Practical Questions

1. Solve the following problem by simplex method:

a. Max. $Z = 60x_1 + 50x_2$
Subject to
$2x_1 + 4x_2 \le 80$
$3x_1 + 2x_2 \le 60$
$x_1, x_2 \ge 0$

c. Max. $Z = 10x_1 + 20x_2$
Subject to:
$4x_1 + 2x_2 \le 60$
$4x_1 + 10x_2 \le 100$
$2x_1 + 3x_2 \le 38$
$x_1, x_2 \ge 0$

e. Max. $P = 3x_1 + 2x_2 + 5x_3$
Subject to:
$x_1 + x_2 + x_3 \le 9$
$2x_1 + 3x_2 + 5x_3 \le 30$
$2x_1 - x_2 - x_3 \le 8$
$x_1, x_2, x_3 \ge 0$

g. Max. $Z = 3x_1 + 2x_2$
Subject to:
$2x_1 + x_2 \le 5$
$3x_1 + 4x_2 \ge 12$
$x_1, x_2 \ge 0$

i. Max. $P = -2x_1 - 8x_2$
Subjected to:
$0 \le x_1 \le 20$
$5x_1 + 10x_2 = 150$
$x_2 \ge 14$

b. Max. $Z = 2x_1 + 3x_2$
Subject to:
$x_1 + 2x_2 \le 13$
$2x_1 + x_2 \le 14$
$x_1 \ge 0, x_2 \ge 0$

d. Max. $Z = 400x_1 + 320x_2$
Subject to
$4x_1 + 10x_2 \le 100$
$20x_1 + 10x_2 \le 300$
$2x_1 + 3x_2 \le 38$
$x_1, x_2 \ge 0$

f. Max $Z = 2x_1 + 4x_2 + 3x_3$
Subject to
$3x_1 + 4x_2 + 2x_3 \le 60$
$2x_1 + x_2 + 2x_3 \le 40$
$x_1 + 3x_2 + 2x_3 \le 80$
$x_1, x_2, x_3 \ge 0$

h. Max. $Z = 8x_1 - 3x_2$
Subject to:
$x_1 + 2x_2 \le 7$
$2x_1 + 4x_2 \ge 14$
$x_1 = 5$ and $x_2 \ge 0$

j. Max. $Z = 4A + 8B + 10C$
Subject to:
$6A + 8B + 5C \le 56$
$2A + 3B + 4C \ge 29$
$2A + B + C \ge 11$ & $A, B, C \ge 0$

2. Solve the following problem by simplex method.

a. Minimize $Z = 4x_1 + 6x_2$
Subject to
$x_1 + 2x_2 \ge 80$
$3x_1 + 2x_2 \ge 75$
$x_1, x_2 \ge 0$

c. Min $Z = 20A + 10B$
Subject to
$A + 2B \le 40$
$4A + 3B \ge 60$
$3A + B \ge 30$
$A, B \ge 0$

e. Min $Z = x_1 + 5x_2 + 6x_3$
Subject to
$x_1 + x_2 + x_3 = 40$
$5x_1 + 6x_2 + 7x_3 = 250$
$x_1, x_2, x_3 \ge 0$

b. Minimize $Z = 20x_1 + 30x_2$
Subject to
$3x_1 + 5x_2 \ge 45$
$2x_1 + x_2 \ge 20$
$x_1, x_2 \ge 0$

d. Min $Z = x_1 + 2x_2 + 3x_3$
Subject to
$x_1 + x_2 + x_3 \ge 30$
$10x_1 + 15x_2 + 20x_3 \le 600$
$5x_2 + 6x_3 \le 120$
$x_1, x_2, x_3 \ge 0$

f. Min $C = 200x + 800y$
Subject to
$x + y = 200$
$0 \le x \le 40$
$0 \le y \ge 30$

3. Use simplex method to solve:

a. Max $Z = 4x_1 + 2x_2$
Subject to:
$x_1 - 2x_2 \ge 2$
$2x_1 + x_2 \ge 8$
$x_1, x_2 \ge 0$

c. Max $Z = 6x_1 - 4x_2$
Subject to:
$2x_1 + 4x_2 \le 4$
$4x_1 + 8x_2 \ge 16$
$x_1, x_2 \ge 0$

e. Min $Z = 4A + 5B$
Subject to
$A + B \le 2$
$2A + B \ge 20$
$A, B \ge 0$

b. Max $Z = 10x_1 + 5x_2$
Subject to:
$x_1 - 2x_2 \le 0$
$x_1 \le 10$
$x_2 \ge 1$ & $x_1, x_2 \ge 0$

d. Max $Z = 12x_1 - 4x_2$
Subject to
$x_1 + 2x_2 \le 2$
$2x_1 + 4x_2 \ge 8$
$x_1 = 6$ and $x_1, x_2 \ge 0$

4. A manufacture makes two types of products P_1 and P_2 using two machines M_1 and M_2. Product P_1 requires 5 hours on machine M_1 and no time on machine on M_2, product P_2 requires 1 hour on machine M_1 and 3 hours on machine M_2. There are 16 hours of time per day available on machine M_1 and 30 hours on machine M_2. Profit margin from P_1 and P_2 is Rs. 2 and Rs. 10 per unit respectively. What should be the daily production mix to maximize the profit?

5. A manufacturing house produces two articles X and Y, each of which is processed by two machines A and B. X requires 2 hours of A and 4 hours of B; Y requires 4 hours of A and 2 hours of B given that both machine can be run all the time in a day. If each article X yield a profit of Rs. 60 and each article Y yield a profit of Rs. 100, find how many of each article should be produced daily for maximum profit.

6. A firm makes two types of furniture; chairs and tables. The contribution for each product as calculated by accounting department is Rs. 20 per chair and Rs. 30 per table. Both the products are processed on three machines M_1, M_2 and M_3. The time required for each product and the total time available per week one each machine are as follows:

Machine	Chair	Table	Available hours
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

a. Formulate this problem as L.P.P.
b. How should the manufacturer schedule his production in order to maximize contribution?

7. A watch manufacturing company produces two types of watches A and B by using three machines M_1, M_2 and M_3. The time required for each watch on each machine and the maximum time available on each machine are given below:

Machine	Time required for each type		Maximum time available
	A	B	
M_1	6	8	[illegible]
M_2	8	4	[illegible]
M_3	12	4	[illegible]

The profit on watch A and B are Rs. 50 and Rs.30 respectively. What combination of watches should be produced to obtain the maximum profit?

8. A firm produces three products that are presented on three different machines. The time required manufacturing one unit of each of the three products and daily capacity of the machines are given in the table below:

Machine	Product (time in minutes) P_1	P_2	P_3	Available time (in minutes)
M_1	2	8	2	440
M_2	4	-	8	470
M_3	2	5	-	430

Determine the daily number of units to be manufactured of each product. The profit per unit for product P_1, P_2, P_3 is Rs. 4, Rs. 3 and Rs. 6 respectively. It is assumed that all the products are consumed in the market.

9. A Gear Manufacturing Company received an order for three special types of gears for regular supply. The management is considering to devote the available excess capacity to one or more of the three types, say A, B and C. the available capacity on the machines which might limit output and the no. of hours required for each unit of the respective gear is also given below:

Machine type	Available machine hour per week	No. of hours required per unit Gear A	Gear B	Gear C
Gear Hobbing Machine	250	8	2	3
Gear Shaping Machine	150	4	3	0
Gear Grinding Machine	50	2	-	1

The unit profit would be Rs. 20, Rs. 6 and Rs. 8 respectively for the gear A, B and C. Find how much of each gear the company should produce in order to maximize the profit?

10. A watch dealer wishes to buy new watch and has two models M_1 and M_2 and cost Rs. 100 and Rs. 200 respectively. In view of showcase of dealer, he wants to buy watches not more than 30 and he can spend up to Rs. 4000. The watch dealer can make a profit of Rs. 20 in M_1 and Rs. 50 in M_2. How many of each model should he buy in order to obtain maximum profit?

11. A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space for at most 20 items. A fan costs him Rs. 360 and a sewing machine costs Rs. 240. His expectation is that he can sell a fan at a profit of Rs. 22 and sewing machine at profit of Rs. 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit?

12. A firm produces two types of clothes A and B and makes a profit of Rs 20 per unit on A and Rs 25 per unit on B. Three types of workers skilled, semi-skilled and un-skilled are available each for 8 hours a day. The production of the one unit of A requires ¼ hour of skilled worker and 2/3 hour of the un-skilled worker. Production of one unit of B requires 1/3 hour of semi-skilled and ½ hour of un-skilled worker. What combination of the two types of clothes should be produced to maximized total profit?

13. A manufacturing company produces three types of leather belts A, B, and C which is produced on three machines M_1, M_2 and M_3. Belt A require 2 hours on machine M_1, 3 hours on machine M_2 and 2 hours on machine M_3. Belt B require 3 hours on machine M_1, 2 hours on machine M_2 and 2 hours on machine M_3. Belt C requires 5 hours on machine M_2 and 4 hours on machine M_3. There are 8 hours of time per day available on machine M_1, 10 hours on machine M_2 and 15 hours on machine M_3. The profit per unit gained from belts A, B and C are respectively Rs 3 , Rs 5 and Rs 4 .What would be the daily production of belt A, B and C to maximize the profit.

14. A manufacturing company manufactures two different products. The demand for both the products is strong enough so that the firm can sell as many units of either product, or both, as it can produce and at such a price as to realize a per unit profit contribution of Rs. 16 on product A and Rs. 10 on product B. Unfortunately, the production capacity of the company's plant is severely limited. This limitation stems from the fact that the manufacture of the products involves the utilization of three scarce resources: raw material, labor and machine time. Each of products A requires 4 units of raw material, 3 units of labor and 2 units of machine time. Each product B requires 2 units of raw material, 3 units of labor and 5 units of machine time. The firm has a daily supply of 24 units of raw material, 21 units of labor and 30 units of machine time. Formulate a LPP model and determine how much of each product should be manufactured to maximize the total profit by using simplex method.

15. Food X contains 6 units of vitamin A and 7 units of vitamin B and costs Rs. 5 per gram. Food Y contains 8 units of vitamin A and 12 units of vitamin B and costs Rs. 18 per gram. The daily minimum requirement of vitamins A and B are respectively 100 units and 138 units respectively.
   a. Construct the problem as a LPP with the objective function minimizing the cost.
   b. Find the minimum cost by using simplex method.

16. The following table gives the three kinds of foods and three kinds of vitamin contained on them. Solve following problem for minimizing costs.

Vitamin	Food F_1	F_2	F_3	Daily Requirements
V_1	20	10	10	300
V_2	10	10	10	200
V_3	10	20	10	240
Cost per food	Rs. 20	Rs. 24	Rs.18	

17. A horticulturist wishes to mix fertilizer that will provide a minimum of 15 units of Potash, 20 units of Niters and 24 units of Phosphate. Brand I provides 3 units of Potash, 1 unit of Nitrates and 3 units of Phosphate; it costs Rs. 120. Brand II provides 1 unit of Potash, 5 units of Niters and 2 units of Phosphate; it costs Rs. 60. Solve by using simplex method to determine the quantities of two brands, which should be mixed such that the cost is minimized.

18. Suppose that 8, 12 and 9 units of Protein, Carbohydrate and Fat respectively are the minimum weekly requirements for a person. Food A contains 2, 6, 1 units of Protein, Carbohydrate and Fat respectively per kg. and Food B contains 1, 1, 3 units of Protein, Carbohydrate and Fat respectively per kg. If A costs Rs. 0.85 per kg. and B costs Rs. 0.40 per kg.. How many kgs of each should be buy per week to minimize the cost and still to meet the minimum requirements.

19. The XYZ company combines factors A and B to form a product which must weight 50 pounds. At least 20 pounds of A and no more than 40 pounds of B can be used. The cost of A is Rs. 25 per pound and of B is Rs 10 per pound. Use the simplex method to find the amount of factors A and B which should be used to minimize total costs.

20. A firm must produce 200 kgs of a mixture consisting of the ingredients P and Q which cost Rs 3 and Rs 8 per kg respectively. Not more than 80 kgs of P and not less than 60 kgs of Q are used. Find how much of each ingredient should be used to minimize total cost.

21. Covert the following primal problem into dual problem.

(a) Max $Z = 5x_1 + 8x_2$

subject to constraint

$3x_1 + 4x_2 \le 12$

$9x_1 + 7x_2 \le 10$

$4x_1 + 6x_2 \le 15$

$x_1, x_2 \ge 0$

(b) Max $Z = 150x_1 + 240x_2 + 300x_3$

subject to constraint

$70x_1 + 50x_2 + 60x_3 \le 1200$

$40x_1 + 15x_2 + 90x_3 \ge 1800$

$x_1, x_2, x_3 \ge 0$

(c) Min $Z = 12x_1 - 7x_2$
subject to constraint
$3x_1 + 8x_2 \geq 15$
$9x_1 - 5x_2 \geq 20$
$x_1, x_2 \geq 0$

(d) Min $Z = x_1 + 3x_2$
subject to constraint
$x_1 + x_2 \geq 4$
$3x_1 + 2x_2 \geq 5$
$3x_1 - 2x_2 \geq 7$
$x_1, x_2 \geq 0$

22. Convert the given primal problem into the dual problem.

(a) Min $C = 7x_1 + 3x_2 + 8x_3$
subject to constraint
$x_1 + x_2 + x_3 \geq 5$
$2x_1 + 3x_2 + 4x_3 \geq 1$
$5x_1 + 3x_2 + 7x_3 \geq 4$
$9x_1 + x_2 - 3x_3 \leq 8$
$x_1, x_2, x_3 \geq 0$

(b) Max $P = 8x_1 + 7x_2$
subject to constraint
$3x_1 + 4x_2 \leq 15$
$2x_1 - 5x_2 \geq 8$
$5x_2 \geq 9$
$x_1, x_2 \geq 0$

(c) Max $Z = 10x_1 + 15x_2 + 20x_3$
Subject to
$x_1 + x_2 + x_3 \leq 40$
$2x_1 + 4x_2 + 6x_3 \geq 180$
$x_1 + 2x_2 + 3x_3 = 80$
$x_1, x_2, x_3 \geq 0$

(d) Max $Z = 6x_1 + 8x_2$
Subject to
$3x_1 + 2x_2 = 5$.
$5x_1 + 4x_2 \leq 9$
$x_1, x_2 \geq 0$

23. Write down the dual of given LPP and obtain its solution by using simplex method. Also give the optimum solution of primal from same simplex table.

Min $Z = 20A + 10B$
Subject to $A + 2B \leq 40$
$4A + 3B \geq 60$
$3A + B \geq 30$ $A, B \geq 0$

24. Write down the dual of the following primal problem and solve by using simplex method. Also produce economic interpretation.

Max. $Z = 6x_1 + 10x_2$
Subject to: $4x_1 + 3x_2 \leq 18$, $4x_2 \leq 20$, $4x_1 + 6x_2 \leq 15$
and $x_1, x_2 \geq 0$

25. Find the optimum solution for the following problem by using simplex.
Min. Z = Rs. 300A + Rs. 800B
Subject to:
$A + B = 200$
$A \leq 40$ $B \geq 30$
where $A, B \geq 0$
Interpret the value of dual from the final table of the primal problem.

26. A firm makes two products A and B. Each product requires production on each of the two machines.

Products	Machines		Profit
	X	Y	(Rs)
A	12	2	6
B	8	4	8
Available time (in minutes)	120	44	

Find the maximum profit by using simplex method. Also write down the dual solution from same table.

