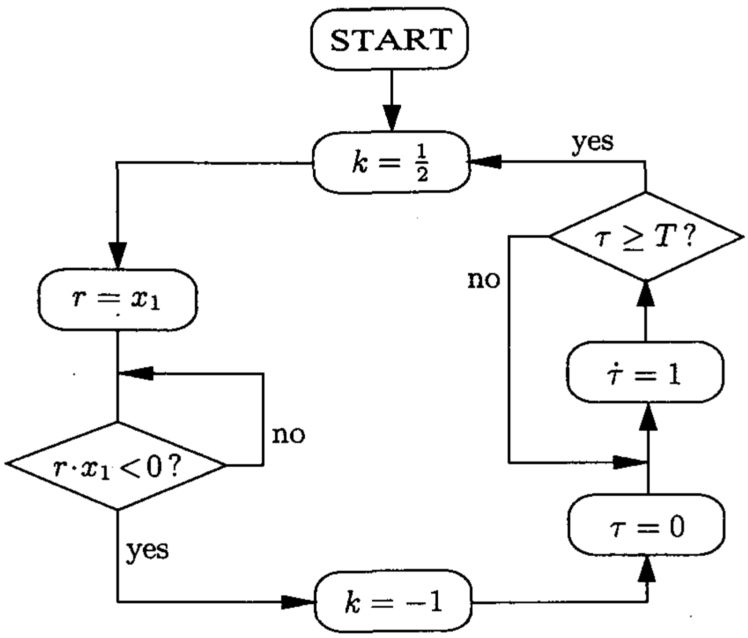

FIGURE 32. The switching logic used in Example 5.2

In the above example, the stability analysis was based on a single Lyapunov function. In view of the results presented in Chapter 3, it may also be possible to use multiple Lyapunov functions. Recall the sufficient conditions expressed by the inequalities (3.23) and (3.24) for the existence of a switching signal that asymptotically stabilizes the switched system defined by two linear subsystems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$. In the present context, we use the formula (5.8) to rewrite these conditions as follows: there exist two numbers $\beta_1, \beta_2 \geq 0$ such that

$$-A^T P_1 - P_1 A - \beta_1(P_2 - P_1) - P_1 B K_1 C - C^T K_1^T B^T P_1 > 0 \tag{5.13}$$

and

$$-A^T P_2 - P_2 A - \beta_2(P_1 - P_2) - P_2 B K_2 C - C^T K_2^T B^T P_2 > 0. \tag{5.14}$$

Here the given data consists of the matrices A, B, and C and the gains K_1 and K_2, whereas positive definite matrices P_1 and P_2 are to be found.

Using a standard technique for elimination of matrix variables, one can show that the inequalities (5.13) and (5.14) are satisfied if and only if there exist some numbers δ_1 and δ_2 (which we can take to be the same with no loss of generality) such that

$$\begin{aligned} &-A^T P_1 - P_1 A - \beta_1(P_2 - P_1) + \delta_1 P_1 B B^T P_1^T > 0 \\ &-A^T P_1 - P_1 A - \beta_1(P_2 - P_1) + \delta_1 C^T C > 0 \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} &-A^T P_2 - P_2 A - \beta_2(P_1 - P_2) + \delta_2 P_2 B B^T P_2^T > 0 \\ &-A^T P_2 - P_2 A - \beta_2(P_1 - P_2) + \delta_2 C^T C > 0. \end{aligned} \tag{5.16}$$

If these inequalities can be solved for P_i, β_i, and δ_i, $i = 1, 2$, then we can find feedback gains K_1 and K_2 that make (5.13) and (5.14) hold. As explained in Section 3.4.2, a stabilizing switching signal can then be explicitly constructed, with switches occurring at those x for which we have

$$x^T P_1 x = x^T P_2 x. \tag{5.17}$$

It is in general not possible to check the condition (5.17) using only the output measurements. However, there are certain situations in which this can be done; for example, in the case of two-dimensional systems, this reduces to calculating the time it takes for the active subsystem to traverse a conic region in $\mathbb{R}^2$ (cf. Example 5.2). One can also envision using some simple observer-like procedure to determine (at least approximately) the times at which (5.17) holds.

To summarize, hybrid control can be a useful tool for stabilization by output feedback. In general, the problem of stabilizing linear systems by hybrid output feedback remains largely open.

5.3 Hybrid control of systems with quantization

5.3.1 Quantizers

In the classical feedback control setting, the output of the process is assumed to be passed directly to the controller, which generates the control input and in turn passes it directly back to the process. In practice, however, this paradigm often needs to be re-examined because the interface between the controller and the process features some additional information-processing devices.

One important aspect to take into account in such situations is *signal quantization*. We think of a quantizer as a device that converts a real-valued signal into a piecewise constant one taking a finite set of values (although in other sources quantizers taking infinitely many values are also considered). Quantization may affect the process output, as in Figure 33(a). This happens, for example, when the output measurements to be used for feedback are obtained by a digital camera, stored or processed in a computer, or transmitted over a limited-communication channel. Quantization may also affect the control input, as in Figure 33(b). Examples include the standard PWM amplifier, the manual transmission in a car, a stepping motor, and a variety of other event-driven actuators.

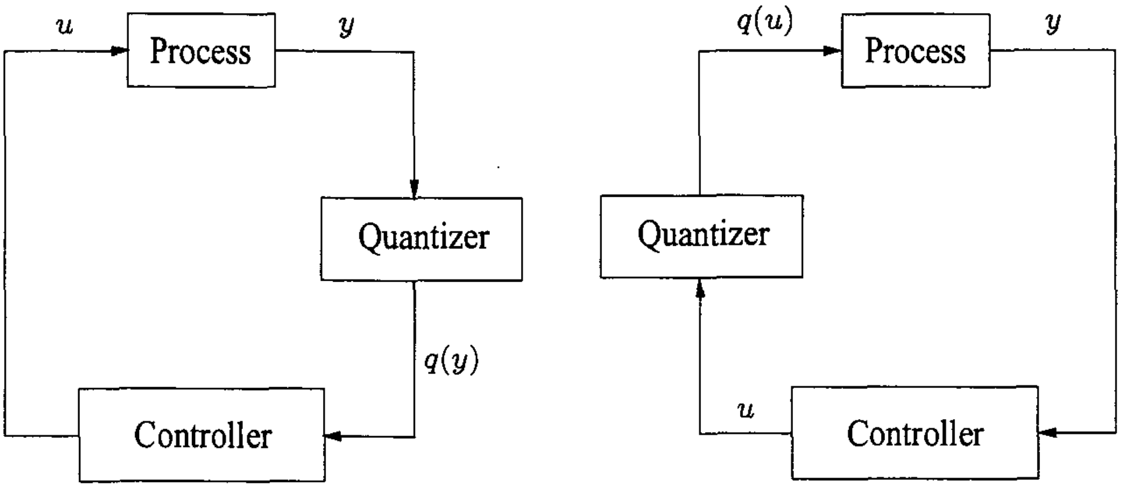

FIGURE 33. A quantizer in the feedback loop: (a) output quantization, (b) input quantization

Let $z \in \mathbb{R}^l$ be the variable being quantized. In the presence of quantization, the space $\mathbb{R}^l$ is divided into a finite number of *quantization regions*, each corresponding to a fixed value of the quantizer. More precisely, let the quantizer be described by a function $q : \mathbb{R}^l \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}^l$. Then the quantization regions are the sets $\{z \in \mathbb{R}^l : q(z) = i\}$, $i \in \mathcal{Q}$. To preserve the equilibrium at the origin, we will let $q(0) = 0$.

In the literature it is usually assumed that the quantization regions are rectilinear and are either of equal size ("uniform quantizer") or get smaller close to the origin and larger far away from the origin ("logarithmic quantizer"). These two most common types of quantizers are illustrated in Figure 34(a) and (b), respectively. In most of what follows, we do not need to make any explicit assumptions—such as rectilinear shape or convexity—regarding the quantization regions. In Figure 34, the dots represent the values that the quantizer takes in each region (although in principle they do not necessarily belong to the respective regions). The complement of the union of all quantization regions of finite size is the infinite quantization region, in which the quantizer saturates (the corresponding value is given by the extra dot).

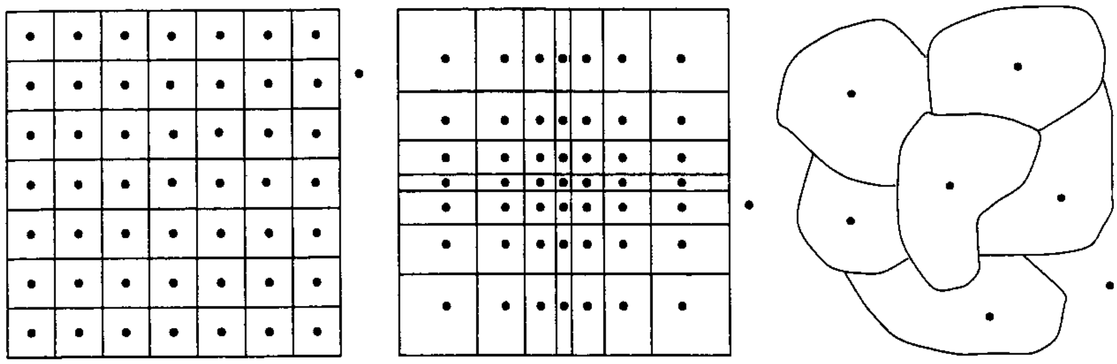

FIGURE 34. Quantization regions: (a) uniform, (b) logarithmic, (c) general

To describe more precisely what is meant by saturation, we assume that there exist positive real numbers M and Δ such that the following two conditions hold:

1. If
$$|z| \leq M \tag{5.18}$$
then
$$|q(z) - z| \leq \Delta. \tag{5.19}$$

2. If
$$|z| > M$$
then
$$|q(z)| > M - \Delta.$$

Condition 1 gives a bound on the quantization error when the quantizer does not saturate. Condition 2 provides a way to detect the possibility of saturation. We will refer to M and Δ as the *range* of the quantizer and the *quantization error*, respectively. Practically every reasonable quantizer satisfies the above requirement. For example, consider a uniform or a logarithmic quantizer whose value in each quantization region belongs to that region, as in Figures 34(a) and (b). Then suitable values of Δ and M can be easily calculated from the maximal size of finite quantization regions and the number of these regions.

Exercise 5.2 Do this calculation for the uniform quantizer defined by N^l equal cubical quantization regions (N in each dimension) of an arbitrary fixed size, with the quantizer values at the centers of the cubes.

We assume that the system evolves in continuous time. At the time of passage from one quantization region to another, the dynamics of the closed-loop system change abruptly. Therefore, systems with quantization can be naturally viewed as switched systems with state-dependent switching. Chattering on the boundaries between the quantization regions is possible, and solutions are to be interpreted in the sense of Filippov if necessary (see Section 1.2.3). However, this issue will not play a significant role in the subsequent stability analysis.[1]

5.3.2 Static state quantization

We concentrate on the case of state quantization for now, postponing the discussion of input quantization until Section 5.3.4 and output quantization until Section 5.3.5. In this section we assume (as is invariably done in the literature on quantized control) that the quantizer function q is fixed. It is well known and easy to see that a feedback law $u = k(x)$ that globally asymptotically stabilizes the given system $\dot{x} = f(x, u)$ in the absence of quantization will in general fail to provide global asymptotic stability of the closed-loop system $\dot{x} = f(x, k(q(x)))$ which arises in the presence of state quantization.

There are two phenomena that account for changes in the system's behavior caused by quantization. The first one is saturation: if the quantized signal is outside the range of the quantizer, then the quantization error is large, and the control law designed for the ideal case of no quantization leads to instability. The second one is deterioration of performance near the equilibrium: as the difference between the current and the desired values of the state becomes small, higher precision is required, and so in the presence of quantization errors asymptotic convergence is impossible.

These phenomena manifest themselves in the existence of two nested invariant regions such that all trajectories of the quantized system starting in the bigger region approach the smaller one, while no further convergence guarantees can be given. The goal of this section is to describe this behavior more precisely, starting with linear systems and then moving on to nonlinear systems.

Linear systems

Consider the linear system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m. \tag{5.20}$$

Suppose that this system is stabilizable, so that for some matrix K the eigenvalues of $A+BK$ have negative real parts. By the standard Lyapunov

[1]This is because we will work with a single C^1 Lyapunov function; cf. Section 2.1.

stability theory, there exist positive definite symmetric matrices P and Q such that

$$(A+BK)^T P + P(A+BK) = -Q. \tag{5.21}$$

(Although the subsequent calculations are done for the general case, a convenient choice is $Q = I$.) We assume that $B \neq 0$ and $K \neq 0$; this is no loss of generality because the case of interest is when A is not Hurwitz. We let $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and the largest eigenvalue of a symmetric matrix, respectively. The inequality

$$\lambda_{\min}(P)|x|^2 \le x^T P x \le \lambda_{\max}(P)|x|^2$$

will be used repeatedly below.

Since only quantized measurements of the state are available, the state feedback law $u = Kx$ is not implementable. Let us consider the "certainty equivalence"[2] quantized feedback control law

$$u = Kq(x).$$

(Everything that follows also applies with minor changes to dynamic feedback laws; cf. Section 5.3.5.) The closed-loop system is given by

$$\dot{x} = Ax + BKq(x) = (A+BK)x + BK(q(x) - x). \tag{5.22}$$

The right-hand side of this system is the sum of an asymptotically stable "nominal" system and a perturbation due to the quantization error.

Whenever the inequality (5.18), and consequently (5.19), hold with $z = x$, the derivative of the quadratic function

$$V(x) := x^T P x$$

along solutions of the system (5.22) satisfies

$$\begin{aligned}\dot{V} &= -x^T Q x + 2x^T PBK(q(x) - x)\\ &\le -\lambda_{\min}(Q)|x|^2 + 2|x|\|PBK\|\Delta\\ &= -|x|\lambda_{\min}(Q)(|x| - \Theta_x \Delta)\end{aligned}$$

where

$$\Theta_x := \frac{2\|PBK\|}{\lambda_{\min}(Q)}$$

and $\|\cdot\|$ denotes the matrix norm induced by the Euclidean norm on $\mathbb{R}^n$. Taking a sufficiently small $\varepsilon > 0$, we have the following formula:

$$\Theta_x \Delta(1+\varepsilon) \le |x| \le M \quad \Rightarrow \quad \dot{V} \le -|x|\lambda_{\min}(Q)\Theta_x \Delta\varepsilon. \tag{5.23}$$

[2]This terminology is common in adaptive control (see Section 6.1); here it refers to the fact that the controller treats quantized state measurements as if they were exact state values.

Define the balls

$$\mathcal{B}_1 := \{x : |x| \leq M\}$$

and

$$\mathcal{B}_2 := \{x : |x| \leq \Theta_x \Delta(1 + \varepsilon)\}$$

and the ellipsoids

$$\mathcal{R}_1 := \{x : x^T P x \leq \lambda_{\min}(P) M^2\} \tag{5.24}$$

and

$$\mathcal{R}_2 := \{x : x^T P x \leq \lambda_{\max}(P) \Theta_x^2 \Delta^2 (1 + \varepsilon)^2\}. \tag{5.25}$$

Suppose that M is large enough compared to Δ so that

$$\sqrt{\lambda_{\min}(P)} M > \sqrt{\lambda_{\max}(P)} \Theta_x \Delta (1 + \varepsilon). \tag{5.26}$$

Then we have

$$\mathcal{B}_2 \subset \mathcal{R}_2 \subset \mathcal{R}_1 \subset \mathcal{B}_1.$$

This situation is illustrated in Figure 35, which will also be useful later.

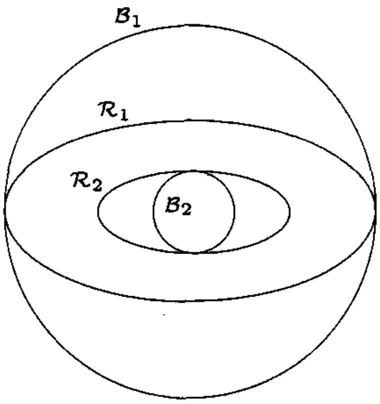

FIGURE 35. The regions used in the proofs

In view of the formula (5.23) and the fact that the ellipsoids $\mathcal{R}_1$ and $\mathcal{R}_2$ are defined in terms of level sets of V, we conclude that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are invariant with respect to the system (5.22). We arrive at the following result.

Lemma 5.1 *Fix an arbitrary $\varepsilon > 0$ and assume that the inequality* (5.26) *holds. Then the ellipsoids $\mathcal{R}_1$ and $\mathcal{R}_2$ defined by* (5.24) *and* (5.25) *are invariant regions for the system* (5.22). *Moreover, all solutions of* (5.22) *that start in the ellipsoid $\mathcal{R}_1$ enter the smaller ellipsoid $\mathcal{R}_2$ in finite time.*

The fact that the trajectories starting in $\mathcal{R}_1$ approach $\mathcal{R}_2$ in finite time follows from the bound on $\dot{V}$ given by (5.23). Indeed, if a time t_0 is given such that $x(t_0)$ belongs to $\mathcal{R}_1$ and if we let

$$T := \frac{\lambda_{\min}(P) M^2 - \lambda_{\max}(P) \Theta_x^2 \Delta^2 (1 + \varepsilon)^2}{\Theta_x^2 \Delta^2 (1 + \varepsilon) \lambda_{\min}(Q) \varepsilon} \tag{5.27}$$

then $x(t_0 + T)$ is guaranteed to belong to $\mathcal{R}_2$.

Nonlinear systems

Consider the system

$$\dot{x} = f(x, u), \qquad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \tag{5.28}$$

with $f(0,0) = 0$. All vector fields and control laws are understood to be sufficiently regular (e.g., C^1) so that existence and uniqueness of solutions are ensured. It is natural to assume that there exists a state feedback law $u = k(x)$ that makes the closed-loop system globally asymptotically stable. (Although we work with static feedback laws, the extension to dynamic feedback laws is straightforward.) Actually, we need to impose the following stronger condition on k.

ASSUMPTION 1. There exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ such that for some class $\mathcal{K}_\infty$ functions[3] $\alpha_1, \alpha_2, \alpha_3, \rho$ and for all $x, e \in \mathbb{R}^n$ we have

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{5.29}$$

and

$$|x| \geq \rho(|e|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, k(x+e)) \leq -\alpha_3(|x|). \tag{5.30}$$

We will examine this assumption in detail later. As we now show, it leads to a direct analog of Lemma 5.1 for nonlinear systems. Since only quantized measurements of the state are available, we again consider the "certainty equivalence" quantized feedback control law, which is now given by

$$u = k(q(x)).$$

The corresponding closed-loop system is

$$\dot{x} = f(x, k(q(x))). \tag{5.31}$$

This can be rewritten as

$$\dot{x} = f(x, k(x+e)) \tag{5.32}$$

where

$$e := q(x) - x \tag{5.33}$$

represents the quantization error. In what follows, $\circ$ denotes function composition.

Lemma 5.2 *Assume that we have*

$$\alpha_1(M) > \alpha_2 \circ \rho(\Delta). \tag{5.34}$$

[3]See Section A.2 for the relevant definitions.

Then the sets

$$\mathcal{R}_1 := \{x : V(x) \le \alpha_1(M)\} \tag{5.35}$$

and

$$\mathcal{R}_2 := \{x : V(x) \le \alpha_2 \circ \rho(\Delta)\} \tag{5.36}$$

are invariant regions for the system (5.31). *Moreover, all solutions of* (5.31) *that start in the set* $\mathcal{R}_1$ *enter the smaller set* $\mathcal{R}_2$ *in finite time.*

PROOF. Whenever the inequality (5.18), and consequently (5.19), hold with $z = x$, the quantization error e given by (5.33) satisfies

$$|e| = |q(x) - x| \le \Delta.$$

Using (5.30), we obtain the following formula for the derivative of V along solutions of the system (5.31):

$$\rho(\Delta) \le |x| \le M \quad \Rightarrow \quad \dot{V} \le -\alpha_3(|x|). \tag{5.37}$$

Define the balls

$$\mathcal{B}_1 := \{x : |x| \le M\}$$

and

$$\mathcal{B}_2 := \{x : |x| \le \rho(\Delta)\}.$$

As before, in view of (5.29) and (5.34) we have

$$\mathcal{B}_2 \subset \mathcal{R}_2 \subset \mathcal{R}_1 \subset \mathcal{B}_1.$$

Combined with (5.37), this implies that the ellipsoids $\mathcal{R}_1$ and $\mathcal{R}_2$ are both invariant. The fact that the trajectories starting in $\mathcal{R}_1$ approach $\mathcal{R}_2$ in finite time follows from the bound on the derivative of V deduced from (5.37). Indeed, if a time t_0 is given such that $x(t_0)$ belongs to $\mathcal{R}_1$ and if we let

$$T := \frac{\alpha_1(M) - \alpha_2 \circ \rho(\Delta)}{\alpha_3 \circ \rho(\Delta)} \tag{5.38}$$

then $x(t_0 + T)$ is guaranteed to belong to $\mathcal{R}_2$. □

Note that for Lemma 5.2 to hold, the quantization error Δ does not need to be small. When Δ is sufficiently small, one can use a standard perturbation argument without relying on Assumption 1. However, it is the more general property expressed by Lemma 5.2 that will be needed below.

The above requirement imposed on the feedback law k, expressed by Assumption 1, is equivalent to input-to-state stability (ISS) of the perturbed closed-loop system (5.32) with respect to the measurement disturbance input e (see Section A.6). For linear systems and linear stabilizing control laws such robustness with respect to measurement errors is automatic, but for nonlinear systems this is far from being the case.

Example 5.3 Consider the equation

$$\dot{x} = -x - x^4 + x^3 u, \qquad x \in \mathbb{R}.$$

The feedback law $u = x$ cancels the nonlinearities and yields the globally asymptotically stable closed-loop system $\dot{x} = -x$. However, in the presence of a measurement error the control law becomes $u = x + e$, which gives the closed-loop system $\dot{x} = -x + x^3 e$. It is not hard to see that this system is not ISS, because bounded e produce unbounded solutions for sufficiently large $|x(0)|$. □

In general, the requirement that the original system (5.28) be input-to-state stabilizable with the respect to the measurement error is rather restrictive for nonlinear systems. In fact, there exist systems that are globally asymptotically stabilizable but not input-to-state stabilizable with respect to measurement errors. The problem of finding control laws that achieve ISS with respect to measurement errors is a nontrivial one, even for systems affine in controls, and continues to be a subject of research efforts. We have thus revealed what appears to be an interesting connection between the problem of quantized feedback stabilization, the theory of hybrid systems, and topics of current interest in nonlinear control design.

5.3.3 Dynamic state quantization

In the preceding discussion, the parameters of the quantizer are fixed in advance and cannot be changed by the control designer. We now consider the situation where it is possible to vary some parameters of the quantizer in real time, on the basis of collected data. For example, if a quantizer is used to represent a digital camera, this corresponds to zooming in and out, i.e., varying the focal length, while the number of pixels of course remains fixed. Other specific examples can be given. More generally, this approach fits into the framework of *control with limited information*: the state of the system is not completely known, but it is only known which one of a fixed number of quantization regions contains the current state at each instant of time. The quantizer can be thought of as a coder that generates an encoded signal taking values in a given finite set. By changing the size and relative position of the quantization regions—i.e., by modifying the coding mechanism—one can learn more about the behavior of the system, without violating the restriction on the type of information that can be communicated to the controller.

The quantization parameters will be updated at discrete instants of time; these switching events will be triggered by the values of a suitable Lyapunov function. This results in a *hybrid quantized feedback control policy*. There are several reasons for adopting a hybrid control approach rather than varying the quantization parameters continuously. First, in specific situations there may be some constraints on how many values these parameters are allowed to take and how frequently they can be adjusted. Thus a discrete adjustment policy is more natural and easier to implement than a continuous one. Second, the analysis of hybrid systems obtained in this way appears to be more tractable than that of systems resulting from continuous parameter tuning. In fact, we will see that invariant regions defined by level sets of a Lyapunov function provide a simple and effective tool for studying the behavior of the closed-loop system. This also implies that precise computation of switching times is not essential, which makes our hybrid control policies robust with respect to certain types of time delays (such as those associated with periodic sampling).

The control strategy will usually be composed of two stages. The first "zooming-out" stage consists of increasing the range of the quantizer until the state of the system can be adequately measured; at this stage, the system is open-loop. The second "zooming-in" stage involves applying feedback and at the same time decreasing the quantization error in such a way as to drive the state to the origin. These two techniques enable us to overcome the two limitations of quantized control mentioned earlier, namely, saturation and loss of precision near the origin. We will show that if a linear system can be stabilized by a linear feedback law, then it can also be globally asymptotically stabilized by a hybrid quantized feedback control policy, and that under certain conditions this result can be generalized to nonlinear systems. (With some abuse of terminology, we call a closed-loop hybrid system globally asymptotically stable if the origin is a globally asymptotically stable equilibrium of the continuous dynamics.)

We formalize the above idea by using quantized measurements of the form

$$q_\mu(z) := \mu q\Big(\frac{z}{\mu}\Big) \tag{5.39}$$

where $\mu > 0$. The range of this quantizer is $M\mu$ and the quantization error is $\Delta\mu$. We can think of μ as the "zoom" variable: increasing μ corresponds to zooming out and essentially obtaining a new quantizer with larger range and quantization error, whereas decreasing μ corresponds to zooming in and obtaining a quantizer with a smaller range but also a smaller quantization error. The variable μ will be the discrete state of the hybrid closed-loop system. In the camera model example, μ corresponds to the inverse of the focal length[4] f. It is possible to introduce more general, nonlinear scaling of the quantized variable, as in $\nu \circ q \circ \nu^{-1}(z)$ where ν is some invertible function from $\mathbb{R}^n$ to $\mathbb{R}^n$; however, this does not seem to yield any immediate advantages.

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[4]We prefer to work with $\mu = 1/f$ rather than with f to avoid system signals that grow unbounded, although this is merely a formal distinction.

Linear systems

Consider again the linear system (5.20), a stabilizing state feedback gain K, and positive definite symmetric matrices P and Q satisfying the Lyapunov equation (5.21). The "certainty equivalence" quantized feedback control law is now

$$u = Kq_\mu(x). \tag{5.40}$$

Assume for the moment that μ is a fixed positive number. The closed-loop system is given by

$$\dot{x} = Ax + BKq_\mu(x) = (A + BK)x + BK\mu\left(q\Big(\frac{x}{\mu}\Big) - \frac{x}{\mu}\right). \tag{5.41}$$

The behavior of trajectories of the system (5.41) for a fixed μ is characterized by the following result, which is a straightforward generalization of Lemma 5.1.

Lemma 5.3 *Fix an arbitrary $\varepsilon > 0$ and assume that we have*

$$\sqrt{\lambda_{\min}(P)}M > \sqrt{\lambda_{\max}(P)}\Theta_x\Delta(1+\varepsilon) \tag{5.42}$$

where

$$\Theta_x := \frac{2\|PBK\|}{\lambda_{\min}(Q)} > 0.$$

Then the ellipsoids

$$\mathcal{R}_1(\mu) := \{x : x^T Px \le \lambda_{\min}(P)M^2\mu^2\} \tag{5.43}$$

and

$$\mathcal{R}_2(\mu) := \{x : x^T Px \le \lambda_{\max}(P)\Theta_x^2\Delta^2(1+\varepsilon)^2\mu^2\} \tag{5.44}$$

are invariant regions for the system (5.41). *Moreover, all solutions of* (5.41) *that start in the ellipsoid $\mathcal{R}_1(\mu)$ enter the smaller ellipsoid $\mathcal{R}_2(\mu)$ in finite time, given by the formula* (5.27).

As we explained before, a hybrid quantized feedback control policy involves updating the value of μ at discrete instants of time. An open-loop zooming-out stage is followed by a closed-loop zooming-in stage, so that the resulting control law takes the form

$$u(t) = \begin{cases} 0 & \text{if } \ 0 \le t < t_0 \\ Kq_{\mu(t)}(x(t)) & \text{if } \ t \ge t_0 \,. \end{cases}$$

Using this idea and Lemma 5.3, it is possible to achieve global asymptotic stability, as we now show.

Theorem 5.4 *Assume that we have*

$$\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M > 2\Delta \max\left\{1, \frac{\|PBK\|}{\lambda_{\min}(Q)}\right\}. \tag{5.45}$$

Then there exists a hybrid quantized feedback control policy that makes the system (5.20) *globally asymptotically stable.*

PROOF. *The zooming-out stage.* Set u equal to zero. Let $\mu(0) = 1$. Then increase μ in a piecewise constant fashion, fast enough to dominate the rate of growth of $\|e^{At}\|$. For example, one can fix a positive number τ and let $\mu(t) = 1$ for $t \in [0, \tau)$, $\mu(t) = \tau e^{2\|A\|\tau}$ for $t \in [\tau, 2\tau)$, $\mu(t) = 2\tau e^{2\|A\|2\tau}$ for $t \in [2\tau, 3\tau)$, and so on. Then there will be a time $t \geq 0$ such that

$$\left|\frac{x(t)}{\mu(t)}\right| \leq \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M - 2\Delta$$

(by (5.45), the right-hand side of this inequality is positive). In view of Condition 1 imposed in Section 5.3.1, this implies

$$\left|q\left(\frac{x(t)}{\mu(t)}\right)\right| \leq \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M - \Delta$$

which is equivalent to

$$|q_\mu(x(t))| \leq \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M\mu(t) - \Delta\mu(t). \tag{5.46}$$

We can thus pick a time t_0 such that (5.46) holds with $t = t_0$. Therefore, in view of Conditions 1 and 2 of Section 5.3.1, we have

$$\left|\frac{x(t_0)}{\mu(t_0)}\right| \leq \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M$$

hence $x(t_0)$ belongs to the ellipsoid $\mathcal{R}_1(\mu(t_0))$ given by (5.43). Note that this event can be detected using only the available quantized measurements.

The zooming-in stage. Choose an $\varepsilon > 0$ such that the inequality (5.42) is satisfied; this is possible because of (5.45). We know that $x(t_0)$ belongs to $\mathcal{R}_1(\mu(t_0))$. We now apply the control law (5.40). Let $\mu(t) = \mu(t_0)$ for $t \in [t_0, t_0 + T)$, where T is given by the formula (5.27). Then $x(t_0 + T)$ belongs to the ellipsoid $\mathcal{R}_2(\mu(t_0))$ given by (5.44). For $t \in [t_0 + T, t_0 + 2T)$, let

$$\mu(t) = \Omega\mu(t_0)$$

where

$$\Omega := \frac{\sqrt{\lambda_{\max}(P)}\Theta_x\Delta(1+\varepsilon)}{\sqrt{\lambda_{\min}(P)}M}.$$

We have $\Omega < 1$ by (5.42), hence $\mu(t_0 + T) < \mu(t_0)$. It is easy to check that $\mathcal{R}_2(\mu(t_0)) = \mathcal{R}_1(\mu(t_0 + T))$. This means that we can continue the analysis for $t \geq t_0 + T$ as before. Namely, $x(t_0 + 2T)$ belongs to the ellipsoid $\mathcal{R}_2(\mu(t_0 + T))$ defined by (5.44). For $t \in [t_0 + 2T, t_0 + 3T)$, let $\mu(t) = \Omega\mu(t_0 + T)$. Repeating this procedure, we obtain the desired control policy. Indeed, we have $\mu(t) \to 0$ as $t \to \infty$, and the above analysis implies that $x(t) \to 0$ as $t \to \infty$. It is also not hard to show that the equilibrium $x = 0$ of the continuous dynamics is stable in the sense of Lyapunov. □

Exercise 5.3 Prove Lyapunov stability.

We see from the proof of Theorem 5.4 that the state of the closed-loop system belongs, at equally spaced instants of time, to ellipsoids whose sizes decrease according to consecutive integer powers of Ω. Therefore, $x(t)$ converges to zero exponentially for $t \geq t_0$.

