Optimal Computation of Avoided Words

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No non-trivial solution was provided [Brendel et al, 1986].

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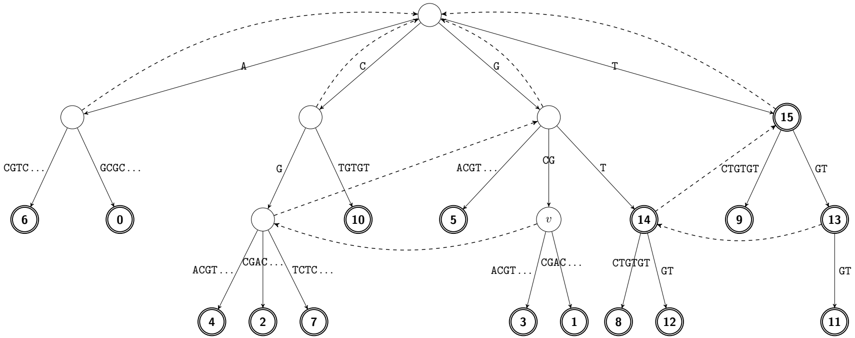
Example



A
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CGTC...
GCGC...
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ACGT...
CG
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CTGTGT
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9
13
ACGT...
CGAC...
TCTC...
ACGT...
CGAC...
CTGTGT
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12
11

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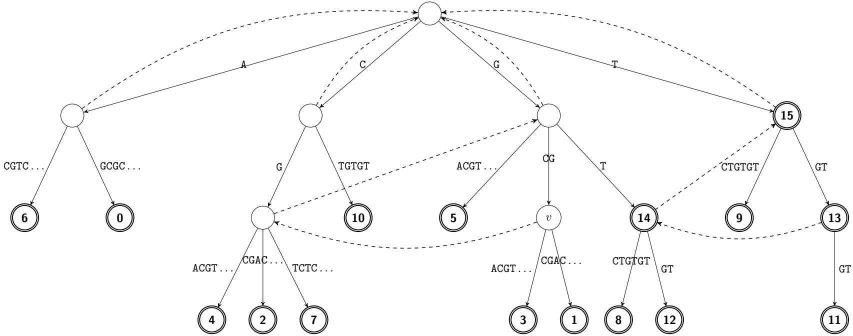

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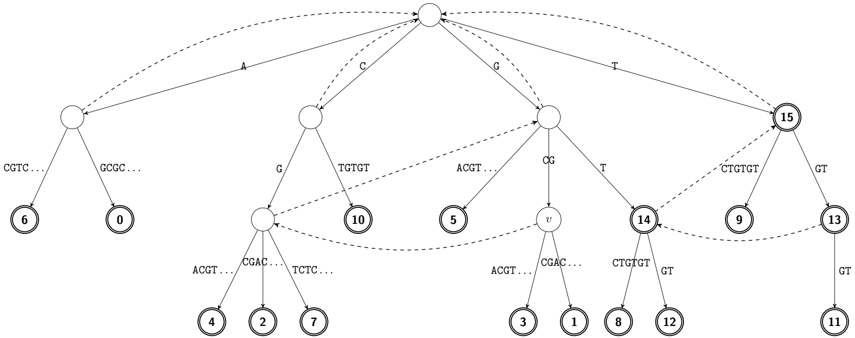

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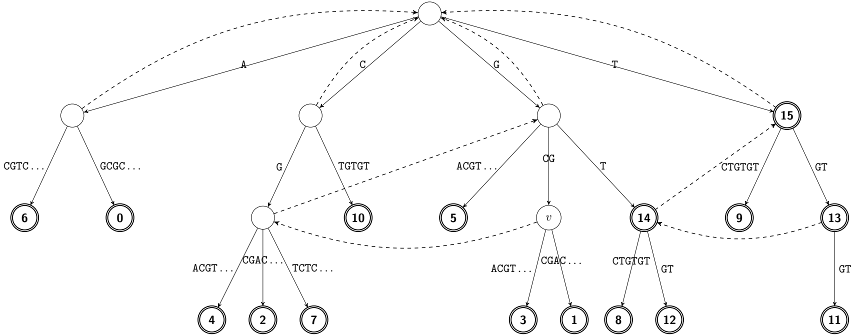

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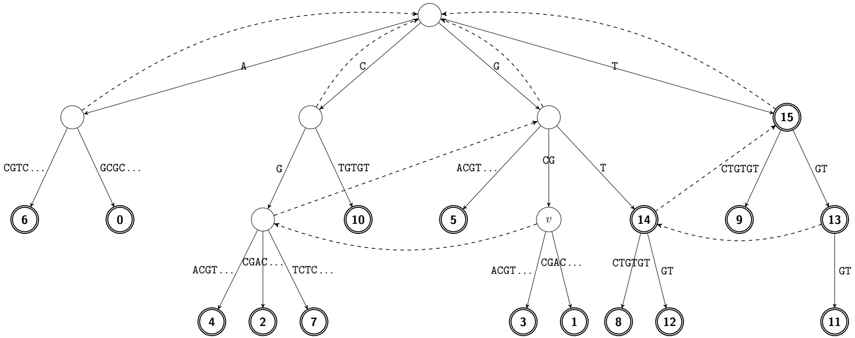

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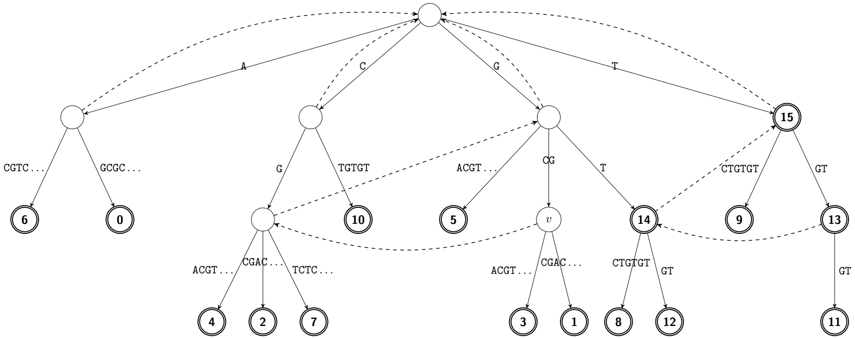

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for each node $v \in \mathcal{T}(x)$ **do**
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$\mathcal{O}(\sigma n)$ time, $\sigma = |\Sigma|$, to compute the words [Crochemore et al, 1998];

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$\mathcal{O}(\sigma n)$ time, $\sigma = |\Sigma|$, to compute the words [Crochemore et al, 1998]; and $\mathcal{O}(1)$ work per word using Weighted Ancestors queries [Gawrychowski et al, 2014].

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Theorem

Alg. $\textsc{AvoidedWords}$ *solves problem* $\textsc{AvoidedWordsComputation}$ *in time and space* $\mathcal{O}(n)$ *for constant-sized alphabets.*
For integer alphabets, the algorithm solves the problem in time $\mathcal{O}(\sigma n)$*.*

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