ANSWERS

1.
a. Max $Z = 1350$, $x_1 = 10$, $x_2 = 15$
b. Max $Z = 22$, $x_1 = 5$, $x_2 = 4$, $S_1 = 0$ & $S_2 = 0$
c. Max $Z = 220$, $x_1 = 10$, $x_2 = 6$, $x_3 = 0$, $S_1 = 8$
d. Max $Z = 6000$, $x_1 = 13$, $x_2 = 4$, $S_2 = 48$
e. Max. $P = 35$, $x_1 = 5$, $x_2 = 0$, $x_3 = 4$
f. Max $Z = 230/3$, $x_1 = 0$, $x_2 = 20/3$, $x_3 = 50/3$
g. Max $Z = 10$, $x_1 = 0$, $x_2 = 5$, $S_1 = 0$, $S_2 = 8$
h. Max $Z = 37$, $x_1 = 5$, $x_2 = 1$, $x_3 = 0$, $S_1 = 0$, $S_2 = 0$
i. Max $P = -116$, $x_1 = 2$, $x_2 = 14$
j. Max $Z = 112$, $A = 0$, $B = 0$, $C = 56/5$, $S_1 = 0$, $S_2 = 79/5$, $S_3 = 1/5$

2.
a. Min. $Z = 240$, $x_1 = 0$ and $x_2 = 40$
b. Min. $Z = 2000/7$, $x_1 = 55/7$ and $x_2 = 30/7$
c. Min $Z = 240$, $A = 6$, $B = 12$,
d. Min $Z = 30$, $x_1 = 30$, $x_2 = 0$, $x_3 = 0$
e. Min $Z = 3$, $x_1 = 0$, $x_2 = 4/3$, $x_3 = 5/3$
f. Min. $C = 140{,}000$, $A = 40$, $B = 160$, $S_2 = 130$

3.
a. Unbounded solution
b. Unbounded solution
c. Infeasible solution
d. Infeasible solution
e. Infeasible solution

4. Max $Z = 512/5$, $x_1 = 6/5$, $x_2 = 10$
5. Max. $Z = 660$, $x_1 = 4$, $x_2 = 4$
6. Max Z = Rs 330, $x_1 = 3$ and $x_2 = 9$
7. Max Z = Rs 2030 $x_1 = 22$ and $x_2 = 31$
8. Max $Z = 3200/3$, $x_1 = 0$; $x_2 = 380/9$; $x_3 = 470/3$
9. Max $Z = 700$, $A = 0$, $B = 50$, $C = 50$
10. Max $Z = 1000$, $x_1 = 0$ and $x_2 = 20$
11. Max P = Rs 392, $x_1 = 8$ and $x_2 = 12$
12. Max. $Z = 400$, $x_1 = 0$ and $x_2 = 16$
13. Max $Z = 775/41$ for $x_1 = 89/41$, $x_2 = 50/41$ & $x_3 = 64/41$
14. Max $Z = 100$, $x_1 = 5$, $x_2 = 2$
15. (a) Min. Cost $= 5x + 18y$
s.t.c. $6x + 8y \geq 100$
$7x + 12y \geq 138$ & $x \geq 0$, $y \geq 0$
(b) Z Min = Rs. 98.55 at (19.71, 0)
16. Min $C = 404$, $x_1 = 10$, $x_2 = 4$, $x_3 = 6$
17. Min C = 780 at Brand I = 2, Brand II = 9
18. Min. cost = Rs. 3.25 at A = 1 kg and B = 6 kg
19. Min. C = Rs 800, $x_1 = 20$ and $x_2 = 30$
20. Min. C = Rs 1200, $x_1 = 80$ and $x_2 = 120$

21.(a) Min $z^* = 12W_1 + 10W_2 + 15W_3$
Stc : $3W_1 + 9W_2 + 4W_3 \geq 5$
$4W_1 + 7W_2 + 6W_3 \geq 8$
$W_1, W_2, W_3 \geq 0$

(b) Min $Z^* = 1200W_1 + 1800W_2$
Stc : $70W_1 + 40W_2 \geq 150$
$50W_1 + 15W_2 \geq 240$
$60W_1 + 90W_2 \geq 300$
$W_1, W_2, W_3 \geq 0$

(c) Max $Z^* = 15W_1 + 20W_2$
Stc : $3W_1 + 9W_2 \leq 12$
$8W_1 - 5W_2 \leq -7$
$W_1, W_2 \geq 0$

(d) Max $Z^* = 4W_1 + 5W_2 + 7W_3$
Stc : $W_1 + 3W_2 + 3W_3 \leq 1$
$W_1 + 2W_2 - 2W_3 \leq 3$
$W_1, W_2, W_3 \geq 0$

22.(a) Max $Z = 5W_1 + W_2 + 4W_3 - 8W_4$
Stc : $W_1 + 2W_2 + 5W_3 - 9W_4 \leq 7$
$W_1 + 3W_2 + 3W_3 - W_4 \leq 3$
$W_1 + 4W_2 + 7W_3 + 3W_4 \leq 8$
$W_1, W_2, W_3, W_4 \geq 0$

(b) Min $Z = 15W_1 - 8W_2 - 9W_3$
Stc.: $3W_1 - 2W_2 \geq 8$
$4W_1 + 5W_2 - 5W_3 \geq 7$
$W_1, W_2, W_3 \geq 0$

(c) Min $Z = 40W_1 - 180W_2 + 80W_3$

Sub to

$W_1 - 2W_2 + W_3 \geq 10$

$W_1 - 4W_2 + 2W_3 \geq 15$

$W_1 - 6W_2 + 3W_3 \geq 20$

$W_1, W_2, W_3 \geq 0$

(d) Min $Z = 5W_1 + 9W_2$

Sub to

$3W_1 + 5W_2 \geq 6$

$2W_1 + 4W_2 \geq 8$

$W_1, W_2, \geq 0$

23. Dual solution: Max $Z^* = 240$, $W_1 = 0$, $W_2 = 2$, $W_3 = 4$

    Primal Solution: Min $Z = 240$, $A = 6$, $B = 12$

24. Dual solution: Min $Z^* = 30$, $W_1 = \frac{5}{3}$, $W_2 = 0$, $W_3 = 0$

    Primal Solution: Max $Z = 30$, $x_1 = 0$, $x_2 = 3$.

25. Primal Solution: Min. $Z = 140000$, $A = 40$, $B = 160$

    Dual solution: Max Z^* = Rs. 140000, $W_1 = 800$, $W_2 = 500$, $W_3 = 0$

26. Primal Solution: Max $Z = 96$, $x_1 = 4$, $x_2 = 9$

    Dual solution: Min $Z^* = 96$, $W_1 = 0.25$, $W_2 = 1.5$

Answers of Multiple Choice Questions

1. (d)	2. (d)	3. (d)	4. (b)	5. (a)	6. (d)	7. (b)	8. (d)	9. (b)	10. (
11. (b)	12. (a)	13. (c)	14. (b)	15. (a)	16. (b)	17. (c)	18. (c)	19. (b)	20. (
21. (c)									

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Demand	[illegible]	$D_2 = 8$	$D_3 = 7$	$D_4 = 14$	

We have to be ensured whether the initial solution is feasible or not. Since the number of occupied cell i.e. allocated cell is equal to $m + n - 1 = 3 + 4 - 1 = 6$ where m is no. of row and no. of column in the transportation table, so the initial solution is feasible.

The total transportation cost of initial solution can be calculated as

Total TC = Σ Unit Cell Cost × Allocated Quantity.

$$= C_{14}X_{14} + C_{21}X_{21} + C_{23}X_{23} + C_{31}X_{31} + C_{32}X_{32} + C_{34}X_{34}$$

$$= 10 \times 7 + 70 \times 2 + 40 \times 7 + 40 \times 3 + 8 \times 8 + 20 \times 7$$

= Rs. 814

Since, the total transportation cost of the initial solution obtained by LCM is less than that of NWCM, so LCM is more efficient than NWCM.

Vogel's Approximation Method (VAM)

The Vogel's Approximation Method commonly known as VAM is another cost based method used for obtaining the initial solution to a transportation problem. It uses the cost difference rather than actual costs to select the appropriate cell to make the allocations. It is preferred over two methods because its initial solution is either optimal or very near to optimal. This may reduce the time for optimal solution. The VAM method involves the following steps:

Step 1: Tabulate all the information related to supply, demand and unit transportation cell cost in transportation table.

Step 2: Find the cost difference of the smallest and second smallest unit cell cost along each row and each column. This cost difference indicates the penalty or extra cost which has to be paid if one fails to allocate the commodity to the cell with least unit transportation cost.

Step 3: Select the row or column with largest cost difference and allocate the maximum possible quantity to the cell having least unit transportation cost in the selected row or column. The allocated quantity should not be exceeding the corresponding supply and demand of selected cell. Sometime, there may be tie between the largest cost differences, select that row or column which has largest cost difference and least cell cost. If there is also tie in the minimum least cell cost of the row or column having largest cost difference, select that cell among these ties where maximum quantity can be allocated. If there is also tie in the maximum quantity that can be allocated, then select any one cell from these ties. Then adjust the supply and demand and cross out all the cell of row or column whose quantity is completely satisfied/exhausted.

Step 4: Continue step 1 and step 2 until the entire available supply at various sources and the demand at various destinations are satisfied. It is noted that the unit transportation cost called cell cost of the crossed cell is not taken into account for calculation of further cost difference.

Step 5: Finally calculate the total transportation cost by adding the product of unit cell cost and quantity allocated to that cell.

Example 3: A Company has 3 production factories A, B and C with production capacity of 7, 9 and 18 units (in '00's) per week of a product, respectively. These units are to be shipped to 4 warehouses P, Q, R and S with requirement of 5, 8, 7 and 14 units (in '00's) per week, respectively. The transportation costs (in rupees) per unit between factories to warehouses are given in the table below. Use Vogel's Approximation Method to find the initial solution.

	P	Q	R	S	Supply
A	19	30	50	10	7
B	70	30	40	60	9
C	40	8	70	20	18
Demand	5	8	7	14	34

Solution:

Here, we have, $\Sigma D = 5 + 8 + 7 + 14 = 34$ and $\Sigma S = 7 + 9 + 18 = 34$

Since, $\Sigma D = \Sigma S$. So, given problem is balanced.

Initial Solution from VAM

Warehouse / Factory	P	Q	R	S	Supply	Cost Difference
A	19	× 30	50	10	$S_1 = 7$	9
B	70	× 30	40	60	$S_2 = 9$	10
C	40	8 (8)	70	20	$S_3 = \cancel{18}$ 10	12
Demand	$D_1 = 5$	$D_2 = \cancel{8}$	$D_3 = 7$	$D_4 = 14$	34	
Cost Difference	21	22↑	10	10		

Calculate the cost difference for each row and each column by subtracting the least cell cost from next least cell cost. For example

For first row,	$19 - 10 = 9$	For first column,	$40 - 19 = 21$
For second row,	$40 - 30 = 10$	For second column,	$30 - 8 = 22$
For third row,	$20 - 8 = 12$	For third column,	$50 - 40 = 10$
		For fourth column,	$20 - 10 = 10$

Since, the cost difference of second column i.e. 22 is largest among all cost differences, the cell (3, 2) having least cell cost (i.e. 8) along second column is selected for allocation. The maximum possible quantity which is minimum of {18, 8} i.e. 8 is assigned to the cell (3, 2) as first allocation which means 8 units of commodity are transported from the third source C to the second destination Q and 18 – 8 =

10 units of supply are left over in the third source/row C and there is no demand left at second destination/column Q, i.e. 8 – 8 = 0. Since, all the demand of second column is satisfied, all remaining cell of second column are crossed out (x).

Initial Solution by VAM

Factory \ Warehouse	P	Q	R	S	Supply	Cost Difference	
A	5 19	× 30	50	10	$S_1 = \cancel{7}\ 2$	9	9
B	× 70	× 30	40	60	$S_2 = 9$	10	20
C	× 40	8 8	70	20	$S_3 = \cancel{18}\ 10$	12	20
Demand	$D_1 = \cancel{5}$	$D_2 = \cancel{8}$	$D_3 = 7$	$D_4 = 14$	34		
Cost Difference	21	22 ↑	10	10			
	21↑	–	10	10			

Again, compute the cost difference for the remaining rows and columns by using remaining cell costs. For example,

For first row,	19 – 10 = 9	For first column,	40 – 19 = 21
For second row,	60 – 40 = 20	For third column,	50 – 40 = 10
For third row,	40 – 20 = 20	For fourth column,	20 – 10 = 10

Since, the cost difference of first column i.e. 21 is largest among all cost differences, the cell (1, 1) having least cell cost (i.e. 19) along first column is selected for next allocation. The maximum possible quantity which is minimum of {7, 5} i.e. 5 is assigned to the cell (1, 1) as second allocation which means 5 units of commodity are transported from the first source A to the first destination P and 7 – 5 = 2 units of supply are left over in the first source A and there is no demand left at first destination/column P i. e. 8 – 8 = 0. Since all the demand of first column is satisfied, all remaining cell of first column are crossed out (x).

Proceed in the same manner for each row and each column until all the demand and the supply are satisfied/exhausted. The initial solution of above transportation problem will be

Initial Solution by VAM

Factory \ Warehouse	P	Q	R	S	Supply	Cost difference			
A	5 19	× 30	× 50	2 10	$S_1 = \cancel{7}\ \cancel{2}$	9	9	40	40
B	× 70	× 30	7 40	2 60	$S_2 = \cancel{9}\ \cancel{2}$	10	20	20	20
C	× 40	8 8	× 70	10 20	$S_3 = \cancel{18}\ \cancel{10}$	12	20	← 50	–
Demand	$D_1 = \cancel{5}$	$D_2 = \cancel{8}$	$D_3 = \cancel{7}$	$D_4 = \cancel{14}\ \cancel{4}\ \cancel{2}$	34				
Cost Difference	21	22 ↑	10	10					
	21↑	–	10	10					
	–	–	10	10					
	–	–	10	50↑					

We have to be ensured whether the initial solution is feasible or not. Since, the number of occupied cell i.e. allocated cell is equal to m + n − 1 = 3 + 4 − 1 = 6 where, m is no. of rows and no. of columns in the transportation table. So, the initial solution is feasible.

The total transportation cost of initial solution can be calculated as

$$
\begin{aligned}
\text{Total TC} &= \Sigma\text{Cell cost} \times \text{Quantity allocated} \\
&= C_{11}X_{11} + C_{14}X_{14} + C_{23}X_{23} + C_{24}X_{24} + C_{32}X_{32} + C_{34}X_{34} \\
&= 19 \times 5 + 10 \times 2 + 40 \times 7 + 60 \times 2 + 8 \times 8 + 20 \times 10 \\
&= \text{Rs. } 779
\end{aligned}
$$

Since, the total transportation cost of the initial solution obtained by VAM is less than that of NWCM and LCM, so VAM is more efficient than NWCM and LCM.

Step 2: Testing the Optimality of Initial Solution

After obtaining the initial feasible solution, we have to test whether it is optimal or not. The optimal solution is one where there is no other set of transportation routes/allocation that will reduce the total transportation cost. There are two methods available to test the optimality of the initial feasible solution:

1. Stepping Stone Method.
2. Modified Distribution (MODI) Method.

Stepping Stone Method

In this method, we calculate the opportunity cost of each unoccupied cell. Opportunity cost measures that what effect on the total transportation cost would be if one unit quantity is assigned to the unoccupied cell. If the value of total transportation cost deceases, the initial solution is not supposed to be optimal and if the value of total transportation cost does not deceases (increases or remains unchanged), the initial solution is optimal. It requires the following steps:

Step 1: Obtain the initial feasible solution by using NWCM/LCM/VAM. But it is recommended to use VAM rather than NWCM and LCM because it is more efficient than others.