The fact that the scaling of μ is performed at $t = t_0 + T, t_0 + 2T, \ldots$ is not crucial: since the ellipsoids considered in the proof are invariant regions for the closed-loop system, we could instead take another sequence of switching times $t_1, t_2, \ldots$ satisfying $t_i - t_{i-1} \geq T$, $i \geq 1$. However, doing this in an arbitrary fashion would sacrifice the exponential rate of decay. The constant T plays the role of a dwell time (cf. Section 3.2.1).

At the zooming-in stage described in the proof of Theorem 5.4, the switching strategy is time-dependent, since the values of the discrete state μ are changed at precomputed times at which the continuous state x is known to belong to a certain region. An alternative approach would be to employ state-dependent switching, by means of relying on the quantized measurements to determine when x enters a desired region. A state-dependent switching strategy relies more on feedback measurements and less on off-line computations than a time-dependent one; it is therefore likely to be more robust with respect to modeling errors. On the other hand, the expressions for switching times are somewhat more straightforward in the case of time-dependent switching.

In the preceding, μ takes a countable set of values which is not assumed to be fixed in advance. In some situations μ may be restricted to take values in a given countable set S. It is not difficult to see that the proposed method, suitably modified, still works in this case, provided that the set S has the following properties:

1. S contains a sequence $\mu_{11}, \mu_{21}, \ldots$ that increases to ∞.

2. Each μ_{i1} from this sequence belongs to a sequence $\mu_{i1}, \mu_{i2}, \ldots$ in S that decreases to zero and is such that we have $\Omega \leq \mu_{i(j+1)}/\mu_{ij}$ for each j.

Theorem 5.4 can be interpreted as saying that to achieve global asymptotic stabilization, the precise information about the state is not necessary:

a digital sensor with sufficiently many bits is enough, provided that we have at our disposal a "smart" algorithm for coding and processing the available information. (In fact, it is not even necessary to require that the sensor have many bits, as we will see in Section 5.3.6.)

Nonlinear systems

Consider again the nonlinear system (5.28). It can be shown via a linearization argument that by using the above approach one can obtain local asymptotic stability for this system, provided that the corresponding linearized system is stabilizable. Here we are concerned with achieving global stability results. (Working with a given nonlinear system directly, one gains an advantage even if only local asymptotic stability is sought, because the linearization of a stabilizable nonlinear system may fail to be stabilizable.) To this end, suppose that there exists a state feedback law $u = k(x)$, with $k(0) = 0$, which satisfies the input-to-state stabilizability assumption (Assumption 1 of Section 5.3.2). The "certainty equivalence" quantized feedback control law in the present case is given by

$$u = k(q_\mu(x)) \tag{5.47}$$

where q_μ is defined by (5.39). For a fixed μ, the closed-loop system is

$$\dot{x} = f(x, k(q_\mu(x))) \tag{5.48}$$

and this takes the form (5.32) with

$$e = q_\mu(x) - x. \tag{5.49}$$

The behavior of trajectories of (5.48) for a fixed value of μ is characterized by the following counterpart of Lemma 5.2.

Lemma 5.5 *Assume that we have*

$$\alpha_1(M\mu) > \alpha_2 \circ \rho(\Delta\mu). \tag{5.50}$$

Then the sets

$$\mathcal{R}_1(\mu) := \{x : V(x) \leq \alpha_1(M\mu)\} \tag{5.51}$$

and

$$\mathcal{R}_2(\mu) := \{x : V(x) \leq \alpha_2 \circ \rho(\Delta\mu)\} \tag{5.52}$$

are invariant regions for the system (5.48)*. Moreover, all solutions of* (5.48) *that start in the set* $\mathcal{R}_1(\mu)$ *enter the smaller set* $\mathcal{R}_2(\mu)$ *in finite time, given by the formula*

$$T_\mu := \frac{\alpha_1(M\mu) - \alpha_2 \circ \rho(\Delta\mu)}{\alpha_3 \circ \rho(\Delta\mu)}. \tag{5.53}$$

Clearly, the effect of μ here is not as straightforward to characterize as in the linear case, where everything scales linearly. In particular, the above upper bound on the time it takes to enter $\mathcal{R}_2(\mu)$ depends on μ.

The unforced system

$$\dot{x} = f(x, 0) \tag{5.54}$$

is called *forward complete* if for every initial state $x(0)$ the solution of (5.54), which we denote by $\xi(x(0), \cdot)$, is defined for all $t \geq 0$. Our goal now is to show that this property, Assumption 1, and a certain additional technical condition allow one to extend the result expressed by Theorem 5.4 to the nonlinear system (5.28).

Theorem 5.6 *Assume that the system* (5.54) *is forward complete and that we have*

$$\alpha_2^{-1} \circ \alpha_1(M\mu) > \max\{\rho(\Delta\mu), \chi(\mu) + 2\Delta\mu\} \qquad \forall\, \mu > 0 \tag{5.55}$$

for some class $\mathcal{K}_\infty$ function χ. Then there exists a hybrid quantized feedback control policy that makes the system (5.28) *globally asymptotically stable.*

PROOF. As in the linear case, the control strategy is divided into two stages.

The zooming-out stage. Set the control equal to zero. Let $\mu(0) = 1$. Increase μ in a piecewise constant fashion, fast enough to dominate the rate of growth of $|x(t)|$. For example, fix a positive number τ and let $\mu(t) = 1$ for $t \in [0, \tau)$, $\mu(t) = \chi^{-1}(2 \max_{|x(0)| \leq \tau,\, s \in [0,\tau]} |\xi(x(0), s)|)$ for $t \in [\tau, 2\tau)$, $\mu(t) = \chi^{-1}(2 \max_{|x(0)| \leq 2\tau,\, s \in [0,2\tau]} |\xi(x(0), s)|)$ for $t \in [2\tau, 3\tau)$, and so on. Then there will be a time $t \geq 0$ such that

$$|x(t)| \leq \chi(\mu(t)) < \alpha_2^{-1} \circ \alpha_1(M\mu(t)) - 2\Delta\mu(t)$$

where the second inequality follows from (5.55). This implies

$$\left|\frac{x(t)}{\mu(t)}\right| < \frac{1}{\mu(t)} \alpha_2^{-1} \circ \alpha_1(M\mu(t)) - 2\Delta.$$

By virtue of Condition 1 of Section 5.3.1 we have

$$\left|q\left(\frac{x(t)}{\mu(t)}\right)\right| \leq \frac{1}{\mu(t)} \alpha_2^{-1} \circ \alpha_1(M\mu(t)) - \Delta$$

which is equivalent to

$$|q_\mu(x(t))| \leq \alpha_2^{-1} \circ \alpha_1(M\mu(t)) - \Delta\mu(t). \tag{5.56}$$

Picking a time t_0 at which (5.56) holds and using Conditions 1 and 2 of Section 5.3.1, we obtain

$$\left|\frac{x(t_0)}{\mu(t_0)}\right| \leq \frac{1}{\mu(t_0)} \alpha_2^{-1} \circ \alpha_1(M\mu(t_0))$$

hence $x(t_0)$ belongs to the set $\mathcal{R}_1(\mu(t_0))$ given by (5.51). This event can be detected solely on the basis of quantized measurements.

The zooming-in stage. We have established that $x(t_0)$ belongs to the set $\mathcal{R}_1(\mu(t_0))$. We will now use the control law (5.47). Let $\mu(t) = \mu(t_0)$ for $t \in [t_0, t_0 + T_{\mu(t_0)})$, where $T_{\mu(t_0)}$ is given by the formula (5.53). Then $x(t_0 + T_{\mu(t_0)})$ will belong to the set $\mathcal{R}_2(\mu(t_0))$ given by (5.52). Calculate $T_{\omega(\mu(t_0))}$ using (5.53) again, where the function ω is defined as

$$\omega(r) := \frac{1}{M}\alpha_1^{-1} \circ \alpha_2 \circ \rho(\Delta r), \qquad r \geq 0.$$

For $t \in [t_0 + T_{\mu(t_0)}, t_0 + T_{\mu(t_0)} + T_{\omega(\mu(t_0))})$, let

$$\mu(t) = \omega(\mu(t_0)).$$

We have $\omega(r) < r$ for all $r > 0$ by (5.55), thus $\mu(t_0 + T_{\mu(t_0)}) < \mu(t_0)$. One easily checks that $\mathcal{R}_2(\mu(t_0)) = \mathcal{R}_1(\mu(t_0 + T_{\mu(t_0)}))$. This means that we can continue the analysis and conclude that $x(t_0 + T_{\mu(t_0)} + T_{\omega(\mu(t_0))})$ belongs to $\mathcal{R}_2(\mu(t_0 + T_{\mu(t_0)}))$. We then repeat the procedure, letting $\mu = \omega(\mu(t_0 + T_{\mu(t_0)}))$ for the next time interval whose length is calculated from (5.53). Lyapunov stability of the equilibrium $x = 0$ of the continuous dynamics follows from the adjustment policy for μ as in the linear case. Moreover, we have $\mu(t) \to 0$ as $t \to \infty$, and the above analysis implies that $x(t) \to 0$ as $t \to \infty$. □

As in the linear case, we could pick a different sequence of switching times $t_1, t_2, \dots$ as long as they satisfy $t_i - t_{i-1} \geq T_{\mu(t_{i-1})}$, $i \geq 1$. We could also implement state-dependent switching instead of time-dependent switching.

Example 5.4 Consider the system

$$\dot{x} = x^3 + xu, \qquad x, u \in \mathbb{R}.$$

This is a simplified version of the system treated in [146], where it is shown how to construct a feedback law k such that the closed-loop system

$$\dot{x} = x^3 + xk(x + e)$$

is ISS with respect to e. It follows from the analysis given there that the inequalities (5.29) and (5.30) hold with $V(x) = x^2/2$, $\alpha_1(r) = \alpha_2(r) = r^2/2$, $\alpha_3(r) = r^2$, and $\rho(r) = cr$ for an arbitrary $c > 1$. We have $(\alpha_2^{-1} \circ \alpha_1)(r) = r$, so (5.55) is valid for every $M > \Delta \max\{c, 2\}$. □

As we explained in Section 5.3.2, Assumption 1 is quite restrictive for nonlinear systems. The technical assumption (5.55) also appears to be restrictive and hard to check. It depends on the relative growth of the functions α_1, α_2, and ρ. For example, if the function $\alpha_1^{-1} \circ \alpha_2 \circ \gamma$, where

$\gamma(r) := \max\{\rho(\Delta r), \chi(r) + 2\Delta r\}$, is globally Lipschitz, then (5.55) is satisfied for every M greater than the Lipschitz constant. However, there is a weaker and more easily verifiable assumption which enables one to prove asymptotic stability in the case when a bound on the magnitude of the initial state is known ("semiglobal asymptotic stability"). To see how this works, take a positive number E_0 such that $|x(0)| \leq E_0$. Suppose that

$$(\alpha_1^{-1} \circ \alpha_2 \circ \rho)'(0) < \infty. \tag{5.57}$$

Then it is an elementary exercise to verify that for M sufficiently large we have

$$\alpha_2^{-1} \circ \alpha_1(M\mu) > \rho(\Delta\mu) \qquad \forall\, \mu \in (0, 1]$$

and also

$$\alpha_2^{-1} \circ \alpha_1(M) \geq E_0.$$

Thus $x(0)$ belongs to the set $\mathcal{R}_1(1)$ defined by (5.51), the zooming-out stage is not necessary, and zooming in can be carried out as in the proof of Theorem 5.6, starting at $t_0 = 0$ and $\mu(0) = 1$. Forward completeness of the unforced system (5.54) is not required here.

If (5.57) does not hold, it is still possible to ensure that all solutions starting in a given compact set approach an arbitrary prespecified neighborhood of the origin ("semiglobal practical stability"). This is not difficult to show by choosing Δ to be sufficiently small, provided that M is sufficiently large and the feedback law k is robust with respect to *small* measurement errors. All continuous stabilizing feedback laws possess such robustness, and discontinuous control laws for a large class of systems can also be shown to have this robustness property. The value of μ can then be kept fixed, and Assumption 1 is not needed.

Every asymptotically stabilizing feedback law is automatically input-to-state stabilizing with respect to the measurement error e locally, i.e., for sufficiently small values of x and e (see Section A.6). This leads at once to local versions of the present results.

5.3.4 Input quantization

We now present results analogous to those obtained in Section 5.3.3 for systems whose input, rather than state, is quantized. These results yield a basis for comparing the effects of input quantization and state quantization on the performance of the system. The proofs are similar to the ones given earlier, and some details will be omitted.

Linear systems

Consider the linear system (5.20). Suppose again that there exists a matrix K such that the eigenvalues of $A + BK$ have negative real parts, so that for some positive definite symmetric matrices P and Q the equation (5.21) holds.

The state feedback law $u = Kx$ is not implementable because only quantized measurements $q_\mu(u)$ of the input u are available, where q_μ is defined by (5.39). We therefore consider the "certainty equivalence" quantized feedback control law

$$u = q_\mu(Kx). \tag{5.58}$$

This yields the closed-loop system

$$\dot{x} = Ax + B\mu q_\mu(x) = (A + BK)x + B\mu\left(q\left(\frac{Kx}{\mu}\right) - \frac{Kx}{\mu}\right). \tag{5.59}$$

The behavior of trajectories of (5.59) for a fixed value of μ is characterized as follows.

Lemma 5.7 *Fix an arbitrary $\varepsilon > 0$ and assume that we have*

$$\sqrt{\lambda_{\min}(P)}M > \sqrt{\lambda_{\max}(P)}\Theta_u\|K\|\Delta(1+\varepsilon) \tag{5.60}$$

where

$$\Theta_u := \frac{2\|PB\|}{\lambda_{\min}(Q)}.$$

Then the ellipsoids

$$\mathcal{R}_1(\mu) := \{x : x^T Px \le \lambda_{\min}(P)M^2\mu^2/\|K\|^2\} \tag{5.61}$$

and

$$\mathcal{R}_2(\mu) := \{x : x^T Px \le \lambda_{\max}(P)\Theta_u^2\Delta^2(1+\varepsilon)^2\mu^2\} \tag{5.62}$$

are invariant regions for the system (5.59). *Moreover, all solutions of* (5.59) *that start in the ellipsoid $\mathcal{R}_1(\mu)$ enter the smaller ellipsoid $\mathcal{R}_2(\mu)$ in finite time, given by the formula*

$$T := \frac{\lambda_{\min}(P)M^2 - \lambda_{\max}(P)\Theta_u^2\|K\|^2\Delta^2(1+\varepsilon)^2}{\Theta_u^2\|K\|^2\Delta^2(1+\varepsilon)\lambda_{\min}(Q)\varepsilon}. \tag{5.63}$$

We now present a hybrid quantized feedback control policy which combines the control law (5.58) with a switching strategy for μ, similarly to the state quantization case studied in Section 5.3.3.

Theorem 5.8 *Assume that we have*

$$\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M > 2\Delta\frac{\|PB\|\|K\|}{\lambda_{\min}(Q)}. \tag{5.64}$$

Then there exists a hybrid quantized feedback control policy that makes the system (5.20) *globally asymptotically stable.*

PROOF. *The zooming-out stage.* Set u equal to zero. Let $\mu(0) = 1$. Increase μ fast enough to dominate the rate of growth of $\|e^{At}\|$, as in the proof of Theorem 5.4. Then there will be a time $t_0 \geq 0$ such that

$$|x(t_0)| \leq \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} \frac{M\mu(t_0)}{\|K\|}$$

which implies that $x(t_0)$ belongs to the ellipsoid $\mathcal{R}_1(\mu(t_0))$ given by (5.61).

The zooming-in stage. For $t \geq t_0$ we use the control law (5.58). Pick a positive ε so that (5.60) holds; this is possible because of (5.64). Let $\mu(t) = \mu(t_0)$ for $t \in [t_0, t_0 + T)$, where T is given by the formula (5.63). Then $x(t_0 + T)$ belongs to the ellipsoid $\mathcal{R}_2(\mu(t_0))$ given by (5.62). For $t \in [t_0 + T, t_0 + 2T)$, let

$$\mu(t) = \Omega\mu(t_0)$$

where

$$\Omega := \frac{\sqrt{\lambda_{\max}(P)}\Theta_u\|K\|\Delta(1+\varepsilon)}{\sqrt{\lambda_{\min}(P)}M}.$$

We have $\mu(t_0 + T) < \mu(t_0)$ by (5.60), and $\mathcal{R}_2(\mu(t_0)) = \mathcal{R}_1(\mu(t_0 + T))$. The rest of the proof follows along the lines of the proof of Theorem 5.4. □

It is interesting to observe that in view of the inequality

$$\|PBK\| \leq \|PB\|\|K\|$$

the condition (5.64) is in general more restrictive than the corresponding condition for the case of state quantization (see Theorem 5.4). On the other hand, the zooming-in stage for input quantization is more straightforward and does not require any additional assumptions. The remarks made after the proof of Theorem 5.4 concerning the exponential rate of convergence, robustness to time delays, and the alternative method of state-dependent switching carry over to the present case without any changes. The above analysis can also be extended in a straightforward manner to the situation where both the state and the input are quantized.

Exercise 5.4 Derive a combination of Lemmas 5.3 and 5.7 for linear systems with quantization affecting both the state and the input. For example, consider the system (5.20) with the control law $u = q^u_\mu(Kq^x_\mu(x))$, where q^x is a state quantizer with range M_x and error Δ_x and q^u is an input quantizer with range M_u and error Δ_u, taking μ in both quantizers to be the same for simplicity.

Nonlinear systems

Consider the nonlinear system (5.28). Assume that there exists a feedback law $u = k(x)$, with $k(0) = 0$, which makes the closed-loop system globally asymptotically stable and, moreover, ensures that for some class $\mathcal{K}_\infty$

functions $\alpha_1, \alpha_2, \alpha_3, \rho$ there exists a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the inequalities (5.29) and

$$|x| \geq \rho(|e|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, k(x) + e) \leq -\alpha_3(|x|) \tag{5.65}$$

for all $x, e \in \mathbb{R}^n$. This is equivalent to saying that the perturbed closed-loop system

$$\dot{x} = f(x, k(x) + e) \tag{5.66}$$

is ISS with respect to the actuator disturbance input e. Take κ to be some class $\mathcal{K}_\infty$ function with the property that

$$\kappa(r) \geq \max_{|x| \leq r} |k(x)| \qquad \forall r \geq 0.$$

Then we have

$$|k(x)| \leq \kappa(|x|) \qquad \forall x.$$

The closed-loop system with the "certainty equivalence" quantized feedback control law

$$u = q_\mu(k(x)) \tag{5.67}$$

becomes

$$\dot{x} = f(x, q_\mu(k(x))) \tag{5.68}$$

and this takes the form (5.66) with

$$e = q_\mu(k(x)) - x. \tag{5.69}$$

The behavior of trajectories of (5.68) for a fixed μ is characterized by the following result.

Lemma 5.9 *Assume that we have*

$$\alpha_1 \circ \kappa^{-1}(M\mu) > \alpha_2 \circ \rho(\Delta\mu). \tag{5.70}$$

Then the sets

$$\mathcal{R}_1(\mu) := \{x : V(x) \leq \alpha_1 \circ \kappa^{-1}(M\mu)\} \tag{5.71}$$

and

$$\mathcal{R}_2(\mu) := \{x : V(x) \leq \alpha_2 \circ \rho(\Delta\mu)\} \tag{5.72}$$

are invariant regions for the system (5.68). *Moreover, all solutions of* (5.68) *that start in the set* $\mathcal{R}_1(\mu)$ *enter the smaller set* $\mathcal{R}_2(\mu)$ *in finite time, given by the formula*

$$T_\mu := \frac{\alpha_1 \circ \kappa^{-1}(M\mu) - \alpha_2 \circ \rho(\Delta\mu)}{\alpha_3 \circ \rho(\Delta\mu)}. \tag{5.73}$$

The next theorem is a counterpart of Theorem 5.6.

Theorem 5.10 *Assume that the system* (5.54) *is forward complete and that we have*

$$\alpha_2^{-1} \circ \alpha_1 \circ \kappa^{-1}(M\mu) > \rho(\Delta\mu) \qquad \forall\, \mu > 0. \tag{5.74}$$

Then there exists a hybrid quantized feedback control policy that makes the system (5.28) *globally asymptotically stable.*

PROOF. *The zooming-out stage.* Set the control to zero, and let $\mu(0) = 1$. Increase μ fast enough to dominate the rate of growth of $|x(t)|$, as in the proof of Theorem 5.6. Then there will be a time $t_0 \geq 0$ such that

$$|x(t_0)| \leq \rho(\Delta\mu(t_0)) < \alpha_2^{-1} \circ \alpha_1 \circ \kappa^{-1}(M\mu(t_0))$$

hence $x(t_0)$ belongs to the set $\mathcal{R}_1(\mu(t_0))$ given by (5.71).

The zooming-in stage. For $t \geq t_0$ apply the control law (5.67). Let $\mu(t) = \mu(t_0)$ for $t \in [t_0, t_0 + T_{\mu(t_0)})$, where $T_{\mu(t_0)}$ is given by the formula (5.73). Then $x(t_0 + T_{\mu(t_0)})$ belongs to the set $\mathcal{R}_2(\mu(t_0))$ given by (5.72). Use (5.73) again to compute $T_{\omega(\mu(t_0))}$, where ω is the function defined by

$$\omega(r) := \frac{1}{M}\kappa \circ \alpha_1^{-1} \circ \alpha_2 \circ \rho(\Delta r), \qquad r \geq 0.$$

For $t \in [t_0 + T_{\mu(t_0)}, t_0 + T_{\mu(t_0)} + T_{\omega(\mu(t_0))})$, let

$$\mu(t) = \omega(\mu(t_0)).$$

We have $\mu(t_0 + T_{\mu(t_0)}) < \mu(t_0)$ by (5.74), and $\mathcal{R}_2(\mu(t_0)) = \mathcal{R}_1(\mu(t_0 + T_{\mu(t_0)}))$. The proof can now be completed exactly as the proof of Theorem 5.6. □

In general, the requirement of ISS with respect to *actuator* errors imposed here is not as severe as the requirement of ISS with respect to *measurement* errors imposed in Section 5.3.3. In fact, if an affine system of the form

$$\dot{x} = f(x) + G(x)u \tag{5.75}$$

is asymptotically stabilizable by a feedback law $u = k_0(x)$, then one can always find a feedback law $u = k(x)$ that makes the system

$$\dot{x} = f(x) + G(x)(k(x) + e) \tag{5.76}$$

ISS with respect to an actuator disturbance e, whereas there might not exist a feedback law that makes the system

$$\dot{x} = f(x) + G(x)k(x + e) \tag{5.77}$$

ISS with respect to a measurement disturbance e. This means that for nonlinear systems, the stabilization problem in the presence of input quantization may be less challenging from the point of view of control design than the corresponding problem for state quantization. When the condition (5.74) is not satisfied, weaker results can be obtained as in Section 5.3.3.

5.3.5 Output quantization

It is possible to extend the approach presented in Section 5.3.3 to linear systems with output feedback. Consider the system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned} \tag{5.78}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$. Suppose that (A, B) is a stabilizable pair and (C, A) is an observable pair. This implies that there exist a state feedback matrix K and an output injection matrix L such that the matrices $A + BK$ and $A + LC$ are Hurwitz. The matrix

$$\overline{A} := \begin{pmatrix} A+BK & -BK \\ 0 & A+LC \end{pmatrix}$$

is then also Hurwitz, and so there exist positive definite symmetric $2n \times 2n$ matrices $\overline{P}$ and $\overline{Q}$ such that

$$\overline{A}^T\overline{P} + \overline{PA} = -\overline{Q}.$$

We are interested in the situation where only quantized measurements $q_\mu(y)$ of the output y are available, where q_μ is defined by (5.39). We therefore consider the following dynamic output feedback law, which is based on the standard Luenberger observer but uses $q_\mu(y)$ instead of y:

$$\begin{aligned}\dot{\widehat{x}} &= (A + LC)\widehat{x} + Bu - Lq_\mu(y) \\ u &= K\widehat{x}\end{aligned} \tag{5.79}$$

where $\widehat{x} \in \mathbb{R}^n$. The closed-loop system takes the form

$$\begin{aligned}\dot{x} &= Ax + BK\widehat{x} \\ \dot{\widehat{x}} &= (A + LC)\widehat{x} + BK\widehat{x} - Lq_\mu(y).\end{aligned}$$

In the coordinates given by

$$\bar{x} := \begin{pmatrix} x \\ x - \widehat{x} \end{pmatrix} \in \mathbb{R}^{2n}$$

we can rewrite this system more compactly as

$$\dot{\bar{x}} = \overline{A}\bar{x} + L\begin{pmatrix} 0 \\ q_\mu(y) - y \end{pmatrix}. \tag{5.80}$$

For a fixed value of μ, the behavior of trajectories of the system (5.80) is characterized by the following result. The proof is similar to that of Lemma 5.1.

Lemma 5.11 *Fix an arbitrary $\varepsilon > 0$ and assume that we have*

$$\sqrt{\lambda_{\min}(\overline{P})}M > \sqrt{\lambda_{\max}(\overline{P})}\Theta_y\|C\|\Delta(1+\varepsilon) \tag{5.81}$$

where

$$\Theta_y := \frac{2\|\overline{P}L\|}{\lambda_{\min}(\overline{Q})}.$$

Then the ellipsoids

$$\mathcal{R}_1(\mu) := \{\bar{x} : \bar{x}^T\overline{P}\bar{x} \leq \lambda_{\min}(\overline{P})M^2\mu^2/\|C\|^2\} \tag{5.82}$$

and

$$\mathcal{R}_2(\mu) := \{\bar{x} : \bar{x}^T\overline{P}\bar{x} \leq \lambda_{\max}(\overline{P})\Theta_y^2\Delta^2(1+\varepsilon)^2\mu^2\} \tag{5.83}$$

are invariant regions for the system (5.80)*. Moreover, all solutions of* (5.80) *that start in the ellipsoid $\mathcal{R}_1(\mu)$ enter the smaller ellipsoid $\mathcal{R}_2(\mu)$ in finite time, given by the formula*

$$T := \frac{\lambda_{\min}(\overline{P})M^2 - \lambda_{\max}(\overline{P})\Theta_y^2\|C\|^2\Delta^2(1+\varepsilon)^2}{\Theta_y^2\|C\|^2\Delta^2(1+\varepsilon)\lambda_{\min}(\overline{Q})\varepsilon}. \tag{5.84}$$

A hybrid quantized feedback control policy described next combines the above dynamic output feedback law with the idea of updating the value of μ at discrete instants of time as in Section 5.3.3.

Theorem 5.12 *Assume that we have*

$$\sqrt{\frac{\lambda_{\min}(\overline{P})}{\lambda_{\max}(\overline{P})}}M > \max\left\{3\Delta, 2\Delta\frac{\|\overline{P}L\|\|C\|}{\lambda_{\min}(\overline{Q})}\right\}. \tag{5.85}$$

Then there exists a hybrid quantized feedback control policy that makes the system (5.78) *globally asymptotically stable.*

PROOF. *The zooming-out stage.* Set u equal to zero. Increase μ in a piecewise constant fashion as before, starting from $\mu(0) = 1$, fast enough to dominate the rate of growth of $\|e^{At}\|$. Then there will be a time $t \geq 0$ such that

$$\left|\frac{y(t)}{\mu(t)}\right| \leq M - 3\Delta$$

(by (5.85), the right-hand side of this inequality is positive). In view of Condition 1 imposed in Section 5.3.1, this implies

$$\left|q\left(\frac{y(t)}{\mu(t)}\right)\right| \leq M - 2\Delta$$

which is equivalent to

$$|q_\mu(y(t))| \leq M\mu(t) - 2\Delta\mu(t). \tag{5.86}$$

We can thus find a time t_0 such that (5.86) holds with $t = t_0$. Define

$$\widehat{x}(t_0) := W^{-1} \int_{t_0}^{t_0+\tau} e^{A^T(t-t_0)} C^T \mu(t_0) q\left(\frac{y(t)}{\mu(t_0)}\right) dt \tag{5.87}$$

where W denotes the (full-rank) observability Gramian $\int_0^\tau e^{A^T t} C^T C e^{At} dt$ and τ is a positive number such that

$$|q_\mu(y(t))| \le M\mu(t_0) - \Delta\mu(t_0) \qquad \forall\, t \in [t_0, t_0 + \tau]. \tag{5.88}$$

Let

$$\Lambda := \max_{0 \le t \le \tau} \|e^{At}\| \ge 1.$$

In view of (5.88) and the equality

$$\int_{t_0}^{t_0+\tau} e^{A^T(t-t_0)} C^T y(t) dt = W x(t_0)$$

we have

$$|x(t_0) - \widehat{x}(t_0)| \le \|W^{-1}\| \tau \Lambda \|C\| \Delta \mu(t_0)$$

(recall that $\|C^T\| = \|C\|$). Defining $\widehat{x}(t_0 + \tau) := e^{A\tau}\widehat{x}(t_0)$, we obtain

$$|x(t_0 + \tau) - \widehat{x}(t_0 + \tau)| \le \|W^{-1}\| \tau \Lambda^2 \|C\| \Delta\mu(t_0)$$

and hence

$$\begin{aligned} |\bar{x}(t_0 + \tau)| &\le |x(t_0 + \tau)| + |x(t_0 + \tau) - \widehat{x}(t_0 + \tau)| \\ &\le |\widehat{x}(t_0 + \tau)| + 2|x(t_0 + \tau) - \widehat{x}(t_0 + \tau)| \\ &\le |\widehat{x}(t_0 + \tau)| + 2\|W^{-1}\| \tau \Lambda^2 \|C\| \Delta\mu(t_0). \end{aligned}$$

Now, choose $\mu(t_0 + \tau)$ large enough to satisfy

$$\sqrt{\frac{\lambda_{\min}(\overline{P})}{\lambda_{\max}(\overline{P})}} \frac{M\mu(t_0 + \tau)}{\|C\|} \ge |\widehat{x}(t_0 + \tau)| + 2\|W^{-1}\| \tau \Lambda^2 \|C\| \Delta\mu(t_0).$$

Then $\bar{x}(t_0 + \tau)$ belongs to the ellipsoid $\mathcal{R}_1(\mu(t_0 + \tau))$ given by (5.82).

The zooming-in stage. Choose an $\varepsilon > 0$ such that the inequality (5.81) is satisfied; this is possible because of (5.85). We know that $x(t_0 + \tau)$ belongs to $\mathcal{R}_1(\mu(t_0 + \tau))$. We now apply the control law (5.79). Let $\mu(t) = \mu(t_0 + \tau)$ for $t \in [t_0 + \tau, t_0 + \tau + T)$, where T is given by the formula (5.84). Then $x(t_0 + \tau + T)$ belongs to the ellipsoid $\mathcal{R}_2(\mu(t_0 + \tau))$ given by (5.83). For $t \in [t_0 + \tau + T, t_0 + \tau + 2T)$, let

$$\mu(t) = \Omega\mu(t_0 + \tau)$$

where

$$\Omega := \frac{\sqrt{\lambda_{\max}(\overline{P})}\Theta_y \|C\| \Delta(1+\varepsilon)}{\sqrt{\lambda_{\min}(\overline{P})} M}.$$

We have $\Omega < 1$ by (5.81), hence $\mu(t_0 + \tau + T) < \mu(t_0 + \tau)$. Moreover, $\mathcal{R}_2(\mu(t_0+\tau)) = \mathcal{R}_1(\mu(t_0+\tau+T))$. Repeating this procedure as before, we have the desired control policy. □

The zooming-out stage in the above proof is somewhat more complicated than in the state quantization case. However, the integral in (5.87) is easy to compute (in closed form) because the function being integrated is the product of a matrix exponential and a piecewise constant function.