Step 2: Confirm the non degeneracy of the initial solution. A solution to a transportation problem is known as non - degenerate/feasible solution if number of occupied cell is equal to m + n − 1 otherwise it is known as degenerate/infeasible solution. Where, m = no. of rows and n = no. of columns.

Step 3: Determination of the improvement index for each unoccupied cell:

We trace a closed path called 'loop' which starts from unoccupied cell, moves horizontally or vertically only, takes turns only from occupied cells and terminates to the cell from where it has started. Assign positive sign (+) and negative sign (–) alternatively at each corners of the loop beginning with (+) sign at unoccupied cell from where loop originates. The quantity should be added to the cells having (+) sign and subtracted from the cells with (–) sign satisfying the row total and column total.

Step 4: Calculate the improvement index as

Improvement index = sum of unit transportation cost of (+) corners of the loop – sum of unit transportation cost of (–) corners of the loop.

Positive index indicates the increase in total transportation cost and negative index indicates the decrease in total transportation cost due to allocation of commodity to that unoccupied cell. If all the improvement indices are non negative (positive or zero) the solution is optimal and if any of the improvement indices of unoccupied cell is negative, the solution is not assumed to be optimal.

Step 5: If the solution is not optimal; reallocate commodities without violating the total supply and demand. For reallocation of commodities, the minimum quantity among the quantities of all negative corners is selected and it is added to positive corners and subtracted from negative corners of the closed path called loop.

Step 6: Repeat the steps 3 and 4 until we get the optimal solution.

Step 7: Finally calculate the total transportation cost of the optimal solution by adding the product of unit cell cost and quantity allocated to that cell i.e.

Total TC = Σ Unit Cell Cost × Allocated Quantity.

Example 4: A Company has 3 production factories A, B and C with production capacity of 7, 9 and 18 units (in '00's) per week of a product, respectively. These units are to be shipped to 4 warehouses P, Q, R and S with requirement of 5, 8, 7 and 14 units (in '00's) per week, respectively. The transportation costs (in rupees) per unit between factories to warehouses are given in the table below. Use Stepping Stone Method to find the optimal solution.

	P	Q	R	S	Supply
A	19	30	50	10	7
B	70	30	40	60	9
C	40	8	70	20	18
Demand	5	8	7	14	34

Solution:

***Step 1:* Calculation of Initial Solution from VAM**

From example 3, the initial solution of the above problem by VAM is

Table – 1

Initial Solution by VAM

Warehouse / Factory	P	Q	R	S	Supply
A	5 19	30	50	2 10	7
B	70	30	7 40	2 60	9
C	40	8 8	70	10 20	18
Demand	5	8	7	14	34

The total transportation cost of initial solution is

Total TC = ΣCell cost × Quantity allocated

$= C_{11}X_{11} + C_{14}X_{14} + C_{23}X_{23} + C_{24}X_{24} + C_{32}X_{32} + C_{34}X_{34}$

$= 19 \times 5 + 10 \times 2 + 40 \times 7 + 60 \times 2 + 8 \times 8 + 20 \times 10 = \text{Rs. } 779$

***Step 2:* Testing Optimality by Stepping Stone Method**

Number of occupied cells = 6 and m + n – 1 = 3 + 4 – 1 = 6 are equal, so the solution is non degenerate. To test optimality, select the unoccupied cell (A, Q) for which the loop is formed by joining the cells (A, S), (C, S), (C,Q) and (A, Q). The improvement index for (A, Q) is (30 – 10 + 20 – 8) = 32 which is positive. Similarly, the improvement indices for other unoccupied cells are

Unoccupied cells	Loops	Improvement index
(A, R)	+(A, R) – (A, S) + (B, S) – (B, R)	50 – 10 + 60 – 40 = 60
(B, P)	+(B, P) – (B, S) + (A, S) – (A, P)	70 – 60 + 10 – 19 = 1
(B, Q)	+(B,Q) – (B, S) + (C, S) – (C,Q)	30 – 60 + 20 – 8 = –18
(C, P)	+(C, P) – (A,P) + (A, S) – (C, S)	40 – 19 + 10 – 20 = 11
(C, R)	+(C, R) – (B, R) + (B, S) – (C,.S)	70 – 40 + 60 – 20 = 70

Since, some of the improvement indices are negative, so the initial solution is not optimal. To get the optimal solution, reallocation of goods is necessary. For reallocation, the unoccupied cell (B, Q) with most negative improvement index is selected. The minimum quantity among the positive corners of (B, S) and (C, Q) i.e. the minimum of {2, 8} = 2 is selected and it is added to quantity of positive corners (B, Q) and (C, S) and subtracted from the quantity of negative corners (B, S) and (C, Q) of the closed path formed at the cell (B, Q). The new transportation routes will be

Table – 2

Testing of Optimality of Initial Solution

Factory \ Warehouse	P	Q	R	S	Supply
A	19 (5)	30	50	10 (2)	7
B	70	30 (2)	40 (7)	60	9
C	40	8 (6)	70	20 (12)	18
Demand	5	8	7	14	34

Again, since number of occupied cells = 6 and m + n – 1 = 3 + 4 – 1 = 6 are equal so the solution is non degenerate. To test the optimality of the above new solution, calculate improvement index for each unoccupied cells as

Unoccupied cells	Loops	Improvement index
(A, Q)	+(A,Q) – (A, S) + (C, S) – (C, Q)	30 – 10 + 20 – 8 = 32
(A, R)	+(A,R) – (A,S) + (C,S) – (C,Q) + (B,Q) – (B,R)	50 – 10 + 20 – 8 + 30 – 40 = 42
(B, P)	+(B,P) – (A,P) + (A,S) – (C,S) + (C,Q) – (B,Q)	70 – 19 + 10 – 20 + 8 – 30 = 19
(B, S)	+(B,S) – (C, S) + (C, Q) – (B, Q)	60 – 20 + 8 – 30 = 18
(C, P)	+(C,P) – (A,P) + (A,S) – (C,S)	40 – 19 + 10 – 20 = 11
(C, R)	+(C,R) – (B, R) + (B, Q) – (C, Q)	70 – 40 + 30 – 8 = 52

Since all the improvement indices are non negative, so the above new transportation routes are optimal. Hence the minimum total transportation cost is

Min TC = ΣCell cost × Quantity allocated = 19 x 5 + 10 x 2 + 30 × 2 + 40 × 7 + 8 × 6 + 20 × 12 = Rs. 743 for the supply of 5 units from factory A to warehouse P, 2 units from factory A to warehouse S, 2 units from factory B to warehouse Q, 7 units from factory B to warehouse R, 6 units from factory C to warehouse Q and 12 units from factory C to warehouse S.

Modified Distribution (MODI) Method

The Modified Distribution (MODI) Method is also very similar to *Stepping Stone Method*. The only difference between these two methods is that modified distribution method does not require the loop (closed path) formation for each unoccupied cells. In this method, we can identify the unoccupied cell with most negative improvement index where the loop formation is necessary. It requires the following steps:

Step 1: Obtain the initial feasible solution by using NWCM/LCM/VAM. But it is recommended to use VAM rather than NWCM and LCM because it is more efficient than them.

Step 2: Confirm the non degeneracy of the initial solution. A solution to a transportation problem is known as non degenerate/feasible solution if number of occupied cell is equal to m + n −1 otherwise it is known as degenerate/infeasible solution where m = no. of row and n = no. of column.

Step 3: Calculate the row values (R_i) and column values (K_j) for each row and each column by the using the relationship, $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

For calculation of row values and column values, value of any row or column has supposed to be zero for reference. For convenience, we can take the row or column having maximum number of occupied cells for such reference.

Step 4: Calculate improvement indices called opportunity cost to all unoccupied cells by using the relation, $\Delta_{ij} = C_{ij} - (R_i + K_j)$

Step 5: If all the improvement indices are non – negative i.e. either positive or zero, the solution is optimal. If any of the improvement indices is negative, the solution is not supposed to be optimal. To get optimal solution, reallocation of goods is necessary. For reallocation, the unoccupied cell with most negative improvement index is identified and a loop is formed for that unoccupied cell. For reallocation of commodities, the minimum quantity among the quantities of all negative corners is selected and it is added to positive corners and subtracted from negative corners of the loop.

Step 6: Repeat the steps 2, 3, 4 and 5 until we get the optimal solution.

Example 5: A Company has 3 production factories A, B and C with production capacity of 7, 9 and 18 units (in '00's) per week of a product, respectively. These units are to be shipped to 4 warehouses P, Q, R and S with requirement of 5, 8, 7 and 14 units (in '00's) per week, respectively. The transportation costs (in rupees) per unit between factories to warehouses are given in the table below. Use MODI Method to find the optimal solution.

	P	Q	R	S	Supply
A	19	30	50	10	7
B	70	30	40	60	9
C	40	8	70	20	18
Demand	5	8	7	14	34

Solution:

Step 1: Calculation of Initial Solution from VAM

From example 3, the initial solution of the above problem by VAM is

Table – 1

Initial Solution by VAM

Factory \ Warehouse	P	Q	R	S	Supply	R_i
A	19 [5]	30	50	10 [2]	7	$R_1 = 10$
B	70	30 +	40 [7]	60 [2] −	9	$R_2 = 60$
C	40	8 [8] −	70	20 [10] +	18	$R_3 = 20$
Demand	5	8	7	14	34	
K_j	$K_1 = 9$	$K_2 = -12$	$K_3 = -20$	$K_4 = 0$		

The total transportation cost of initial solution is

Total TC $= \Sigma$Cell cost × Quantity allocated

$= C_{11}X_{11} + C_{14}X_{14} + C_{23}X_{23} + C_{24}X_{24} + C_{32}X_{32} + C_{34}X_{34}$

$= 19 \times 5 + 10 \times 2 + 40 \times 7 + 60 \times 2 + 8 \times 8 + 20 \times 10$

= Rs. 779

Step 2: **Testing of Optimality by MODI Method**

Number of occupied cells = 6 and $m + n - 1 = 3 + 4 - 1 = 6$ are equal so the solution is non degenerate.

To calculate the row value and column value, let us assume $K_4 = 0$ for reference. We can take any row or column for reference. Here, fourth column has maximum no. of occupied cell, so it is selected just for convenience. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$C_{14} = R_1 + K_4 \Rightarrow 10 = R_1 + 0 \Rightarrow R_1 = 10$

$C_{24} = R_2 + K_4 \Rightarrow 60 = R_2 + 0 \Rightarrow R_2 = 60$

$C_{34} = R_3 + K_4 \Rightarrow 20 = R_3 + 0 \Rightarrow R_3 = 20$

$C_{11} = R_1 + K_1 \Rightarrow 19 = 10 + K_1 \Rightarrow K_1 = 9$

$C_{23} = R_2 + K_3 \Rightarrow 40 = 60 + K_3 \Rightarrow K_3 = -20$

$C_{32} = R_3 + K_2 \Rightarrow 8 = 20 + K_2 \Rightarrow K_2 = -12$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are,

$\Delta_{12} = C_{12} - (R_1 + K_2) = 30 - (10 - 12) = 32$

$\Delta_{13} = C_{13} - (R_1 + K_3) = 50 - (10 - 20) = 60$

$\Delta_{21} = C_{21} - (R_2 + K_1) = 70 - (60 + 9) = 1$

$\Delta_{22} = C_{22} - (R_2 + K_2) = 30 - (60 - 12) = -18$

$\Delta_{31} = C_{31} - (R_3 + K_1) = 40 - (20 + 9) = 11$

$\Delta_{33} = C_{33} - (R_3 + K_3) = 70 - (20 - 20) = 70$

Since, the improvement index $\Delta_{22} = -18 < 0$, the solution is not optimal. To get optimal solution, reallocation of goods is necessary. For reallocation, a loop starts from the cell (2, 2) and passes through the cells (2, 4), (3, 4) and (3, 2). The minimum quantity (i.e. 2) among the quantities {2 and 8} of all negative corners is selected and it is added to positive corners (2, 2) and (3, 4) and subtracted from negative corners (2, 4) and (3, 2) of the loop. The modified solution becomes

Table – 2

Testing of Optimality of Initial Solution

Factory \ Warehouse	P	Q	R	S	Supply	R_i
A	19 [5]	30	50	10 [2]	7	$R_1 = 0$
B	70	30 [2]	40 [7]	60	9	$R_2 = 32$
C	40	8 [6]	70	20 [12]	18	$R_3 = 10$
Demand	5	8	7	14	34	
K_j	$K_1 = 19$	$K_2 = -2$	$K_3 = 8$	$K_4 = 10$		

Since no. of occupied cells = m + n – 1 = 3 + 4 – 1 = 6, so the solution is still non – degenerate. To calculate the row value and column value, let us assume $R_1 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$$C_{11} = R_1 + K_1 \Rightarrow 19 = 0 + K_1 \Rightarrow K_1 = 19$$
$$C_{14} = R_1 + K_4 \Rightarrow 10 = 0 + K_4 \Rightarrow K_4 = 10$$
$$C_{34} = R_3 + K_4 \Rightarrow 20 = R_3 + 10 \Rightarrow R_3 = 10$$
$$C_{32} = R_3 + K_2 \Rightarrow 8 = 10 + K_2 \Rightarrow K_2 = -2$$
$$C_{22} = R_2 + K_2 \Rightarrow 30 = R_2 - 2 \Rightarrow R_2 = 32$$
$$C_{23} = R_2 + K_3 \Rightarrow 40 = 32 + K_3 \Rightarrow K_3 = 8$$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$$\Delta_{12} = C_{12} - (R_1 + K_2) = 30 - (0 - 2) = 32$$
$$\Delta_{13} = C_{13} - (R_1 + K_3) = 50 - (0 + 8) = 42$$
$$\Delta_{21} = C_{21} - (R_2 + K_1) = 70 - (32 + 19) = 19$$
$$\Delta_{24} = C_{24} - (R_2 + K_4) = 60 - (32 + 10) = 18$$
$$\Delta_{31} = C_{31} - (R_3 + K_1) = 40 - (10 + 19) = 11$$
$$\Delta_{33} = C_{33} - (R_3 + K_3) = 70 - (10 + 8) = 52$$

Since all the improvement indices are positive; so the solution is optimal. The minimum total transportation cost is given by

$$\text{Min TC} = \Sigma\text{Cell cost} \times \text{Quantity allocated}$$
$$= C_{11}X_{11} + C_{14}X_{14} + C_{22}X_{22} + C_{23}X_{23} + C_{32}X_{32} + C_{34}X_{34}$$
$$= 19 \times 5 + 10 \times 2 + 30 \times 2 + 40 \times 7 + 8 \times 6 + 20 \times 12 = \text{Rs. } 743$$

For the supply of 5 units from factory A to warehouse P, 2 units from factory A to warehouse S, 2 units from factory B to warehouse Q, 7 units from factory B to warehouse R, 6 units from factory C to warehouse Q and 12 units from factory C to warehouse S.

Degeneracy in Transportation Problem

The solution in which no. of occupied cell is equal to m + n – 1 where m is no. of rows and n is no. of columns in transportation table is called non – degenerate problem. The solution in which no. of occupied cell is not equal to m + n – 1 in transportation table is called degenerate problem. The degeneracy arises due to following two ways:

(a) There may be an excessive number of occupied cells in a solution i.e. the number of occupied cells is greater than m + n – 1. This type of degeneracy arises only in developing the initial solution and is caused by an improper assignment or an error in formulating the problem. In such cases, one must modify the initial solution in order to get a solution which satisfies the condition of non degeneracy.

(b) There may an insufficient number of occupied cells in a solution i.e. the number of occupied cells is less than m + n – 1. Degeneracy of this type may occur either in the initial solution or in subsequent solutions. It is the type of degeneracy which requires special procedures to resolve. With an insufficient number of occupied cells in a solution, it would be impossible to trace a closed path/loop for each unoccupied cells, and with the MODI method it would be impossible to compute the values of rows and columns.

Resolving Degeneracy

To resolve degeneracy an infinitely small amount denoted by θ (theta) is assigned to one of the unoccupied cell. Although there is a great deal of flexibility in choosing the unoccupied cell for the allocation of θ, the general procedure, when using the northwest corner rule, is to assign it to a cell in such a way that it maintains an unbroken chain of occupied cells. However where the Vogel's Approximation Method is used, θ is allocated in a least cost independent cell. An independent cell in this context means that a cell which will not lead to a closed–path on such allocation. In other words, θ must be allocated to the unoccupied cell having almost least cost where a loop cannot be formed.

Example 6: Nepal Hardware Company's has three plants in Kathmandu, Banepa and Nuwakot and three warehouses at Bhairhawa, Birgunj and Biratnagar. The quantities available at the plants are respectively 60, 70 and 80 where as the demand at the warehouses are 50, 80 and 80 respectively. The unit cost of transportation is observed as follows:

From \ To	Bhairhawa	Birgunj	Biratnagar
Kathmandu	2	5	7
Banepa	2	3	4
Nuwakot	5	8	11

Find an optimal allocation that minimizes the transportation cost. **[TU 2059]**

Solution:

Here, we have, ΣD = 50 + 80 + 80 = 210 and ΣS = 60 + 70 + 80 = 210

Since, ΣD = ΣS, So, given problem is balanced.

***Step 1:* Calculation of Initial Solution from VA**

Ta

Initial Solu

From \ To	Bhairahawa	Birgunj
Kathmandu	2 [50]	5 ×
Banepa	2 ×	3 ×
Nuwakot	5 ×	8 [80]
Demand	~~50~~	~~80~~
I	0	2
II	–	2
III	–	3

The total transportation cost of initial solutio

Total TC = ΣCell cost × Quantity allo

= 2 × 50 + 7 × 10 + 4 × 70

= 100 + 70 + 280 + 640 =

***Step 2:* Testing of Optimality by MODI**

The initial solution from VAM is

From \ To	Bhairahawa
Kathmandu	2 [50]
Banepa	2
Nuwakot	5
Demand	50
K_j	$K_1 = 2$

Since, number of occupied cells = 4 is

Therefore we assign a very small q

cannot be formed at this stage.

To calculate the row value and colu

row value (R_i) and column value (

occupied cells where C_{ij} is the unit

$C_{11} = R_1 + K_1 \Rightarrow$

$C_{13} = R_1 + K_3 \Rightarrow$

$C_{23} = R_2 + K_3 \Rightarrow$

$C_{22} = R_2 + K_2 \Rightarrow$

$C_{32} = R_3 + K_2 \Rightarrow$

Degeneracy in Transportation Problem

The solution in which no. of occupied cell is equal to m + n – 1 where m is no. of rows and n is no. of columns in transportation table is called non – degenerate problem. The solution in which no. of occupied cell is not equal to m + n – 1 in transportation table is called degenerate problem. The degeneracy arises due to following two ways:

(a) There may be an excessive number of occupied cells in a solution i.e. the number of occupied cells is greater than m + n – 1. This type of degeneracy arises only in developing the initial solution and is caused by an improper assignment or an error in formulating the problem. In such cases, one must modify the initial solution in order to get a solution which satisfies the condition of non degeneracy.

(b) There may an insufficient number of occupied cells in a solution i.e. the number of occupied cells is less than m + n – 1. Degeneracy of this type may occur either in the initial solution or in subsequent solutions. It is the type of degeneracy which requires special procedures to resolve. With an insufficient number of occupied cells in a solution, it would be impossible to trace a closed path/loop for each unoccupied cells, and with the MODI method it would be impossible to compute the values of rows and columns.

Resolving Degeneracy

To resolve degeneracy an infinitely small amount denoted by θ (theta) is assigned to one of the unoccupied cell. Although there is a great deal of flexibility in choosing the unoccupied cell for the allocation of θ, the general procedure, when using the northwest corner rule, is to assign it to a cell in such a way that it maintains an unbroken chain of occupied cells. However where the Vogel's Approximation Method is used, θ is allocated in a least cost independent cell. An independent cell in this context means that a cell which will not lead to a closed–path on such allocation. In other words, θ must be allocated to the unoccupied cell having almost least cost where a loop cannot be formed.

Example 6: Nepal Hardware Company's has three plants in Kathmandu, Banepa and Nuwakot and three warehouses at Bhairhawa, Birgunj and Biratnagar. The quantities available at the plants are respectively 60, 70 and 80 where as the demand at the warehouses are 50, 80 and 80 respectively. The unit cost of transportation is observed as follows:

From \ To	Bhairhawa	Birgunj	Biratnagar
Kathmandu	2	5	7
Banepa	2	3	4
Nuwakot	5	8	11

Find an optimal allocation that minimizes the transportation cost. [TU 2059]

Solution:

Here, we have, $\Sigma D = 50 + 80 + 80 = 210$ and $\Sigma S = 60 + 70 + 80 = 210$

Since, $\Sigma D = \Sigma S$, So, given problem is balanced.

Step 1: Calculation of Initial Solution from VAM

Table – 1

Initial Solution from VAM

From \ To	Bhairahawa	Birgunj	Biratnagar	Supply	I	II	III
Kathmandu	2 [50]	5 ×	7 [10]	~~10~~ ~~60~~	← 3	2	2
Banepa	2 ×	3 ×	4 [70]	70	1	1	–
Nuwakot	5 ×	8 [80]	11 ×	80	3	3	3
Demand	~~50~~	80	~~80~~ ~~10~~	210			
I	0	2	3				
II	–	2	3↑				
III	–	3	4↑				

The total transportation cost of initial solution can be calculated as

Total TC $= \Sigma$Cell cost × Quantity allocated

$= 2 \times 50 + 7 \times 10 + 4 \times 70 + 8 \times 80$

$= 100 + 70 + 280 + 640 =$ Rs 1090

***Step 2:* Testing of Optimality by MODI Method**

The initial solution from VAM is

Table – 2

From \ To	Bhairahawa	Birgunj	Biratnagar	Supply	R_i
Kathmandu	2 [50]	5 +	7 [10] –	60	$R_1 = 0$
Banepa	2	3 [θ] –	4 [70] +	70	$R_2 = -3$
Nuwakot	5	8 [80]	11	80	$R_3 = 2$
Demand	50	80	80	210	
K_j	$K_1 = 2$	$K_2 = 6$	$K_3 = 7$		

Since, number of occupied cells = 4 is less than $m + n - 1 = 3 + 3 - 1 = 5$, so the solution is degenerate. Therefore we assign a very small quantity 'θ' ≈ 0 to the cell (2, 2) having least cell cost where loop cannot be formed at this stage.

To calculate the row value and column value, let us assume $R_1 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$C_{11} = R_1 + K_1 \Rightarrow 2 = 0 + K_1 \Rightarrow K_1 = 2$

$C_{13} = R_1 + K_3 \Rightarrow 7 = 0 + K_3 \Rightarrow K_3 = 7$

$C_{23} = R_2 + K_3 \Rightarrow 4 = R_2 + 7 \Rightarrow R_2 = -3$

$C_{22} = R_2 + K_2 \Rightarrow 3 = -3 + K_2 \Rightarrow K_2 = 6$

$C_{32} = R_3 + K_2 \Rightarrow 8 = R_3 + 6 \Rightarrow R_3 = 2$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$\Delta_{12} \quad = C_{12} - (R_1 + K_2) \qquad = 5 - (0 + 6) \quad = -1$

$\Delta_{21} \quad = C_{21} - (R_2 + K_1) \qquad = 2 - (-3 + 2) = 3$

$\Delta_{31} \quad = C_{31} - (R_3 + K_1) \qquad = 5 - (2 + 2) \quad = 1$

$\Delta_{33} \quad = C_{33} - (R_3 + K_3) \qquad = 11 - (2 + 7) \ = 2$

Since, the improvement index $\Delta_{12} = -1 < 0$, the solution is not optimal. To get optimal solution, reallocation of goods is necessary. For reallocation, a loop starts from the cell (1, 2) and passes through the cells (1, 4), (2, 4) and (2, 2). The minimum quantity (i.e. θ) among the quantities (θ and 10) of all negative corners is selected and it is added to positive corners (1, 2) and (2, 4) and subtracted from negative corners (1, 4) and (2, 2) of the loop. The new solution becomes

Table – 3

From \ To	Bhairahawa	Birgunj	Biratnagar	Supply	R_i
Kathmandu	2 [50]	5 [θ]	7 [10]	60	$R_1 = 0$
Banepa	2	3	4 [70]	70	$R_2 = -3$
Nuwakot	5	8 [80]	11	80	$R_3 = 3$
Demand	50	80	80	210	
K_j	$K_1 = 2$	$K_2 = 5$	$K_3 = 7$		

Since, number of occupied cells = 5 is equal to m + n – 1 = 3 + 3 – 1 = 5, so the solution is non-degenerate.

To calculate the row value and column value, let us assume $R_1 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$C_{11} = R_1 + K_1 \quad \Rightarrow \quad 2 = 0 + K_1 \quad \Rightarrow K_1 = 2$

$C_{12} = R_1 + K_2 \quad \Rightarrow \quad 5 = 0 + K_2 \quad \Rightarrow K_2 = 5$

$C_{13} = R_1 + K_3 \quad \Rightarrow \quad 7 = 0 + K_3 \quad \Rightarrow K_3 = 7$

$C_{23} = R_2 + K_3 \quad \Rightarrow \quad 4 = R_2 + 7 \quad \Rightarrow R_2 = -3$

$C_{32} = R_3 + K_2 \quad \Rightarrow \quad 8 = R_3 + 5 \quad \Rightarrow R_3 = 3$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$\Delta_{21} \ = C_{21} - (R_2 + K_1) \ = 2 - (-3 + 2) \ = 3$

$\Delta_{22} \ = C_{22} - (R_2 + K_2) \ = 3 - (-3 + 5) \ = 1$

$\Delta_{31} \ = C_{31} - (R_3 + K_1) \ = 5 - (3 + 2) \quad = 0$

$\Delta_{33} \ = C_{33} - (R_3 + K_3) \ = 11 - (3 + 7) \ = 1$

Since, all the improvement indices are zero or positive, so the solution is optimal. The minimum total transportation cost is given by

Min. TC $= \Sigma$Cell cost × Quantity allocated

$= C_{11}X_{11} + C_{12}X_{12} + C_{13}X_{13} + C_{23}X_{23} + C_{32}X_{32}$

$= 2 \times 50 + 5 \times 0 + 7 \times 10 + 4 \times 70 + 8 \times 80$

$=$ Rs 1090 for $X_{11} = 50$, $X_{13} = 10$, $X_{23} = 70$, $X_{32} = 80$

Unbalanced Trans

The transportation problem is
supply exceeds the demand
transportation problem, we h
creating dummy destination
create a dummy destination
cells. If the demand exceeds
unit transportation cost for a
problem.

Example 7: Determine t

Ware
De

Solution:

Here, we have, ΣD

Since, ΣD ≠ ΣS, So,

To make balanced transpo
180) 40 and cell cost '0' for

***Step 1:* Calculation of In**

Store / Warehouse	P_1
W_1	× 45
W_2	× 35
W_3	[60] 30
Demand	~~60~~
I	5
II	5
III	5
IV	–

The total transportation c

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$$\Delta_{12} = C_{12} - (R_1 + K_2) = 5 - (0 + 6) = -1$$
$$\Delta_{21} = C_{21} - (R_2 + K_1) = 2 - (-3 + 2) = 3$$
$$\Delta_{31} = C_{31} - (R_3 + K_1) = 5 - (2 + 2) = 1$$
$$\Delta_{33} = C_{33} - (R_3 + K_3) = 11 - (2 + 7) = 2$$

Since, the improvement index $\Delta_{12} = -1 < 0$, the solution is not optimal. To get optimal solution reallocation of goods is necessary. For reallocation, a loop starts from the cell (1, 2) and passes through the cells (1, 4), (2, 4) and (2, 2). The minimum quantity (i.e. θ) among the quantities (θ and 10) of all negative corners is selected and it is added to positive corners (1, 2) and (2, 4) and subtracted from negative corners (1, 4) and (2, 2) of the loop. The new solution becomes

Table – 3

From \ To	Bhairahawa	Birgunj	Biratnagar	Supply	R_i
Kathmandu	2 [50]	5 [θ]	7 [10]	60	$R_1 = 0$
Banepa	2	3	4 [70]	70	$R_2 = -3$
Nuwakot	5	8 [80]	11	80	$R_3 = 3$
Demand	50	80	80	210	
K_j	$K_1 = 2$	$K_2 = 5$	$K_3 = 7$		

Since, number of occupied cells = 5 is equal to m + n – 1 = 3 + 3 – 1 = 5, so the solution is non - degenerate.

To calculate the row value and column value, let us assume $R_1 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$$C_{11} = R_1 + K_1 \Rightarrow 2 = 0 + K_1 \Rightarrow K_1 = 2$$
$$C_{12} = R_1 + K_2 \Rightarrow 5 = 0 + K_2 \Rightarrow K_2 = 5$$
$$C_{13} = R_1 + K_3 \Rightarrow 7 = 0 + K_3 \Rightarrow K_3 = 7$$
$$C_{23} = R_2 + K_3 \Rightarrow 4 = R_2 + 7 \Rightarrow R_2 = -3$$
$$C_{32} = R_3 + K_2 \Rightarrow 8 = R_3 + 5 \Rightarrow R_3 = 3$$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$$\Delta_{21} = C_{21} - (R_2 + K_1) = 2 - (-3 + 2) = 3$$
$$\Delta_{22} = C_{22} - (R_2 + K_2) = 3 - (-3 + 5) = 1$$
$$\Delta_{31} = C_{31} - (R_3 + K_1) = 5 - (3 + 2) = 0$$
$$\Delta_{33} = C_{33} - (R_3 + K_3) = 11 - (3 + 7) = 1$$

Since, all the improvement indices are zero or positive, so the solution is optimal. The minimum total transportation cost is given by

$$\text{Min. TC} = \Sigma\text{Cell cost} \times \text{Quantity allocated}$$
$$= C_{11}X_{11} + C_{12}X_{12} + C_{13}X_{13} + C_{23}X_{23} + C_{32}X_{32}$$
$$= 2 \times 50 + 5 \times 0 + 7 \times 10 + 4 \times 70 + 8 \times 80$$
$$= \text{Rs } 1090 \text{ for } X_{11} = 50,\ X_{13} = 10,\ X_{23} = 70,\ X_{32} = 80$$

Unbalanced Transportation Problem

The transportation problem is called an unbalanced problem if Total supply ≠ Total demand i.e. either supply exceeds the demand or demand exceeds the supply. While solving the unbalanced transportation problem, we have to convert the unbalanced problem into the balanced problem by creating dummy destination or dummy source as per required. If the supply exceeds the demand we create a dummy destination for excess units having zero unit transportation cost for all the dummy cells. If the demand exceeds the supply, we create a dummy source for excess demands having zero unit transportation cost for all the dummy cells. Then the remaining process is same as the balanced problem.

Example 7: Determine the minimum transportation cost from the following matrix.

Warehouse	Stores				Supply
	P_1	P_2	P_3	P_4	
	Cost per unit				
W_1	45	60	45	30	70
W_2	35	15	35	35	60
W_3	30	25	45	55	90
					220
Demand	60	40	60	20	180

Solution:

Here, we have, $\Sigma D = 60 + 40 + 60 + 20 = 180$ and $\Sigma S = 70 + 60 + 90 = 220$

Since, $\Sigma D \neq \Sigma S$, So, given problem is unbalanced.