5.3.6 *Active probing for information*

The conditions that we imposed on the state quantizer in Sections 5.3.2 and 5.3.3, namely, the inequalities (5.26), (5.34), (5.42), and (5.50), can all be thought of as saying that the range M of the quantizer is sufficiently large compared to the quantization error Δ. This basically means that the quantizer takes a large number of values, thereby providing reasonably accurate information about the state within its range. However, it is often of interest to consider very "coarse" quantizers, i.e., quantizers with a small number of values and consequently a small range-to-quantization-error ratio. For example, the function $q_1 : \mathbb{R} \to \{-2, 0, 2\}$ defined by

$$q_1(x) := \begin{cases} -2 & \text{if } x < -1 \\ 0 & \text{if } -1 \le x < 1 \\ 2 & \text{if } x \ge 1 \end{cases} \tag{5.89}$$

is a quantizer satisfying Conditions 1 and 2 of Section 5.3.1 with $\Delta = 1$ and an arbitrary $M < 3$. (We can also pick a $\Delta < 1$, but then we have to let $M = \Delta$.) This quantizer can be used to model a sensor which determines whether the temperature of a certain object is "normal," "too high," or "too low." Zooming in and out in the sense of Section 5.3.3 corresponds to adjusting the threshold settings. We use the function (5.89) to define a quantizer q_μ on $\mathbb{R}^n$ componentwise by

$$(q_\mu(x))_i := q_1\left(\frac{x_i}{\mu_i}\right) \tag{5.90}$$

where $\mu_i > 0$, $i = 1, \dots, n$ are the zoom variables. (Unlike in the formula (5.39), we do not normalize by multiplying by the zoom variable, because in this case only the sign of the quantized measurement is important and not its magnitude.)

It turns out that we can compensate for the small amount of information that this quantizer provides, by means of varying quantization parameters more frequently than in the hybrid quantized feedback control policies described in Section 5.3.3 (and using a different control law). In other words, there is a trade-off between how much static information is provided by the quantizer and how actively we need to probe the system for information by changing the quantization dynamically. The procedure is based on the following idea: if the state of the system at a given instant of time is known to belong to a certain rectilinear box, and if we position the quantization regions so as to divide this box into smaller boxes, then on the basis of the corresponding quantized measurement we can immediately determine which one of these smaller boxes contains the state of the system, thereby improving our state estimate. The control law will only use quantized measurements at discrete sampling times. The sampling frequency will depend on how unstable the open-loop system is.

Linear systems

We now demonstrate how this works in the context of the linear system (5.20).

Proposition 5.13 *For the quantizer q_μ given by* (5.89) *and* (5.90), *there exists a hybrid quantized feedback control policy that makes the system* (5.20) *globally asymptotically stable.*

PROOF. It is convenient to use the norm $\|x\|_\infty := \max\{|x_i| : 1 \leq i \leq n\}$ on $\mathbb{R}^n$ and the induced matrix norm $\|A\|_\infty := \max\{\sum_{j=1}^n |A_{ij}| : 1 \leq i \leq n\}$ on $\mathbb{R}^{n\times n}$.

The zooming-out stage. Set u equal to zero. Let $\mu_i(0) = 1$, $i = 1, \dots, n$. Increase each μ_i fast enough relative to the rate of growth of $\|e^{At}\|_\infty$ (cf. the proof of Theorem 5.4). Then there will be a time $t \geq 0$ such that

$$|x_i(t)| < \mu_i(t), \qquad i = 1, \dots, n$$

which implies

$$q_\mu(x(t)) = 0. \tag{5.91}$$

Picking a time t_0 such that (5.91) holds with $t = t_0$, we have

$$\|x(t_0)\|_\infty \leq E_0 := \max_{1\leq i\leq n} \mu_i(t_0). \tag{5.92}$$

The zooming-in stage. As a result of the zooming-out stage, $\widehat{x}(t_0) := 0$ can be viewed as an estimate of $x(t_0)$ with estimation error of norm at most E_0. Our goal is to generate state estimates with estimation errors approaching zero as $t \to \infty$, while at the same time using these estimates to compute the feedback law. Pick a $\tau > 0$ such that we have

$$\Lambda := \max_{0\leq t\leq \tau} \|e^{At}\|_\infty < 2. \tag{5.93}$$

For $t \in [t_0, t_0 + \tau)$, let $u(t) = 0$. From (5.92) and (5.93) we know that

$$\|x(t)\|_\infty \leq \Lambda\|x(t_0)\|_\infty \leq \Lambda E_0, \qquad t_0 \leq t \leq t_0 + \tau.$$

At time $t_0+\tau$, let $\mu_i(t_0+\tau) = \Lambda E_0/3$, $i = 1, \ldots, n$. The quantized measurement $q_\mu(x(t_0+\tau))$ singles out a rectilinear box with edges at most $2\Lambda E_0/3$ which contains $x(t_0+\tau)$. Denoting the center of this box by $\widehat{x}(t_0+\tau)$, we obtain

$$\|\widehat{x}(t_0+\tau) - x(t_0+\tau)\|_\infty \leq \Lambda E_0/3. \tag{5.94}$$

For $t \in [t_0+\tau, t_0+2\tau)$, let

$$u(t) = K\widehat{x}(t) \tag{5.95}$$

where

$$\widehat{x}(t) := e^{(A+BK)(t-t_0-\tau)}\widehat{x}(t_0+\tau)$$

and K is chosen so that $A+BK$ is a Hurwitz matrix. From the equations $\dot{\widehat{x}} = A\widehat{x} + Bu$ and $\dot{x} = Ax + Bu$ and the formulas (5.93) and (5.94) we conclude that

$$\|\widehat{x}(t)-x(t)\|_\infty \leq \Lambda\|\widehat{x}(t_0+\tau)-x(t_0+\tau)\|_\infty \leq \Lambda^2 E_0/3, \quad t_0+\tau \leq t < t_0+2\tau.$$

We will use the notation $\widehat{x}(t_0+t^-) := \lim_{s\to t^-} \widehat{x}(t_0+s)$. At time $t_0+2\tau$, let

$$\mu_i(t_0+2\tau) = \begin{cases} \widehat{x}_i(t_0+2\tau) & \text{if } \widehat{x}_i(t_0+2\tau^-) \neq 0 \\ \Lambda^2 E_0/9 & \text{if } \widehat{x}_i(t_0+2\tau^-) = 0 \end{cases}$$

for $i = 1, \ldots, n$. The quantized measurement $q_\mu(x(t_0+2\tau))$ singles out a rectilinear box which contains $x(t_0+2\tau)$. Denoting the center of this box by $\widehat{x}(t_0+2\tau)$, we have

$$|\widehat{x}_i(t_0+2\tau) - x_i(t_0+2\tau)| \leq \begin{cases} \Lambda^2 E_0/6 & \text{if } \widehat{x}_i(t_0+2\tau^-) \neq 0 \\ \Lambda^2 E_0/9 & \text{if } \widehat{x}_i(t_0+2\tau^-) = 0\,. \end{cases}$$

For $t \in [t_0+2\tau, t_0+3\tau)$, define the control by the formula (5.95), where

$$\widehat{x}(t) := e^{(A+BK)(t-t_0-2\tau)}\widehat{x}(t_0+2\tau).$$

We have

$$\|\widehat{x}(t)-x(t)\|_\infty \leq \Lambda\|\widehat{x}(t_0+2\tau)-x(t_0+2\tau)\|_\infty \leq \Lambda^3 E_0/6, \quad t_0+2\tau \leq t < t_0+3\tau.$$

Continuing this process, we see that the upper bound on $\|\widehat{x}(t) - x(t)\|_\infty$ is divided by at least 2 at the switching times $t_0+\tau, t_0+2\tau, \ldots$ and grows by the factor of $\Lambda < 2$ on every interval between the switching times. This clearly implies that $\|\widehat{x}(t) - x(t)\|_\infty$ converges to zero as $t \to \infty$, thus the closed-loop system can be written as

$$\dot{x} = (A+BK)x + e$$

where $e := BK(\widehat{x} - x) \to 0$. Asymptotic stability follows by the same reasoning as in the proofs of the previous results. $\square$

If the quantizer does not have to be centered at the origin, i.e., if we can use quantized measurements of the form $q_\mu(x - \widehat{x})$, then we can ensure that the bound on $\|\widehat{x}(t) - x(t)\|_\infty$ is divided by 3 at each switching time. The quantizer (5.90) takes 3^n different values. More generally, if a quantizer with N^n values is available, where $N \geq 3$ is an integer, then the uncertainty is decreased by the factor of N at each switching time. In this case, instead of (5.93) we need to impose the following condition on the dwell time τ between the switches:

$$\max_{0 \leq t \leq \tau} \|e^{At}\|_\infty < N. \tag{5.96}$$

This inequality characterizes the trade-off between the amount of static information provided by the quantizer and the required sampling frequency. This relationship depends explicitly on a measure of instability of the open-loop system. We see, for instance, that if a lower bound on τ is given, then N needs to be sufficiently large for asymptotic stabilization to be possible.

The above technique can also be extended to the output feedback setting of Section 5.3.5. The number of values of the quantizer in this case is N^p, where $N \geq 3$ is the number of quantization regions for each component of the quantizer. In place of (5.96), the dwell time τ will have to satisfy the inequality

$$\|W^{-1}\|_\infty \|C\|_\infty^2 \tau \Big(\max_{0 \leq t \leq \tau} \|e^{At}\|_\infty \Big)^3 < N$$

where W is the observability Gramian used in the proof of Theorem 5.12.

Nonlinear systems

There is a relatively straightforward way to extend the above control scheme to the nonlinear system (5.28). Assume as before that there exists a feedback law $u = k(x)$ that satisfies $k(0) = 0$ and provides ISS with respect to measurement errors. This time we write this condition in the time domain and in terms of the infinity norm: there exist functions $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that along every solution of the system (5.32) with e a piecewise continuous signal (this is no loss of generality for our purposes) we have the inequality

$$\|x(t)\|_\infty \leq \beta(\|x(0)\|_\infty, t) + \gamma\Big(\sup_{s \in [0,t]} \|e(s)\|_\infty \Big) \qquad \forall t \geq 0.$$

Proposition 5.14 *Assume that the system* (5.54) *is forward complete. For the quantizer* q_μ *given by* (5.89) *and* (5.90), *there exists a hybrid quantized feedback control policy that makes the system* (5.28) *globally asymptotically stable.*

PROOF. Combining the zooming-out techniques used in the proofs of Proposition 5.13 and Theorem 5.6, we can find a time t_0 at which the bound (5.92) is valid. Next, let L be the Lipschitz constant for the function f corresponding to the region

$$\{(x,u) : \|x\|_\infty \le D,\ \|u\|_\infty \le \kappa(D)\} \tag{5.97}$$

where

$$D := \beta(E_0, 0) + \gamma(2E_0) + 2E_0$$

and κ is a class $\mathcal{K}_\infty$ function satisfying (cf. Section 5.3.4)

$$\|k(x)\|_\infty \le \kappa(\|x\|_\infty) \qquad \forall\, x.$$

Pick a $\tau > 0$ such that we have

$$\Lambda := e^{L\tau} < 2.$$

Similarly to the linear case, we apply the control law

$$u(t) = k(\widehat{x}(t))$$

where $\widehat{x}(t_0) := 0$ and $\widehat{x}$ between the switching times $t_0 + \tau, t_0 + 2\tau, \ldots$ is the solution of the "copy" of the system (5.28), given by

$$\dot{\widehat{x}} = f(\widehat{x}, u).$$

At the switching times, $\widehat{x}$ is updated discontinuously as follows. As long as $\|x\|_\infty$ and $\|\widehat{x}\|_\infty$ do not exceed D, from the equation $\dot{\widehat{x}} - \dot{x} = f(\widehat{x}, u) - f(x, u)$ and the formula $\|f(\widehat{x}, u) - f(x, u)\|_\infty \le L\|\widehat{x} - x\|_\infty$ we obtain an upper bound on $\|\widehat{x}(t) - x(t)\|_\infty$ which grows by the factor of Λ on every interval between the switching times. This implies that at each switching time $t_0 + k\tau$, $k = 1, 2, \ldots$, we have a rectilinear box centered at $\widehat{x}(t_0 + k\tau^-)$ which contains $x(t_0 + k\tau)$. As before, we use the quantized measurement to divide this uncertainty box into smaller boxes and determine which one of these smaller boxes contains the state. We let $\widehat{x}(t_0 + k\tau)$ be the center of the new box, and the upper bound on $\|\widehat{x} - x\|_\infty$ is divided by at least 2.

Since $\|\widehat{x}(t_0) - x(t_0)\|_\infty \le E_0$ and $\Lambda < 2$, it is not difficult to see that (x, u) and $(\widehat{x}, u)$ never leave the region (5.97), hence $e(t) := \widehat{x}(t) - x(t)$ and consequently $x(t)$ converge to zero as $t \to \infty$ (cf. Exercise A.2). Lyapunov stability of the origin is also not hard to show, keeping in mind that $f(0,0) = 0$ and $k(0) = 0$ (cf. Exercise 5.3). □

6

Systems with Large Modeling Uncertainty

6.1 Introductory remarks

This chapter is devoted to control problems for uncertain systems. Modeling uncertainty is typically divided into structured uncertainty (unknown parameters ranging over a known set) and unstructured uncertainty (unmodeled dynamics). When we say that the uncertainty is "large," we usually refer to the structured uncertainty. Informally, this means that the parametric uncertainty set is so large that robust control design tools are inapplicable and thus an adaptive control approach is required. It is probably impossible to formally distinguish between an adaptive control law and a nonadaptive dynamic control law. However, one can in principle draw such a distinction if one knows how the control law in question was designed. To this end, the following circular "definition" is sometimes given: a control law is adaptive if it involves adaptation.

By adaptation one usually means a combination of on-line estimation and control, whereby a suitable controller is selected on the basis of the current estimate for the uncertain process. More precisely, one designs a parameterized family of candidate controllers, where the parameter varies over a continuum which corresponds to the parametric uncertainty range of the process in a suitable way (for example, the candidate controllers may be in 1-to-1 correspondence with admissible process models). One then runs an estimation procedure, which provides time-varying estimates of the unknown parameters of the process model. According to the *certainty equivalence* principle, at each instant of time one applies a candidate controller designed for the process model that corresponds to the current estimate.

This traditional approach to adaptive control has some inherent limitations which have been well recognized in the literature. Most notably, if unknown parameters enter the process model in complicated ways, it may be very difficult to construct a continuously parameterized family of candidate controllers. Parameter estimation over a continuum is also a challenging task. These issues become especially severe if robustness and high performance are sought. An alternative approach to control of uncertain systems, which we describe here and refer to as *supervisory control*, seeks to overcome some of the above difficulties while retaining the fundamental ideas on which adaptive control is based. The main feature that distinguishes it from conventional adaptive control is that controller selection is carried out by means of logic-based switching rather than continuous tuning. Switching among candidate controllers is orchestrated by a high-level decision maker called a *supervisor*. This situation is sketched in Figure 20 on page 75.

When the controller selection is performed in a discrete fashion, one is no longer forced to construct a continuously parameterized family of controllers (which may be a difficult task, especially when using advanced controllers). It is also not necessary to use continuous methods, such as gradient descent, to generate parameter estimates. This allows one to handle process models that are nonlinearly parameterized over nonconvex sets and to avoid loss of stabilizability of the estimated model, which are well-known difficulties in adaptive control. The ability of supervisory control to overcome obstructions that are present in continuously tuned adaptive control algorithms is perhaps not surprising, especially if we recall for comparison the ability of hybrid control laws to overcome obstructions to continuous stabilization (see Chapter 4).

Another important aspect of supervisory control is modularity: as we will see, the analysis of the overall system relies on certain basic properties of its individual parts, but not on the particular ways in which these parts are implemented. As a result, one gains the advantage of being able to use "off-the-shelf" control laws and estimators, rather than tailoring the design to the specifics of an adaptive algorithm. This provides greater flexibility in applications (where there is often pressure to utilize existing control structures) and facilitates the use of advanced techniques for difficult problems.

Moreover, in many cases the stability analysis of the resulting switched system appears to be more tractable than that of the time-varying systems arising in conventional adaptive control. For example, if the process and the controllers are linear time-invariant, then on every interval between consecutive switching times the closed-loop system is linear time-invariant, regardless of the complexity of the estimation procedure and the switching mechanism. The analysis tools studied in Part II are directly applicable in this context.

Although we occasionally refer to concepts from adaptive control in order to highlight analogies and differences, the reader does not need to have any familiarity with adaptive control to be able to follow this chapter.

6.2 First pass: basic architecture

In this section we describe, in general terms, the basic building blocks of a supervisory control system. Let $\mathbb{P}$ be the uncertain process to be controlled, with input u and output y, possibly perturbed by a bounded disturbance input d and a bounded output noise n. We assume that the model of $\mathbb{P}$ is a member of some family of admissible process models

$$\mathcal{F} = \bigcup_{p \in \mathcal{P}} \mathcal{F}_p \tag{6.1}$$

where $\mathcal{P}$ is a compact index set and each $\mathcal{F}_p$ denotes a family of systems "centered" around some known *nominal* process model ν_p. We think of the set $\mathcal{P}$ as representing the range of parametric uncertainty while for each fixed $p \in \mathcal{P}$ the subfamily $\mathcal{F}_p$ accounts for unmodeled dynamics. A standard problem of interest is to design a feedback controller that achieves state or output regulation of $\mathbb{P}$.

Example 6.1 As a simple example that helps illustrate the ideas, consider the system

$$\dot{y} = y^2 + p^* u, \qquad y \in \mathbb{R}$$

where p^* is an unknown element of the set $\mathcal{P} := [-10, -0.1] \cup [0.1, 10]$. The origin is excluded from $\mathcal{P}$ to preserve controllability. We consider the case of no unmodeled dynamics, so that for each $p \in \mathcal{P}$ the subfamily $\mathcal{F}_p$ consists of just the nominal process model $\dot{y} = y^2 + pu$. Since the sign of p^* is unknown, it is intuitively clear that no single control law is capable of driving y to the origin and logical decisions are necessary. □

We thus consider a parameterized family of *candidate controllers* $\{\mathbb{C}_q : q \in \mathcal{Q}\}$, where $\mathcal{Q}$ is some index set, and switch among them on the basis of observed data. The understanding here is that the controller family is sufficiently rich so that every admissible process model from $\mathcal{F}$ can be stabilized by placing in the feedback loop the controller $\mathbb{C}_q$ for some $q \in \mathcal{Q}$. Figure 20 should only be regarded as a conceptual illustration. In practice it is undesirable to run each candidate controller separately, because this leads to the presence of out-of-the-loop signals which may blow up, and because for the case of an infinite set $\mathcal{Q}$ this is not implementable. Instead, one designs a suitable switched system (whose finite dimension is independent of the size of $\mathcal{Q}$) which is capable of generating all necessary control input signals.

It seems natural to identify the index sets $\mathcal{P}$ and $\mathcal{Q}$, although in some situations it may be advantageous to take $\mathcal{Q}$ to be different from $\mathcal{P}$. If $\mathcal{P}$ is a discrete set, there is no need to embed it in a continuum (as one would normally do in the context of adaptive control); instead, one can switch among a discrete family of corresponding controllers. If $\mathcal{P}$ is a continuum, then one has the choice of working with a continuum of controllers (e.g., by taking $\mathcal{Q} = \mathcal{P}$) or a discrete—typically finite—family of controllers. For the system in Example 6.1 we can let $\mathcal{Q} = \mathcal{P}$ and consider the static candidate control laws $u_p = -\frac{1}{p}(y^2 + y)$, $p \in \mathcal{P}$. However, it is possible to find just two control laws such that every process model is asymptotically stabilized (at least locally) by one of them. It is often easier to work with a finite controller family. In fact, one may even want to replace the set $\mathcal{P}$ by its finite subset and absorb the remaining parameter values into unmodeled dynamics. However, in this case one needs to ensure that the resulting control algorithm is sufficiently robust to cover all of the original admissible process models, which in general might not be true. We are thus beginning to see various design options and challenges associated with them. To remain focused, we put these issues aside for the moment.

The switching is orchestrated by a high-level supervisor. We now begin to describe how this task is performed. The supervisor consists of three subsystems, as shown in Figure 36. The first subsystem is called the *multi-estimator*. This is a dynamical system whose inputs are the input u and the output y of the process $\mathbb{P}$ and whose outputs are denoted by y_p, $p \in \mathcal{P}$. The understanding is that for each $p \in \mathcal{P}$, the signal y_p provides an approximation of y (in a suitable asymptotic sense) if $\mathbb{P}$ belongs to $\mathcal{F}_p$, no matter what control input u is applied to the process. In particular, we can require that y_p would converge to y asymptotically if $\mathbb{P}$ were equal to the nominal process model ν_p and there were no noise or disturbances. This property can be restated in terms of asymptotic convergence to zero of one of the *estimation errors*

$$e_p := y_p - y, \qquad p \in \mathcal{P}. \tag{6.2}$$

Consider again the system from Example 6.1. We can let the estimator equations be

$$\dot{y}_p = -(y_p - y) + y^2 + pu, \qquad p \in \mathcal{P}. \tag{6.3}$$

Then the estimation error $e_{p^*} = y_{p^*} - y$ satisfies $\dot{e}_{p^*} = -e_{p^*}$ and hence converges to zero exponentially fast, for an arbitrary control u.

One concern (as with the candidate controllers) is that realizing the multi-estimator simply as a parallel connection of individual estimator equations for $p \in \mathcal{P}$ is not efficient and actually impossible if $\mathcal{P}$ is an infinite set. The estimator equations (6.3) can be implemented differently as follows. Consider the system

$$\begin{aligned} \dot{z}_1 &= -z_1 + y + y^2 \\ \dot{z}_2 &= -z_2 + u \end{aligned} \tag{6.4}$$

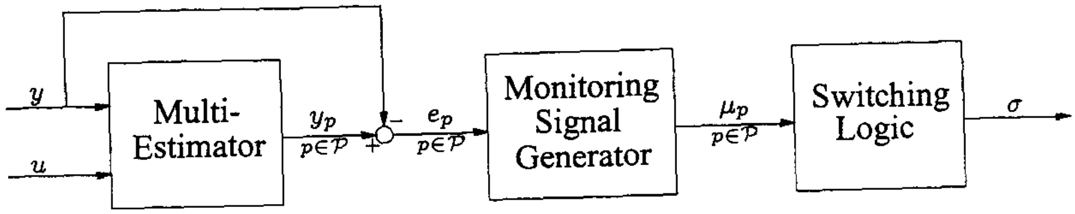

FIGURE 36. The structure of the supervisor

together with the outputs

$$y_p := z_1 + pz_2, \qquad p \in \mathcal{P}. \tag{6.5}$$

It is straightforward to check that these outputs satisfy the same equations (6.3), with the dynamics of the multi-estimator now being described by the two-dimensional system (6.4). This idea is known as *state sharing*. State sharing is always possible if the estimator equations are "affinely separable" in the unknown parameters, as in $\dot{y}_p = -\lambda y_p + f_1(p) f_2(y, u)$. Note that it is not directly relevant whether or not the unknown parameters enter linearly. The family of signals (6.5) is of course still infinite, but at each particular time we can look any one of them up or perform mathematical operations—such as computing the minimum—with the entire family.

If the multi-estimator is constructed properly, then the estimation error e_{p^*} is small in some sense, whereas there is no a priori reason for the other estimation errors to be small. Therefore, it seems intuitively reasonable (although not justified formally in any way) to pick as a current estimate of p^* the index of the smallest estimation error. Rather than basing decisions on the instantaneous values of the estimation errors, however, we would like to take their past behavior into account. Thus we need to implement an appropriate filter, which we call the *monitoring signal generator*. This is a dynamical system whose inputs are the estimation errors and whose outputs μ_p, $p \in \mathcal{P}$ are suitably defined integral norms of the estimation errors, called *monitoring signals*. For example, we can simply work with the squared $\mathcal{L}_2$ norm

$$\mu_p(t) := \int_0^t |e_p(\tau)|^2 d\tau, \qquad p \in \mathcal{P}. \tag{6.6}$$

These monitoring signals can be generated by the differential equations

$$\dot{\mu}_p = |e_p|^2, \qquad \mu_p(0) = 0, \quad p \in \mathcal{P}. \tag{6.7}$$

(This definition of the monitoring signals is actually not satisfactory because it does not involve any "forgetting factor" and the signals μ_p may grow unbounded; it will be refined later.)

Again, we do not want to generate each monitoring signal individually. The idea of state sharing can be applied here as well. To see how this works, let us revisit the multi-estimator for Example 6.1, given by (6.4) and (6.5).

Each estimation error can be equivalently expressed as $e_p = z_1 + pz_2 - y$, so that we have

$$e_p^2 = (z_1 - y)^2 + 2pz_2(z_1 - y) + p^2 z_2^2, \qquad p \in \mathcal{P}.$$

If we now define the monitoring signal generator via

$$\begin{aligned}
\dot\eta_1 &= (z_1 - y)^2 \\
\dot\eta_2 &= 2z_2(z_1 - y) \\
\dot\eta_3 &= z_2^2 \\
\mu_p &= \eta_1 + p\eta_2 + p^2\eta_3, \qquad p \in \mathcal{P}
\end{aligned}$$

then the equations (6.7) still hold.

Exercise 6.1 Suppose that we take the $\mathcal{L}_1$ norm instead of the squared $\mathcal{L}_2$ norm in (6.6). Can we still use state sharing?

The last component of the supervisor is the *switching logic*. This is a dynamical system whose inputs are the monitoring signals μ_p, $p \in \mathcal{P}$ and whose output is a piecewise constant *switching signal* σ, taking values in $\mathcal{Q}$. The switching signal determines the actual control law $u = u_\sigma$ applied to the process, where u_q, $q \in \mathcal{Q}$ are the control signals generated by the candidate controllers. The underlying strategy basically consists of selecting, from time to time, the candidate controller known to stabilize the process model whose corresponding monitoring signal is currently the smallest. Specific ways of doing this will be studied below. The resulting closed-loop system is a hybrid system, with discrete state σ.

The supervision paradigm outlined above is estimator-based, as opposed, for example, to approaches relying on a prespecified controller changing sequence. Another alternative, not considered here, is to incorporate an explicit performance-based criterion into the controller selection process.

6.3 An example: linear supervisory control

Consider the problem of driving to zero, by means of output feedback, the state of a stabilizable and detectable linear system

$$\begin{aligned}
\dot x &= A_{p^*}x + B_{p^*}u \\
y &= C_{p^*}x
\end{aligned} \tag{6.8}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^k$, $\{A_p, B_p, C_p : p \in \mathcal{P}\}$ is a given finite family of matrices, and $p^* \in \mathcal{P}$ is unknown.

If the value of p^* were available, then we could apply, for example, the standard observer-based linear dynamic output feedback law to stabilize the system. Since p^* is not known, we adopt the supervisory control approach. In this example we can use the Luenberger observer in designing both the multi-estimator and the candidate controllers. To this end, consider a family of observer-based estimators parameterized by $\mathcal{P}$ of the form

$$\begin{aligned} \dot{x}_p &= (A_p + K_p C_p)x_p + B_p u - K_p y \\ y_p &= C_p x_p \end{aligned} \tag{6.9}$$

and the corresponding candidate control laws

$$u_p = F_p x_p, \qquad p \in \mathcal{P}. \tag{6.10}$$

Here the matrices K_p and F_p are such that the eigenvalues of $A_p + K_p C_p$ and $A_p + B_p F_p$ have negative real parts for each $p \in \mathcal{P}$ (such matrices exist because each system in the family (6.8) is stabilizable and detectable by assumption). We also consider the estimation errors defined by the formulas (6.2) and the monitoring signals generated by the differential equations (6.7). Each $\mu_p(t)$ is thus given by the equation (6.6). We will design a hybrid feedback control law of the form $u = u_\sigma$, where $\sigma : [0, \infty) \to \mathcal{P}$ is a switching signal. In the present case this means $u(t) = F_{\sigma(t)} x_{\sigma(t)}$.

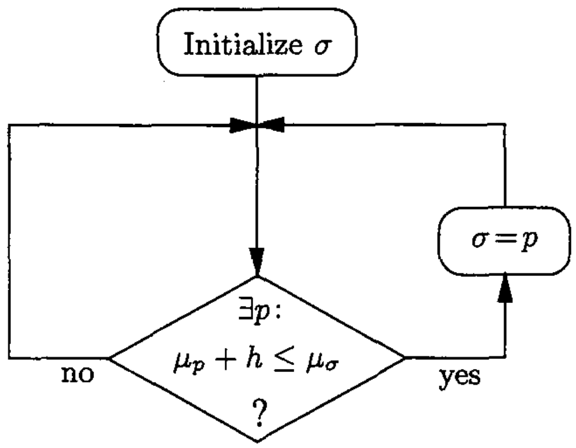

FIGURE 37. The hysteresis switching logic

One way to generate the switching signal σ is by means of the so-called *hysteresis switching logic*, illustrated via the computer-like diagram in Figure 37. Fix a positive number h called the *hysteresis constant.* Set $\sigma(0) = \arg\min_{p \in \mathcal{P}} \mu_p(0)$. Now, suppose that at a certain time σ has just switched to some $q \in \mathcal{P}$. The value of σ is then held fixed until we have $\min_{p \in \mathcal{P}} \mu_p(t) + h \le \mu_q(t)$. If and when that happens, we set σ equal to $\arg\min_{p \in \mathcal{P}} \mu_p(t)$. When the indicated $\arg\min$ is not unique, a particular value for σ among those that achieve the minimum can be chosen arbitrarily. Repeating this procedure, we obtain a piecewise constant switching signal which is continuous from the right everywhere.

It follows from (6.2), (6.8), and (6.9) that the estimation error e_{p^*} converges to zero exponentially fast, regardless of the control u that is applied.

The formula (6.6) then implies that $\mu_{p^*}(t)$ is bounded from above by some number K for all $t \geq 0$. In addition, all monitoring signals μ_p are nondecreasing by construction. Using these two facts and the definition of the hysteresis switching logic, it is not hard to prove that the switching must stop in finite time. Indeed, each μ_p has a limit (possibly ∞) as $t \to \infty$. Since $\mathcal{P}$ is finite, there exists a time T such that for each $p \in \mathcal{P}$ we either have $\mu_p(T) > K$ or $\mu_p(t_2) - \mu_p(t_1) < h$ for all $t_2 > t_1 \geq T$. Then for $t \geq T$ at most one more switch can occur. We conclude that there exists a time T^* such that $\sigma(t) = q^* \in \mathcal{P}$ for all $t \geq T^*$. Moreover, μ_{q^*} is bounded because μ_{p^*} is, hence $e_{q^*} \in \mathcal{L}_2$ by virtue of (6.6).

After the switching stops, the closed-loop system (excluding out-of-the-loop signals) can be written as

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{x}_{q^*} \end{pmatrix} &= \overline{A} \begin{pmatrix} x \\ x_{q^*} \end{pmatrix} \\ e_{q^*} &= \overline{C} \begin{pmatrix} x \\ x_{q^*} \end{pmatrix} \end{aligned} \tag{6.11}$$

where

$$\overline{A} := \begin{pmatrix} A_{p^*} & B_{p^*} F_{q^*} \\ -K_{q^*} C_{p^*} & A_{q^*} + K_{q^*} C_{q^*} + B_{q^*} F_{q^*} \end{pmatrix}$$

and

$$\overline{C} := \begin{pmatrix} -C_{p^*} & C_{q^*} \end{pmatrix}.$$

If we let

$$\overline{K} := \begin{pmatrix} K_{p^*} \\ K_{q^*} \end{pmatrix}$$

then it is straightforward to check that

$$\overline{A} - \overline{K}\,\overline{C} = \begin{pmatrix} A_{p^*} + K_{p^*} C_{p^*} & B_{p^*} F_{q^*} - K_{p^*} C_{q^*} \\ 0 & A_{q^*} + B_{q^*} F_{q^*} \end{pmatrix}.$$

The matrix on the right-hand side is Hurwitz, which shows that the system (6.11) is detectable with respect to e_{q^*}. It now remains to apply the standard output injection argument. Namely, write

$$\begin{pmatrix} \dot{x} \\ \dot{x}_{q^*} \end{pmatrix} = (\overline{A} - \overline{K}\,\overline{C}) \begin{pmatrix} x \\ x_{q^*} \end{pmatrix} + \overline{K} e_{q^*}$$

and observe that x and x_{q^*} converge to zero in view of stability of $\overline{A} - \overline{K}\,\overline{C}$ and the fact that $e_{q^*} \in \mathcal{L}_2$.