To make balanced transportation problem, we create a dummy destination (P_5) with demand (220 – 180) 40 and cell cost '0' for each dummy cells.

Step 1: **Calculation of Initial Solution from VAM**

Table – 1

Initial Solution from VAM

Store / Warehouse	P_1	P_2	P_3	P_4	P_5	Supply	I	II	III	IV
W_1	× 45	× 60	$\boxed{10}$ 45	$\boxed{20}$ 30	$\boxed{40}$ 0	~~30~~ ~~10~~ ~~70~~	← 30	15	15	← 15
W_2	× 35	$\boxed{40}$ 15	$\boxed{20}$ 35	× 35	× 0	~~20~~ ~~60~~	15	← 20	0	0
W_3	$\boxed{60}$ 30	× 25	$\boxed{30}$ 45	× 55	× 0	~~30~~ ~~90~~	25	5	← 15	10
Demand	~~60~~	~~40~~	~~60~~ ~~20~~ ~~10~~	~~20~~	~~40~~	220				
I	5	10	10	5	0					
II	5	10	10	5	–					
III	5	–	10	5	–					
IV	–	–	10	5	–					

The total transportation cost of initial solution can be calculated as

Total TC = ΣCell cost × Quantity allocated

$= C_{13}X_{13} + C_{14}X_{14} + C_{15}X_{15} + C_{22}X_{22} + C_{23}X_{23} + C_{31}X_{31} + C_{33}X_{33}$

$= 45 \times 10 + 30 \times 20 + 0 \times 40 + 15 \times 40 + 35 \times 20 + 30 \times 60 + 45 \times 30$

= Rs 5,500

Step 2: **Testing of Optimality by MODI Method**

The initial solution from VAM is

Table – 2

From \ To	P_1	P_2	P_3	P_4	P_5	Supply	R_i
W_1	45	60	45 [10]	30 [20]	0 [40]	70	$R_1 = 0$
W_2	35	15 [40]	35 [20]	35	0	60	$R_2 = -10$
W_3	30 [60]	25	45 [30]	55	0	90	$R_3 = 0$
Demand	60	40	60	20	40	220	
K_j	$K_1 = 30$	$K_2 = 25$	$K_3 = 45$	$K_4 = 30$	$K_5 = 0$		

Since, number of occupied cells = 7 is equal to m + n – 1 = 3 + 5 – 1 = 7, So, the solution is non-degenerate.

To calculate the row value and column value, let us assume $R_1 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$C_{13} = R_1 + K_3 \Rightarrow 45 = 0 + K_3 \Rightarrow K_3 = 45$

$C_{14} = R_1 + K_4 \Rightarrow 30 = 0 + K_4 \Rightarrow K_4 = 30$

$C_{15} = R_1 + K_5 \Rightarrow 0 = 0 + K_5 \Rightarrow K_5 = 0$

$C_{23} = R_2 + K_3 \Rightarrow 35 = R_2 + 45 \Rightarrow R_2 = -10$

$C_{22} = R_2 + K_2 \Rightarrow 15 = -10 + K_2 \Rightarrow K_2 = 25$

$C_{33} = R_3 + K_3 \Rightarrow 45 = R_3 + 45 \Rightarrow R_3 = 0$

$C_{31} = R_3 + K_1 \Rightarrow 30 = 0 + K_1 \Rightarrow K_1 = 30$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$\Delta_{11} = C_{11} - (R_1 + K_1) = 45 - (0 + 30) = 15$

$\Delta_{12} = C_{12} - (R_1 + K_2) = 60 - (0 + 25) = 35$

$\Delta_{21} = C_{21} - (R_2 + K_1) = 35 - (-10 + 30) = 15$

$\Delta_{24} = C_{24} - (R_2 + K_4) = 35 - (-10 + 30) = 15$

$\Delta_{25} = C_{25} - (R_2 + K_5) = 0 - (-10 + 0) = 10$

$\Delta_{32} = C_{32} - (R_3 + K_2) = 25 - (0 + 25) = 0$

$\Delta_{34} = C_{34} - (R_3 + K_4) = 55 - (0 + 30) = 25$

$\Delta_{35} = C_{35} - (R_3 + K_5) = 0 - (0 + 0) = 0$

Since, all the improvement indices are positive, so the solution is optimal. The minimum total transportation cost is given by

Min TC = ΣCell cost × Quantity allocated

$= C_{13}X_{13} + C_{14}X_{14} + C_{15}X_{15} + C_{22}X_{22} + C_{23}X_{23} + C_{31}X_{31} + C_{33}X_{33}$

$= 45 \times 10 + 30 \times 20 + 0 \times 40 + 15 \times 40 + 35 \times 20 + 30 \times 60 + 45 \times 30$

= Rs 5500 for $X_{13} = 10$, $X_{14} = 20$, $X_{22} = 40$, $X_{23} = 20$, $X_{31} = 60$, $X_{33} = 30$.

Maximization of Transportation Problem

For maximization of transportation problem, first of all we convert the given profit matrix into *opportunity loss matrix* by subtracting each entry from the highest cell value. Then we proceed the same process proceeded in the minimization of Transportation problem. For calculation of final maximum value, we should consider the original table. If the transportation problem is unbalanced maximization problem, first of all make it balanced by creating dummy source or destination as per required with profit zero for all the dummy cells and then convert it into opportunity loss matrix by subtracting each elements from highest element. We proceed the same process proceeded in the minimization one.

Example 8: From the following profit matrix, find the maximum profit by using transportation model.

Plant	Units Available	Project	Units Demanded
A	1700	W	1300
B	2500	X	2000
C	1000	Y	1900

Profit in '000' rupees

From	To Project W	To Project X	To Project Y
Plant A	12	8	5
Plant B	11	15	10
Plant C	2	17	6

Solution:

Here, we have, $\Sigma D = 1300 + 2000 + 1900 = 5200$ and $\Sigma S = 1700 + 2500 + 1000 = 5200$

Since, $\Sigma D = \Sigma S$, So, given problem is balanced.

The given profit table is

Table – 1
Profit Table

From \ To	W	X	Y	Supply
A	12	8	5	1700
B	11	15	10	2500
C	2	17	6	1000
Demand	1300	2000	1900	5200

Since, it is the case of profit maximization, we should construct opportunity loss table. It can be obtained by subtracting each element from the highest element 17; we get the loss table as

Table – 2
Loss Table

From \ To	W	X	Y	Supply
A	5	9	12	1700
B	6	2	7	2500
C	15	0	11	1000
Demand	1300	2000	1900	5200

***Step 1:* Calculation of Initial Solution from VAM**

Table – 3

Initial Solution from VAM

From \ To	W	X	Y	Supply	I	II	III
A	5 [1300]	9 ×	12 [400]	~~1700~~ 400	4	4	7
B	6 ×	2 [1000]	7 [1500]	~~2500~~ 1500	4	4	1
C	15 ×	0 [1000]	11 ×	1000	11 ←	–	–
Demand	~~1300~~	~~2000~~ ~~1000~~	~~1900~~ ~~1500~~	5200			
I	1	2	4				
II	1	7 ↑	5				
III	1	–	5				

***Step 2:* Testing of Optimality by MODI Method**

The initial solution from VAM is

Table – 4

From \ To	W	X	Y	Supply	R_i
A	5 [1300]	9	12 [400]	1700	$R_1 = 0$
B	6	2 [1000]	7 [1500]	2500	$R_2 = -5$
C	15	0 [1000]	11	1000	$R_3 = -7$
Demand	1300	2000	1900	5200	
K_j	$K_1 = 5$	$K_2 = 7$	$K_3 = 12$		

Since number of occupied cells = 5 is equal to m + n – 1 = 3 + 3 – 1 = 5, so the solution is non-degenerate.

To calculate the row value and column value, let us assume $R_1 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$$C_{11} = R_1 + K_1 \Rightarrow 5 = 0 + K_1 \Rightarrow K_1 = 5$$
$$C_{13} = R_1 + K_3 \Rightarrow 12 = 0 + K_3 \Rightarrow K_3 = 12$$
$$C_{23} = R_2 + K_3 \Rightarrow 7 = R_2 + 12 \Rightarrow R_2 = -5$$
$$C_{22} = R_2 + K_2 \Rightarrow 2 = -5 + K_2 \Rightarrow K_2 = 7$$
$$C_{32} = R_3 + K_2 \Rightarrow 0 = R_3 + 7 \Rightarrow R_3 = -7$$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$$\Delta_{12} = C_{12} - (R_1 + K_2) = 9 - (0 + 7) = 2$$
$$\Delta_{21} = C_{21} - (R_2 + K_1) = 6 - (-5 + 5) = 6$$
$$\Delta_{31} = C_{31} - (R_3 + K_1) = 15 - (-7 + 5) = 17$$
$$\Delta_{33} = C_{33} - (R_3 + K_3) = 11 - (-7 + 12) = 6$$

Since all the improvement indices are positive, so the solution is optimal. The maximum profit can be obtained by taking the sum of product of profit of cell from table – 1 and allocated commodity to that cell in table – 4. The total maximum profit is given by

$$\begin{aligned}\text{Max Profit} &= \Sigma\text{Cell profit} \times \text{Quantity allocated}\\ &= P_{11}X_{11} + P_{13}X_{13} + P_{22}X_{22} + P_{23}X_{23} + P_{32}X_{32}\\ &= 12 \times 1300 + 5 \times 400 + 15 \times 1000 + 10 \times 1500 + 17 \times 1000\\ &= \text{Rs } 64600 \text{ for } X_{11} = 1300,\ X_{13} = 400,\ X_{22} = 1000,\ X_{23} = 1500,\ X_{32} = 1000\end{aligned}$$

Alternative Optimal Solution of a Transportation Problem

While solving the transportation problem by using the MODI (Modified Distribution) Method, the solution is supposed to be optimal if the opportunity cost called improvement index of all the unallocated cells are zero or positive. If all the improvement indices are positive, solution is unique. But if any one of the improvement indices is zero, it means the transportation problem has alternative optimal solution with another set of allocations without increasing the total transportation cost. The alternative optimal solution can be obtained by forming a loop to the unoccupied cell having zero improvement index. For example let's consider the **Example 6.**

Since, the improvement index $\Delta_{31} = 0$, for the reallocation, we form a loop starting from the cell (3, 1). The loop moves through the cell (1, 1), (1, 2) and (3, 2). The minimum quantity (i.e. 50) among the quantities {50 and 80} of all negative corners is selected and it is added to positive corners (3, 1) and (1, 2) and subtracted from negative corners (1, 1) and (3, 2) of the loop. The new solution becomes

Table – 3 (Continue from Example 6)

From \ To	Bhairahawa	Birgunj	Biratnagar	Supply	R_i
Kathmandu	− 50 2	θ + 5	10 7	60	$R_1 = 0$
Banepa	2	3	70 4	70	$R_2 = -3$
Nuwakot	+ 5	80 − 8	11	80	$R_3 = 3$
Demand	50	80	80	210	
K_j	$K_1 = 2$	$K_2 = 5$	$K_3 = 7$		

Table – 4

From \ To	Bhairahawa	Birgunj	Biratnagar	Supply	R_i
Kathmandu	2	50 5	10 7	60	$R_1 = 0$
Banepa	2	3	70 4	70	$R_2 = -3$
Nuwakot	50 5	30 8	11	80	$R_3 = 3$
Demand	50	80	80	210	
K_j	$K_1 = 2$	$K_2 = 5$	$K_3 = 7$		

Since, number of occupied cells = 5 is equal to $m + n - 1 = 3 + 3 - 1 = 5$, so the solution is non-degenerate.

To calculate the row value and column value, let us assume $R_1 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$C_{12} = R_1 + K_2 \quad \Rightarrow \quad 5 = 0 + K_2 \quad \Rightarrow \quad K_2 = 5$

$C_{13} = R_1 + K_3 \quad \Rightarrow \quad 7 = 0 + K_3 \quad \Rightarrow \quad K_3 = 7$

$C_{23} = R_2 + K_3 \quad \Rightarrow \quad 4 = R_2 + 7 \quad \Rightarrow \quad R_2 = -3$

$C_{32} = R_3 + K_2 \quad \Rightarrow \quad 8 = R_3 + 5 \quad \Rightarrow \quad R_3 = 3$

$C_{31} = R_3 + K_1 \quad \Rightarrow \quad 5 = 3 + K_1 \quad \Rightarrow \quad K_1 = 2$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$\Delta_{11} = C_{11} - (R_1 + K_1) = 2 - (0 + 2) = 0$

$\Delta_{21} = C_{21} - (R_2 + K_1) = 2 - (-3 + 2) = 3$

$\Delta_{22} = C_{22} - (R_2 + K_2) = 3 - (-3 + 5) = 1$

$\Delta_{33} = C_{33} - (R_3 + K_3) = 11 - (3 + 7) = 1$

Since all the improvement indices are zero or positive, so the solution is optimal. The minimum total transportation cost is given by

Min TC $= \Sigma$Cell cost $\times$ Quantity allocated

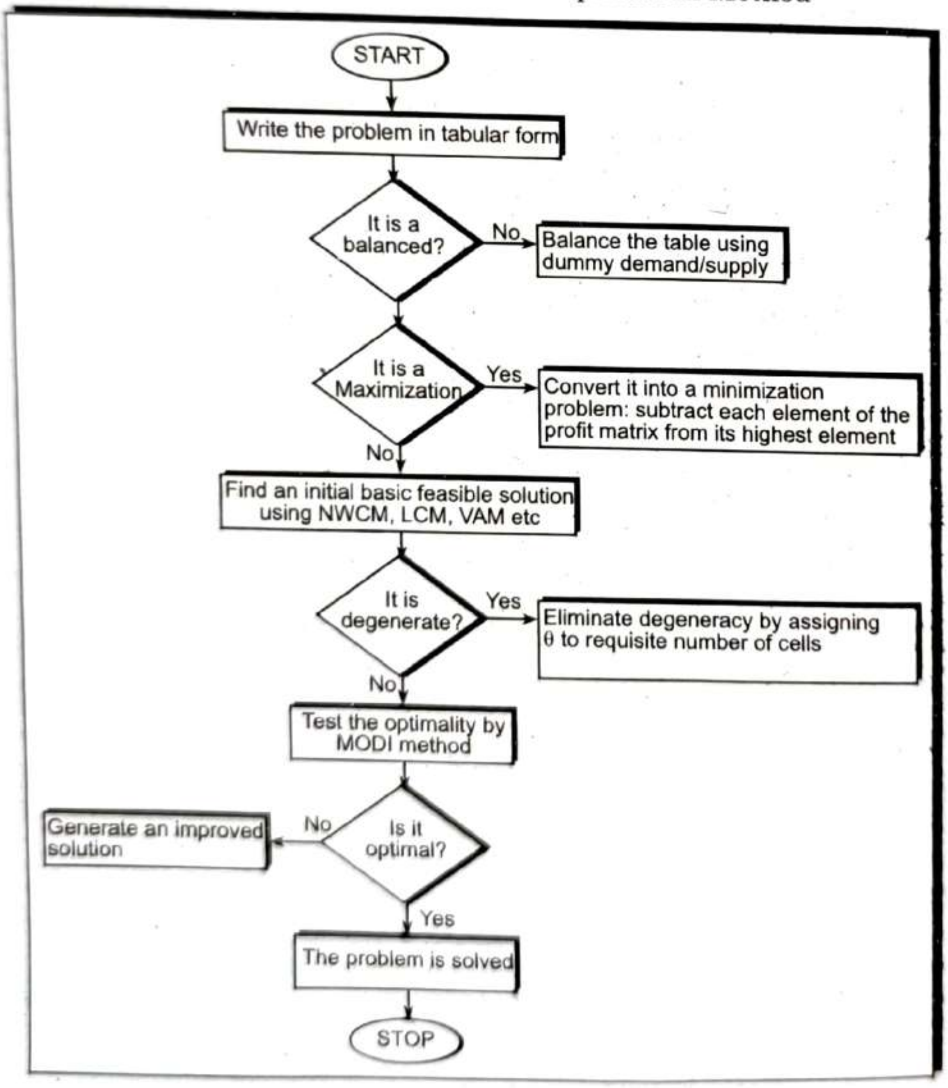
$= C_{12}X_{12} + C_{13}X_{13} + C_{23}X_{23} + C_{31}X_{31} + C_{32}X_{32}$

$= 5 \times 50 + 7 \times 10 + 4 \times 70 + 5 \times 50 + 8 \times 30$

$=$ Rs 1090 for $X_{12} = 50, X_{13} = 10, X_{23} = 70, X_{31} = 50, X_{32} = 30$

It is seen that the total transportation cost is Rs. 1090 for the transportation schedule $X_{12} = 50$ $X_{13} = 10$, $X_{23} = 70$, $X_{31} = 50$, $X_{32} = 30$ is same as the total transportation cost for the transportation schedule $X_{11} = 50$, $X_{13} = 10$, $X_{23} = 70$, $X_{32} = 80$.