We emphasize that the particular choice of candidate control laws given by (6.10) is by no means crucial. Assume, for example, that every system in the family (6.8) is stabilizable by a static linear output feedback. In other words, assume that for each $p \in \mathcal{P}$ there exists a matrix G_p such that the eigenvalues of $A_p + B_p G_p C_p$ have negative real parts. A straightforward modification of the above argument shows that if we keep the estimators (6.9) but replace the control laws (6.10) by $u_p = G_p y$, we still achieve state regulation.

The above problem is rather special, and the solution and the method of proof have several drawbacks so they are only given here as an illustration of the main ideas. Nonetheless, the essential features of this example are present in the supervisory control strategies developed in the subsequent sections.

6.4 Second pass: design objectives

In this section we explain, informally, the basic requirements that need to be placed on the different parts of the supervisory control system. More precise descriptions of these requirements, and techniques for fulfilling them, will be discussed in Section 6.5. For simplicity, we first consider the case when the process parameter set $\mathcal{P}$ and the controller parameter set $\mathcal{Q}$ are equal, and after that explain what modifications are needed to handle the general situation.

As we already mentioned in Section 6.2, the multi-estimator should be designed so that each particular y_p provides a "good" approximation of the process output y—and therefore e_p is "small"—whenever the actual process model is inside the corresponding subfamily $\mathcal{F}_p$. Since the process is assumed to match one of the models in the family $\mathcal{F}$, we should then expect at least one of the estimation errors, say e_{p^*}, to be small in some sense. For example, we may require that in the absence of unmodeled dynamics, noise, and disturbances, e_{p^*} converge to zero exponentially fast for every control input u (cf. Section 6.3). It is also desirable to have an explicit characterization of e_{p^*} in the presence of unmodeled dynamics, noise, and disturbances.

The property that we impose on the candidate controllers is that for every fixed $q \in \mathcal{Q}$, the closed-loop system consisting of the process, the multi-estimator, and the controller $\mathbb{C}_q$ (in other words, the system obtained when the value of the switching signal is frozen at q) must be *detectable* with respect to the estimation error e_q. Detectability for systems with outputs can be defined in several different ways which are not equivalent in the nonlinear context, but it invariably means that the smallness of e_q in a suitable sense should imply the smallness of the state of the system. We already saw the relevance of detectability in Section 6.3.

There are two more required properties which concern the switching logic; they become especially significant when, unlike in Section 6.3, the switching does not stop. First, we need to have a bound on e_σ in terms of the smallest of the signals $e_p, p \in \mathcal{P}$. This bound is usually stated in terms of suitable "energy" expressions for the estimation errors, which are related to the monitoring signals. Second, the switching signal σ should preserve the detectability property of the closed-loop system, i.e., the overall switched system should remain detectable (in the same sense as in the previous paragraph) with respect to the output e_σ. These two properties are typically at odds: on one hand, to keep e_σ small we should switch to the index of the smallest estimation error; on the other hand, too much switching may destroy detectability.

Exercise 6.2 Consider a finite (or infinite but compact) family of detectable linear systems $\dot{x} = A_p x$, $y = C_p x$, $p \in \mathcal{P}$. Prove that for every switching signal σ with a sufficiently large (average) dwell time, the state of the switched linear system $\dot{x} = A_\sigma x$ converges to zero whenever the output $y = C_\sigma x$ converges to zero. What if the detectability hypothesis is strengthened to observability?

To summarize, the main requirements placed on the individual components of the supervisory control system can be qualitatively expressed as follows:

1. At least one of the estimation errors is small.
2. For each fixed controller, the closed-loop system is detectable through the corresponding estimation error.
3. The signal e_σ is bounded in terms of the smallest of the estimation errors.
4. The switched closed-loop system is detectable through e_σ provided that detectability holds for every frozen value of σ.

It is not difficult to see now, at least conceptually, how the above properties of the various blocks of the supervisory control system can be put together to analyze its behavior. Because of Property 1, there exists some $p^* \in \mathcal{P}$ for which e_{p^*} is small. Property 3 implies that e_σ is small. Properties 2 and 4 then guarantee that the state of the closed-loop system is small. Proceeding in this fashion, it is possible to analyze stability and robustness of supervisory control algorithms for quite general classes of uncertain systems, as will be shown below.

When the sets $\mathcal{P}$ and $\mathcal{Q}$ are different, we need to have a *controller assignment map* $\chi : \mathcal{P} \to \mathcal{Q}$. Let us say that a piecewise constant signal ζ taking values in $\mathcal{P}$ is *σ-consistent* (with respect to this map χ) if $\chi(\zeta(t)) = \sigma(t)$ for all t and the set of discontinuities of ζ is a subset of the set of discontinuities of σ (on a time interval of interest). More generally, we can consider a set-valued map $\chi : \mathcal{P} \to 2^{\mathcal{Q}}$, where $2^{\mathcal{Q}}$ stands for the set of all subsets of $\mathcal{Q}$, and replace the first property in the definition of σ-consistency by $\chi(\zeta(t)) \supset \sigma(t)$ for all t (this is natural when some process models are stabilized by more than one candidate controller, which is actually very often the case). Properties 2 and 4 need to be strengthened to guarantee that the closed-loop switched system is detectable with respect to the output e_ζ, for every σ-consistent signal ζ. Property 3, on the other hand, can be relaxed as follows: there must exist a σ-consistent signal ζ such that e_ζ is bounded in terms of the smallest of the signals e_p, $p \in \mathcal{P}$. Then the analysis sketched above goes through with suitable minor modifications, as we will see in Section 6.6.2.

Not surprisingly, the four properties that were just introduced for supervisory control have direct counterparts in conventional adaptive control. Property 1 is usually implicit in the derivation of error model equations, where one assumes that, for a specific value of the parameter, the output estimate matches the true output. Property 2 is known as *tunability*. Properties 3 and 4 are pertinent to the tuning algorithms, being typically stated in terms of the smallness (most often in the $\mathcal{L}_2$ sense) of the estimation error and the derivative of the parameter estimate, respectively.

6.5 Third pass: achieving the design objectives

6.5.1 *Multi-estimators*

Linear systems

As we saw in Section 6.3, the observer-based approach to the estimator design in the linear case yields exponential convergence of the estimation error associated with the actual unknown parameter value. We would like to have a bound on this estimation error in the presence of noise, disturbances, and unmodeled dynamics. Let us restrict our attention to single-input, single-output (SISO) controllable and observable linear processes. With some abuse of notation, we identify the process models with their transfer functions.

There are several ways to specify allowable unmodeled dynamics around the nominal process model transfer functions ν_p, $p \in \mathcal{P}$. For example, take two arbitrary numbers $\delta \geq 0$ and $\lambda_u > 0$. Then we can define

$$\mathcal{F}_p := \{\nu_p(1+\delta_p^m) + \delta_p^a : \|\delta_p^m\|_{\infty,\lambda_u} \leq \delta,\ \|\delta_p^a\|_{\infty,\lambda_u} \leq \delta\}, \qquad p \in \mathcal{P} \quad (6.12)$$

where $\|\cdot\|_{\infty,\lambda_u}$ denotes the $e^{\lambda_u t}$-weighted $\mathcal{H}_\infty$ norm of a transfer function: $\|\nu\|_{\infty,\lambda_u} = \sup_{\omega\in\mathbb{R}} |\nu(j\omega - \lambda_u)|$. Alternatively, one can define $\mathcal{F}_p$ to be the ball of radius δ around ν_p with respect to an appropriate metric. Another possible definition is

$$\mathcal{F}_p := \left\{ \frac{N_p + \delta_p^n}{M_p + \delta_p^m} : \|\delta_p^n\|_{\infty,\lambda_u} \leq \delta,\ \|\delta_p^m\|_{\infty,\lambda_u} \leq \delta \right\}, \qquad p \in \mathcal{P} \qquad (6.13)$$

where $\nu_p = N_p/M_p$ is a (normalized) coprime factorization of ν_p for each $p \in \mathcal{P}$; here N_p and M_p are stable transfer functions. This setting is more general than (6.12) in that it allows for uncertainty about the pole locations of the nominal process model transfer functions. With allowable unmodeled dynamics specified in one of the aforementioned ways, we will refer to the positive parameter δ as the *unmodeled dynamics bound.*

Coprime factorizations are actually suggestive of how one can go about designing the multi-estimator. Suppose that the admissible process models are described according to the formula (6.13). If there is a common upper bound k on the McMillan degrees of ν_p, $p \in \mathcal{P}$, then we can assume that for each $p \in \mathcal{P}$ the transfer functions N_p and M_p take the form $N_p = n_p/\beta$ and $M_p = m_p/\beta$, where n_p and m_p are polynomials and β is a fixed (independent of p) polynomial of degree k with all roots in the open left half-plane. Here and elsewhere, we denote by p^* an (unknown) element of $\mathcal{P}$ such that the transfer function of $\mathbb{P}$ belongs to $\mathcal{F}(p^*)$, i.e., the "true" parameter value; due to possibly overlapping sets of unmodeled dynamics, p^* may not be unique. The input-output behavior of the process is then defined by

$$y = \frac{N_{p^*} + \delta^n_{p^*}}{M_{p^*} + \delta^m_{p^*}}(u + d) + n.$$

We can rewrite this as

$$M_{p^*}y + \delta^m_{p^*}y = N_{p^*}u + \delta^n_{p^*}u + N_{p^*}d + \delta^n_{p^*}d + M_{p^*}n + \delta^m_{p^*}n$$

which is equivalent to

$$y = \frac{\beta - m_{p^*}}{\beta}y - \delta^m_{p^*}y + \frac{n_{p^*}}{\beta}u + \delta^n_{p^*}u + \frac{n_{p^*}}{\beta}d + \delta^n_{p^*}d + \frac{m_{p^*}}{\beta}n + \delta^m_{p^*}n.$$

We can now introduce the multi-estimator equations

$$y_p = \frac{\beta - m_p}{\beta}y + \frac{n_p}{\beta}u, \qquad p \in \mathcal{P}.$$

The last two equations imply

$$e_{p^*} = \delta^m_{p^*}y - \delta^n_{p^*}u - \frac{n_{p^*}}{\beta}d - \delta^n_{p^*}d - \frac{m_{p^*}}{\beta}n - \delta^m_{p^*}n.$$

We see, in particular, that if $n = d \equiv 0$ and $\delta = 0$, then e_{p^*} decays exponentially fast. This multi-estimator is readily amenable to a state-shared realization of the form

$$\begin{aligned}
\dot{z}_1 &= A_{\mathrm{E}}z_1 + b_{\mathrm{E}}u \\
\dot{z}_2 &= A_{\mathrm{E}}z_2 + b_{\mathrm{E}}y \\
y_p &= c_p^T z, \qquad p \in \mathcal{P}.
\end{aligned}$$

The above discussion is only given for the purpose of illustration, and we do not formalize it here. It turns out that, using standard techniques from realization theory, it is possible to design the multi-estimator in a state-shared fashion and guarantee that the following condition holds: for a sufficiently small $\lambda > 0$ there exist positive constants δ_1, δ_2 that only depend on the unmodeled dynamics bound δ and go to zero as δ goes to zero, positive constants B_1, B_2 that only depend on the noise and disturbance bounds and go to zero as these bounds go to zero, and positive constants C_1, C_2 that only depend on the system's parameters and on initial conditions, such that along all solutions of the closed-loop system we have

$$\int_0^t e^{2\lambda\tau} e_{p^*}^2(\tau)d\tau \leq B_1 e^{2\lambda t} + C_1 + \delta_1 \int_0^t e^{2\lambda\tau} u^2(\tau)d\tau \tag{6.14}$$

and

$$|e_{p^*}(t)| \leq B_2 + C_2 e^{-\lambda t} + \delta_2 e^{-\lambda t}\sqrt{\int_0^t e^{2\lambda\tau} u^2(\tau)d\tau}. \tag{6.15}$$

These two inequalities provide a precise quantitative description of Property 1 from Section 6.4.

Nonlinear systems

A general methodology for designing multi-estimators for nonlinear systems, which would enable one to extend the results available for the linear case, does not seem to exist. However, for certain classes of nonlinear systems it is not difficult to build multi-estimators that at least ensure exponential convergence of e_{p^*} to zero when there are no noise, disturbances, or unmodeled dynamics. We already discussed one such example in Section 6.2 (Example 6.1). In that example, the entire state of the process was available for measurement. The class of systems whose output and state are equal is actually quite tractable from the viewpoint of multi-estimator design. Indeed, if the process model takes the form

$$\dot{y} = f(y, u, p^*)$$

where $p^* \in \mathcal{P}$, then we can use the estimator equations

$$\dot{y}_p = A_p(y_p - y) + f(y, u, p), \qquad p \in \mathcal{P}$$

where $\{A_p : p \in \mathcal{P}\}$ is a set of Hurwitz matrices. This implies $\dot{e}_{p^*} = A_{p^*}e_{p^*}$ which gives the desired behavior. Using the idea of state sharing explained in Section 6.2, it is often possible to implement the above multi-estimator via a finite-dimensional system even if $\mathcal{P}$ is an infinite set.

Another case in which achieving exponential convergence of e_{p^*} (in the absence of noise, disturbances, and unmodeled dynamics) is straightforward is when the process is "output injection away" from an exponentially stable linear system. Namely, suppose that the process model belongs to a family of systems

$$\dot{x} = A_p x + f(C_p x, u, p)$$
$$y = C_p x$$

where $p \in \mathcal{P}$ and the matrices A_p , $p \in \mathcal{P}$ are Hurwitz. Then the estimator equations can be defined, for each $p \in \mathcal{P}$, as

$$\dot{x}_p = A_p x_p + f(y, u, p)$$
$$y_p = C_p x.$$

This multi-estimator design also applies in the case when each process model is modified by a state coordinate transformation, possibly dependent on p.

6.5.2 Candidate controllers

We now discuss Property 2 from Section 6.4 in more detail. This is a property of the system shown in Figure 38, where q is an arbitrary fixed element of $\mathcal{Q}$.

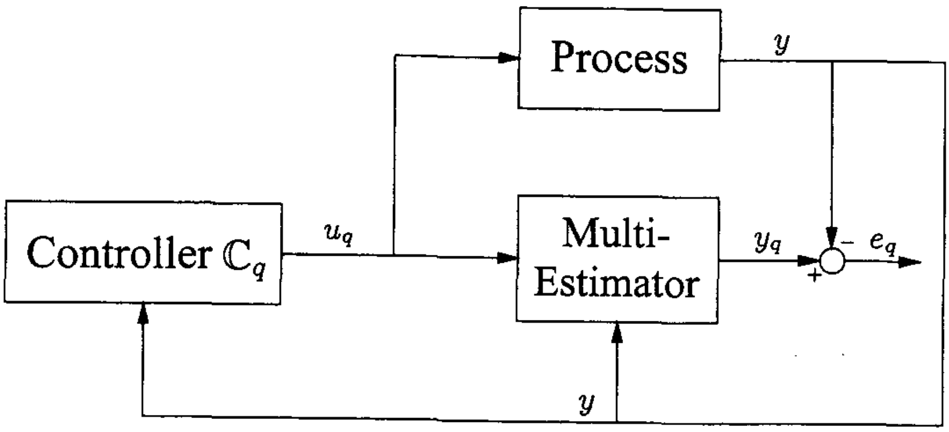

FIGURE 38. The closed-loop system (6.17)

Let us write the (state-shared) multi-estimator as

$$\dot{x}_{\mathrm{E}} = F(x_{\mathrm{E}}, y, u)$$
$$y_p = h_p(x_{\mathrm{E}}), \qquad p \in \mathcal{P}$$

where $h_p(0) = 0$ for each $p \in \mathcal{P}$. Here and below, all functions are assumed to be sufficiently regular (e.g., C^1).

The input to the controller $\mathbb{C}_q$ in the figure is the output y of the process. However, when writing state-space equations, we assume that $\mathbb{C}_q$ takes the more general form

$$\dot{x}_{\mathrm{C}} = g_q(x_{\mathrm{C}}, x_{\mathrm{E}}, e_q)$$
$$u = r_q(x_{\mathrm{C}}, x_{\mathrm{E}}, e_q)$$

with $r_q(0,0,0) = 0$. Assuming the entire state x_{E} of the multi-estimator to be available for control is reasonable, because the multi-estimator is implemented by the control designer. Since $y = h_q(x_{\mathrm{E}}) - e_q$ by virtue of (6.2), this set-up includes the situation depicted in the figure as a special case. A particular choice of inputs to the candidate controllers may vary.

A convenient way of thinking about the closed-loop system is facilitated by Figure 39. This figure displays the same system as Figure 38, but the block diagram is drawn differently in order to separate the process from the rest of the system. The subsystem enclosed in the dashed box is the feedback interconnection of the controller $\mathbb{C}_q$ and the multi-estimator, which we view as a system with input e_q. This system is called the *injected system.* Its state-space description is

$$\begin{aligned} \dot{x}_{\mathrm{C}} &= g_q(x_{\mathrm{C}}, x_{\mathrm{E}}, e_q) \\ \dot{x}_{\mathrm{E}} &= F(x_{\mathrm{E}}, h_q(x_{\mathrm{E}}) - e_q, r_q(x_{\mathrm{C}}, x_{\mathrm{E}}, e_q)). \end{aligned} \tag{6.16}$$

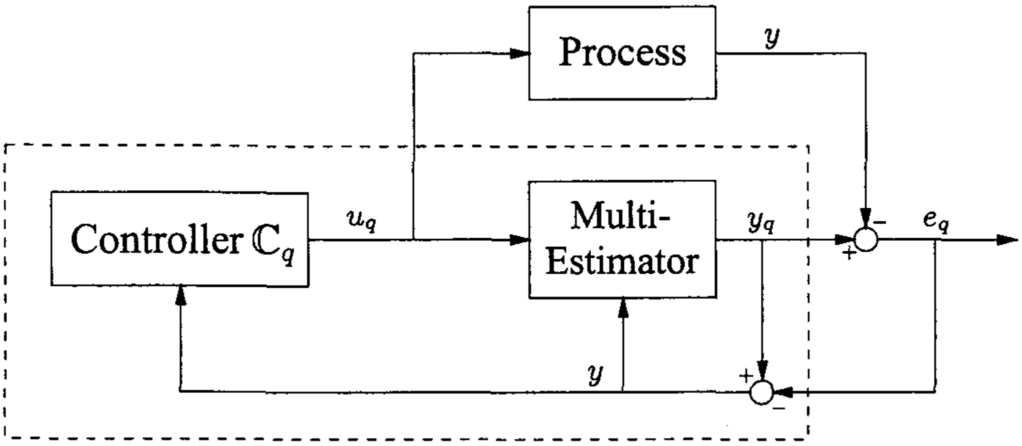

FIGURE 39. A different representation of the closed-loop system (6.17)

If the uncertain process $\mathbb{P}$ has dynamics of the form

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned}$$

then the overall closed-loop system shown in the figures is given by

$$\begin{aligned} \dot{x} &= f(x, r_q(x_{\mathrm{C}}, x_{\mathrm{E}}, h_q(x_{\mathrm{E}}) - h(x))) \\ \dot{x}_{\mathrm{C}} &= g_q(x_{\mathrm{C}}, x_{\mathrm{E}}, h_q(x_{\mathrm{E}}) - h(x)) \\ \dot{x}_{\mathrm{E}} &= F(x_{\mathrm{E}}, h(x), r_q(x_{\mathrm{C}}, x_{\mathrm{E}}, h_q(x_{\mathrm{E}}) - h(x))). \end{aligned} \tag{6.17}$$

Let us denote the state $(x; x_{\mathrm{C}}; x_{\mathrm{E}})$ of this system by $\mathbf{x}$ and assign the output of this system to be the estimation error $e_q = h_q(x_{\mathrm{E}}) - h(x)$. We would like to establish conditions under which the above system is detectable with respect to e_q. For linear systems, one set of such conditions is provided by the so-called *certainty equivalence stabilization theorem* from adaptive control. Adopted to the present context, this theorem says that the closed-loop system is detectable through the estimation error if the controller (internally) asymptotically stabilizes the multi-estimator and the process is detectable.

The next result is a natural generalization of the certainty equivalence stabilization theorem to nonlinear systems. Detectability now has to be interpreted in a nonlinear sense, and the effect of the input e_q on the injected system needs to be explicitly taken into account. See Section A.6 for precise definitions of the properties used below.

Theorem 6.1 *Assume that the process* $\mathbb{P}$ *is input/output-to-state stable (IOSS).*

1. *If the injected system* (6.16) *is input-to-state stable (ISS) with respect to* e_q*, then the closed-loop system* (6.17) *is output-to-state stable (OSS) with respect to* e_q.

2. *If the injected system* (6.16) *is integral-input-to-state stable (iISS) with respect to* e_q*, then the closed-loop system* (6.17) *is integral-input-to-state stable (iOSS) with respect to* e_q.

SKETCH OF PROOF. The result can be proved by straightforward (although tedious) manipulations, using the definitions of the relevant input-to-state and output-to-state properties. We do not give a complete proof here, but provide an informal signal-chasing argument which should convince the reader that the result is true. What we need to show is that if e_q is small, then $\mathbf{x}$ is small (here the smallness of e_q is defined in the supremum norm sense or in an integral sense, while the smallness of $\mathbf{x}$ is always understood in the supremum norm sense). So, assume that e_q is small. The standing assumption on the injected system implies that $x_{\mathbb{E}}$ and $x_{\mathbb{C}}$ are small, hence so are u and y_q. It follows that $y = y_q - e_q$ is also small. Finally, use the detectability property of the process to deduce that x is small. □

Exercise 6.3 Write down the injected system for the process from Example 6.1 and the candidate controllers and the multi-estimator proposed for that example. Show that it has the iISS property required in the second statement of Theorem 6.1. (Hint: start by working with (6.3) rather than (6.4), and use Lemma A.4 and the fact that the state of an exponentially stable linear system with an $\mathcal{L}_1$ input converges to zero.)

Conditions such as those provided by Theorem 6.1 are very useful because they decouple the properties that need to be satisfied by the parts of the system constructed by the designer from the properties of the unknown process. Note that the above result does not rely on any explicit assumptions regarding the structure of the process modeling uncertainty.

As a direct extension, one can treat the situation where the process has external disturbance inputs, as long as the IOSS assumption is still satisfied. By virtue of Theorem 6.1, the design of candidate controllers is reduced to the (integral-)ISS disturbance attenuation problem, which has been studied in the nonlinear control literature. One aspect that substantially simplifies this problem in the current setting is that the values of the disturbance (whose role is played by the estimation error e_q) are known and can be used by the controller.

Since (integral-)input-to-state stabilization of the multi-estimator is often a challenging task, it is of interest to obtain alternative conditions for detectability, which demand less from the injected system at the expense of more restrictive hypotheses on the process. A result along these lines, known as the *certainty equivalence output stabilization theorem*, is used in model reference adaptive control. Here it can be stated as follows: in the linear case, the detectability property of the closed-loop system still holds if the controller only asymptotically output-stabilizes the multi-estimator (with respect to the output y_q), the controller and the multi-estimator are detectable, and the process is minimum-phase. To see the reason behind this, assume that e_q is small (or just set $e_q = 0$). Then y_q, and consequently $y = y_q - e_q$, are small in view of output stability of the injected system. From the minimum-phase property of the process it follows that x and u are also small. Detectability of the controller and the multi-estimator now imply that x_{E} and x_{C} are small as well.

There is a nonlinear counterpart of the certainty equivalence output stabilization theorem, which relies on a variant of the minimum-phase property for nonlinear systems and involves several other concepts. Its discussion is beyond the scope of this book.

6.5.3 *Switching logics*

The sets $\mathcal{P}$ and $\mathcal{Q}$ are the same

Let us first consider the case when $\mathcal{P} = \mathcal{Q}$. As we mentioned in Section 6.2, the intuitive goal of the switching logic is to realize the relationship

$$\sigma(t) = \arg\min_{p\in\mathcal{P}} \mu_p(t) \tag{6.18}$$

because the right-hand side can be viewed as an estimate of the actual parameter value p^*. However, there is no guarantee that the estimates converge anywhere, let alone to the correct value (note, in particular, that no persistently exciting probing signals are being used). Moreover, defining σ according to (6.18) may lead to chattering. A better understanding of how the switching logic should be designed can be reached through examining Properties 3 and 4 of Section 6.4.

Suppose for the moment that σ is defined by (6.18), so that we have $\mu_\sigma(t) \le \mu_p(t)$ for all t and all $p \in \mathcal{P}$. Using this inequality, it is often possible to arrive at a suitable representation of Property 3. For example, take $\mathcal{P}$ to be a finite set with m elements and the monitoring signals to be the squared $\mathcal{L}_2$ norms of the estimation errors as in (6.6). Ignoring the chattering issue, suppose that σ is a piecewise constant switching signal. Then Property 3 is satisfied in the $\mathcal{L}_2$ sense, namely,

$$\int_0^t |e_\sigma(\tau)|^2 d\tau \leq m \min_{p\in\mathcal{P}} \int_0^t |e_p(\tau)|^2 d\tau.$$

(A more general result along these lines for hysteresis-based switching will be proved shortly.)

Ensuring Property 4, on the other hand, is more problematic. Just as stability may be lost due to switching, detectability for systems with outputs may be destroyed by switching; for example, just think of asymptotically stable linear systems with zero outputs which do not share a common Lyapunov function. The simplest way around this problem is to guarantee that the switching either stops in finite time (as in Section 6.3) or is slow enough. In the linear case, detectability under sufficiently slow switching among detectable systems can be shown with the help of the standard output injection argument (see Exercise 6.2). Alternatively, one can ensure that the switching does not destabilize the injected system (i.e., the interconnection of the switched controller and the multi-estimator), in which case detectability of the closed-loop system can be established as in Section 6.5.2. This latter line of reasoning will be implicitly used in the analysis given in Section 6.6. (Instead of relying on slow switching conditions, one may be able to utilize specific structure of the systems being switched, but this approach is not pursued here.)

There are essentially two ways to slow the switching down. One is to introduce a dwell time τ_d and set σ equal to the index of the smallest monitoring signal only every τ_d units of time. This results in the *dwell-time switching logic*, which is illustrated by Figure 40. Under suitable assumptions, Property 4 is automatic if the dwell time τ_d is chosen to be sufficiently large. Characterizing Property 3 is more difficult, because if the switching is only allowed at event times separated by τ_d, then μ_σ is not always guaranteed to be small compared to the other monitoring signals. Although there are ways to handle this problem, the above observation actually reveals a significant disadvantage of dwell-time switching. With a prespecified dwell time, the performance of the currently active controller might deteriorate to an unacceptable level before the next switch is permitted. If the uncertain process is nonlinear, the trajectories may even escape to infinity in finite time! In view of these considerations, we drop the discussion of the dwell-time switching logic altogether and explore an alternative direction.

A different way to slow the switching down is by means of hysteresis. We already explored this idea in Section 6.3. Hysteresis means that we do not switch every time $\min_{p\in\mathcal{P}} \mu_p(t)$ becomes smaller than $\mu_\sigma(t)$, but switch only when it becomes "significantly" smaller. The threshold

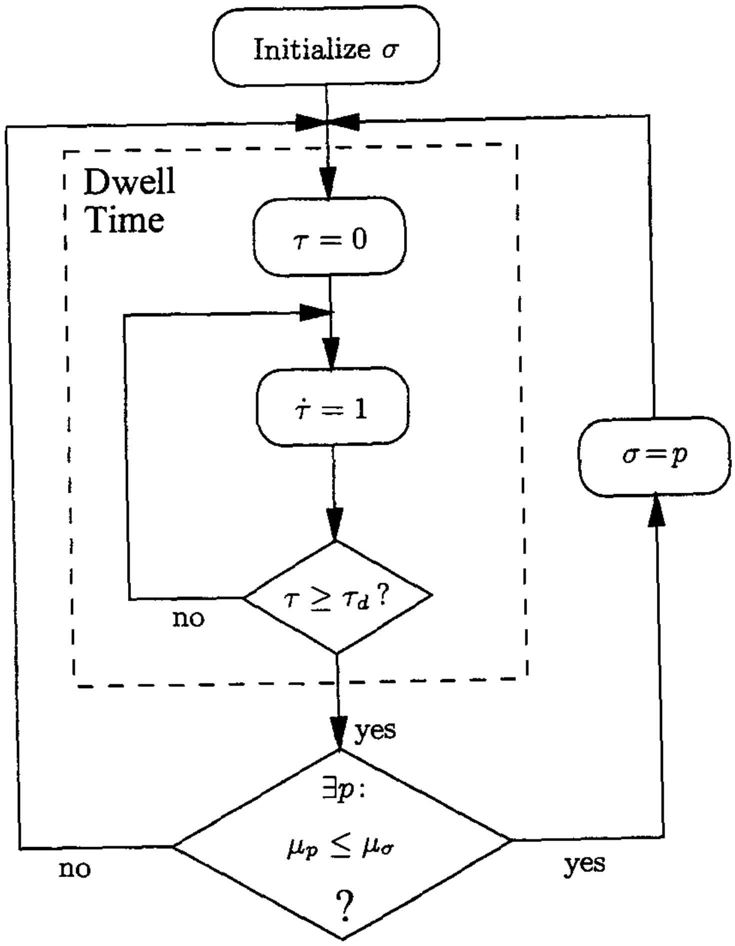

FIGURE 40. The dwell-time switching logic

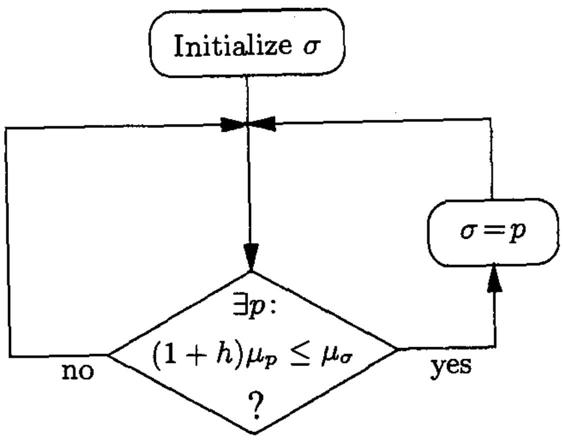

FIGURE 41. The scale-independent hysteresis switching logic

of tolerance is determined by a *hysteresis constant* $h > 0$. Let $\sigma(0) = \arg\min_{p\in\mathcal{P}} \mu_p(0)$. Suppose that at a certain time t_i the value of σ has just switched to some $q \in \mathcal{P}$. We then keep σ fixed until a time $t_{i+1} > t_i$ such that $(1+h)\min_{p\in\mathcal{P}} \mu_p(t_{i+1}) \le \mu_q(t_{i+1})$, at which point we let $\sigma(t_{i+1}) = \arg\min_{p\in\mathcal{P}} \mu_p(t_{i+1})$. The accompanying diagram is shown in Figure 41. When the indicated arg min is not unique, a particular candidate for the value of σ is selected arbitrarily. Repeating the above steps, we generate a piecewise constant switching signal σ which is continuous from the right everywhere. If the signals μ_p, $p \in \mathcal{P}$ are uniformly bounded away from zero, i.e., if for some $\varepsilon_\mu > 0$ we have $\mu_p(t) \ge \varepsilon_\mu$ for all $p \in \mathcal{P}$ and all $t \ge 0$, then chattering is avoided: there can only be a finite number of switches on every bounded time interval.[1]

The only difference between this switching logic and the one used in Section 6.3 is that multiplicative hysteresis is used here instead of additive hysteresis. The advantage gained is that with multiplicative hysteresis, the output σ of the switching logic would not be affected if we replaced the signals μ_p, $p \in \mathcal{P}$ by their scaled versions

$$\bar{\mu}_p(t) := \Theta(t)\mu_p(t), \qquad p \in \mathcal{P} \tag{6.19}$$

where Θ is some positive function of time. For this reason, the above switching logic is called the *scale-independent hysteresis switching logic*. For analysis purposes, it is very useful to work with the scaled signals $\bar{\mu}_p$, $p \in \mathcal{P}$, where Θ is chosen so that these signals have some additional properties (such as monotonicity). The actual monitoring signals being implemented and used for the switching logic are still the original ones μ_p, $p \in \mathcal{P}$ (because the presence of monotonically increasing signals in a supervisory control system is undesirable).