Schematic Presentation of Transportation Method



Workedout Examples

Example 9: The Rent a car company rents car trailers to individuals making one way moves. Occasionally the company has to redistribute the trailers in order to eliminate a surplus build up in some cities and a shortage in others. The company currently has 4 trailers in A, 3 in B, 6 in C and 1 in D they would like to shift 5 trailers to E, 3 to F and 6 to G. the following tables gives the miles between the various sources and destinations.

Source	Destinations		
	E	F	G
A	60	50	80
B	30	60	90
C	60	30	100
D	90	70	30

How should company redistribute the trailers so that a total mile travelled is minimized? [TU 2052]

Solution:

Here, we have, $\Sigma D = 4 + 3 + 6 + 1 = 14$ and $\Sigma S = 5 + 3 + 6 = 14$

Since, $\Sigma D = \Sigma S$, So, given problem is balanced.

Step 1: **Calculation of Initial Solution from VAM**

Table – 1

Initial Solution from VAM

Sources / Destinations	A	B	C	D	Supply	I	II	III	IV
E	× 60	3 30	2 60	× 90	~~5~~ ~~2~~	30	← 30	0	0
F	× 50	× 60	3 30	× 70	~~3~~	20	20	20	–
G	4 80	× 90	1 100	1 30	~~6~~ ~~5~~ ~~1~~	← 50	10	20	20
Demand	4	~~3~~	~~6~~ ~~3~~ ~~1~~	~~1~~	14				
I	10	30	30	40					
II	10	30	30	–					
III	10	–	30 ↑	–					
IV	20	–	40 ↑	–					

The total travelled miles from initial solution is

Total Distance $= \Sigma$Cell cost × Quantity allocated

$= C_{12}X_{12} + C_{13}X_{13} + C_{23}X_{23} + C_{31}X_{31} + C_{33}X_{33} + C_{34}X_{34}$

$= 30 \times 3 + 60 \times 2 + 30 \times 3 + 80 \times 4 + 100 \times 1 + 30 \times 1$

$= 90 + 120 + 90 + 320 + 100 + 30 = 750$ miles

Step 2: **Testing of Optimality by MODI Method**

The initial solution from VAM is

Table – 2

Sources / Destinations	A	B	C	D	Supply	R_i
E	60	30 [3]	60 [2]	90	5	$R_1 = 0$
F	50	60	30 [3]	70	3	$R_2 = -30$
G	80 [4]	90	100 [1]	30 [1]	6	$R_3 = 40$
Demand	4	3	6	1	14	
K_j	$K_1 = 40$	$K_2 = 30$	$K_3 = 60$	$K_4 = -10$		

Since, number of occupied cells = 6 is equal to m + n – 1 = 3 + 4 – 1 = 6, so the solution is non-degenerate.

To calculate the row value and column value, let us assume $R_1 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$$C_{12} = R_1 + K_2 \Rightarrow 30 = 0 + K_2 \Rightarrow K_2 = 30$$
$$C_{13} = R_1 + K_3 \Rightarrow 60 = 0 + K_3 \Rightarrow K_3 = 60$$
$$C_{23} = R_2 + K_3 \Rightarrow 30 = R_2 + 60 \Rightarrow R_2 = -30$$
$$C_{33} = R_3 + K_3 \Rightarrow 100 = R_3 + 60 \Rightarrow R_3 = 40$$
$$C_{31} = R_3 + K_1 \Rightarrow 80 = 40 + K_1 \Rightarrow K_1 = 40$$
$$C_{34} = R_3 + K_4 \Rightarrow 30 = 40 + K_4 \Rightarrow K_4 = -10$$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$$\Delta_{11} = C_{11} - (R_1 + K_1) = 60 - (0 + 40) = 20$$
$$\Delta_{14} = C_{14} - (R_1 + K_4) = 90 - [0 + (-10)] = 100$$
$$\Delta_{21} = C_{21} - (R_2 + K_1) = 50 - (-30 + 40) = 40$$
$$\Delta_{22} = C_{22} - (R_2 + K_2) = 60 - (-30 + 30) = 60$$
$$\Delta_{24} = C_{24} - (R_2 + K_4) = 70 - (-30 - 10) = 110$$
$$\Delta_{32} = C_{32} - (R_3 + K_2) = 90 - (40 + 30) = 20$$

Since, all the improvement indices are positive, so the solution is optimal. The minimum total miles travelled is given by

Min. Distance = ΣCell cost × Quantity allocated

$$= C_{12}X_{12} + C_{13}X_{13} + C_{23}X_{23} + C_{31}X_{31} + C_{33}X_{33} + C_{34}X_{34}$$
$$= 30 \times 3 + 60 \times 2 + 30 \times 3 + 80 \times 4 + 100 \times 1 + 30 \times 1$$
$$= 90 + 120 + 90 + 320 + 100 + 30$$
$$= 750 \text{ miles for } X_{12} = 3, X_{13} = 2, X_{23} = 3, X_{31} = 4, X_{33} = 1, X_{34} = 1.$$

Example 10: The table given below has been taken from the solution procedure of a transportation problem, involving minimization of cost (in rupees):

Factory	Stockiest X	Y	Z	Monthly capacity
A	4	8	8	56
B	16	24	16	82
C	8	16	24	77
Monthly demand	72	102	41	

(i) Find the optimum solution of the above transportation problem.

(ii) If the transport cost from factory A to stockiest Z is increased by Rs 5 per unit, will the solution change? If so, find the new solution.

Solution:

Here, we have, $\Sigma D = 72 + 102 + 41 = 215$ and $\Sigma S = 56 + 82 + 77 = 215$

Since, $\Sigma D = \Sigma S$, So, given problem is balanced.

***Step 1:* Calculation of Initial Solution from VAM**

Table – 1

Initial Solution from VAM

Factory \ Stockiest	X	Y	Z	Supply	Cost difference I	II	III
A	× 4	56 8	× 8	~~56~~	4	0	–
B	× 16	41 24	41 16	~~82~~ ~~41~~	0	8	← 8
C	72 8	5 16	× 24	~~77~~ ~~5~~	← 8	8	8
Demand	~~72~~	~~102~~ ~~46~~ ~~5~~	~~41~~	215			
Cost difference I	4	8	8				
Cost difference II	–	8↑	8				
Cost difference III	–	8	8				

The total transportation cost of initial solution is

Total TC $= \Sigma$Cell cost × Quantity allocated

$= 8 \times 56 + 24 \times 41 + 16 \times 41 + 8 \times 72 + 16 \times 5$

$= 448 + 984 + 656 + 576 + 80 =$ Rs 2744

***Step 2:* Testing of Optimality by MODI Method**

The initial solution from VAM is

Table 2

Factory \ Stockiest	X	Y	Z	Supply	R_i
A	4	8 [56]	8	56	$R_1 = 8$
B	16	24 [41]	16 [41]	82	$R_2 = 24$
C	8 [72]	16 [5]	24	77	$R_3 = 16$
Demand	72	102	41	215	
K_j	$K_1 = -8$	$K_2 = 0$	$K_3 = -8$		

Since, number of occupied cells = 5 is equal to m + n − 1 = 3 + 3 − 1 = 5, so the solution is non-degenerate.

To calculate the row value and column value, let us assume $K_2 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$$C_{12} = R_1 + K_2 \Rightarrow 8 = R_1 + 0 \Rightarrow R_1 = 8$$
$$C_{22} = R_2 + K_2 \Rightarrow 24 = R_2 + 0 \Rightarrow R_2 = 24$$
$$C_{32} = R_3 + K_2 \Rightarrow 16 = R_3 + 0 \Rightarrow R_3 = 16$$
$$C_{23} = R_2 + K_3 \Rightarrow 16 = 24 + K_3 \Rightarrow K_3 = -8$$
$$C_{31} = R_3 + K_1 \Rightarrow 8 = 16 + K_1 \Rightarrow K_1 = -8$$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$$\Delta_{11} = C_{11} - (R_1 + K_1) = 4 - (8 - 8) = 4$$
$$\Delta_{13} = C_{13} - (R_1 + K_3) = 8 - (8 - 8) = 8$$
$$\Delta_{21} = C_{21} - (R_2 + K_1) = 16 - (24 - 8) = 0$$
$$\Delta_{33} = C_{33} - (R_3 + K_3) = 24 - (16 - 8) = 16$$

Since, all the improvement indices are positive, so the solution is optimal. The minimum total transportation cost is given by

$$\text{Min TC} = \Sigma\text{Cell cost} \times \text{Quantity allocated}$$
$$= 8 \times 56 + 24 \times 41 + 16 \times 41 + 8 \times 72 + 16 \times 5$$
$$= \text{Rs } 2744 \text{ for } X_{12} = 56, X_{22} = 41, X_{23} = 41, X_{31} = 72, X_{32} = 5$$

(ii) When the transportation cost from factory A to stockiest Z is increased by Rs 5 per unit. The transportation cost at that cell will be 8 + 5 = 13 i.e. $C_{13} = 13$. The improvement index for that cell is $\Delta_{13} = C_{13} - (R_1 + K_3) = 13 - (8 - 8) = 13$ which is positive, so there is no necessary to transport to the cell (1, 3). Thus there is no change in solution.

Example 11: Obtain the minimum transportation cost for the following transportation problem.

Cost of shipping and physical units

Source / Destination	P	Q	R	Units demanded
A	10	14	8	20
B	13	10	8	20
C	9	10	10	5
Units available	20	15	10	45 / 45

Solution:

Here, we have, $\Sigma D = 20 + 20 + 5 = 45$ and $\Sigma S = 20 + 15 + 10 = 45$

Since, $\Sigma D = \Sigma S$. So, given problem is balanced.

Step 1: **Calculation of Initial Solution from VAM**

Table – 1

Initial Solution from VAM

Stock / Demand	P	Q	R	Demand	I	II	III
A	10 [10]	14 ×	8 [10]	~~20~~ ~~10~~	← 2	← 4	–
B	13 [5]	10 [15]	8 ×	~~20~~ ~~15~~	2	3	3
C	9 [5]	10 ×	10 ×	~~5~~	1	1	1
Supply	~~20~~ ~~10~~ ~~5~~	~~15~~	~~10~~	45			
I	1	0	0				
II	1	0	–				
III	4↑	0	–				

The total transportation cost of initial solution is

Total TC $= \Sigma$Cell cost × Quantity allocated

$= 10 \times 10 + 8 \times 10 + 13 \times 5 + 10 \times 15 + 9 \times 5 =$ Rs 440

Step 2: **Testing of Optimality by MODI Method**

The initial solution from VAM is

Table – 2

Stock / Demand	P	Q	R	Demand	R_i
A	10 [+ 10]	14	8 [10 –]	20	$R_1 = 10$
B	13 [– 5]	10 [15]	8 +	20	$R_2 = 13$
C	9 [5]	10	10	5	$R_3 = 9$
Supply	20	15	10	45	
K_j	$K_1 = 0$	$K_2 = -3$	$K_3 = -2$		

Since, number of occupied cells = 5 is equal to $m + n - 1 = 3 + 3 - 1 = 5$, so the solution is non-degenerate.

To calculate the row value and column value, let us assume $K_1 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$C_{11} = R_1 + K_1 \Rightarrow 10 = R_1 + 0 \Rightarrow R_1 = 10$

$C_{13} = R_1 + K_3 \Rightarrow 8 = 10 + K_3 \Rightarrow K_3 = -2$

$C_{21} = R_2 + K_1 \Rightarrow 13 = R_2 + 0 \Rightarrow R_2 = 13$

$C_{31} = R_3 + K_1 \Rightarrow 9 = R_3 + 0 \Rightarrow R_3 = 9$

$C_{22} = R_2 + K_2 \Rightarrow 10 = 13 + K_2 \Rightarrow K_2 = -3$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$\Delta_{12} = C_{12} - (R_1 + K_2) = 14 - (10 - 3) = 7$

$\Delta_{23} = C_{23} - (R_2 + K_3) = 8 - (13 - 2) = -3$

$\Delta_{32} = C_{32} - (R_3 + K_2) = 10 - (9 - 3) = 4$

$\Delta_{33} = C_{33} - (R_3 + K_3) = 10 - (9 - 2) = 3$

Since, the improvement index $\Delta_{23} = -3 < 0$, the solution is not optimal. To get optimal solution reallocation of goods is necessary. For reallocation, a loop starts from the cell (2, 3) and passes through the cells (1, 3), (1, 1) and (2, 1). The minimum quantity (i.e. 5) among the quantities {5 and 10} of all negative corners is selected and it is added to positive corners (2, 3) and (1, 1) and subtracted from negative corners (1, 3) and (2, 1) of the loop. The new solution becomes

Table – 3

Stock \ Demand	P	Q	R	Demand	R_i
A	10 [15]	14	8 [5]	20	$R_1 = 10$
B	13	10 [15]	8 [5]	20	$R_2 = 10$
C	9 [5]	10	10	5	$R_3 = 9$
Supply	20	15	10	45	
K_j	$K_1 = 0$	$K_2 = 0$	$K_3 = -2$		

Since, number of occupied cells = 5 is equal to $m + n - 1 = 3 + 3 - 1 = 5$, so the solution is non-degenerate.

To calculate the row value and column value, let us assume $K_1 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$C_{11} = R_1 + K_1 \Rightarrow 10 = R_1 + 0 \Rightarrow R_1 = 10$

$C_{31} = R_3 + K_1 \Rightarrow 9 = R_3 + 0 \Rightarrow R_3 = 9$

$C_{13} = R_1 + K_3 \Rightarrow 8 = 10 + K_3 \Rightarrow K_3 = -2$

$C_{23} = R_2 + K_3 \Rightarrow 8 = R_2 - 2 \Rightarrow R_2 = 10$

$C_{22} = R_2 + K_2 \Rightarrow 10 = 10 + K_2 \Rightarrow K_2 = 0$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$$\Delta_{12} = C_{12} - (R_1 + K_2) = 14 - (0 + 10) = 4$$
$$\Delta_{21} = C_{21} - (R_2 + K_1) = 13 - (0 + 10) = 3$$
$$\Delta_{32} = C_{32} - (R_3 + K_2) = 10 - (9 + 0) = 1$$
$$\Delta_{33} = C_{33} - (R_3 + K_3) = 10 - (9 - 2) = 3$$

Since, all the improvement indices are positive, so the solution is optimal. The minimum total transportation cost is given by

$$\text{Min. TC} = \Sigma\text{Cell cost} \times \text{Quantity allocated}$$
$$= 10 \times 15 + 8 \times 5 + 10 \times 15 + 8 \times 5 + 9 \times 5$$
$$= \text{Rs } 425 \text{ for } X_{11} = 15,\ X_{13} = 5,\ X_{22} = 15,\ X_{23} = 5,\ X_{31} = 5$$

Example 12: The Sunshine Transportation Company ships trucks loads of grain from three regions to four mills. The supply (truck load) and the demand truck load and the per unit transportation cost in hundred rupees are given below.