Unlike with the dwell-time switching logic, a slow switching property is not explicitly ensured by the presence of hysteresis and needs to be proved. Such a property will follow from the first statement of the next lemma. The second statement of the lemma provides a basis for characterizing Property 3 for the scale-independent hysteresis switching logic. (The precise connection will become clear later; here the result is stated as a general property of the switching logic, independent of a particular way in which the monitoring signals are obtained from the estimation errors.) As in Section 3.2.2, we denote by $N_\sigma(t, t_0)$ the number of discontinuities of σ on an interval (t_0, t).

Lemma 6.2 *Let $\mathcal{P}$ be a finite set consisting of m elements. Suppose that the signals $\bar{\mu}_p$, $p \in \mathcal{P}$ are continuous and monotonically nondecreasing, and*

---

[1] To be precise, σ is defined on some maximal interval $[0, T_{\max})$, where $T_{\max} \le \infty$. However, in the supervisory control systems studied below we will always have $T_{\max} = \infty$. We will not mention this fact explicitly, but it can be easily established.

that there exists a number $\varepsilon_\mu > 0$ such that $\bar{\mu}_p(0) \geq \varepsilon_\mu$ for all $p \in \mathcal{P}$. Then for every index $l \in \mathcal{P}$ and arbitrary numbers $t > t_0 \geq 0$ we have

$$N_\sigma(t,t_0) \leq 1 + m + \frac{m}{\log(1+h)} \log\left(\frac{\bar{\mu}_l(t)}{\min_{p\in\mathcal{P}} \bar{\mu}_p(t_0)}\right) \tag{6.20}$$

and

$$\sum_{k=0}^{N_\sigma(t,t_0)} \Big(\bar{\mu}_{\sigma(t_k)}(t_{k+1}) - \bar{\mu}_{\sigma(t_k)}(t_k)\Big) \leq m\Big((1+h)\bar{\mu}_l(t) - \min_{p\in\mathcal{P}} \bar{\mu}_p(t_0)\Big) \tag{6.21}$$

where $t_1 < t_2 < \cdots < t_{N_\sigma(t,t_0)}$ are the discontinuities of σ on (t_0, t) and $t_{N_\sigma(t,t_0)+1} := t$.

Remark 6.1 The left-hand side of the inequality (6.21) can be thought of as the variation of $\bar{\mu}_\sigma$ over the interval $[t_0, t]$. If the signals $\bar{\mu}_p$, $p \in \mathcal{P}$ are absolutely continuous, then the left-hand side of (6.21) equals the integral $\int_{t_0}^t \dot{\bar{\mu}}_{\sigma(\tau)}(\tau)d\tau$, which is to be interpreted as the sum of integrals over intervals on which σ is constant. □

Proof of Lemma 6.2. For each $k \in \{0, 1, \ldots, N_\sigma(t,t_0)\}$, the value of σ remains constant and equal to $\sigma(t_k)$ on $[t_k, t_{k+1})$. Taking continuity of $\bar{\mu}_{\sigma(t_k)}$ into account, we have

$$\bar{\mu}_{\sigma(t_k)}(t) \leq (1+h)\bar{\mu}_p(t) \quad \forall t \in [t_k, t_{k+1}],\ \forall k \in \{0, 1, \ldots, N_\sigma(t,t_0)\},\ \forall p \in \mathcal{P}. \tag{6.22}$$

Since σ switched to $\sigma(t_k)$ at time t_k, $k \in \{1, 2, \ldots, N_\sigma(t,t_0)\}$, we also have

$$\bar{\mu}_{\sigma(t_k)}(t_k) \leq \bar{\mu}_p(t_k) \qquad \forall k \in \{1, 2, \ldots, N_\sigma(t,t_0)\},\quad \forall p \in \mathcal{P}. \tag{6.23}$$

Moreover, σ switched from $\sigma(t_k)$ to $\sigma(t_{k+1})$ at time t_{k+1}, $k < N_\sigma(t,t_0) - 1$, which implies that

$$\bar{\mu}_{\sigma(t_k)}(t_{k+1}) = (1+h)\bar{\mu}_{\sigma(t_{k+1})}(t_{k+1}) \quad \forall k \in \{0, 1, \ldots, N_\sigma(t,t_0) - 1\}. \tag{6.24}$$

Since $\mathcal{P}$ has m elements, there must be a $q \in \mathcal{P}$ such that $\sigma = q$ on at least[2] $\overline{N} := \left\lceil \frac{N_\sigma(t,t_0)-1}{m} \right\rceil$ of the intervals

$$[t_1, t_2),\ [t_2, t_3),\ \ldots,\ [t_{N_\sigma(t,t_0)-1}, t_{N_\sigma(t,t_0)}). \tag{6.25}$$

If $\overline{N} \leq 1$, then we must have $N_\sigma(t,t_0) \leq 1 + m$ and therefore (6.20) automatically holds. Suppose now that $\overline{N} \geq 2$ and let

$$[t_{k_1}, t_{k_1+1}),\ [t_{k_2}, t_{k_2+1}),\ \cdots,\ [t_{k_{\overline{N}}}, t_{k_{\overline{N}}+1})$$

[2]Given a scalar a, we denote by $\lceil a \rceil$ the smallest integer larger than or equal to a.

be $\overline{N}$ intervals on which $\sigma = q$. Pick an arbitrary $i \in \{1, 2, \ldots, \overline{N} - 1\}$. In view of (6.24), the assumed monotonicity of $\bar{\mu}_{\sigma(t_{k_i+1})}$, and (6.23) we conclude that

$$\bar{\mu}_q(t_{k_i+1}) = (1+h)\bar{\mu}_{\sigma(t_{k_i+1})}(t_{k_i+1}) \geq (1+h)\bar{\mu}_{\sigma(t_{k_i+1})}(t_{k_i}) \geq (1+h)\bar{\mu}_q(t_{k_i}). \tag{6.26}$$

Since the intervals are nonoverlapping, we know that $t_{k_{i+1}} \geq t_{k_i+1}$, hence $\bar{\mu}_q(t_{k_{i+1}}) \geq \bar{\mu}_q(t_{k_i+1})$. From this and (6.26) we obtain

$$(1+h)\bar{\mu}_q(t_{k_i}) \leq \bar{\mu}_q(t_{k_{i+1}}), \qquad i \in \{1, 2, \ldots, \overline{N} - 1\}.$$

Iterating the above inequality from $i = 1$ to $i = \overline{N} - 1$ yields

$$(1+h)^{\overline{N}-1}\bar{\mu}_q(t_{k_1}) \leq \bar{\mu}_q(t_{k_{\overline{N}}})$$

and therefore

$$(1+h)^{\overline{N}-1}\bar{\mu}_q(t_{k_1}) \leq \bar{\mu}_l(t_{k_{\overline{N}}}) \qquad \forall l \in \mathcal{P}$$

because of (6.23). Using monotonicity of $\bar{\mu}_q$, we conclude that

$$(1+h)^{\overline{N}-1}\bar{\mu}_q(t_0) \leq \bar{\mu}_l(t) \qquad \forall l \in \mathcal{P}$$

and therefore

$$(1+h)^{\overline{N}-1}\min_{p\in\mathcal{P}}\bar{\mu}_p(t_0) \leq \bar{\mu}_l(t) \qquad \forall l \in \mathcal{P}$$

from which the inequality (6.20) follows by virtue of the fact that $\overline{N} \geq \frac{N_\sigma(t,t_0)-1}{m}$.

It remains to show that the inequality (6.21) holds. Grouping all terms in the summation on the left-hand side of (6.21) for which $\sigma(t_k)$ is the same, we obtain

$$\sum_{k=0}^{N_\sigma(t,t_0)} \Big(\bar{\mu}_{\sigma(t_k)}(t_{k+1}) - \bar{\mu}_{\sigma(t_k)}(t_k)\Big) = \sum_{q\in\mathcal{P}} \sum_{\substack{k=0 \\ \sigma(t_k)=q}}^{N_\sigma(t,t_0)} \Big(\bar{\mu}_q(t_{k+1}) - \bar{\mu}_q(t_k)\Big). \tag{6.27}$$

Take some value $q \in \mathcal{P}$ that σ takes on the interval (t_0, t). Since the intervals (6.25) are nonoverlapping, it follows from monotonicity of $\bar{\mu}_q$ that

$$\sum_{\substack{k=0 \\ \sigma(t_k)=q}}^{N_\sigma(t,t_0)} \Big(\bar{\mu}_q(t_{k+1}) - \bar{\mu}_q(t_k)\Big) \leq \bar{\mu}_q(t_{k_q+1}) - \bar{\mu}_q(t_0)$$

where k_q denotes the largest index $k \in \{0, 1, \ldots, N_\sigma(t, t_0)\}$ for which $\sigma(t_k) = q$. By virtue of (6.22), we have

$$\bar{\mu}_q(t_{k_q+1}) \leq (1+h)\bar{\mu}_l(t_{k_q+1}) \qquad \forall l \in \mathcal{P}.$$

Using monotonicity again, we conclude that

$$\sum_{\substack{k=0\\ \sigma(t_k)=q}}^{N_\sigma(t,t_0)} \Big(\bar{\mu}_q(t_{k+1}) - \bar{\mu}_q(t_k)\Big) \le (1+h)\bar{\mu}_l(t) - \min_{p\in\mathcal{P}} \bar{\mu}_p(t_0) \qquad \forall\, l\in\mathcal{P}.$$

The inequality (6.21) now follows from (6.27) and the fact that $\mathcal{P}$ has m elements. □

If at least one of the signals $\bar{\mu}_p$, $p\in\mathcal{P}$, say $\bar{\mu}_{p^*}$, is bounded, then the inequality (6.20) applied with $l = p^*$ implies that $N_\sigma(t,t_0)$ is upper-bounded by a fixed constant for all $t > t_0 \ge 0$. This means that the switching stops in finite time, i.e., there exist a time T^* and an index $q^*\in\mathcal{P}$ such that $\sigma(t) = q^*$ for $t \ge T^*$. Moreover, the signal $\bar{\mu}_{q^*}$ is bounded, as is apparent from the definition of the switching logic and also from the formula (6.21). This special case of Lemma 6.2 will be used several times below, so we state it formally as a corollary. (Its counterpart for the case of additive hysteresis was already used in Section 6.3.)

Corollary 6.3 *If the hypotheses of Lemma 6.2 are satisfied and there exists an index $p^*\in\mathcal{P}$ such that the signal $\bar{\mu}_{p^*}$ is bounded, then the switching stops in finite time at some index $q^*\in\mathcal{Q}$ and the signal $\bar{\mu}_{q^*}$ is bounded.*

Exercise 6.4 Prove Corollary 6.3 directly (without using Lemma 6.2).

Since the scale-independent hysteresis switching logic is defined for an arbitrary $\mathcal{P}$, it seems natural to ask whether Lemma 6.2 is also valid when $\mathcal{P}$ is an infinite compact set. The answer to this question is negative, as demonstrated by the following counterexample. Let

$$\bar{\mu}_p(t) := \begin{pmatrix} p & 1-p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix}, \qquad p\in[0,1].$$

These signals have all of the properties required by Lemma 6.2. (Signals of this form arise in linear supervisory control; see Section 6.6.) Note that $\bar{\mu}_1$ is bounded while $\bar{\mu}_p$, $p<1$ are all unbounded. Moreover,

$$\arg\min_{p\in\mathcal{P}} \bar{\mu}_p(t) = \frac{e^t}{1+e^t} < 1 \qquad \forall\, t\ge 0.$$

It is easy to deduce from these facts that the switching never stops. In view of Corollary 6.3, this implies that Lemma 6.2 does not hold for infinite $\mathcal{P}$.

The sets $\mathcal{P}$ and $\mathcal{Q}$ are different

The hysteresis switching paradigm can also be applied in situations where $\mathcal{P}\neq\mathcal{Q}$. For example, a switching signal can be generated by means of composing the scale-independent hysteresis switching logic described above with a controller assignment map $\chi : \mathcal{P} \to \mathcal{Q}$. In other words, we can let $\sigma = \chi(\zeta)$, where ζ is the output of the scale-independent hysteresis switching logic. However, the question arises as to whether or not a switching signal generated in this way still has desired properties. The situation of particular interest is when $\mathcal{P}$ is a continuum and $\mathcal{Q}$ is a finite set (see the discussion in Section 6.2). Since Lemma 6.2 does not apply when the set $\mathcal{P}$ is infinite, some modifications to the switching logic are in fact necessary. One way to proceed is described next

Assume that we are given a family of closed subsets $\mathcal{D}_q, q \in \mathcal{Q}$ of $\mathcal{P}$, whose union is the entire $\mathcal{P}$. Pick a hysteresis constant $h > 0$. First, we select some $q_0 \in \mathcal{Q}$ such that $\mathcal{D}_{q_0}$ contains $\arg\min_{p\in\mathcal{P}} \mu_p(0)$, and set $\sigma(0) = q_0$. Suppose that at a certain time t_i the value of σ has just switched to some $q_i \in \mathcal{Q}$. We then keep σ fixed until a time $t_{i+1} > t_i$ such that the following inequality is satisfied:

$$(1+h)\min_{p\in\mathcal{P}} \mu_p(t_{i+1}) \le \min_{p\in\mathcal{D}_{q_i}} \mu_p(t_{i+1}).$$

At this point, we select some $q_{i+1} \in \mathcal{Q}$ such that the set $\mathcal{D}_{j+1}$ contains $\arg\min_{p\in\mathcal{P}} \mu_p(t_{i+1})$, and set $\sigma(t_{i+1}) = q_{i+1}$. See Figure 42.

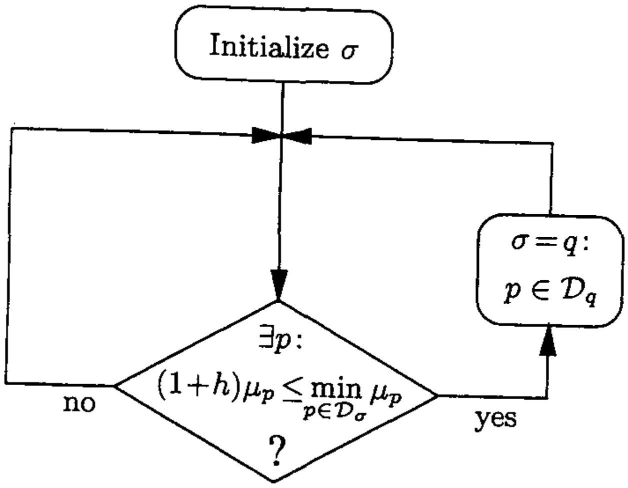

FIGURE 42. The hierarchical hysteresis switching logic

The above procedure yields a piecewise constant switching signal σ which is continuous from the right everywhere. We call this switching logic the *hierarchical hysteresis switching logic*. The name is motivated by the fact that the minimization of the monitoring signals is carried out on two levels: first, the smallest one is taken for each of the subsets that form the partition of $\mathcal{P}$, and then the smallest signal among these is chosen. This switching logic is also scale-independent, i.e., its output would not be affected if we replaced the signals μ_p, $p \in \mathcal{P}$ by their scaled versions (6.19), where Θ is some positive function of time. In the supervisory control context, we will arrange matters in such a way that a suitable function Θ makes the scaled signals $\bar{\mu}_p$, $p \in \mathcal{P}$ strictly positive and monotonically nondecreasing. For analysis purposes we will always use the scaled signals $\bar{\mu}_p$, $p \in \mathcal{P}$ with these properties, while the actual inputs to the switching logic are the original signals μ_p, $p \in \mathcal{P}$.

For each $p \in \mathcal{P}$, let $\chi(p)$ be the set of all $q \in \mathcal{Q}$ such that $p \in \mathcal{D}_q$. Recall that in Section 6.4 we introduced the notion of a σ-consistent signal, in terms of a switching signal σ and a (possibly set-valued) controller assignment map χ. In the present context, a piecewise constant signal ζ taking values in $\mathcal{P}$ is σ-consistent on an interval $[t_0, t]$ if and only if

1. For all $s \in [t_0, t]$ we have $\zeta(s) \in \mathcal{D}_{\sigma(s)}$.
2. The set of discontinuities of ζ on $[t_0, t]$ is a subset of the set of discontinuities of σ.

We now note the following fact, which follows immediately from the definitions of the switching logics given in this section.

Remark 6.2 The signal σ produced by the hierarchical hysteresis switching logic coincides with the signal that would be produced by the scale-independent hysteresis switching logic with inputs $\min_{p\in\mathcal{D}_q} \mu_p(t)$, $q \in \mathcal{Q}$. □

The following counterpart of Lemma 6.2 is a consequence of this observation. As before, $N_\sigma(t, t_0)$ denotes the number of discontinuities of σ on an interval (t_0, t).

Lemma 6.4 *Let $\mathcal{Q}$ be a finite set consisting of m elements. Suppose that the signals $\bar{\mu}_p$, $p \in \mathcal{P}$ are continuous and monotonically nondecreasing, and that there exists a number $\varepsilon_\mu > 0$ such that $\bar{\mu}_p(0) \geq \varepsilon_\mu$ for all $p \in \mathcal{P}$. Then, for every index $l \in \mathcal{P}$ and arbitrary numbers $t > t_0 \geq 0$ we have*

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m}{\log(1+h)} \log\left(\frac{\bar{\mu}_l(t)}{\min_{p\in\mathcal{P}} \bar{\mu}_p(t_0)}\right). \tag{6.28}$$

In addition, there exists a signal ζ which is σ-consistent on $[t_0, t]$ and such that

$$\sum_{k=0}^{N_\sigma(t,t_0)} \Big(\bar{\mu}_{\zeta(t_k)}(t_{k+1}) - \bar{\mu}_{\zeta(t_k)}(t_k)\Big) \leq m\Big((1+h)\bar{\mu}_l(t) - \min_{p\in\mathcal{P}} \bar{\mu}_p(t_0)\Big) \tag{6.29}$$

where $t_1 < t_2 < \cdots < t_{N_\sigma(t,t_0)}$ are the discontinuities of σ on (t_0, t) and $t_{N_\zeta(t,t_0)+1} := t$.

PROOF. The inequality (6.28) follows at once from Lemma 6.2 and Remark 6.2. A signal ζ that satisfies the second statement of the lemma can be defined as follows: for each $s \in [t_0, t]$, let $\zeta(s) := \arg\min_{p\in\mathcal{D}_{\sigma(s)}} \bar{\mu}_p(t_{k+1})$, where k is the largest index in the set $\{0, 1, ..., N_\sigma(t, t_0)\}$ for which $\sigma(t_k) =$

$\sigma(s)$. In other words, $\zeta(s) = \arg\min_{p\in\mathcal{D}_{\sigma(s)}} \bar{\mu}_p(\tau)$, where τ is the right endpoint of the last subinterval of $[t_0, t]$ on which σ equals $\sigma(s)$. Then ζ is σ-consistent on $[t_0, t]$ by construction. Grouping all terms in the summation on the left-hand side of (6.29) for which $\sigma(t_k)$ is the same, and reasoning exactly as in the proof of Lemma 6.2, we arrive at (6.29). □

Remark 6.3 The signal ζ depends on the choice of the time t. As before, if the signals $\bar{\mu}_p$, $p \in \mathcal{P}$ are absolutely continuous, then the left-hand side of the inequality (6.29) equals the integral $\int_{t_0}^{t} \dot{\bar{\mu}}_{\zeta(\tau)}(\tau)d\tau$, which is to be interpreted as the sum of integrals over intervals on which ζ is constant. □

We close this section by pointing out that the problem of computing $\arg\min_{p\in\mathcal{P}} \mu_p(t)$ requires attention. Carrying out the minimization over $\mathcal{P}$ is a trivial task if $\mathcal{P}$ is a finite set. If $\mathcal{P}$ is a continuum, in many cases of interest this problem reduces to solving a polynomial equation in p (see, e.g., Section 6.8 below). In the context of hierarchical hysteresis switching, the understanding is that minimization over $\mathcal{D}_q$, $q \in \mathcal{Q}$ is computationally tractable if these sets are sufficiently small.

6.6 Linear supervisory control revisited

Having discussed design objectives and ways to achieve them, we are now in position to present in some more detail and generality the analysis of supervisory control algorithms for uncertain linear systems. Suppose that the uncertain process $\mathbb{P}$ to be controlled admits the model of a SISO finite-dimensional stabilizable and detectable linear system, and that the modeling uncertainty is of the kind described in Section 6.2. The problem of interest is to design a feedback controller that performs state regulation, namely:

1. Ensures boundedness of all signals in response to arbitrary bounded noise and disturbance signals and sufficiently small unmodeled dynamics.
2. Drives the state x of $\mathbb{P}$ to zero whenever the noise and disturbance signals equal (or converge to) zero.

As q ranges over $\mathcal{Q}$, let

$$\begin{aligned} \dot{x}_C &= A_q x_C + b_q y \\ u &= k_q^T x_C + r_q y \end{aligned}$$

be realizations of the transfer functions of the candidate controllers $\mathbb{C}_q$, $q \in \mathcal{Q}$, all sharing the same state x_C. We assume that for each $p \in \mathcal{P}$ there exists an index $q \in \mathcal{Q}$ such that the controller $\mathbb{C}_q$ stabilizes the nominal process model ν_p with stability margin λ_0, where λ_0 is a fixed positive number; i.e., all closed-loop poles of the feedback interconnection of $\mathbb{C}_q$ with ν_p have real parts smaller than $-\lambda_0$. For each $p \in \mathcal{P}$, the set of such indices q is given by $\chi(p)$, where χ is a controller assignment map (see Section 6.4).

We assume that allowable unmodeled dynamics are described by the formula (6.12) or by the formula (6.13). Fix a number $\lambda \in (0, \min\{\lambda_u, \lambda_0\})$. Then it is possible to design the multi-estimator in a state-shared fashion, as given by

$$\begin{aligned} \dot{x}_{\mathrm{E}} &= A_{\mathrm{E}} x_{\mathrm{E}} + b_{\mathrm{E}} y + d_{\mathrm{E}} u \\ y_p &= c_p^T x_{\mathrm{E}}, \qquad p \in \mathcal{P} \end{aligned}$$

with A_{E} a Hurwitz matrix, which has the properties expressed by the inequalities (6.14) and (6.15). We henceforth assume that these properties hold.

Let $x_{\mathrm{CE}} := (x_{\mathrm{C}}; x_{\mathrm{E}})$. Using the definition (6.2) of the estimation errors, we can write the interconnection of the switched controller and the multi-estimator (i.e., the injected system) in the form

$$\dot{x}_{\mathrm{CE}} = A_{\sigma l} x_{\mathrm{CE}} + d_\sigma e_l \tag{6.30}$$

$$y = (0 \; c_{p^*}^T)\, x_{\mathrm{CE}} - e_{p^*} \tag{6.31}$$

$$u = f_\sigma^T x_{\mathrm{CE}} + g_\sigma e_{p^*} \tag{6.32}$$

where for l we can substitute an arbitrary signal taking values in $\mathcal{P}$. Here p^* denotes the (unknown) "true" parameter value as before, and the matrix A_{qp} is Hurwitz whenever $q \in \chi(p)$; in fact, all eigenvalues of such matrices can be made to have real parts smaller than $-\lambda_0$.

The constant λ will play the role of a "weighting" design parameter in the definition of the monitoring signals. Let ε_μ be some nonnegative number (its role will become clear later). We generate the monitoring signals μ_p, $p \in \mathcal{P}$ by the equations

$$\begin{aligned} \dot{W} &= -2\lambda W + \begin{pmatrix} x_{\mathrm{E}} \\ y \end{pmatrix} \begin{pmatrix} x_{\mathrm{E}} \\ y \end{pmatrix}^T, \qquad W(0) \geq 0 \\ \mu_p &:= \begin{pmatrix} c_p^T & -1 \end{pmatrix} W \begin{pmatrix} c_p^T & -1 \end{pmatrix}^T + \varepsilon_\mu, \qquad p \in \mathcal{P}. \end{aligned} \tag{6.33}$$

Here $W(t)$ is a nonnegative definite symmetric $k \times k$ matrix, $k := \dim(x_{\mathrm{E}}) + 1$. Since $c_p^T x_{\mathrm{E}} - y = e_p$ for all $p \in \mathcal{P}$ in light of (6.2), we have

$$\mu_p(t) = e^{-2\lambda t} \widetilde{\mu}_p(0) + \int_0^t e^{-2\lambda(t-\tau)} e_p^2(\tau) d\tau + \varepsilon_\mu, \qquad p \in \mathcal{P}$$

where $\widetilde{\mu}_p(0) := \begin{pmatrix} c_p^T & -1 \end{pmatrix} W(0) \begin{pmatrix} c_p^T & -1 \end{pmatrix}^T$.

We will distinguish between two cases: the one when the parametric uncertainty set $\mathcal{P}$ is finite and the one when $\mathcal{P}$ is infinite. As we mentioned in Section 6.2, the nature of $\mathcal{P}$ is determined not only by the original process model parameterization but also by choices made at the control design stage; in particular, one may wish to replace a given infinite set $\mathcal{P}$ by its finite subset and incorporate the remaining parameter values into unmodeled dynamics. The main difference between the two cases will be in the switching logic used. When $\mathcal{P}$ is finite, we will use the scale-independent hysteresis switching logic. When $\mathcal{P}$ is infinite, we will employ the hierarchical hysteresis switching logic; the subsets in the partition needed to define the logic will be taken to be sufficiently small so that there exists a robustly stabilizing controller for each subset. Analysis steps in the two cases will be very similar.

6.6.1 *Finite parametric uncertainty*

We first consider the case when $\mathcal{P}$ is a finite set (consisting of m elements). We also let $\mathcal{Q} = \mathcal{P}$ and $\chi(p) = p$ for all $p \in \mathcal{P}$. In this case, we define the switching signal σ using the scale-independent hysteresis switching logic described in Section 6.5.3.

Setting $\Theta(t) := e^{2\lambda t}$ in the formula (6.19), we see that the scaled signals $\bar{\mu}_p$, $p \in \mathcal{P}$ are monotonically nondecreasing, because they satisfy

$$\bar{\mu}_p(t) = \widetilde{\mu}_p(0) + \int_0^t e^{2\lambda\tau} e_p^2(\tau)d\tau + \varepsilon_\mu e^{2\lambda t}, \qquad p \in \mathcal{P}. \tag{6.34}$$

Moreover, it is easy to ensure that μ_p, $p \in \mathcal{P}$ are uniformly bounded away from zero, by setting $\varepsilon_\mu > 0$ and/or by requiring $W(0)$ to be positive definite. Therefore, we can apply Lemma 6.2 and conclude that the inequalities (6.20) and (6.21) are valid. Since in this case the signals $\bar{\mu}_p$, $p \in \mathcal{P}$ are smooth, the left-hand side of the inequality (6.21) equals $\int_{t_0}^t \dot{\bar{\mu}}_{\sigma(\tau)}(\tau)d\tau$ (see Remark 6.1). From (6.34) we have the following formula for the expression under this integral for each fixed value of σ:

$$\dot{\bar{\mu}}_p(t) = e^{2\lambda t} e_p^2(t) + 2\lambda\varepsilon_\mu e^{2\lambda t}, \qquad p \in \mathcal{P}. \tag{6.35}$$

We now turn to stability analysis, which centers around the system (6.30). Consider the system obtained from (6.30) by substituting σ for l:

$$\dot{x}_{\text{CE}} = A_{\sigma\sigma} x_{\text{CE}} + d_\sigma e_\sigma. \tag{6.36}$$

We know that for every fixed time $s \geq 0$, all eigenvalues of the matrix $A_{\sigma(s)\sigma(s)}$ have real parts smaller than $-\lambda_0$. Note that this is true even if the controller $\mathbb{C}_{\sigma(s)}$ does not stabilize the process (this fact would be reflected, however, in the estimation error $e_{\sigma(s)}$ being large, which is a consequence of detectability of the overall closed-loop system; cf. Section 6.5.2).

No noise, disturbances, or unmodeled dynamics

We first consider the simple situation where there are no unmodeled dynamics ($\delta = 0$), i.e., the process $\mathbb{P}$ exactly matches one of the m nominal process models, and where the noise and disturbance signals are zero ($n = d \equiv 0$). In this case, the constants $B_1, B_2, \delta_1, \delta_2$ in (6.14) and (6.15) are all zero. Let us take ε_μ in the definition of the monitoring signals to be zero as well; $W(0)$ must then be positive definite. The inequality (6.14) gives $\int_0^t e^{2\lambda\tau} e_{p^*}^2(\tau)d\tau \le C_1$, which together with (6.34) implies $\bar{\mu}_{p^*} \le \widetilde{\mu}_{p^*}(0) + C_1$. It follows from Corollary 6.3 that the switching stops in finite time, i.e., there exist a time T^* and an index $q^* \in \mathcal{P}$ such that $\sigma(t) = q^*$ for $t \ge T^*$, and $\bar{\mu}_{q^*}$ is a bounded signal. In view of (6.34), we have in particular $e_{q^*} \in \mathcal{L}_2$. Since $A_{q^*q^*}$ is a Hurwitz matrix, we see from (6.36) that $x_{\text{CE}} \to 0$. Moreover, $e_{p^*} \to 0$ by virtue of (6.15), and we conclude using (6.31) and (6.32) that $u, y \to 0$. Detectability of $\mathbb{P}$ then implies that $x \to 0$, so the regulation problem is solved. In light of (6.33), we also have $\mu_p \to 0$ for all $p \in \mathcal{P}$. Since the evolution of x and x_{CE} for $t \ge T^*$ is described by a linear time-invariant system, the rate of convergence is actually exponential. We summarize our conclusions as follows.

Proposition 6.5 *Suppose that the noise and disturbance signals are zero and there are no unmodeled dynamics, and set $\varepsilon_\mu = 0$. Then all signals in the supervisory control system remain bounded for every set of initial conditions such that $W(0) > 0$. Moreover, the switching stops in finite time, after which all continuous states converge to zero exponentially fast.*

Noise and disturbances, no unmodeled dynamics

We now assume that bounded noise n and disturbance d are present but there are no unmodeled dynamics. In this case the switching typically will not stop in finite time. The inequalities (6.14) and (6.15) hold with some unknown but finite constants B_1, B_2. The parameters δ_1 and δ_2 are still zero, and C_1 and C_2 are positive constants as before. Let us take ε_μ to be positive. From (6.14) and (6.34) we have

$$\bar{\mu}_{p^*}(t) \le \widetilde{\mu}_{p^*}(0) + B_1 e^{2\lambda t} + C_1 + \varepsilon_\mu e^{2\lambda t}. \tag{6.37}$$

Applying (6.20) with $l = p^*$ and using the bound $\bar{\mu}_p(t_0) \ge \varepsilon_\mu e^{2\lambda t_0}$ provided by (6.34) yields

$$N_\sigma(t, t_0) \le N_0 + \frac{t - t_0}{\tau_a}$$

where

$$N_0 = 1 + m + \frac{m}{\log(1+h)} \log\left(\frac{\widetilde{\mu}_{p^*}(0) + B_1 + C_1 + \varepsilon_\mu}{\varepsilon_\mu}\right)$$

and

$$\tau_a = \frac{\log(1+h)}{2\lambda m}. \tag{6.38}$$

That is, under the present assumptions the switching signal σ produced by the scale-independent hysteresis switching logic has average dwell time τ_a given by (6.38).

Since $0 < \lambda < \lambda_0$, Theorem 3.2 from Section 3.2.2 implies that the switched system

$$\dot{x}_{\text{CE}} = A_{\sigma\sigma} x_{\text{CE}} \tag{6.39}$$

is exponentially stable with stability margin λ, uniformly over all σ with sufficiently large average dwell time (see Remarks 3.2 and 3.3). The corresponding lower bound on the average dwell time, which we denote by τ^*, can be calculated explicitly as shown in the proof of that theorem. A corollary of this result—stated below as a lemma—is that the system (6.36) has a finite $e^{\lambda t}$-weighted $\mathcal{L}_2$-to-$\mathcal{L}_\infty$ induced norm, uniform over all σ with average dwell time no smaller than τ^*.

Lemma 6.6 *There exist positive constants g and g_0 such that for every switching signal σ with average dwell time $\tau_a \geq \tau^*$ all solutions of the system* (6.36) *satisfy*

$$e^{2\lambda t}|x_{\text{CE}}(t)|^2 \leq g \int_0^t e^{2\lambda\tau} e^2_{\sigma(\tau)}(\tau) d\tau + g_0 |x_{\text{CE}}(0)|^2. \tag{6.40}$$

As seen from (6.38), we can guarantee that $\tau_a \geq \tau^*$ by increasing the hysteresis constant h and/or decreasing the weighting constant λ if necessary. In the sequel, we assume that h and λ have been chosen in this way. Using (6.21) with $l = p^*$ and $t_0 = 0$, (6.35), and (6.37), we obtain

$$\int_0^t e^{2\lambda\tau} e^2_{\sigma(\tau)}(\tau) d\tau \leq m(1+h)(\widetilde{\mu}_{p^*}(0) + B_1 e^{2\lambda t} + C_1 + \varepsilon_\mu e^{2\lambda t})$$

which together with (6.40) gives

$$|x_{\text{CE}}(t)|^2 \leq \big(gm(1+h)(\widetilde{\mu}_{p^*}(0)+C_1)+g_0|x_{\text{CE}}(0)|^2\big)e^{-2\lambda t}+gm(1+h)(B_1+\varepsilon_\mu).$$

This formula implies that x_{CE} is bounded, and it is not hard to deduce from (6.15), (6.31)–(6.33), and detectability of $\mathbb{P}$ that all other system signals remain bounded. Note that the choice of the design parameters λ, h, and ε_μ did not depend on the noise or disturbance bounds; in other words, explicit knowledge of these bounds is not necessary (we are merely requiring that such bounds exist). We arrive at the following result.

Proposition 6.7 *Suppose that the noise and disturbance signals are bounded and there are no unmodeled dynamics. Then for an arbitrary $\varepsilon_\mu > 0$ all signals in the supervisory control system remain bounded for every set of initial conditions.*

Remark 6.4 A close inspection of the above calculations reveals that if the noise and disturbance signals converge to zero, then the continuous states of the supervisory control system eventually enter a neighborhood around zero whose size can be made as small as desired by choosing ε_μ sufficiently small. Thus by decreasing ε_μ (e.g., in a piecewise constant fashion) it is possible to recover asymptotic convergence. Note, however, that we cannot simply let $\varepsilon_\mu = 0$, as this would invalidate the above analysis even if $W(0) > 0$. □

Noise, disturbances, and unmodeled dynamics

If unmodeled dynamics are present, i.e., if the parameter δ is positive, then δ_1 and δ_2 in (6.14) and (6.15) are also positive. In this case, the analysis becomes more complicated, because we can no longer deduce from (6.14) that the switching signal possesses an average dwell time. However, it is possible to prove that the above control algorithm, without any modification, is robust with respect to unmodeled dynamics in the following "semi-global" sense: for arbitrary bounds on the noise and disturbance signals, for every positive value of ε_μ, and every number $E > 0$ there exists a number $\bar{\delta} > 0$ such that if the unmodeled dynamics bound δ does not exceed $\bar{\delta}$, then all signals remain bounded for every set of initial conditions such that $|x(0)|, |x_{\text{CE}}(0)| \le E$. Remark 6.4 applies in this case as well.

6.6.2 *Infinite parametric uncertainty*

Now suppose that the set $\mathcal{P}$ is infinite. We still work with a finite set $\mathcal{Q}$. We assume that a partition $\mathcal{P} = \bigcup_{q\in\mathcal{Q}} \mathcal{D}_q$ is given, such that the matrices A_{qp}, $q \in \mathcal{Q}$, $p \in \mathcal{P}$ have the following property: for every $q \in \mathcal{Q}$ and every $p \in \mathcal{D}_q$ the matrix $A_{qp} + \lambda_0 I$ is Hurwitz, where λ_0 is a fixed positive number. It can be shown that such a partition exists, provided that the sets $\mathcal{D}_q$, $q \in \mathcal{Q}$ are sufficiently small and each $\mathbb{C}_q$ stabilizes the pth nominal process model whenever $p \in \mathcal{D}_q$. (Without loss of generality, we assume that all parameters of the system (6.30)–(6.32) depend continuously on p.) We take the sets $\mathcal{D}_q$, $q \in \mathcal{Q}$ to be closed. The switching signal σ can then be generated using these sets via the hierarchical hysteresis switching logic, as explained in Section 6.5.3. Note that the overall supervisory control system still has finite-dimensional dynamics.

Instead of (6.36), we now consider the system obtained from (6.30) by substituting ζ for l, where ζ is some σ-consistent signal:

$$\dot{x} = A_{\sigma\zeta} x + d_\sigma e_\zeta .$$

This system has a finite $e^{\lambda t}$-weighted $\mathcal{L}_2$-to-$\mathcal{L}_\infty$ induced norm, uniform over all σ with sufficiently large average dwell time and all σ-consistent signals ζ; this is a straightforward generalization of Lemma 6.6. Applying Lemma 6.4 in place of Lemma 6.2, we can carry out stability analysis of the supervisory control system in much the same way as before and prove that all of the results stated for the case of a finite $\mathcal{P}$ remain valid. The details are left to the reader.

6.7 Nonlinear supervisory control

The goal of this section is to address the problem of global state or output regulation for uncertain nonlinear systems. Throughout this section we take $\mathcal{P}$ to be a finite set. We also let $\mathcal{Q} = \mathcal{P}$. As before, we will write p^* for the actual parameter value, which exists but is not known.

In Section 6.5.2 we described the nonlinear supervisory control architecture and established conditions for detectability of the closed-loop system (for a frozen controller index) in the OSS and iOSS sense. These conditions provide a characterization of Property 2 from Section 6.4. In what follows, we work with the weaker iOSS notion (which is also somewhat more natural in the present context). Assume that for each fixed $q \in \mathcal{P}$, the system (6.17) is iOSS with respect to the estimation error e_q. This means that along solutions of (6.17) we have the inequality

$$\widetilde{\alpha}_q(|\mathbf{x}(t)|) \leq \widetilde{\beta}_q(|\mathbf{x}(0)|, t) + \int_0^t \widetilde{\gamma}_q(|e_q(\tau)|)d\tau \qquad (6.41)$$

where $\widetilde{\alpha}_q, \widetilde{\gamma}_q \in \mathcal{K}_\infty$, $\widetilde{\beta}_q \in \mathcal{KL}$, and $\mathbf{x} = (x; x_{\mathrm{C}}; x_{\mathrm{E}})$.

The next assumption characterizes Property 1 from Section 6.4. It effectively restricts us to considering the situation where there are no unmodeled dynamics, noise, or disturbances, so that the unknown process $\mathbb{P}$ exactly matches one of a finite number of nominal process models.

ASSUMPTION 1. There exists a positive number λ with the property that for arbitrary initial conditions $x(0)$, $x_{\mathrm{C}}(0)$, $x_{\mathrm{E}}(0)$ there exists a constant C such that we have $\int_0^t e^{\lambda\tau}\widetilde{\gamma}_{p^*}(|e_{p^*}(\tau)|)d\tau \leq C$ for all t. Here $\widetilde{\gamma}_{p^*}$ is the function from the formula (6.41) for $q = p^*$.

It is not hard to see that in the case when $\widetilde{\gamma}_{p^*}$ is locally Lipschitz, the integral $\int_0^t e^{\lambda\tau}\widetilde{\gamma}_{p^*}(|e_{p^*}(\tau)|)d\tau$ is bounded if e_{p^*} and $\int_0^t e^{\lambda\tau}|e_{p^*}(\tau)|d\tau$ are bounded. Then Assumption 1 is satisfied with λ small enough if the multi-estimator is designed so that e_{p^*} converges to zero exponentially fast for every control signal u. Some examples of such multi-estimator design for nonlinear systems (in the absence of noise, disturbances, and unmodeled dynamics) were mentioned in Section 6.5.1.

We generate the monitoring signals using the differential equations

$$\dot{\mu}_p = -\lambda\mu_p + \widetilde{\gamma}_p(|e_p|), \qquad \mu_p(0) > 0, \quad p \in \mathcal{P} \qquad (6.42)$$

with the same $\lambda > 0$ as in Assumption 1. Let us define the switching signal σ using the scale-independent hysteresis switching logic. The overall supervisory control system is a hybrid system with continuous states $\mathbf{x}$ and

μ_p, $p \in \mathcal{P}$ and discrete state σ. We have the following result, whose proof parallels that of Proposition 6.5.

Proposition 6.8 *Let $\mathcal{P}$ be a finite set. Suppose that*

1. *For each $q \in \mathcal{P}$, the system* (6.17) *is iOSS with respect to e_q, as given by* (6.41).
2. *Assumption 1 holds.*
3. *The monitoring signals are defined by* (6.42).

Then the switching stops in finite time and all continuous states of the supervisory control system converge to zero for arbitrary initial conditions.

PROOF. Let us define (for analysis purposes only) the scaled monitoring signals

$$\bar{\mu}_p(t) := e^{\lambda t}\mu_p(t), \qquad p \in \mathcal{P}. \tag{6.43}$$

In view of (6.42) we have

$$\bar{\mu}_p(t) = \bar{\mu}_p(0) + \int_0^t e^{\lambda\tau}\widetilde{\gamma}_p(|e_p(\tau)|)d\tau, \qquad p \in \mathcal{P}. \tag{6.44}$$

The scale independence property of the switching logic implies that replacing μ_p by $\bar{\mu}_p$ for all $p \in \mathcal{P}$ would have no effect on σ. From (6.44) we see that each $\bar{\mu}_p$ is nondecreasing. The finiteness of $\mathcal{P}$ and the fact that $\bar{\mu}_p(0) > 0$ for each $p \in \mathcal{P}$ guarantee the existence of a positive number ε_μ such that $\bar{\mu}_p(0) \geq \varepsilon_\mu$ for all $p \in \mathcal{P}$. Observe also that $\bar{\mu}_{p^*}$ is bounded by virtue of (6.44) and Assumption 1. It follows that the signals $\bar{\mu}_p$, $p \in \mathcal{P}$ satisfy the hypotheses of Corollary 6.3, which enables us to deduce that the switching stops in finite time. More precisely, there exists a time T^* such that $\sigma(t) = q^* \in \mathcal{P}$ for all $t \geq T^*$. In addition, $\bar{\mu}_{q^*}$ is bounded, which in view of (6.44) means that the integral $\int_0^\infty \widetilde{\gamma}_{q^*}(|e_{q^*}(\tau)|)d\tau$ is finite. Using (6.41) and applying Lemma A.4, we conclude that $\mathbf{x} \to 0$. Finally, in light of (6.42) we also have $\mu_p \to 0$ for all $p \in \mathcal{P}$. □

Theorem 6.1 from Section 6.5.2 guarantees that the first hypothesis of Proposition 6.8 (the iOSS property) holds when the process $\mathbb{P}$ is IOSS and the injected system (6.16) is iISS with respect to e_q for each fixed $q \in \mathcal{Q}$. The latter condition means that solutions of (6.16) satisfy the inequality

$$\alpha_q(|x_{\text{CE}}(t)|) \leq \beta_q(|x_{\text{CE}}(0)|, t) + \int_0^t \gamma_q(|e_q(\tau)|)d\tau \tag{6.45}$$

where $\alpha_q, \gamma_q \in \mathcal{K}_\infty$, $\beta_q \in \mathcal{KL}$, and $x_{\text{CE}} := (x_{\text{C}}; x_{\text{E}})$. The function $\widetilde{\gamma}_q$ appearing in the formula (6.41) depends on the functions that express the IOSS property of the uncertain process $\mathbb{P}$. Therefore, it is rather restrictive to assume the knowledge of $\tilde{\gamma}_q$ or its upper bound for each $q \in \mathcal{P}$. The alternative construction presented below allows us to work directly with the functions γ_q from (6.45), but requires a somewhat different convergence proof. Let us replace Assumption 1 by the following.

ASSUMPTION 1′. There exists a positive number λ with the property that for arbitrary initial conditions $x(0)$, $x_C(0)$, $x_E(0)$ there exist constants C_1, C_2 such that we have $|e_{p^*}(t)| \le C_1$ and $\int_0^t e^{\lambda\tau}\gamma_{p^*}(|e_{p^*}(\tau)|)d\tau \le C_2$ for all t. Here γ_{p^*} is the function from the formula (6.45) for $q = p^*$.

Instead of using the equations (6.42), we now generate the monitoring signals by

$$\dot{\mu}_p = -\lambda\mu_p + \gamma_p(|e_p|), \qquad \mu_p(0) > 0, \quad p \in \mathcal{P} \tag{6.46}$$

with the same $\lambda > 0$ as in Assumption 1′. We use the scale-independent hysteresis switching logic as before. Then the following result holds.

Proposition 6.9 *Let $\mathcal{P}$ be a finite set. Suppose that*

1. *The process $\mathbb{P}$ is IOSS.*
2. *For each $q \in \mathcal{P}$, the injected system* (6.16) *is iISS with respect to e_q, as given by* (6.45).
3. *Assumption 1′ holds.*
4. *The monitoring signals are defined by* (6.46).

Then the switching stops in finite time and all continuous states of the supervisory control system converge to zero for arbitrary initial conditions.

PROOF. Exactly as in the proof of Proposition 6.8, using the scaled monitoring signals (6.43), we prove that the switching stops in finite time at some index $q^* \in \mathcal{P}$ and that the integral $\int_0^\infty \gamma_{q^*}(|e_{q^*}(\tau)|)d\tau$ is finite. In view of (6.45) and Lemma A.4 this implies that x_C and x_E converge to zero. Thus $u \to 0$ and $y_p \to 0$ for all $p \in \mathcal{P}$. Since e_{p^*} is bounded by Assumption 1′, it follows that $y = y_{p^*} - e_{p^*}$ remains bounded as well. Therefore, x is bounded because $\mathbb{P}$ is IOSS. The derivative $\dot{e}_{q^*} = \dot{y}_{q^*} - \dot{y}$ is then also bounded. Boundedness of e_{q^*}, $\dot{e}_{q^*}$, and of the integral $\int_0^\infty \gamma_{q^*}(|e_{q^*}(\tau)|)d\tau$ is well known to imply that $e_{q^*} \to 0$ (this is a version of the so-called Barbalat's lemma). Thus we have $y = y_{q^*} - e_{q^*} \to 0$, hence $x \to 0$ because $\mathbb{P}$ is IOSS, and $\mu_p \to 0$ for each $p \in \mathcal{P}$ as before. □

If one is only concerned with output regulation and not state regulation, a close examination of the proof of Proposition 6.9 reveals that the assumptions can be weakened even further and the following result can be proved by the same argument.

Proposition 6.10 *Let $\mathcal{P}$ be a finite set. Suppose that*

1. *The state x of $\mathbb{P}$ is bounded if the control input u and the output y are bounded.*

2. *For each $q \in \mathcal{P}$ the injected system* (6.16) *has the property that if $\int_0^\infty \gamma_q(|e_q(\tau)|)d\tau < \infty$ then x_C and x_E remain bounded and $y_q = h_q(x_E)$ converges to zero.*

3. *Assumption 1′ holds.*

4. *The monitoring signals are defined by* (6.46).

Then all signals in the supervisory control system remain bounded for arbitrary initial conditions, the switching stops in finite time, and the output y of the process converges to zero.

A specific supervisory control system satisfying the hypotheses of Proposition 6.10 is studied in the next section.

6.8 An example: a nonholonomic system with uncertainty

We now return to the unicycle example studied in Section 4.2.1 and consider a more general process model, given by

$$\begin{aligned}\dot{x}_1 &= p_1^* u_1 \cos\theta \\ \dot{x}_2 &= p_1^* u_1 \sin\theta \\ \dot{\theta} &= p_2^* u_2\end{aligned}$$

where p_1^* and p_2^* are positive parameters determined by the radius of the rear wheels and the distance between them. The case of interest to us here is when the actual values of p_1^* and p_2^* are unknown. We assume that $p^* := (p_1^*, p_2^*)$ belongs to some compact subset $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$ of $(0,\infty)\times(0,\infty)$. The control task, as in Section 4.2.1, is to park the vehicle at the origin—i.e., make x_1, x_2, and θ tend to zero—by means of applying state feedback.

What makes this problem especially interesting is that for every set of values for the unknown parameters, the corresponding system is nonholonomic and so cannot be stabilized by any continuous state feedback law (as explained in Chapter 4). Thus even the design of candidate controllers is a challenging problem. One option is to use the simple switching control laws of the kind described in Section 4.2.1, which make every trajectory approach the origin after at most one switch. In the presence of parametric modeling uncertainty, it is then natural to proceed with the supervisory control design, using these switching candidate controllers and a supervisor that orchestrates the switching among them. The switching will thus occur at two levels with distinct purposes: at the lower level, each candidate controller utilizes switching to overcome the obstruction to continuous stabilizability of the nonholonomic process, and at the higher level, the supervisor switches to deal with the parametric modeling uncertainty. This solution, presented in some more detail below, illustrates the flexibility of supervisory control in incorporating advanced control techniques developed in the nonadaptive control literature (cf. the remarks made in Section 6.1).

The supervisory control design for this example is based on the developments of Sections 4.2.1 and 6.7. Consider the state coordinate transformation

$$\begin{aligned} x &= x_1 \cos\theta + x_2 \sin\theta \\ y &= \theta \\ z &= 2(x_1 \sin\theta - x_2\cos\theta) - \theta(x_1\cos\theta + x_2\sin\theta) \end{aligned}$$

as well as the family of control transformations given, for each $p \in \mathcal{P}$, by

$$\begin{aligned} u_p &= p_1 u_1 - p_2 u_2 (x_1 \sin\theta - x_2\cos\theta) \\ v_p &= p_2 u_2. \end{aligned}$$

In what follows, when we write u_p, v_p, we always mean the functions of the original state and control variables defined by the above formulas. For $p = p^*$, the transformed equations are those of Brockett's nonholonomic integrator

$$\begin{aligned} \dot{x} &= u_{p^*} \\ \dot{y} &= v_{p^*} \\ \dot{z} &= x v_{p^*} - y u_{p^*}. \end{aligned} \tag{6.47}$$

If we can make x, y, and z approach zero, then the regulation problem under consideration will be solved.

Since the state of the system (6.47) is accessible, the multi-estimator design is straightforward (cf. Section 6.5.1). For each $p \in \mathcal{P}$, we introduce the estimator equations

$$\begin{aligned} \dot{x}_p &= -(x_p - x) + u_p \\ \dot{y}_p &= -(y_p - y) + v_p \\ \dot{z}_p &= -(z_p - z) + x v_p - y u_p \end{aligned}$$

together with the estimation errors

$$\widetilde{x}_p := x_p - x, \qquad \widetilde{y}_p := y_p - y, \qquad \widetilde{z}_p := z_p - z.$$

When $\mathcal{P}$ is an infinite set, the multi-estimator can be easily realized using state sharing, due to the simple way in which the unknown parameters enter the process model. Let $e_p := (\widetilde{x}_p, \widetilde{y}_p, \widetilde{z}_p)^T$. For all control laws we have

$$e_{p^*}(t) = e^{-t} e_{p^*}(0). \tag{6.48}$$

For each $p \in \mathcal{P}$, when $x_p^2 + y_p^2 \neq 0$, the equations

$$x_p = r_p \cos\psi_p, \qquad y_p = r_p \sin\psi_p$$

define the cylindrical coordinates r_p and ψ_p. We now describe the candidate controllers, using an auxiliary logical variable s taking values 0 and 1. Take an arbitrary $q \in \mathcal{P}$ (here we take $\mathcal{Q} = \mathcal{P}$). If $r_q = 0$, let $s = 0$ and apply the control law $u_q = v_q = 1$ for a certain amount of time T. Then let $s = 1$ and apply the control law

$$\begin{aligned} u_q &= -x_q r_q + z_q \sin\psi_q + \widetilde{x}_q \\ v_q &= -y_q r_q - z_q \cos\psi_q + \widetilde{y}_q. \end{aligned}$$

The idea behind this control law is to make the multi-estimator reproduce the desired behavior given by (4.10) as closely as possible. Indeed, for the corresponding injected system we have, in particular,

$$\begin{aligned} \dot{r}_q &= -r_q^2 \\ \dot{z}_q &= -z_q(r_q - \widetilde{x}_q \cos\psi_q - \widetilde{y}_q \sin\psi_q) + (x_q \widetilde{y}_q - \widetilde{x}_q y_q)(1 - r_q) - \widetilde{z}_q. \end{aligned}$$

These equations would precisely match the first two equations in (4.10) if e_q were equal to zero.

We leave it to the reader to write the injected system in full detail and verify that it satisfies Condition 2 of Proposition 6.10 with $\gamma_q(r) = r + r^2 + r^6$. In view of (6.48), Assumption 1$'$ of Section 6.7 holds: we can take an arbitrary $\lambda \in (0, 1)$. We thus generate the monitoring signals by

$$\dot{\mu}_p = -\lambda\mu_p + \gamma_p(|e_p|), \qquad \mu_p(0) > 0, \quad p \in \mathcal{P}$$

where λ is some number satisfying $0 < \lambda < 1$. This monitoring signal generator can be efficiently realized using state sharing. Let us define the switching signal via the scale-independent hysteresis switching logic. The minimization procedure used in the switching logic reduces to finding roots of polynomials in one variable of degree at most 5, so its computational tractability is not an issue.

Exercise 6.5 As a continuation of Exercise 4.2, implement the supervisory control strategy described here via computer simulation.

Although the above supervisory control algorithm can be implemented when the set $\mathcal{P}$ is infinite, the analysis tools from the previous section can only be used when $\mathcal{P}$ is finite. So, let us restrict $\mathcal{P}$ to be a finite set. Note that here the process is trivially IOSS since its output is the entire state. We can now apply Proposition 6.10 and conclude that all signals remain bounded and the state of the system (6.47) converges to zero.

Part IV

Supplementary Material

Appendix A
Stability

In this appendix we briefly review basic facts from Lyapunov's stability theory. We restrict our attention to the time-invariant system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n \tag{A.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function. Extensions to time-varying and particularly switched systems are mentioned in appropriate places in the main chapters. We also assume that the origin is an (isolated) equilibrium point of the system (A.1), i.e., $f(0) = 0$, and confine our attention to stability properties of this equilibrium.

A.1 Stability definitions

Since the system (A.1) is time-invariant, we let the initial time be $t_0 = 0$ without loss of generality. The origin is said to be a *stable* equilibrium of (A.1), in the sense of Lyapunov, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that we have

$$|x(0)| \leq \delta \quad \Rightarrow \quad |x(t)| \leq \varepsilon \qquad \forall\, t \geq 0.$$

In this case we will also simply say that the system (A.1) is *stable*. A similar convention will apply to other stability concepts introduced below.

The system (A.1) is called *asymptotically stable* if it is stable and δ can be chosen so that

$$|x(0)| \leq \delta \quad \Rightarrow \quad x(t) \to 0 \ \text{ as } \ t \to \infty.$$

The set of all initial states from which the trajectories converge to the origin is called the *region of attraction.* If the above condition holds for all δ, i.e., if the origin is a stable equilibrium and its region of attraction is the entire state space, then the system (A.1) is called *globally asymptotically stable.*

If the system is not necessarily stable but has the property that all solutions with initial conditions in some neighborhood of the origin converge to the origin, then it is called (locally) *attractive.* We say that the system is *globally attractive* if its solutions converge to the origin from all initial conditions.

The system (A.1) is called *exponentially stable* if there exist positive constants δ, c, and λ such that all solutions of (1.3) with $|x(0)| \leq \delta$ satisfy the inequality

$$|x(t)| \leq c|x(0)|e^{-\lambda t} \qquad \forall\, t \geq 0. \tag{A.2}$$

If this exponential decay estimate holds for all δ, the system is said to be *globally exponentially stable.* The constant λ in (A.2) is occasionally referred to as a *stability margin.*

A.2 Function classes $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$

A function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of *class* $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, then it is said to be of *class* $\mathcal{K}_\infty$. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be of *class* $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r,t)$ is decreasing to zero as $t \to \infty$ for each fixed $r \geq 0$. We will write $\alpha \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$ to indicate that α is a class $\mathcal{K}_\infty$ function and β is a class $\mathcal{KL}$ function, respectively.

As an immediate application of these function classes, we can rewrite the stability definitions of the previous section in a more compact way. Indeed, stability of the system (A.1) is equivalent to the property that there exist a $\delta > 0$ and a class $\mathcal{K}$ function α such that all solutions with $|x(0)| \leq \delta$ satisfy

$$|x(t)| \leq \alpha(|x(0)|) \qquad \forall\, t \geq 0.$$

Asymptotic stability is equivalent to the existence of a $\delta > 0$ and a class $\mathcal{KL}$ function β such that all solutions with $|x(0)| \leq \delta$ satisfy

$$|x(t)| \leq \beta(|x(0)|, t) \qquad \forall\, t \geq 0.$$

Global asymptotic stability amounts to the existence of a class $\mathcal{KL}$ function β such that the inequality

$$|x(t)| \leq \beta(|x(0)|, t) \qquad \forall\, t \geq 0$$

holds for all initial conditions. Exponential stability means that the function β takes the form $\beta(r,s) = cre^{-\lambda s}$ for some $c, \lambda > 0$.

A.3 Lyapunov's direct (second) method

Consider a C^1 (i.e., continuously differentiable) function $V : \mathbb{R}^n \to \mathbb{R}$. It is called *positive definite* if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. If $V(x) \to \infty$ as $|x| \to \infty$, then V is said to be *radially unbounded.* If V is both positive definite and radially unbounded, then there exist two class $\mathcal{K}_\infty$ functions α_1, α_2 such that V satisfies

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \qquad \forall x. \tag{A.3}$$

We write $\dot{V}$ for the derivative of V along solutions of the system (A.1), i.e.,

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x).$$

The main result of Lyapunov's stability theory is expressed by the following statement.

Theorem A.1 (Lyapunov) *Suppose that there exists a positive definite C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ whose derivative along solutions of the system* (A.1) *satisfies*

$$\dot{V} \leq 0 \qquad \forall x. \tag{A.4}$$

Then the system (A.1) *is stable. If the derivative of V satisfies*

$$\dot{V} < 0 \qquad \forall x \neq 0 \tag{A.5}$$

then (A.1) *is asymptotically stable. If in the latter case V is also radially unbounded, then* (A.1) *is globally asymptotically stable.*

We refer to a positive definite C^1 function V as a *weak Lyapunov function* if it satisfies the inequality (A.4) and a *Lyapunov function* if it satisfies the inequality (A.5). The conclusions of the theorem remain valid when V is merely continuous and not necessarily C^1, provided that the inequalities (A.4) and (A.5) are replaced by the conditions that V is nonincreasing and strictly decreasing along nonzero solutions, respectively (this can be seen from the proof outlined below).

Sketch of proof of Theorem A.1. First assume that (A.4) holds. Consider the ball around the origin of a given radius $\varepsilon > 0$. Pick a positive number $b < \min_{|x|=\varepsilon} V(x)$. Denote by δ the radius of some ball around the origin which is inside the set $\{x : V(x) \leq b\}$ (see Figure 43). Since V is nonincreasing along solutions, each solution starting in the smaller ball of radius δ satisfies $V(x(t)) \leq b$, hence it remains inside the bigger ball of radius ε. This proves stability.

To prove the second statement of the theorem, take an arbitrary initial condition satisfying $|x(0)| \leq \delta$, where δ is as defined above (for some ε). Since V is positive and decreasing along the corresponding solution, it has a

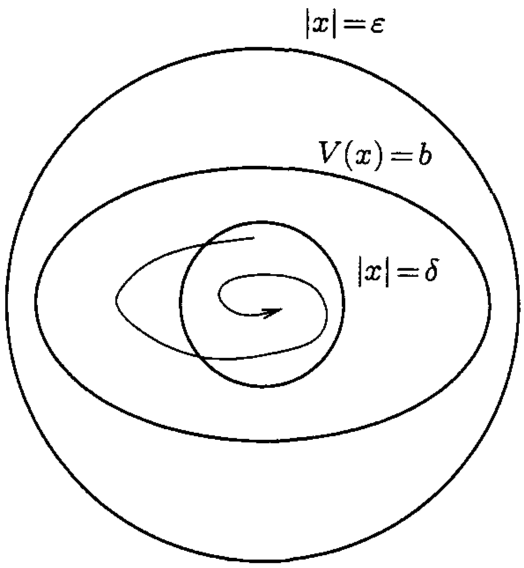

FIGURE 43. Proving Lyapunov stability

limit $c \geq 0$ as $t \to \infty$. If we can prove that $c = 0$, then we have asymptotic stability (in view of positive definiteness of V and the fact that x stays bounded in norm by ε). Suppose that c is positive. Then the solution cannot enter the set $\{x : V(x) < c\}$. In this case the solution evolves in a compact set that does not contain the origin. For example, we can take this set to be $S := \{x : r \leq |x| \leq \varepsilon\}$ for a sufficiently small $r > 0$. Let $d := \max_{x \in S} \dot{V}(x)$; this number is well defined and negative in view of (A.5) and compactness of S. We have $\dot{V} \leq d$, hence $V(t) \leq V(0) + dt$. But then V will eventually become smaller than c, which is a contradiction.

The above argument is valid locally around the origin, because the level sets of V may not all be bounded and so δ may stay bounded as we increase ε to infinity. If V is radially unbounded, then all its level sets are bounded. Thus we can have $\delta \to \infty$ as $\varepsilon \to \infty$, and global asymptotic stability follows. □

Exercise A.1 Assuming that V is radially unbounded and using the functions α_1 and α_2 from the formula (A.3), write down a possible definition of δ as a function of ε which can be used in the above proof.

Various converse Lyapunov theorems show that the conditions of Theorem A.1 are also necessary. For example, if the system is asymptotically stable, then there exists a positive definite C^1 function V that satisfies the inequality (A.5).

Example A.1 It is well known that for the linear time-invariant system

$$\dot{x} = Ax \tag{A.6}$$

asymptotic stability, exponential stability, and their global versions are all equivalent and amount to the property that A is a *Hurwitz* matrix, i.e., all eigenvalues of A have negative real parts. Fixing an arbitrary positive definite symmetric matrix Q and finding the unique positive definite symmetric matrix P that satisfies the Lyapunov equation

$$A^T P + PA = -Q$$

one obtains a quadratic Lyapunov function $V(x) = x^T Px$ whose derivative along solutions is $\dot{V} = -x^T Qx$. The explicit formula for P is

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt.$$

Indeed, we have

$$A^T P + PA = \int_0^\infty \frac{d}{dt}\left(e^{A^T t} Q e^{At}\right) = -Q$$

because A is Hurwitz. □

A.4 LaSalle's invariance principle

With some additional knowledge about the behavior of solutions, it is possible to prove asymptotic stability using a weak Lyapunov function, which satisfies the nonstrict inequality (A.4). This is facilitated by *LaSalle's invariance principle.*

A set M is called (positively) *invariant* with respect to the given system if all solutions starting in M remain in M for all future times. We now state a version of LaSalle's theorem which is the most useful one for our purposes.

Theorem A.2 (LaSalle) *Suppose that there exists a positive definite C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ whose derivative along solutions of the system* (A.1) *satisfies the inequality* (A.4). *Let M be the largest invariant set contained in the set $\{x : \dot{V}(x) = 0\}$. Then the system* (A.1) *is stable and every solution that remains bounded for $t \geq 0$ approaches M as $t \to \infty$. In particular, if all solutions remain bounded and $M = \{0\}$, then the system* (A.1) *is globally asymptotically stable.*

To deduce global asymptotic stability with the help of this result, one needs to check two conditions. First, all solutions of the system must be bounded. This property follows automatically from the inequality (A.4) if V is chosen to be radially unbounded; however, radial unboundedness of V is not necessary when boundedness of solutions can be established by other means.[1] The second condition is that $\dot{V}$ is not identically zero

[1]When just local asymptotic stability is of interest, it suffices to note that boundedness of solutions starting sufficiently close to the origin is guaranteed by the first part of Theorem A.1.

along any nonzero solution. We also remark that if one only wants to prove asymptotic convergence of bounded solutions to zero and is not concerned with Lyapunov stability of the origin, then positive definiteness of V is not needed (this is in contrast with Theorem A.1).