Regions	Mills				Supply
	1	2	3	4	
1	10	2	20	11	15
2	12	7	9	20	25
3	4	4	16	18	10
Demand	5	15	15	15	50

Obtain the minimum cost of shipping schedule.

Solution:

Here, we have, $\Sigma D = 5 + 15 + 15 + 15 = 50$ and $\Sigma S = 15 + 25 + 10 = 50$

Since, $\Sigma D = \Sigma S$, So, given problem is balanced.

Step 1: **Calculation of Initial Solution from VAM**

Table 1

Initial Solution from VAM

Regions \ Mills	1	2	3	4	Supply	I	II	III
1	× 10	15 2	× 20	× 11	~~15~~	← 8	–	–
2	× 12	× 7	15 9	10 20	~~25~~ ~~10~~	2	3	← 11
3	5 4	× 4	× 16	5 18	~~10~~ ~~5~~	0	← 12	2
Demand	~~5~~	~~15~~	~~15~~	~~15~~ ~~10~~	50			
I	6	2	7	7				
II	8	–	7	2				
III	–	–	7	2				

The total transportation cost of initial solution is

Total TC $= \Sigma$Cell cost $\times$ Quantity allocated

$= 2 \times 15 + 9 \times 15 + 20 \times 10 + 4 \times 5 + 18 \times 5$

$=$ Rs 475

***Step 2:* Testing of Optimality by MODI Method**

The initial solution from VAM is

Table – 2

Regions \ Mills	1	2	3	4	Supply	R_i
1	10	2 [– 15]	20	11 [+]	15	$R_1 = 0$
2	12	7	9 [15]	20 [10]	25	$R_2 = 4$
3	4 [5]	4 [+ θ]	16	18 [5 –]	10	$R_3 = 2$
Demand	5	15	15	15	50	
K_j	$K_1 = 2$	$K_2 = 2$	$K_3 = 5$	$K_4 = 16$		

Number of occupied cells = 5 is less than $m + n - 1 = 3 + 4 - 1 = 6$, so the solution is degenerate. Therefore we assign a very small quantity 'θ' ≈ 0 to the cell (3, 2) where loop cannot be formed at this stage.

To calculate the row value and column value, let us assume $R_1 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$C_{12} = R_1 + K_2 \Rightarrow 2 = 0 + K_2 \Rightarrow K_2 = 2$

$C_{32} = R_3 + K_2 \Rightarrow 4 = R_3 + 2 \Rightarrow R_3 = 2$

$C_{34} = R_3 + K_4 \Rightarrow 18 = 2 + K_4 \Rightarrow K_4 = 16$

$C_{31} = R_3 + K_1 \Rightarrow 4 = 2 + K_1 \Rightarrow K_1 = 2$

$C_{24} = R_2 + K_4 \Rightarrow 20 = R_2 + 16 \Rightarrow R_2 = 4$

$C_{23} = R_2 + K_3 \Rightarrow 9 = 4 + K_3 \Rightarrow K_3 = 5$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$\Delta_{11} = C_{11} - (R_1 + K_1) = 10 - (0 + 2) = 8$

$\Delta_{13} = C_{13} - (R_1 + K_3) = 20 - (0 + 5) = 15$

$\Delta_{14} = C_{14} - (R_1 + K_4) = 11 - (0 + 16) = -5$

$\Delta_{21} = C_{21} - (R_2 + K_1) = 12 - (4 + 2) = 6$

$\Delta_{22} = C_{22} - (R_2 + K_2) = 7 - (4 + 2) = 1$

$\Delta_{33} = C_{33} - (R_3 + K_3) = 16 - (2 + 5) = 9$

Since, the improvement index $\Delta_{14} = -5 < 0$, the solution is not optimal. To get optimal solution, reallocation of goods is necessary. For reallocation, a loop starts from the cell (1, 4) and passes through the cells (3, 4), (3, 2) and (1, 2). The minimum quantity (i.e. 5) among the quantities {5 and 15} of all negative corners is selected and it is added to positive corners (1, 4) and (3, 2) and subtracted from negative corners (3, 4) and (1, 2) of the loop. The new solution becomes

Table – 3

Regions \ Mills	1	2	3	4	Supply	R_i
1	10	2 [– 10]	20	11 [5 +]	15	$R_1 = 0$
2	12	7 [+]	9 [15]	20 [10 –]	25	$R_2 = 9$
3	4 [5]	4 [5]	16	18	10	$R_3 = 2$
Demand	5	15	15	15	50	
K_j	$K_1 = 2$	$K_2 = 2$	$K_3 = 0$	$K_4 = 11$		

Since, number of occupied cells = 6 is equal to $m + n - 1 = 3 + 4 - 1 = 6$, so the solution is non-degenerate.

To calculate the row value and column value, let us assume $R_1 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$$C_{12} = R_1 + K_2 \quad\Rightarrow\quad 2 = 0 + K_2 \quad\Rightarrow\quad K_2 = 2$$
$$C_{14} = R_1 + K_4 \quad\Rightarrow\quad 11 = 0 + K_4 \quad\Rightarrow\quad K_4 = 11$$
$$C_{32} = R_3 + K_2 \quad\Rightarrow\quad 4 = R_3 + 2 \quad\Rightarrow\quad R_3 = 2$$
$$C_{24} = R_2 + K_4 \quad\Rightarrow\quad 20 = R_2 + 11 \quad\Rightarrow\quad R_2 = 9$$
$$C_{31} = R_3 + K_1 \quad\Rightarrow\quad 4 = 2 + K_1 \quad\Rightarrow\quad K_1 = 2$$
$$C_{23} = R_2 + K_3 \quad\Rightarrow\quad 9 = 9 + K_3 \quad\Rightarrow\quad K_3 = 0$$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$$\Delta_{11} = C_{11} - (R_1 + K_1) = 10 - (0 + 2) = 8$$
$$\Delta_{13} = C_{13} - (R_1 + K_3) = 20 - (0 + 0) = 20$$
$$\Delta_{21} = C_{21} - (R_2 + K_1) = 12 - (9 + 2) = 1$$
$$\Delta_{22} = C_{22} - (R_2 + K_2) = 7 - (9 + 2) = -4$$
$$\Delta_{33} = C_{33} - (R_3 + K_3) = 16 - (2 + 0) = 14$$
$$\Delta_{34} = C_{34} - (R_3 + K_4) = 18 - (2 + 11) = 5$$

Since, the improvement index $\Delta_{22} = -4 < 0$, the solution is not optimal. To get optimal solution, reallocation of goods is necessary. For reallocation, a loop starts from the cell (2, 2) and passes through the cells (2, 4), (1, 4) and (1, 2). The minimum quantity (i.e. 10) among the quantities {10 and 10} of all negative corners is selected and it is added to positive corners (2, 2) and (1, 4) and subtracted from negative corners (1, 2) and (2, 4) of the loop. The new solution becomes

Table – 4

Regions \ Mills	1	2	3	4	Supply	R_i
1	10	2 [θ]	20	11 [15]	15	$R_1 = 2$
2	12	7 [10]	9 [15]	20	25	$R_2 = 7$
3	4 [5]	4 [5]	16	18	10	$R_3 = 4$
Demand	5	15	15	15	50	
K_j	$K_1 = 0$	$K_2 = 0$	$K_3 = 2$	$K_4 = 9$		

Number of occupied cells = 5 is less than $m + n - 1 = 3 + 4 - 1 = 6$, so the solution is degenerate. Therefore we assign a very small quantity 'θ' = 0 to the cell (1, 2) where loop cannot be formed at this stage.

To calculate the row value and column value, let us assume $K_2 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$C_{12} = R_1 + K_2 \Rightarrow 2 = R_1 + 0 \Rightarrow R_1 = 2$

$C_{22} = R_2 + K_2 \Rightarrow 7 = R_2 + 0 \Rightarrow R_2 = 7$

$C_{32} = R_3 + K_2 \Rightarrow 4 = R_3 + 0 \Rightarrow R_3 = 4$

$C_{14} = R_1 + K_4 \Rightarrow 11 = 2 + K_4 \Rightarrow K_4 = 9$

$C_{23} = R_2 + K_3 \Rightarrow 9 = 7 + K_3 \Rightarrow K_3 = 2$

$C_{31} = R_3 + K_1 \Rightarrow 4 = 4 + K_1 \Rightarrow K_1 = 0$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$\Delta_{11} = C_{11} - (R_1 + K_1) = 10 - (2 + 0) = 8$

$\Delta_{13} = C_{13} - (R_1 + K_3) = 20 - (2 + 2) = 16$

$\Delta_{21} = C_{21} - (R_2 + K_1) = 12 - (7 + 0) = 5$

$\Delta_{24} = C_{24} - (R_2 + K_4) = 20 - (7 + 9) = 4$

$\Delta_{33} = C_{33} - (R_3 + K_3) = 16 - (4 + 2) = 10$

$\Delta_{34} = C_{34} - (R_3 + K_4) = 18 - (4 + 9) = 5$

Since, all the improvement indices are positive, so the solution is optimal. The minimum total transportation cost is given by

Min TC $= \Sigma$Cell cost × Quantity allocated

$= 2 \times 0 + 11 \times 15 + 7 \times 10 + 9 \times 15 + 4 \times 5 + 4 \times 5$

$=$ Rs. 410 for $X_{14} = 15, X_{22} = 10, X_{23} = 15, X_{31} = 5, X_{32} = 5$

Example 13: Find optimal transportation schedule to minimize the cost from the following unbalanced transportation problem.

Plant	Stores				Available quantity
	S_1	S_2	S_3	S_4	
W	29	32	29	26	70
X	27	23	27	27	60
Y	26	25	29	31	90
Demand units	60	40	60	20	180 / 220

Solution:

Here, we have, $\Sigma D = 60 + 40 + 60 + 20 = 180$ and $\Sigma S = 70 + 60 + 90 = 220$

Since, $\Sigma D \neq \Sigma S$, So, given problem is unbalanced.

Since, the total demand (180) is not equal to total supply (220), the given problem is unbalanced. So, we create a dummy destination (S_5) with demand ($220 - 180 = 40$) and cell cost '0' for each dummy cells.

***Step 1:* Calculation of Initial Solution from VAM**

Table – 1
Initial Solution from VAM

Plant \ Stores	S_1	S_2	S_3	S_4	S_5	Available Quantity	I	II	III	IV
W	29 ×	32 ×	29 [10]	26 [20]	0 [40]	~~70~~ ~~30~~ 10	28	3	3	3
X	27 ×	23 [40]	27 [20]	27 ×	0 ×	~~60~~ 20	23	4	0	0
Y	26 [60]	25 ×	29 [30]	31 ×	0 ×	~~90~~ 30	25	1	3	2
Demand	60	40	~~60~~ ~~30~~ 10	20	40	220				
I	1	2	2	1	0					
II	1	2	2	1	–					
III	1	–	2	1	–					
IV	–	–	2	1	–					

The total transportation cost of initial solution is

Total TC $= \Sigma$Cell cost $\times$ Quantity allocated

$= 29 \times 10 + 26 \times 20 + 0 \times 40 + 23 \times 40 + 27 \times 20 + 26 \times 60 + 29 \times 30$

$=$ Rs 4700

Step 2: **Testing of Optimality by MODI Method**

The initial solution from VAM is

Table – 2

Plant \ Stores	S_1	S_2	S_3	S_4	S_5	Supply	R_i
W	29	32	29 [10]	26 [20]	0 [40]	70	$R_1 = 0$
X	27	23 [40]	27 [20]	27	0	60	$R_2 = -2$
Y	26 [60]	25	29 [30]	31	0	90	$R_3 = 0$
Demand	60	40	60	20	40	220	
K_j	$K_1 = 26$	$K_2 = 25$	$K_3 = 29$	$K_4 = 26$	$K_5 = 0$		

Since, number of occupied cells = 7 is equal to $m + n - 1 = 3 + 5 - 1 = 7$, so the solution is non–degenerate.

To calculate the row value and column value, let us assume $R_1 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$C_{13} = R_1 + K_3 \Rightarrow 29 = 0 + K_3 \rightarrow K_3 = 29$

$C_{14} = R_1 + K_4 \Rightarrow 26 = 0 + K_4 \rightarrow K_4 = 26$

$C_{15} = R_1 + K_5 \Rightarrow 0 = 0 + K_5 \rightarrow K_5 = 0$

$C_{23} = R_2 + K_3 \Rightarrow 27 = R_2 + 29 \rightarrow R_2 = -2$

$C_{22} = R_2 + K_2 \Rightarrow 23 = -2 + K_2 \rightarrow K_2 = 25$

$C_{33} = R_3 + K_3 \Rightarrow 29 = R_3 + 29 \rightarrow R_3 = 0$

$C_{31} = R_3 + K_1 \Rightarrow 26 = 0 + K_1 \rightarrow K_1 = 26$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$\Delta_{11} = C_{11} - (R_1 + K_1) = 29 - (0 + 26) = 3$

$\Delta_{12} = C_{12} - (R_1 + K_2) = 32 - (0 + 25) = 7$

$\Delta_{21} = C_{21} - (R_2 + K_1) = 27 - (-2 + 26) = 3$

$\Delta_{24} = C_{24} - (R_2 + K_4) = 27 - (-2 + 26) = 3$

$\Delta_{25} = C_{25} - (R_2 + K_5) = 0 - (-2 + 0) = 2$

$\Delta_{32} = C_{32} - (R_3 + K_2) = 25 - (0 + 25) = 0$

$\Delta_{34} = C_{34} - (R_3 + K_4) = 31 - (0 + 26) = 5$

$\Delta_{35} = C_{35} - (R_3 + K_5) = 0 - (0 + 0) = 0$

Since, all the improvement indices are zero or positive, so the solution is optimal. The minimum total transportation cost is given by

Min. TC $= \Sigma$Cell cost × Quantity allocated

$= 29 \times 10 + 26 \times 20 + 0 \times 40 + 23 \times 40 + 27 \times 20 + 26 \times 60 + 29 \times 30$

$=$ Rs 4700 for $X_{13} = 10, X_{14} = 20, X_{22} = 40, X_{23} = 20, X_{31} = 60, X_{33} = 30$

Example 14: A company manufacturing mobile sets has four plants with a capacity of 25, 50, 35 and 20 units respectively. The company supplies mobile sets to its four show rooms which have a demand of 20, 80, 18 and 12 units respectively. Due to the differences in the raw material cost and the transportation cost, the profits per unit (in rupees) differ which are given in the following table.

Plants	Showrooms			
	P	Q	R	S
A	450	550	600	550
B	500	525	650	585
C	555	545	550	600
D	650	625	540	565

The demand at showroom P must be supplied from plant A. Determine the maximum total profit.