Example A.2 Consider the two-dimensional system

$$\ddot{x} + a\dot{x} + f(x) = 0$$

where $a > 0$ and the function f satisfies $f(0) = 0$ and $xf(x) > 0$ for all $x \neq 0$. Systems of this form frequently arise in models of mechanical systems with damping or electrical circuits with nonlinear capacitors or inductors. The equivalent first-order state-space description is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - f(x_1). \end{aligned} \tag{A.7}$$

Consider the function

$$V(x_1, x_2) := \frac{x_2^2}{2} + F(x_1) \tag{A.8}$$

where $F(x_1) := \int_0^{x_1} f(x)dx$. Assume that f is such that F is positive definite and radially unbounded (this is true, for example, under the sector condition $k_1 x_1^2 \leq x_1 f(x_1) \leq k_2 x_1^2$, $0 < k_1 < k_2 \leq \infty$). The derivative of the function (A.8) along solutions of (A.7) is given by

$$\dot{V} = -ax_2^2 \leq 0.$$

Moreover, $x_2 \equiv 0$ implies that x_1 is constant, and the second equation in (A.7) then implies that $x_1 \equiv 0$ as well. Therefore, the system (A.7) is globally asymptotically stable by Theorem A.2. □

While Lyapunov's stability theorem readily generalizes to time-varying systems, for LaSalle's invariance principle this is not the case. Instead, one usually works with the weaker property that all solutions approach the set $\{x : \dot{V}(x) = 0\}$.

A.5 Lyapunov's indirect (first) method

Lyapunov's indirect method allows one to deduce stability properties of the nonlinear system (A.1), where f is C^1, from stability properties of its *linearization*, which is the linear system (A.6) with

$$A := \frac{\partial f}{\partial x}(0). \tag{A.9}$$

By the mean value theorem, we can write

$$f(x) = Ax + g(x)x$$

where g is given componentwise by $g_i(x) := \frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)$ for some point z_i on the line segment connecting x to the origin, $i = 1, \ldots, n$. Since $\frac{\partial f}{\partial x}$ is continuous, we have $g(x) \to 0$ as $x \to 0$. From this it follows that if the matrix A is Hurwitz (i.e., all its eigenvalues are in the open left half of the complex plane), then a quadratic Lyapunov function for the linearization serves—locally—as a Lyapunov function for the original nonlinear system. Moreover, its rate of decay in a neighborhood of the origin can be bounded from below by a quadratic function, which implies that stability is in fact exponential. This is summarized by the following result.

Theorem A.3 *If f is C^1 and the matrix* (A.9) *is Hurwitz, then the system* (A.1) *is locally exponentially stable.*

It is also known that if the matrix A has at least one eigenvalue with a positive real part, the nonlinear system (A.1) is not stable. If A has eigenvalues on the imaginary axis but no eigenvalues in the open right half-plane, the linearization test is inconclusive. However, in this *critical* case the system (A.1) cannot be exponentially stable, since exponential stability of the linearization is not only a sufficient but also a necessary condition for (local) exponential stability of the nonlinear system.

A.6 Input-to-state stability

It is of interest to extend stability concepts to systems with disturbance inputs. In the linear case represented by the system

$$\dot{x} = Ax + Bd$$

it is well known that if the matrix A is Hurwitz, i.e., if the unforced system $\dot{x} = Ax$ is asymptotically stable, then bounded inputs d lead to bounded states while inputs converging to zero produce states converging to zero. Now, consider a nonlinear system of the form

$$\dot{x} = f(x, d) \tag{A.10}$$

where d is a measurable locally essentially bounded[2] disturbance input. In general, global asymptotic stability of the unforced system $\dot{x} = f(x, 0)$ does

[2]The reader not familiar with this terminology may assume that d is piecewise continuous.

not guarantee input-to-state properties of the kind mentioned above. For example, the scalar system

$$\dot{x} = -x + xd \tag{A.11}$$

has unbounded trajectories under the bounded input $d \equiv 2$. This motivates the following important concept, introduced by Sontag.

The system (A.10) is called *input-to-state stable* (ISS) with respect to d if for some functions $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$, for every initial state $x(0)$, and every input d the corresponding solution of (A.10) satisfies the inequality

$$|x(t)| \le \beta(|x(0)|, t) + \gamma(\|d\|_{[0,t]}) \qquad \forall\, t \ge 0 \tag{A.12}$$

where $\|d\|_{[0,t]} := \operatorname{ess\,sup}\{|d(s)| : s \in [0,t]\}$ (supremum norm on $[0,t]$ except for a set of measure zero). Since the system (A.10) is time-invariant, the same property results if we write

$$|x(t)| \le \beta(|x(t_0)|, t - t_0) + \gamma(\|d\|_{[t_0,t]}) \qquad \forall\, t \ge t_0 \ge 0.$$

The ISS property admits the following Lyapunov-like equivalent characterization: the system (A.10) is ISS if and only if there exists a positive definite radially unbounded C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ such that for some class $\mathcal{K}_\infty$ functions α and χ we have

$$\frac{\partial V}{\partial x} f(x,d) \le -\alpha(|x|) + \chi(|d|) \qquad \forall\, x, d.$$

This is in turn equivalent to the following "gain margin" condition:

$$|x| \ge \rho(|d|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x,d) \le -\bar{\alpha}(|x|)$$

where $\bar{\alpha}, \rho \in \mathcal{K}_\infty$. Such functions V are called *ISS-Lyapunov functions.*

Exercise A.2 Prove that if the system (A.10) is ISS, then $d(t) \to 0$ implies $x(t) \to 0$.

The system (A.10) is said to be *locally input-to-state stable* (locally ISS) if the bound (A.12) is valid for solutions with sufficiently small initial conditions and inputs, i.e., if there exists a $\delta > 0$ such that (A.12) is satisfied whenever $|x(0)| \le \delta$ and $\|u\|_{[0,t]} \le \delta$. It turns out (local) asymptotic stability of the unforced system $\dot{x} = f(x, 0)$ implies local ISS.

For systems with outputs, it is natural to consider the following notion which is dual to ISS. A system

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned} \tag{A.13}$$

is called *output-to-state stable* (OSS) if for some functions $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ and every initial state $x(0)$ the corresponding solution of (A.13) satisfies the inequality

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|y\|_{[0,t]})$$

as long as it is defined. While ISS is to be viewed as a generalization of stability, OSS can be thought of as a generalization of detectability; it does indeed reduce to the standard detectability property in the linear case. Given a system with both inputs and outputs

$$\begin{aligned} \dot{x} &= f(x, d) \\ y &= h(x) \end{aligned} \tag{A.14}$$

one calls it *input/output-to-state stable* (IOSS) if for some functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$, for every initial state $x(0)$, and every input d the corresponding solution of (A.14) satisfies the inequality

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_1(\|d\|_{[0,t]}) + \gamma_2(\|y\|_{[0,t]})$$

as long as it exists.

Replacing supremum norms by integral norms, one arrives at integral versions of the above concepts. The system (A.10) is called *integral-input-to-state stable* (iISS) if for some functions $\alpha, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$, for every initial state $x(0)$, and every input d the inequality

$$\alpha(|x(t)|) \leq \beta(|x(0)|, t) + \int_0^t \gamma(|d(\tau)|)d\tau$$

holds on the domain of the corresponding solution of (A.10). It turns out that iISS is a weaker property than ISS: every ISS system is iISS, but the converse is not true. For example, the system (A.11) can be shown to be iISS, but it is not ISS as we saw earlier.

The following characterization of iISS is useful.

Lemma A.4 *The system* (A.10) *is iISS if and only if the system* $\dot{x} = f(x, 0)$ *is Lyapunov stable and there exists a function* $\gamma \in \mathcal{K}_\infty$ *such that for every initial state* $x(0)$ *and every input* d *satisfying* $\int_0^\infty \gamma(|d(\tau)|)d\tau < \infty$ *we have* $x(t) \to 0$ *as* $t \to \infty$.

Necessity is easy to show using the same function γ as in the definition of iISS, while sufficiency is not at all obvious.

Similarly, the system (A.13) is *integral-output-to-state stable* (iOSS) if for some functions $\alpha, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ and every initial state $x(0)$ the inequality

$$\alpha(|x(t)|) \leq \beta(|x(0)|, t) + \int_0^t \gamma(|y(\tau)|)d\tau$$

holds on the domain of the corresponding solution of (A.13). Again, this is a weaker property than OSS.

Appendix B
Lie Algebras

This appendix provides an overview of basic properties of Lie algebras which are of relevance in the developments of Section 2.2.

B.1 Lie algebras and their representations

A *Lie algebra* $\mathfrak{g}$ is a finite-dimensional vector space equipped with a *Lie bracket*, i.e., a bilinear, skew-symmetric map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$. Any Lie algebra $\mathfrak{g}$ can be identified with a tangent space at the identity of a Lie group $\mathcal{G}$ (an analytic manifold with a group structure). If $\mathfrak{g}$ is a matrix Lie algebra, then the elements of $\mathcal{G}$ are given by products of the exponentials of the matrices from $\mathfrak{g}$. In particular, each element $A \in \mathfrak{g}$ generates the one-parameter subgroup $\{e^{At} : t \in \mathbb{R}\}$ in $\mathcal{G}$. For example, if $\mathfrak{g}$ is the Lie algebra $gl(n,\mathbb{R})$ of all real $n \times n$ matrices with the standard Lie bracket $[A,B] = AB - BA$, then the corresponding Lie group is given by the invertible matrices.

Given an abstract Lie algebra $\mathfrak{g}$, one can consider its (matrix) representations. A *representation* of $\mathfrak{g}$ on an n-dimensional vector space V is a homomorphism (i.e., a linear map that preserves the Lie bracket) $\phi : \mathfrak{g} \to gl(V)$. It assigns to each element $g \in \mathfrak{g}$ a linear operator $\phi(g)$ on V, which can be described by an $n \times n$ matrix. A representation ϕ is called *irreducible* if V contains no nontrivial subspaces invariant under the action of all $\phi(g)$, $g \in \mathfrak{g}$. A particularly useful representation is the *adjoint* one, denoted by "ad." The vector space V in this case is $\mathfrak{g}$ itself, and for $g \in \mathfrak{g}$ the operator ad$g$ is defined by ad$g(a) := [g, a]$, $a \in \mathfrak{g}$. There is also *Ado's theorem* which says that every Lie algebra is isomorphic to a subalgebra of $gl(V)$ for some finite-dimensional vector space V (compare this with the adjoint representation which is in general not injective).

B.2 Example: $sl(2,\mathbb{R})$ and $gl(2,\mathbb{R})$

The *special linear Lie algebra* $sl(2,\mathbb{R})$ consists of all real 2×2 matrices of trace 0. A canonical basis for this Lie algebra is given by the matrices

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{B.1}$$

They satisfy the relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$, and form what is sometimes called an $sl(2)$-*triple.*

One can also consider other representations of $sl(2,\mathbb{R})$. Although all irreducible representations of $sl(2,\mathbb{R})$ can be classified by working with the Lie algebra directly, it is sometimes useful to exploit the corresponding Lie group $SL(2,\mathbb{R}) = \{S \in \mathbb{R}^{n\times n} : \det S = 1\}$. Let $P^k[x, y]$ denote the space of polynomials in two variables x and y that are homogeneous of degree k (where k is a positive integer). A homomorphism ϕ that makes $SL(2,\mathbb{R})$ act on $P^k[x, y]$ can be defined as follows:

$$\phi(S)p\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = p\left(S^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

where $S \in SL(2,\mathbb{R})$ and $p \in P^k[x, y]$. The corresponding representation of the Lie algebra $sl(2,\mathbb{R})$, which we denote also by ϕ with slight abuse of notation, is obtained by considering the one-parameter subgroups of $SL(2,\mathbb{R})$ and differentiating the action defined above at $t = 0$. For example, for e as in (B.1) we have

$$\phi(e)p\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \frac{d}{dt}\bigg|_{t=0} p\left(\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right) = -y\frac{\partial}{\partial x}p\left(\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

Similarly, $\phi(f)p = -x\frac{\partial}{\partial y}p$ and $\phi(h)p = (-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})p$. With respect to the basis in $P^k[x, y]$ given by the monomials y^k, $-ky^{k-1}x$, $k(k-1)y^{k-2}x^2$, $\ldots, (-1)^k k! x^k$, the corresponding differential operators are realized by the matrices

$$h \mapsto \begin{pmatrix} k & \cdots & \cdots & 0 \\ \vdots & k-2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & -k \end{pmatrix}, e \mapsto \begin{pmatrix} 0 & \mu_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mu_k \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, f \mapsto \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \tag{B.2}$$

where $\mu_i = i(k-i+1)$, $i = 1, \dots, k$. It turns out that any irreducible representation of $sl(2,\mathbb{R})$ of dimension $k+1$ is equivalent (under a linear change of coordinates) to the one just described. An arbitrary representation of $sl(2,\mathbb{R})$ is a direct sum of irreducible ones.

When working with $gl(2,\mathbb{R})$ rather than $sl(2,\mathbb{R})$, one also has the 2×2 identity matrix $I_{2\times 2}$. It corresponds to the operator $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on $P^k[x,y]$, whose associated matrix is $kI_{(k+1)\times(k+1)}$. One can thus naturally extend the above representation to $gl(2,\mathbb{R})$. The complementary subalgebras $\mathbb{R}I$ and $sl(2,\mathbb{R})$ are invariant under the resulting action.

B.3 Nilpotent and solvable Lie algebras

If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are linear subspaces of a Lie algebra $\mathfrak{g}$, one writes $[\mathfrak{g}_1, \mathfrak{g}_2]$ for the linear space spanned by the products $[g_1, g_2]$ with $g_1 \in \mathfrak{g}_1$ and $g_2 \in \mathfrak{g}_2$. Given a Lie algebra $\mathfrak{g}$, the sequence $\mathfrak{g}^{(k)}$ is defined inductively as follows: $\mathfrak{g}^{(1)} := \mathfrak{g}$, $\mathfrak{g}^{(k+1)} := [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] \subset \mathfrak{g}^{(k)}$. If $\mathfrak{g}^{(k)} = 0$ for k sufficiently large, then $\mathfrak{g}$ is called *solvable*. This happens if and only if $\dim \mathfrak{g}^{(i+1)} < \dim \mathfrak{g}^{(i)}$ for all $i \geq 1$. Similarly, one defines the sequence $\mathfrak{g}^k$ by $\mathfrak{g}^1 := \mathfrak{g}$, $\mathfrak{g}^{k+1} := [\mathfrak{g}, \mathfrak{g}^k] \subset \mathfrak{g}^k$, and calls $\mathfrak{g}$ *nilpotent* if $\mathfrak{g}^k = 0$ for k sufficiently large. For example, if $\mathfrak{g}$ is a Lie algebra generated by two matrices A and B, we have $\mathfrak{g}^{(1)} = \mathfrak{g}^1 = \mathfrak{g} = \text{span}\{A, B, [A,B], [A,[A,B]], \dots\}$, $\mathfrak{g}^{(2)} = \mathfrak{g}^2 = \text{span}\{[A,B], [A,[A,B]], \dots\}$, $\mathfrak{g}^{(3)} = \text{span}\{[[A,B],[A,[A,B]]], \dots\} \subset \mathfrak{g}^3 = \text{span}\{[A,[A,B]], [B,[A,B]], \dots\}$, and so on. Every nilpotent Lie algebra is solvable, but the converse is not true.

The *Killing form* on a Lie algebra $\mathfrak{g}$ is the symmetric bilinear form K given by $K(a,b) := \text{tr}(\text{ad}a \circ \text{ad}b)$ for $a, b \in \mathfrak{g}$. *Cartan's first criterion* says that $\mathfrak{g}$ is solvable if and only if its Killing form vanishes identically on $[\mathfrak{g}, \mathfrak{g}]$. Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field, and let ϕ be a representation of $\mathfrak{g}$ on a vector space V. *Lie's theorem* states that there exists a basis for V with respect to which the matrices $\phi(g)$, $g \in \mathfrak{g}$ are all upper-triangular.

B.4 Semisimple and compact Lie algebras

A subalgebra $\bar{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ is called an *ideal* if $[g, \bar{g}] \in \bar{\mathfrak{g}}$ for all $g \in \mathfrak{g}$ and $\bar{g} \in \bar{\mathfrak{g}}$. Any Lie algebra has a unique maximal solvable ideal $\mathfrak{r}$, the *radical*. A Lie algebra $\mathfrak{g}$ is called *semisimple* if its radical is 0. *Cartan's second criterion* says that $\mathfrak{g}$ is semisimple if and only if its Killing form is nondegenerate (meaning that if for some $g \in \mathfrak{g}$ we have $K(g,a) = 0$ for all $a \in \mathfrak{g}$, then g must be 0).

A semisimple Lie algebra is called *compact* if its Killing form is negative definite. A general *compact Lie algebra* is a direct sum of a semisimple compact Lie algebra and a commutative Lie algebra (with the Killing form vanishing on the latter). This terminology is justified by the facts that the tangent algebra of any compact Lie group is compact according to this definition, and that for any compact Lie algebra $\mathfrak{g}$ there exists a connected compact Lie group $\mathcal{G}$ with tangent algebra $\mathfrak{g}$. Compactness of a semisimple matrix Lie algebra $\mathfrak{g}$ amounts to the property that all matrices in $\mathfrak{g}$ are diagonalizable and their eigenvalues lie on the imaginary axis. If $\mathcal{G}$ is a compact Lie group, one can associate to any continuous function $f : \mathcal{G} \to \mathbb{R}$ a real number $\int_{\mathcal{G}} f(G)dG$ so as to have $\int_{\mathcal{G}} 1dG = 1$ and $\int_{\mathcal{G}} f(AGB)dG = \int_{\mathcal{G}} f(G)dG$ for all $A, B \in \mathcal{G}$ (left and right invariance). The measure dG is called the *Haar measure.*

An arbitrary Lie algebra $\mathfrak{g}$ can be decomposed into the semidirect sum $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$, where $\mathfrak{r}$ is the radical, $\mathfrak{s}$ is a semisimple subalgebra, and $[\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{r}$ because $\mathfrak{r}$ is an ideal. This is known as a *Levi decomposition.* To compute $\mathfrak{r}$ and $\mathfrak{s}$, switch to a basis in which the Killing form K is diagonalized. The subspace on which K is not identically zero corresponds to $\mathfrak{s} \oplus (\mathfrak{r} \bmod \mathfrak{n})$, where $\mathfrak{n}$ is the maximal nilpotent subalgebra of $\mathfrak{r}$. Construct the Killing form $\overline{K}$ for the factor algebra $\mathfrak{s} \oplus (\mathfrak{r} \bmod \mathfrak{n})$. This form will vanish identically on $(\mathfrak{r} \bmod \mathfrak{n})$ and will be nondegenerate on $\mathfrak{s}$. The subalgebra $\mathfrak{s}$ identified in this way is compact if and only if $\overline{K}$ is negative definite on it.

B.5 Subalgebras isomorphic to $sl(2, \mathbb{R})$

Let $\mathfrak{g}$ be a real, noncompact, semisimple Lie algebra. Our goal here is to show that $\mathfrak{g}$ has a subalgebra isomorphic to $sl(2, \mathbb{R})$. To this end, consider a *Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ is its orthogonal complement with respect to K. The Killing form K is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. Let $\mathfrak{a}$ be a maximal commuting subalgebra of $\mathfrak{p}$. Then it is easy to check using the Jacobi identity that the operators $\mathrm{ad}a$, $a \in \mathfrak{a}$ are commuting. These operators are also symmetric with respect to a suitable inner product on $\mathfrak{g}$ (for $a, b \in \mathfrak{g}$ this inner product is given by $-K(a, \Theta b)$, where Θ is the map sending $k + p$, with $k \in \mathfrak{k}$ and $p \in \mathfrak{p}$, to $k - p$), hence they are simultaneously diagonalizable. Thus $\mathfrak{g}$ can be decomposed into a direct sum of subspaces invariant under $\mathrm{ad}a$, $a \in \mathfrak{a}$, on each of which every operator $\mathrm{ad}a$ has exactly one eigenvalue. The unique eigenvalue of $\mathrm{ad}a$ on each of these invariant subspaces is given by a linear function λ on $\mathfrak{a}$, and accordingly the corresponding subspace is denoted by $\mathfrak{g}_\lambda$. Since $\mathfrak{p} \neq 0$ (because $\mathfrak{g}$ is not compact) and since K is positive definite on $\mathfrak{p}$, the subspace $\mathfrak{g}_0$ associated with λ being identically zero cannot be the entire $\mathfrak{g}$. Summarizing, we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda \right)$$

where Σ is a finite set of nonzero linear functions on $\mathfrak{a}$ (which are called the *roots*) and $\mathfrak{g}_\lambda = \{g \in \mathfrak{g} : \mathrm{ad}a(g) = \lambda(a)g \; \forall\, a \in \mathfrak{a}\}$. Using the Jacobi identity, one can show that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu]$ is a subspace of $\mathfrak{g}_{\lambda+\mu}$ if $\lambda+\mu \in \Sigma \cup \{0\}$, and equals 0 otherwise. This implies that the subspaces $\mathfrak{g}_\lambda$ and $\mathfrak{g}_\mu$ are orthogonal with respect to K unless $\lambda + \mu = 0$. Since K is nondegenerate on $\mathfrak{g}$, it follows that if λ is a root, then so is $-\lambda$. Moreover, the subspace $[\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}]$ of $\mathfrak{g}_0$ has dimension 1, and λ is not identically zero on it. This means that there exist some elements $e \in \mathfrak{g}_\lambda$ and $f \in \mathfrak{g}_{-\lambda}$ such that $h := [e, f] \neq 0$. It is now easy to see that, multiplying e, f, and h by constants if necessary, we obtain an $sl(2)$-triple. Alternatively, we could finish the argument by noting that if $g \in \mathfrak{g}_\lambda$ for some $\lambda \in \Sigma$, then the operator $\mathrm{ad}g$ is nilpotent (because it maps each $\mathfrak{g}_\mu$ to $\mathfrak{g}_{\mu+\lambda}$ to $\mathfrak{g}_{\mu+2\lambda}$ and eventually to zero since Σ is a finite set), and the existence of a subalgebra isomorphic to $sl(2, \mathbb{R})$ is guaranteed by the Jacobson-Morozov theorem.

B.6 Generators for $gl(n, \mathbb{R})$

This section is devoted to showing that in an arbitrarily small neighborhood of any pair of $n \times n$ matrices, one can find another pair of matrices that generate the entire Lie algebra $gl(n, \mathbb{R})$. This fact demonstrates that the Lie-algebraic stability criteria presented in Section 2.2 are not robust with respect to small perturbations of the matrices that define the switched linear system.

We begin by finding some matrices B_1, B_2 that generate $gl(n, \mathbb{R})$. Let B_1 be a diagonal matrix $B_1 = \mathrm{diag}(b_1, b_2, \dots, b_n)$ satisfying the following two properties:

1. $b_i - b_j \neq b_k - b_l$ if $(i, j) \neq (k, l)$
2. $\sum_{i=1}^n b_i \neq 0$.

Denote by $od(n, \mathbb{R})$ the space of matrices with zero elements on the main diagonal. Let B_2 be any matrix in $od(n, \mathbb{R})$ such that all its off-diagonal elements are nonzero. It is easy to check that if $E_{i,j}$ is a matrix whose ijth element is 1 and all other elements are 0, where $i \neq j$, then $[B_1, E_{i,j}] = (b_i - b_j)E_{i,j}$. Thus it follows from Property 1 above that B_2 does not belong to any proper subspace of $od(n, \mathbb{R})$ that is invariant with respect to the operator $\mathrm{ad}B_1$. Therefore, the linear space spanned by the iterated brackets $ad^k B_1(B_2)$ is the entire $od(n, \mathbb{R})$. Taking brackets of the form $[E_{i,j}, E_{j,i}]$, we generate all traceless diagonal matrices (cf. $[e, f] = h$ in Section B.2). Since B_1 has a nonzero trace by Property 2 above, we conclude that $\{B_1, B_2\}_{LA} = gl(n, \mathbb{R})$.

Now, let A_1 and A_2 be two arbitrary $n \times n$ matrices. Using the matrices B_1 and B_2 just constructed, we define $A_1(\alpha) := A_1 + \alpha B_1$ and $A_2(\alpha) := A_2 + \alpha B_2$, where $\alpha \geq 0$. We claim that these two matrices $A_1(\alpha)$ and $A_2(\alpha)$

generate $gl(n, \mathbb{R})$ for every sufficiently small α. Indeed, consider a basis for $gl(n, \mathbb{R})$ formed by B_1, B_2, and their suitable Lie brackets. Replacing B_1 and B_2 in these expressions by $A_1(\alpha)$ and $A_2(\alpha)$, respectively, and writing the coordinates of the resulting elements relative to the above basis, we obtain a square matrix $\Delta(\alpha)$. Its determinant is a polynomial in α whose value tends to ∞ as $\alpha \to \infty$, and therefore it is not identically zero. Thus $\Delta(\alpha)$ is nondegenerate for all but finitely many values of α. In particular, $A_1(\alpha)$, $A_2(\alpha)$, and their suitable Lie brackets provide a desired basis for $gl(n, \mathbb{R})$ if α is sufficiently small.

Notes and References

Part I

There exist very general formulations, such as the theory of hereditary systems described in [111], which can in particular be applied to hybrid dynamical systems. However, it is clearly of interest to develop models specifically designed to capture interactions between continuous and discrete dynamics. An early model of a hybrid system, which closely corresponds to the state-dependent switched system model described in Section 1.1, appeared in [305]. Since then, more general models have been developed, incorporating impulse effects, controlled dynamics, different continuous state spaces for different discrete states, etc. [7, 224, 63, 50, 187, 290]. There has also been interest in models for hybrid systems that allow more general types of continuous dynamics, such as differential inclusions [82], stochastic differential equations [138], and differential-algebraic equations [296]. The paper [313] develops a model that uses general concepts of a time space and a motion. A very general behavioral definition of a hybrid system is presented in [296]. See also the work of [284] on abstract control systems.

When solving specific problems, the generic models mentioned above are often difficult to work with, and there have been many attempts to develop more concrete models of hybrid systems that are still general enough to cover applications of interest. The paper [55] proposes several models where symbolic strings trigger transitions between continuous dynamics; they are developed with the view toward modeling motion control systems but have a wider applicability. A similar model is considered in [24]. The model discussed in [27] is designed for the situations where a continuous plant is connected with a hybrid feedback controller. *Complementarity systems* studied in [296], where two variables are subject to inequality constraints and at all times at least one of these constraints is satisfied with equality, provide another example of a framework that covers many examples of hybrid systems. *Piecewise linear systems* are readily suitable for implementation and analysis and can be used as approximations of more general nonlinear and hybrid systems [67, 265]. *Mixed logical dynamical systems* defined in [37] are linear differential equations with inequality constraints involving real and integer variables, amenable to systematic control synthesis via techniques from mixed-integer quadratic programming. (Equivalences among the last three classes are discussed in [113].)

Hybrid dynamical systems are pervasive in today's technology and generate tremendous research interest. In the report [167], they were identified as one of the major challenges in control theory. In the last few years, every major control conference has had several sessions on hybrid systems. Workshops and symposia devoted specifically to this topic are regularly taking place [107, 20, 9, 21, 22, 213, 114, 294, 188, 40]. Almost every major technical control journal has had a special issue on hybrid systems [240, 23, 216, 86, 19, 39, 119]. Textbooks and research monographs on the subject are beginning to appear [296, 196, 282, 87].

Modeling tools frequently used for hybrid systems are those of discrete-event system theory, such as finite-state machines and Petri nets. Purely discrete abstractions of hybrid systems, which preserve temporal properties, are of interest in this context [8]. The references cited above contain many papers illustrating these approaches, which are beyond the scope of this book. These references also discuss various issues regarding solutions of hybrid systems; another body of literature which is relevant—and closer in spirit to systems theory—is the work on control of discrete-event systems [249]. In this book, we are concerned primarily with properties of the continuous state, the main research issues being stability analysis of switched systems and synthesis of switching control laws. For analysis purposes, it often makes sense to assume that the switching signal belongs to a certain class, ignoring the details of the discrete dynamics. The results presented in Chapters 2 and 3 are of this flavor.

For background on existence and uniqueness theorems for solutions of ordinary differential equations, the reader can consult standard textbooks such as [25, 152, 298]. The bouncing ball example is taken from [296]; see, e.g., [262] for more examples and further discussion of Zeno behavior. Filippov's generalized solution concept described in [92, 94] provides a general methodology for modeling (or approximating) fast switching by continuous sliding modes; there is a vast literature on the use of sliding modes in control theory, going back to the book [293] and the references discussed there. Another approach is to approximate discrete transitions by very fast continuous dynamics, arriving at a singularly perturbed system (see [49]).

When continuous dynamics for different discrete states of a hybrid system evolve on different state spaces, the analysis can be simplified if it is possible to paste these spaces together, obtaining a single switched system evolving on a "hybrifold" [262, 166].

Switched systems have numerous applications in robotics, automotive industry, aircraft and air traffic control, power systems, and many other fields. The book [213] contains reports on various developments in some of these areas. Example 1.1 points to just one of the many places in which switching occurs in modern automobiles. As automotive technology evolves, new ways to explore switching will undoubtedly appear. One advantage of switching is that it helps avoid the use of overly multi-functional devices which sacrifice performance. For example, automobile tires are designed to maintain the desired contact with the road surface when the car is in motion. However, when it comes to braking, they may not provide the most efficient solution. Arguably, humans have the ability to come to a complete stop more abruptly than wheeled vehicles. On a science-fiction note, one can imagine a two-legged (or multi-legged) device which extends from the bottom of a car to the ground to facilitate the braking when necessary. This would be an example of a switching control mechanism.

Part II

Restricting our attention in Part II to Problems 1 and 2, we are of course ignoring other interesting problem formulations. For example, we are assuming that the switching signal is either arbitrary or belongs with certainty to some class of signals. An intermediate case, not studied here, is when the switching conforms to some probability law, so it is neither completely arbitrary nor precisely known (see [145, 109, 36, 291, 77], Chapter 9 of [261], and the references therein for some results on systems in which the switching is governed by a Markovian jump process). An important topic which is complementary to equilibrium analysis is stability of limit cycles in switched systems; see, e.g., [196, 104]. Other concepts are of interest besides stability, especially for systems with inputs and/or outputs; these include input/output (or input/state) properties [129, 127, 173, 192, 309, 118], controllability [184, 185, 88, 2, 278, 101], switched observer design [242, 183, 6], passivity [241, 315], and output tracking [225, 148].

Chapter 2

In defining uniform stability concepts for switched systems in Section 2.1.1, we used the notion of a class $\mathcal{KL}$ function; the same approach is taken in [191]. Alternatively, these stability concepts can be defined using the more familiar ε-δ formalism (see, e.g., [78]). The equivalence between the two formulations is proved (in the context of systems with disturbances) in [181].

An early converse Lyapunov theorem for switched systems was proved in [198] (derived in the setting of differential inclusions, the results of that paper can be easily adopted to switched systems). It is shown in [198] that for linear systems, GUAS implies the existence of a common Lyapunov function satisfying quadratic bounds; this result is then extended to nonlinear systems that are uniformly asymptotically stable in the first approximation. (Incidentally, the second part of the same paper [198] describes the control design technique that we now know as *integrator backstepping* [161].) In [203] it is shown that for asymptotically stable linear differential inclusions, one can always find a common Lyapunov function that is homogeneous of degree 2 or piecewise quadratic; several algebraic criteria for asymptotic stability are also derived in that paper. Detailed proofs of these converse Lyapunov theorems in the context of switched systems are also given in [78]. In [181], a smooth converse Lyapunov theorem is proved for nonlinear systems with disturbances. Associating to a switched system a system with disturbances and applying the result of [181], one can arrive at the converse Lyapunov theorem for switched nonlinear systems which we stated as Theorem 2.2; this is done in [191]. The converse Lyapunov theorem for differential inclusions proved in [288] also gives this result as a corollary.

The relaxation theorem for locally Lipschitz differential inclusions, known as the Filippov-Ważewski theorem, says that a solution of the "relaxed" differential inclusion (i.e., the one obtained by passing to the closed convex hull) defined on a finite interval can be approximated by a solution of the original differential inclusion on that interval, with the same initial condition. For details see, e.g., [30, 94]. A more recent result proved in [139] guarantees approximation on infinite intervals, but possibly with different initial conditions for the two solutions. (The same paper also gives a counterexample demonstrating that approximation on infinite intervals with matching initial conditions may not be possible.) The latter result is more suitable for studying global asymptotic convergence, while the former is useful for investigating uniform Lyapunov stability. Computational aspects of verifying stability of all convex combinations for linear systems are addressed in [43] and the references therein.

It has been known for some time that for switched linear systems, uniform asymptotic stability is equivalent to GUES (under an appropriate compactness assumption); see, e.g., [203] and the references therein. The equivalence between these properties and attractivity, expressed by Theorem 2.4, is proved in the recent paper [14] which deals with the larger class of switched homogeneous systems. The proof relies on the fact that local uniform stability plus global attractivity imply GUAS, established for general nonlinear systems in [275].

A general reference on LMIs is [45]. The infeasibility condition (2.12), which can be used to check that a quadratic common Lyapunov function does not exist, is based on the duality principle of convex optimization; for a discussion and references, see [45, pp. 9 and 29]. The system discussed in Section 2.1.5 is actually a specific member of a family of switched linear systems described in [78]. A different example illustrating the same point, namely, that the existence of a quadratic common Lyapunov function is not necessary for GUES of a switched linear system, is given in [246]. A sharp result on the existence of quadratic common Lyapunov functions in the discrete-time case is obtained in [13].

Theorem 2.7 was proved in [178], and the details of the arguments given in Section 2.2.2 (including the treatment of complex matrices) can be found there. Lie's theorem and the triangular structure that it provides were also used in the derivation of the well-known Wei-Norman equations [300]; further relations (if any) between the two results remain to be understood. The fact that a family of linear systems with Hurwitz triangular matrices has a quadratic common Lyapunov function, used in the proof of Theorem 2.7, is relatively well known; see, e.g., [70, 205, 259]. The paper [259] summarizes various conditions for simultaneous triangularizability of linear systems (also see, e.g., [134, Chapter 2]). For an extension to the case of pairwise triangularizable matrix families, see [257]. Some related results on diagonal quadratic common Lyapunov functions are discussed in [230].

The results presented in Section 2.2.3 are taken from [3]. More material, including complete proofs, can be found in that paper. For examples of P. Hall bases for nilpotent Lie algebras (and their use in control theory), see [220]. The perturbation argument given in Section 2.2.4 is quite standard; it is presented in a more general setting in [152, Example 5.1] and applied to switched linear systems in [205].

The material of Section 2.3.1 is taken from the paper [17], which can be consulted for proofs and extensions. Nonlinear versions of Lie's theorem, which provide Lie-algebraic conditions under which a family of nonlinear systems can be simultaneously triangularized, are developed in [74, 151, 194]. However, in view of the need to satisfy coordinate-dependent ISS conditions such as those of Theorem 2.14, these results are not directly applicable to the stability problem for switched nonlinear systems. Moreover, the methods described in these papers require that the Lie algebra spanned by a given family of vector fields have full rank (LARC), which is not true in the case of a common equilibrium.

Passivity is a special case of *dissipativity*, which is defined as the property that a certain function of the input and the output (called the *supply rate*) has a nonnegative integral along all trajectories. Characterizations of dissipativity in terms of storage functions are discussed, e.g., in [132]. Many results and references on passivity and its use in nonlinear control can be found in the book [254]. See [193] for an application of passivity to switching control design for cascade nonlinear systems.

The original references on the Kalman-Yakubovich-Popov lemma—and its use in the context of the absolute stability problem posed by Lurie in the 1940s—are [244], [311], and [150]. The connection between positive realness and Lyapunov functions containing integrals of the nonlinearities was also investigated in [51]. A proof of the circle criterion using loop transformations and the Kalman-Yakubovich-Popov lemma is given, e.g., in [105, 152]; an alternative treatment can be found in [52]. For a detailed discussion of Popov's criterion, see [152, 298] and the references therein. The converse result that for two-dimensional systems the existence of a quadratic common Lyapunov function for all nonpositive feedback gains implies strict positive realness of the open-loop transfer function is proved in [156]. A different frequency-domain condition for the existence of a quadratic common Lyapunov function is presented in [60].

A proof of the necessary and sufficient small-gain condition for the existence of a quadratic common Lyapunov function for the family of feedback systems (2.36) can be found in [237, 153]; the result is originally due to Popov. The application to stability of switched linear systems was pointed out to us by Andy Teel. Further results on connections between small-gain conditions, passivity, and the existence of quadratic common Lyapunov functions, as well as numerical algorithms for finding the latter, are presented in [46]. For additional references and historical remarks on the developments surrounding passivity and small-gain theorems, see [157]. Methods for reducing passivity and small-gain conditions to solving Riccati equations and LMIs, as well as computational aspects, are discussed in [45, pp. 25–27 and 34–35]. Many additional relevant references are provided in that book (see, in particular, pp. 72–73).

The results on coordinate changes and realizations of stabilizing controllers are taken from [116, 131]. Incidentally, the proof of Theorem 2.15 relies on the existence of a quadratic common Lyapunov function for a family of triangular asymptotically stable linear systems. Further results that utilize the feedback structure of subsystems being switched are discussed in [212, Section VIII] and [117].

Before a complete solution to the problem of finding necessary and sufficient conditions for GUES of two-dimensional switched homogeneous systems was obtained in [93], the case of two-dimensional switched linear systems in feedback form was settled in [246]. These results were subsequently generalized in [247], yielding in particular necessary and sufficient conditions for GUES of three-dimensional switched linear systems in feedback form as well as general two-dimensional switched linear systems. More recently, alternative necessary and sufficient conditions for GUES in two dimensions were derived for a pair of linear systems in [44] and for a finite family of homogeneous systems in [133]. The proofs given in the last three references are based on the method of worst-case switching analysis, instantiated also by the example considered in Section 2.1.5.

Proposition 2.16 is taken from [69, Section IV]; see also [260]. The preliminary result mentioned before the statement of Proposition 2.16, namely, that a quadratic common Lyapunov function exists if all convex combinations have negative real eigenvalues, was obtained in [258]. Necessary and sufficient conditions for the existence of a quadratic common Lyapunov function for a family of three linear systems are also available; the systems must satisfy the hypothesis of Proposition 2.16 pairwise, and some additional conditions must hold [260]. The same paper notes that by Helly's theorem, a family of more than three linear systems in the plane has a quadratic common Lyapunov function if and only if so does every triple of systems from this family; in general, for a family of linear systems in $\mathbb{R}^n$ one must check all subfamilies containing $n(n+1)/2$ systems. Further results for the planar case involving nonquadratic common Lyapunov functions are reported in [308].

The problem of characterizing the GUAS property of switched systems continues to attract a lot of research interest. We have attempted to describe the main research avenues. Other relevant results include the ones using the concepts of matrix measure [297, 88, 168], spectral radius [34, 76, 109, 13], connective stability [261], and miscellaneous algebraic conditions [228, 229, 206]. The paper [12] deals with the related problem of characterizing the sets of linear systems with the property that they share a quadratic common Lyapunov function which does not serve as a Lyapunov function for any system outside the set. Most of the results discussed here can be carried over to the case when the individual subsystems evolve in discrete time, by associating the discrete-time subsystems with suitable continuous-time ones, although there are some notable differences (as discussed, e.g., in [207, 5]). Discrete-time switched systems are closely related to discrete multi-dimensional systems, whose study is motivated by discretization of PDEs; see [100] where results on asymptotic stability of such systems in the case of commuting matrices—which parallel those discussed in Section 2.2.1—are presented. The recent paper [144] describes how a switched linear system can be used to model mobile autonomous agent coordination, both in discrete and in continuous time, and discusses stability results that utilize the special structure of the matrices arising in that problem.

Chapter 3

When discussing stability of switched systems, we assume that there are no impulse effects, i.e., that the state trajectory remains continuous at switching times. Results dealing specifically with stability of systems with impulse effects can be found, e.g., in [235, 33, 312, 164, 251, 110, 31, 131, 169]. Although not studied here, such systems are important and arise, for example, in models of switched electrical circuits due to jumps in currents and voltages.

The stability results for switched systems using multiple Lyapunov functions, such as Theorem 3.1, are close in spirit to the aforementioned results on impulsive systems. The idea first appeared in [236]. In that paper, a result similar to Theorem 3.1 was proved, but under the stronger condition that at switching times, the values of Lyapunov functions corresponding to the active subsystems form a decreasing sequence (this was called the "sequence nonincreasing condition" in [49]). As shown in [49] (see also the earlier conference papers [47, 48]), this condition gives Lyapunov stability in the case of a general compact index set $\mathcal{P}$ under a suitable continuity assumption. The same paper also introduced a weaker condition of the kind we used in Theorem 3.1, whereby one only needs to compare the values of each Lyapunov function at the times when the corresponding subsystem becomes active, separately for each subsystem index. (According to the terminology of [49], functions satisfying such conditions are called "Lyapunov-like.") For extensions allowing a Lyapunov function to grow during the period on which the corresponding subsystem is active, provided that this growth is bounded via a positive definite function, see [135, 239] (as well as [313], where the results of [135] are presented in a more general setting).

The paper [112] develops a stability result based on multiple Lyapunov functions for systems with switching generated by a finite automaton, whose structure can be used to reduce the class of switching signals for which the conditions need to be checked. In [117], multiple weak Lyapunov functions are used to obtain an extension of LaSalle's invariance principle to switched linear systems; a nonlinear counterpart which relies on a suitable observability notion for nonlinear systems is discussed in [125]. (Another version of LaSalle's theorem for hybrid systems, which uses a single Lyapunov function, appeared in [318].) The idea of using multiple Lyapunov functions is also exploited in [315], where a notion of passivity for hybrid systems which involves multiple storage functions is studied.

Multiple Lyapunov functions provide a useful tool for switching control design. We mention the "min-switching" strategies with an application to pendulum swing-up described in [189], the stability criteria for systems with different equilibria and the stabilization results for systems with changing dynamics developed in [316], and the switching control scheme incorporating multiple Lyapunov functions in an optimal control framework proposed in [159].

Slow switching assumptions greatly simplify stability analysis and are, in one form or another, ubiquitous in the literature on switched systems and control. The results on stability under slow switching discussed in Section 3.2 parallel the ones on stability of slowly time-varying systems (cf. [264] and the references therein). The motivation comes in particular from adaptive control: loosely speaking, the counterpart of a dwell-time switching signal is a tuning signal with bounded derivative, while the counterpart of an average dwell-time switching signal is a "nondestabilizing" tuning signal in the sense of [208]. The term "dwell time" was apparently coined in [210]. For a proof that a sufficiently large dwell time guarantees exponential stability of a switched linear system generated by a compact family of Hurwitz matrices, see, e.g., [212, Lemma 2]; an explicit bound—very similar to (3.5)—is obtained there using direct manipulations with matrix exponentials. Theorem 3.2, which generalizes the above result, and its proof using multiple Lyapunov functions are taken from [129]. The fact that taking $W_p(x) = \lambda_p V_p(x)$ in (3.11) does not lead to a loss of generality is proved, e.g., in [245]. An extension of the average dwell time analysis to the case when some of the subsystems are unstable is described in [317]. It is interesting to note that the concept of average dwell time has a counterpart in information theory, which has been used to characterize average rate of traffic flow in communication networks [75] (this reference was kindly provided by Bruce Hajek).

The material of Section 3.3 is based on [149] and [239]. More details on LMI formulations of the search for multiple Lyapunov functions for state-dependent switched linear systems are given in these papers and in the survey article [79]. See also the related work reported in [103] where global asymptotic stability of state-dependent switched linear systems is studied with the help of Lyapunov functions defined on the switching surface rather than on the entire state space, and stability conditions are presented in the form of LMIs. References and historical remarks on the S-procedure are provided in [45, pp. 33–34].

Theorem 3.4 was proved in [303] (see also the earlier conference paper [302]). Other stabilization techniques relying on Hurwitz convex combinations (namely, rapid time-dependent switching and sliding mode control), as well as an algorithm for finding such convex combinations, are also discussed there. The latter topic is further addressed in [170]. The NP-hardness of the problem of identifying stable convex combinations is proved in [42]. The term "quadratic stability" is quite standard in the robust control literature, where it means the existence of a single quadratic Lyapunov function for all admissible uncertainties.

Conic switching laws such as the one described in Example 3.3 are studied in [310]. The remainder of Section 3.4.2—devoted to multiple Lyapunov functions—is based on [189], where the same "min-switching" scheme is presented in the more general context of nonlinear systems, and [301], where additional algebraic sufficient conditions for stabilizability are derived. An example of asymptotic stabilization of a switched linear system via state-dependent switching and multiple Lyapunov functions is given in [236]. The same paper discusses the problem of choosing a quadratic function that decreases along solutions of a given linear system in some nonempty conic region. An earlier reference which presents an application of state-dependent stabilizing switching strategies is [143].

Part III

We remark that there is no general theory of switching control, nor is there a standard set of topics to be covered in this context. Therefore, the material in Part III is to be viewed merely as a collection of illustrative examples. In particular, we only address stabilization and set-point control problems, although there are many other control tasks of interest, such as trajectory tracking and disturbance rejection.

Chapter 4

The exposition in Section 4.1.1 is based largely on the survey article [271]; additional examples and discussion can be found there, as well as in [270, Section 5.9]. Early references on topological obstructions to asymptotic stabilization—and ways to overcome them—are [280, 272, 26] (these papers discuss discontinuous controls, continuous time-varying controls, and relaxed controls, respectively). See also the paper [227]. The book [202] provides a good quick introduction to some of the mathematical concepts encountered in Section 4.1.

Brockett's paper [54], which established the necessary condition for feedback stabilization (Theorem 4.1), is certainly one of the most widely referenced works in the nonlinear control literature. (The well-known nonholonomic integrator example studied in Section 4.2.1 was also introduced there.) This paper has spurred a lot of research activity, which can be loosely classified into three categories. First, there was the issue of filling in some technical details in the proof. Brouwer's fixed point theorem states that a smooth (or at least continuous) function from a ball to itself has a fixed point. Brockett's original argument, paraphrased in Section 4.1.2, cited the result from [304] that the level sets of Lyapunov functions are homotopically equivalent to spheres. However, this fact is not enough to justify an application of a fixed point theorem, because in R^n with $n = 4$ or 5 it is not known whether these level sets are diffeomorphic to spheres (for $n = 4$ this is closely related to the Poincaré conjecture; in fact, it is not even known whether the level sets are homeomorphic to a sphere in this case). For details and references on this issue, see [108]. As pointed out in [271], a fixed point theorem can indeed be applied because the sublevel set $\mathcal{R}$ is a retract of $\mathbb{R}^n$. An altogether different proof of Brockett's condition can be given using degree theory. This is mentioned briefly in Brockett's original proof; for details, see [314, 267] and [270, Section 5.9] (in [267], Sontag attributes the argument to Roger Nussbaum). This proof actually yields a stronger conclusion, namely, that the degree of the map (4.4) is $(-1)^n$ near zero. As explained in [271], there are some systems, such as the one considered in [26, Counterexample 6.5], which satisfy Brockett's condition but violate the degree condition. Actually, Brockett's condition was originally stated in a slightly weaker form than even his proof automatically provides:

it was not specified that the image of every neighborhood of zero—no matter how small—under the map (4.3) contains a neighborhood of zero (i.e., that this map is open at zero). As demonstrated in Section 4.2.1, this more precise condition is required for treating systems such as the unicycle (it is easy to restate the result to accommodate the case when $k(0) \neq 0$). Another necessary condition for feedback stabilizability which is close in spirit to Brockett's but stronger—stated in terms of homology groups—is established using degree theory in [72]. See [62] for a generalization of Brockett's condition to systems with parameters (whose proof is also based on degree theory).

The second research direction that originated from Brockett's result was to understand if it also holds for more general, not necessarily smooth and static, feedback laws. As shown in [314], Brockett's condition still applies if one allows continuous feedback laws, as long as they result in unique trajectories. (The same paper proves that if Brockett's condition fails to hold, then the system cannot be rendered globally attractive by continuous feedback, even without requiring asymptotic stability, and for systems of dimension $n \leq 2$, Lyapunov stability is also impossible.) It is not difficult to see that Brockett's condition extends to systems with continuous dynamic feedback, if the overall closed-loop system is required to be asymptotically stable. A much deeper result is that the existence of a discontinuous stabilizing feedback also implies Brockett's condition if solutions are interpreted in Filippov's sense, at least for systems affine in controls; see [252, 73, 64]. Thus one is left with two possibilities for stabilizing systems that violate Brockett's condition. One is to use continuous time-varying feedback or, more generally, dynamic feedback that stabilizes the original system states only. The other is to use discontinuous feedback and interpret solutions in the sense other than Filippov's (for example, via the approach developed in [68]) or, more generally, use hybrid feedback.

Stabilization of systems that violate Brockett's condition, and so cannot be asymptotically stabilized by continuous static feedback, constitutes the third major research direction. It was recognized that general classes of nonholonomic systems fall into this category (cf. [243, 41]), and various stabilizing control laws of the aforementioned kinds have been developed for these systems. The switching stabilizing control law for the nonholonomic integrator described in Section 4.2 is similar to the one presented in [85]. More advanced switching control strategies which are close in spirit to these are developed in [234] and [130]. Since every kinematic completely nonholonomic system with three states and two control inputs can be converted to the nonholonomic integrator by means of a state and control transformation [220], a fairly general class of nonholonomic systems can be treated by similar methods. Many other relevant results are referenced and discussed in the survey articles [158] and [197]; the first article is devoted specifically to nonholonomic systems and reviews various control techniques for these systems, while the second one concentrates on hybrid control design. Another topic of interest for nonholonomic systems is motion planning, which is not affected by Brockett's condition but still is challenging.

For details on the Lie bracket computations and the controllability analysis of the nonholonomic integrator illustrated in Figures 23 and 26 in Section 4.2, see, e.g., [58]. The relationship between Lie algebras and controllability for general nonlinear systems is a classical topic in control theory, addressed in many books such as [142, 226, 270]. References to early works on the subject can be found in [53, 115, 281]. An overview of results particularly relevant to nonholonomic systems is given in [219, Chapter 7].

Stabilization of an inverted pendulum is a popular benchmark problem in nonlinear control, studied by many researchers. The material of Section 4.3 is drawn primarily from [29]. The argument proving that a continuous globally stabilizing feedback does not exist was shown to us by David Angeli; the same proof also appears in [256]. The control law (4.16) is a special case of the one proposed and analyzed in [286, Example 4.1]. The problem of designing a continuous stabilizing feedback with a large domain of attraction is addressed in [15] and [250]. The above sources, in particular [29], can be consulted for additional references on the subject.

Chapter 5

A more detailed exposition of the time-optimal control problem and the bang-bang principle for (time-varying) linear systems can be found, e.g., in [154] or [270, Chapter 10]. For an insightful discussion of applicability of the bang-bang principle to nonlinear systems, see [281].

The treatment of the dynamic output feedback stabilization problem for stabilizable and detectable linear systems can be found in many textbooks; see, e.g., [306, Section 6.4]. The static output feedback stabilization problem, and the difficulties associated with it, are thoroughly discussed in the survey article [283]. It is shown in [182] that every controllable and observable linear system admits a stabilizing hybrid output feedback law that uses a countable number of discrete states. The finite-state hybrid output feedback stabilization problem, considered in Section 5.2, was posed in [27]. Example 5.2 was also examined in that paper and later revisited in [174]. For two-dimensional systems, this problem is further studied in [38] and [137]. Note that a special case is the problem of finite-state hybrid *state* feedback stabilization for linear systems that are not stabilizable by static state feedback (cf. [263]).

The matrix variable elimination procedure which we used in arriving at the inequalities (5.15) and (5.16) is described in [45, pp. 22–23 and 32–33]. For $\beta_1 = \beta_2 = 0$ these inequalities decouple into two pairs of inequalities which are equivalent to the LMIs that characterize stabilizability of each subsystem by static output feedback, given in [283, Theorem 3.8]. A condition involving a set of LMIs which guarantees the existence of a stabilizing switching strategy based on output measurements and observer design (for a switched system with no control inputs) is derived in [91]; using this result and the above elimination procedure, one can obtain conditions for stabilizability of the system (5.7) via observer-based switched output feedback. In contrast with the methods considered in Section 5.2, one could employ time-dependent switching among a family of output gains, for example, periodic switching (cf. [1]). This is related to the problem of stabilization by means of static time-varying output feedback, whose solution is not known except in special cases [57, 204].

For some examples of problems where considerations mentioned at the beginning of Section 5.3 arise, see the articles in [61] and the references therein. Effects of quantization on the performance of control systems, and control design techniques for systems with quantized signals, have been major topics of study in control theory. Only a few relevant references are listed below, and they can be consulted for further information. Stochastic methods have played a role [81], but here we confine ourselves to deterministic nonlinear analysis. A standard assumption made in the literature is that the quantization regions (typically rectilinear) are fixed in advance and cannot be changed by the control designer; see, among many sources, [201, 81, 248, 90, 279]. Some work concerned with the question of how the choice of quantization parameters affects the behavior of the system is reported in [28, 307, 84, 141, 177, 155]. Quantization regions of general shapes are considered in [186].

The result of Lemma 5.1 is well known (see, e.g., [201, Theorem 3] and [81, Proposition 2.3] for results along these lines). The generalization to the nonlinear case using the ISS assumption was obtained in [175]. In [172] these results were extended to more general types of quantizers, with quantization regions having arbitrary shapes. This extension is useful in several situations. For example, in the context of vision-based feedback control, the image plane of the camera is divided into rectilinear regions, but the shapes of the quantization regions in the state space which result from computing inverse images of these rectangles can be rather complicated. So-called Voronoi tessellations (see, e.g., [83]) suggest that, at least in two dimensions, it may be beneficial to use hexagonal quantization regions rather than more familiar rectangular ones.

The idea of using variable state quantization was introduced and explored in [59] in the context of linear systems and subsequently extended to nonlinear systems in [175] and [172]. For details on how one can relax the assumptions in the nonlinear case and obtain weaker versions of the results, see [175]. The material of Sections 5.3.4 and 5.3.5 is taken from [172]. The control law presented in Section 5.3.6 is a slightly modified version of the one considered in [59] and is somewhat similar to the control scheme described in [80, 81]; the results of [285, 238, 32, 221, 176], as well as the earlier work [307] which deals with memoryless quantizers and time delays, are also relevant. Several related developments not covered here, such as quantized control of discrete-time systems, state-dependent switching, time sampling, and sliding mode control, are discussed in [59]. See also [136] where these methods are explored in the context of sampled-data control and local asymptotic stabilization of nonlinear systems is treated using linearization. The results presented here can be readily extended to the setting of [155] where quantization is combined with hysteresis, since Conditions 1 and 2 of Section 5.3.1 are still satisfied.

It is shown in [266] that if the affine system (5.75) is asymptotically stabilizable, then the system (5.76) is input-to-state stabilizable with respect to an actuator disturbance e. The fact that a similar statement does not hold for the system (5.77) with a measurement disturbance was demonstrated by way of counterexamples in [95] and later in [89]. Thus the problem of finding control laws that achieve ISS with respect to measurement disturbances is a nontrivial one, even for systems affine in controls. This problem has recently attracted considerable attention in the literature [97, 89, 146]. In particular, it is shown in [97, Chapter 6] that the class of systems that are input-to-state stabilizable with respect to measurement disturbances includes single-input plants in strict feedback form. As pointed out in [268], the above problem also has a solution for systems that admit globally Lipschitz control laws achieving ISS with respect to actuator disturbances. Of course, for linear systems with linear feedback laws all three concepts (asymptotic stabilizability, input-to-state stabilizability with respect to actuator errors, and input-to-state stabilizability with respect to measurement errors) are equivalent.

An important research direction is to develop performance-based theory of control with limited information. Performance of hybrid quantized feedback control policies of the kind considered here can be evaluated with respect to standard criteria, such as control effort, transient behavior, or time optimality. Of particular significance, however, are performance measures that arise specifically in the context of quantized control. An interesting optimal control problem obtained in this way consists of minimizing the number of crossings of the boundaries between the quantization regions—and the consequent changes in the control value—per unit time, which can be interpreted as the problem of minimizing the "attention" needed for implementing a given control law [56, 177]. Another relevant problem is that of stabilizing the system using the coarsest possible quantizer [84], which corresponds to reducing the data rate; this is related to the developments of Section 5.3.6.

Chapter 6

For background on adaptive control, the reader can consult textbooks such as [140]. The idea of using switching in the adaptive control context has been around for some time, and various results have been obtained; see, e.g., [195, 98, 200, 223, 160] and the overview and additional references in [211]. In the form considered here, supervisory control originates in the work of

Morse [212, 214]. For a comparison of supervisory control and traditional adaptive control and examples illustrating the advantages of supervisory control mentioned in Section 6.1, see [123]. Specific references on which the presentation in this chapter is based are given below. Related work by other researchers includes [163], [232], and the papers in the special issue [119]. We also point out that the term "supervisory control" has other meanings in the literature (cf. [249]).

The problem of constructing a finite controller family for an infinite family of process models is studied in [231, 233, 10]. For more on state sharing for nonlinear systems, see [126]. Some methods for defining monitoring signals by processing the estimation errors in a more sophisticated way than done here are discussed in [163, 218, 11]. The example of Section 6.3 is borrowed from the appendix of [120]. The additive hysteresis switching logic used in that example is studied in [199, 217].

For details on the multi-estimator design for linear systems presented in Section 6.5.1, see [212]. In particular, the inequalities (6.14) and (6.15) are immediate consequences of the equation (28) in that paper. The formula (6.12) describes the class of admissible process models treated in [214], while the class defined by (6.13) was investigated in [124]. More information on coprime factorizations can be found, e.g., in [319]. Multi-estimators for nonlinear systems are further discussed in [116, 126]. The work on coprime factorizations for nonlinear systems described in [295] is also relevant in this regard.

The developments of Section 6.5.2 are adopted from [126] and [121]. The linear prototypes of these results—the certainty equivalence stabilization and certainty equivalence output stabilization theorems from adaptive control—were established in [209]. The earlier paper [208] introduced the concept of detectability (tunability) in the adaptive control context. In [126], an extension of the certainty equivalence stabilization theorem to nonlinear systems (Part 1 of Theorem 6.1) was obtained using the Lyapunov-like characterization of ISS. The iISS case was treated in [121] using more direct time-domain techniques. The related problems of input-to-state and integral-input-to-state disturbance attenuation are addressed in [161, 273, 299, 96, 292, 162, 287, 173, 180]. For a nonlinear version of the certainty equivalence output stabilization theorem, based on a novel variant of the minimum-phase property for nonlinear systems, see [179].

The dwell-time switching logic is studied in the context of supervisory control in [212]. That paper also explains how one can remove the slow switching requirement with the help of additional analysis tools. The scale-independent hysteresis switching logic was proposed in [116, 128]. These references provide technical details on why the switching signal is well defined and no chattering occurs, as well as a proof of Corollary 6.3; the more general result expressed by Lemma 6.2 was derived later. The present treatment, including the definition of the hierarchical hysteresis logic and the proof of Lemma 6.4, is based on [122]. A different hysteresis-based switching logic for dealing with a continuum, called the "local priority hysteresis switching logic," is described in [124].

For details on state-space realizations of the linear supervisory control system described in Section 6.6, see [212]. The analysis of Section 6.6.1 for the case of scale-independent hysteresis, using the result on stability of switched systems with average dwell time, is taken from [124]. In that paper, robustness with respect to unmodeled dynamics is also established, and an alternative approach which involves "normalized" estimation errors is sketched. Results on induced norms of switched systems with average dwell time (such as Lemma 6.6) are discussed in [129]. (In fact, it is exactly with analysis of hysteresis-based switching control algorithms in mind that the concept of average dwell time was developed in [129].) The extension to infinite process model sets, outlined in Section 6.6.2, is presented in [122]. Robustness of supervisory control algorithms for linear systems which rely on the dwell-time switching logic is thoroughly investigated in [214, 215].

The exposition of Section 6.7 follows [121]. For the version of Barbalat's lemma used in the proof of Proposition 6.9 see, e.g., [4, p. 58]. An extension of these results to nonlinear systems with infinite parametric uncertainty, using hierarchical hysteresis switching and a finite controller family along the lines of Section 6.6.2, is also possible, provided that each controller yields a suitable robust iISS property of the injected system. The example treated in Section 6.8 is studied in greater detail in [120] (actually using slightly different monitoring signals and switching logic). A parking movie generated with MATLAB Simulink which illustrates the corresponding motion of the unicycle is available from the author's webpage [171]. Some other results on control of nonholonomic systems with uncertainty can be found in [289, 147, 277, 71].

One advantage of supervisory control is its rapid adaptation ability, which suggests that supervisory control algorithms may yield better performance than more traditional adaptive control techniques. While the quantitative results obtained so far (see especially [215]) and successful use of supervisory control in applications [66, 99, 65, 123] are encouraging, a performance-based comparative study of adaptive control methods based on logic-based switching and on continuous tuning is yet to be carried out. Another important direction for future research is to develop provably correct supervisory control algorithms for nonlinear systems which do not rely on the termination of switching and are thus applicable in more general settings than the one treated in Section 6.7.

Part IV

Appendix A

Basic textbooks on nonlinear systems, such as [152] or [298], can be consulted for a more general treatment of Lyapunov stability and additional information. The original paper on LaSalle's invariance principle is [165]; the asymptotic stability part (for the case when $M = \{0\}$) was proved earlier in [35].

The concept of ISS was proposed in [266]. Its Lyapunov characterization was established in [274]. Further characterizations are given in [275], and in particular it is shown there that asymptotic stability under the zero input implies local ISS (see also [152, Section 5.3]). Corresponding notions for systems with outputs (OSS and IOSS) were defined in [276]. The integral variant (iISS) was introduced in [269]. It was subsequently studied in [18], where the iOSS property is also considered. The necessity part of Lemma A.4 was noted in [269], while sufficiency was proved more recently in [16].

Appendix B

This appendix on Lie algebras is borrowed from [3]. Most of the material is adopted from [106, 253], and the reader is referred to these and other standard references for further information. For more details and examples on the construction of a Levi decomposition, see [102, pp. 256–258].

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Notation

$\lvert \cdot \rvert$	Euclidean norm on $\mathbb{R}^n$
$\lVert \cdot \rVert$	matrix norm induced by Euclidean norm
$:=$	equal by definition
$\equiv$	identically equal
$\square$	end of proof, example, or remark
$\circ$	function composition
$\lambda_{\max}$ ($\lambda_{\min}$)	largest (smallest) eigenvalue
$\sigma_{\max}$	largest singular value
tr	trace
$[\cdot, \cdot]$	Lie bracket
$\{\cdots\}_{LA}$	Lie algebra generated by given matrices
t^+ (t^-)	time right after (right before) t
x_i, $(x)_i$	ith component of vector x
$(x; y; z)$	stack vector $(x^T, y^T, z^T)^T$
$N_\sigma(t, t_0)$	number of discontinuities of signal σ on interval (t_0, t)
$\Phi(t, t_0)$	transition matrix of a linear system on interval (t_0, t)
$\mathcal{L}_1$ ($\mathcal{L}_2$, $\mathcal{L}_\infty$)	space of integrable (square-integrable, bounded) functions
$\mathcal{C}^1$	space of continuously differentiable functions
Re	real part
cl	closure
co	convex hull
∂	boundary; partial derivative

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