Solution:

Since, the demand (20) at the showroom P must be supplied from the plant A, so the showroom (P) can be removed from the calculation table because its demand has been fulfilled. The supply at the plant A remains 25 – 20 = 5. Thus the profit table will be

Table 1

Plant \ Showroom	Q	R	S	Supply
A	550	600	550	5
B	525	650	585	50
C	545	550	600	35
D	625	540	565	20
Demand	80	18	12	110

Here, we have, $\Sigma D = 80 + 18 + 12 = 110$ and $\Sigma S = 5 + 50 + 35 + 20 = 110$

Since, $\Sigma D = \Sigma S$, So, given problem is balanced.

The profit table can be converted into loss table by subtracting each element from highest element (650). The loss table will be

Table 2

Plant \ Showroom	Q	R	S	Supply
A	100	50	100	5
B	125	0	65	50
C	105	100	50	35
D	25	110	85	20
Demand	80	18	12	110

Step 1: **Calculation of Initial Solution from VAM**

Table – 3

Initial Solution from VAM

Plant \ Showroom	Q	R	S	Supply	Cost difference I	II	III
A	[5] 100	× 50	× 100	5	50	50	0
B	[20] 125	[18] 0	[12] 65	~~50~~ ~~32~~ ~~20~~	65	← 65	← 60
C	[35] 105	× 100	× 50	~~35~~	50	50	55
D	[20] 25	× 110	× 85	~~20~~	60	–	–
Demand	~~80~~ ~~60~~	~~18~~	~~12~~	110			
Cost difference I	75↑	50	15				
Cost difference II	5	50	15				
Cost difference III	5	–	15				

Step 2: **Testing of Optimality by MODI Method**

The initial solution from VAM is

Table – 4

Plant \ Showroom	Q	R	S	Supply	R_i
A	[5] 100	50	100	5	$R_1 = 100$
B	[20] 125	[18] 0	[12] 65	50	$R_2 = 125$
C	[35] 105	100	50	35	$R_3 = 105$
D	[20] 25	110	85	20	$R_4 = 25$
Demand	80	18	12	110	
K_j	$K_1 = 0$	$K_2 = -125$	$K_3 = -60$		

Since, number of occupied cells = 6 is equal to $m + n - 1 = 4 + 3 - 1 = 6$, so the solution is non-degenerate.

To calculate the row value and column value, let us assume $K_1 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$$C_{11} = R_1 + K_1 \Rightarrow 100 = R_1 + 0 \Rightarrow R_1 = 100$$
$$C_{21} = R_2 + K_1 \Rightarrow 125 = R_2 + 0 \Rightarrow R_2 = 125$$
$$C_{31} = R_3 + K_1 \Rightarrow 105 = R_3 + 0 \Rightarrow R_3 = 105$$
$$C_{41} = R_4 + K_1 \Rightarrow 25 = R_4 + 0 \Rightarrow R_4 = 25$$
$$C_{22} = R_2 + K_2 \Rightarrow 0 = 125 + K_2 \Rightarrow K_2 = -125$$
$$C_{23} = R_2 + K_3 \Rightarrow 65 = 125 + K_3 \Rightarrow K_3 = -60$$

The improvement index for each unoccupied cells can be calculated by using the relation $\Delta_{ij} = C_{ij} - (R_i + K_j)$. The improvement indices are

$$\Delta_{12} = C_{12} - (R_1 + K_2) = 50 - (100 - 125) = 75$$
$$\Delta_{13} = C_{13} - (R_1 + K_3) = 100 - (100 - 60) = 60$$
$$\Delta_{32} = C_{32} - (R_3 + K_2) = 100 - (105 - 125) = 120$$
$$\Delta_{33} = C_{33} - (R_3 + K_3) = 50 - (105 - 60) = 5$$
$$\Delta_{42} = C_{42} - (R_4 + K_2) = 110 - (25 - 125) = 210$$
$$\Delta_{43} = C_{43} - (R_4 + K_3) = 85 - (25 - 60) = 120$$

Since, all the improvement indices are positive, so the solution is optimal. The maximum total profit is given by

Max. Profit $= \Sigma$Cell profit × Quantity allocated

$= 450 \times 20 + 550 \times 5 + 525 \times 20 + 545 \times 35 + 625 \times 20 + 650 \times 18 + 585 \times 12$

$=$ Rs. 72,545

For plant A to showroom P = 20 sets according to restriction of the problem.

plant A to showroom Q = 5 sets

plant B to showroom Q = 20 sets

plant C to showroom Q = 35 sets

plant D to showroom Q = 20 sets

plant B to showroom R = 18 sets

plant B to showroom S = 12 sets

Example 15: Given the transportation frame work, find the optimal transportation cost.

From \ To	A	B	C	Plant capacity
W	40	80	80	55
X	160	–	160	25
Y	80	160	240	35
Project Re out	35	45	35	115

Solution:

Here, we have, $\Sigma D = 35 + 45 + 35 = 115$ and $\Sigma S = 55 + 25 + 35 = 115$

Since, $\Sigma D = \Sigma S$, So, given problem is balanced.

Step 1: **Calculation of Initial Solution from VAM**

Table – 1

Initial Solution from VAM

To / From	A	B	C	Supply	I	II
W	40 ×	80 **45**	80 **10**	~~55~~ ~~10~~	40	40
X	160 ×	– ×	160 **25**	~~25~~	0	0
Y	80 **35**	160 ×	240 ×	~~35~~	80	← 160
Demand	~~35~~	~~45~~	~~35~~ ~~10~~	115		
I	40	80 ↑	80			
II	40	–	80			

The total transportation cost of initial solution is

Total TC = ΣCell cost × Quantity allocated

$= 80 \times 45 + 80 \times 10 + 160 \times 25 + 80 \times 35$

$= 3600 + 800 + 4000 + 2800 = \text{Rs } 11200$

Step 2: **Testing of Optimality by MODI Method**

The initial solution from VAM is

Table – 2

To / From	A	B	C	Supply	R_i
W	40 **θ**	80 **45**	80 **10**	55	$R_1 = 0$
X	160	–	160 **25**	25	$R_2 = 80$
Y	80 **35**	160	240	35	$R_3 = 40$
Demand	35	45	35	115	
K_j	$K_1 = 40$	$K_2 = 80$	$K_3 = 80$		

Since, number of occupied cells = 4 is not equal to $m + n - 1 = 3 + 3 - 1 = 5$, so the solution is degenerate. To make it non degenerate we assign a very small quantity $\theta \approx 0$ to the cell (1, 1) where loop cannot be formed at this stage.

To calculate the row value and column value, let us assume $R_1 = 0$ for reference. Then the remaining row value (R_i) and column value (K_j) can be calculated by using the relation $C_{ij} = R_i + K_j$ for each occupied cells where C_{ij} is the unit transportation cost for i^{th} row and j^{th} column.

$C_{11} = R_1 + K_1 \Rightarrow 40 = 0 + K_1 \Rightarrow K_1 = 40$

$C_{12} = R_1 + K_2 \Rightarrow 80 = 0 + K_2 \Rightarrow K_2 = 80$

$C_{13} = R_1 + K_3 \Rightarrow 80 = 0 + K_3 \Rightarrow K_3 = 80$

$C_{23} = R_2 + K_3 \Rightarrow 160 = R_2 + 80 \Rightarrow R_2 = 80$

$C_{31} = R_3 + K_1 \Rightarrow 80 = R_3 + 40 \Rightarrow R_3 = 40$

Practical Questions

1. A company has to transport its production from three plants to three warehouses. The unit transportation cost is given in the table given below. Determine the optimal solution of the transportation problem.

Plant \ Warehouse	W_1	W_2	W_3	Supply
P_1	46	50	52	27
P_2	34	42	24	22
P_3	58	60	38	26
Demand	24	18	33	

Find the minimum cost for the transportation problem.

2. A company has factories A, B, C and D which supply to warehouses X, Y and Z. The information related to supply from factories, requirement to warehouses and unit transportation cost from various factories to various warehouse are given below:

From \ To	X	Y	Z	Supply
A	40	20	50	10
B	45	35	45	80
C	30	25	40	15
D	20	25	40	40
Demand	75	20	50	145

Determine the optimal distribution for this company to minimize the total transportation cost.

3. 'Nepal Transport' own several trucks used to deliver crushed stones to road projects in the region. The company has received the delivery schedule for next week as follows:

Project	Requirement per week	Plant	Available per week
A	50	W	55
B	75	X	60
C	50	Y	60

The cost matrix for above schedule is given as follows:

Plant	Cost information ('000 Rs.)		
	Project A	Project B	Project C
W	4	8	3
X	6	7	9
Y	8	2	5

Find an optimum solution to minimize the transportation cost.

4. Obtain the minimum transportation cost for the following transportation problem.

Cost of shipping and physical units

Source / Determination	S_1	S_2	S_3	Units demanded
W_1	90	100	100	5
W_2	100	140	80	20
W_3	130	100	80	20
Units available	20	15	10	45

5. Obtain an optimal solution of the transportation problem having three sources and four destinations. The table given below represents the unit transportation cost.

From / To	P	Q	R	S	Supply
A	19	30	50	10	70
B	70	30	40	60	90
C	40	8	70	20	180
Demand	50	80	70	140	

6. Find the optimal transportation schedule from following with the objective of minimizing the cost.

Factory	Quantity requirements per day in kg.	Warehouse	Quantity available per day in kg.
P	45	X	35
Q	50	Y	40
R	20	Z	40

Cost of transportation per kg is given in the following table.

From	To factory		
	P	Q	R
Warehouse X	10	20	20
Warehouse Y	40	60	40
Warehouse Z	10	16	24

7. Obtain the minimum transportation cost for the following transportation problem.

From / To	P	Q	R	Supply
A	40	80	80	55
B	160	240	160	25
C	80	160	240	35
Demand	35	45	35	

8. Goods have to be transported from sources S_1, S_2 and S_3 to destinations D_1, D_2 and D_3. The transportation cost per unit, capacities of the sources and requirement of the destinations are given in the following table. Find the optimal solution of the problem.

Sources	Destinations			Supply
	D_1	D_2	D_3	
S_1	16	10	12	120
S_2	30	20	24	80
S_3	6	18	20	80
Demand	150	80	50	280

9. Nepal Hardware Company's has plants in Kathmandu, Banepa and Nuwakot and three warehouses at Bhairhawa, Birgunj and Biratnagar. The quantities available at the plants are respectively 60, 70 and 80 where as the demand at the warehouses are 50, 80 and 80 respectively. The unit cost of transportation is observed as follows:

	Bhairhawa	Birgunj	Biratnagar
Kathmandu	2	5	7
Banepa	2	3	4
Nuwakot	5	8	11

Find an optimal allocation that minimizes the transportation cost.

10. The Sunshine Transportation Company ships trucks loads of grain from three regions to four mills. The supply (truck load) and the demand truck load and the per unit transportation cost in hundred rupees are given below.

Regions	Mills				Supply
	1	2	3	4	
1	10	2	20	11	15
2	12	7	9	20	25
3	4	4	16	18	10
Demand	5	15	15	15	50

11. Find the optimum transportation schedule from the following in order to minimize transportation costs.

Plant	Market			Supply (units)
	X	Y	Z	
A	5	2	8	150
B	4	3	5	150
C	2	4	–	200
D	6	3	4	250
Demand (units)	250	200	175	750 / 625

12. A company has three plants W, X and Y and three warehouses A, B and C. The supplies are transported from the plants to the warehouses which are located at varying distances from the plants. On account of these varying distances, the transportation costs from plant to warehouses are given below in the table. Determine the transportation schedule to optimize the cost.

Plants	Warehouses			Supply (units)
	A	B	C	
W	60	40	90	400
X	100	50	80	350
Y	120	70	60	150
Demand	500	200	300	900 / 1000

13. A company has four factories manufacturing the same commodity, which are required to be transported to meet the demands in four warehouses. The supplies, demands and the cost per transportation from factory to warehouse in rupees per unit of product are given in the following table.

Factory	Warehouses X	Y	Z	W	Supply
A	25	55	40	60	60
B	35	30	50	40	140
C	36	45	26	66	150
D	35	30	41	50	50
Demand	90	100	120	140	450 / 400

(a) Determine an optimal strategy of the transportation of goods from factories to warehouses and assess the optimal cost.

(b) If a new transporter agrees to transport goods from factory C to warehouse W at a unit cost of Rs 50, analyze the impact of this on your current optimal solution.

14. From the following profit matrix, find the maximum profit by using transportation model.

Supply and Demand

Plant	Unit available	Project	Unit Demanded
A	1700	W	1300
B	2500	X	2000
C	1000	Y	1900

Profit in '000 rupees

From	To project W	To project X	To project Y
Plant A	120	80	50
Plant B	110	150	100
Plant C	20	170	60

15. Using transportation problem, determine the maximum profit from the information given below.

Warehouse	Factory (profit in rupees) X	Y	Z	Capacity
A	130	170	180	30
B	110	140	180	30
C	150	120	150	40
D	200	130	120	50
Demand	20	60	70	

16. The following table gives the profit of a company; find the maximum profit by using transportation model.

Profit table

Destination \ Source	S_1	S_2	S_3	Units demanded
A	490	480	480	5
B	480	440	500	20
C	450	480	500	20
Units available	20	15	10	45

17. A company has four factories A, B, C and D manufacturing the same products. Production and raw materials cost differ from factory to factory and are given in following table in the first two rows. The transportation costs from the factory to the sales depots P, Q and R also given. The last two columns in the table give the sales price and the total requirements at each depot. The production capacity of each factory is given in the last row.

Factory		A	B	C	D	Sales price Per unit	Requirement
Production cost/unit		15	18	14	13		
Raw material cost/unit		10	9	12	9		
Transport cost/unit	P	3	9	5	4	34	80
	Q	1	7	4	5	32	120
	R	5	8	3	6	31	150
Supply		10	150	50	100		

Determine the most profitable production and distribution schedule.

18. A company has three plants W, X and Y and three warehouses A, B and C. The supplies are transported from the plants to the warehouses which are located at varying distances from the plants. On account of these varying distances, the profits from plant to warehouses vary from Rs. 800 to Rs. 2400. The company wishes to maximize the profit. The profits from plants to the warehouses are given below in the table. Determine the transportation schedule to optimize the profit.

Plants	Warehouses A	B	C	Supply
W	1200	800	1800	400
X	2000	1000	1600	350
Y	2400	1400	1200	150
Demand	500	200	300	1000 / 900

19. Find the maximum profit for the following transportation problem with the following profit table.

Plants	Warehouses A	B	C	Supply (units)
W	12	15	14	120
X	5	10	8	80
Y	20	11	10	50
Demand (units)	150	80	70	300 / 250

ANSWERS

1. Min. Cost = Rs. 2980
2. Min. Cost = Rs. 4950
3. Min. Cost = Rs. 665 ('000)
4. Min. Cost = Rs. 4250
5. Min. Cost = Rs 7430
6. Min. Cost = Rs 2690
7. Min. Cost = Rs 11200
8. Min. Cost = Rs 3800
9. Min. Cost = Rs 1090
10. Min. Cost = Rs 410
11. Min. Cost = Rs 1750
12. Min. Cost = Rs 55000
13. Min. Cost = (a) Rs 12300
14. Max. Cost = Rs 6460 ('00000)
15. Max. Profit = Rs 24400
16. Max. Profit = Rs 21850
17. Max. Profit = Rs 480
18. Max. Profit = Rs 1680000
19. Max. Profit = Rs 3380

Answers of Multiple Choice Questions

1. (a)	2. (b)	3. (a)	4. (b)	5. (a)	6. (c)	7. (b)	8. (c)	9. (a)	10. (a
11. (a)	12. (d)	13. (d)	14. (b)	15. (d)	16. (d)	17. (a)	18. (b)	19. (a)	20. (b
21. (c)	22. (b)	23. (a)	24. (c)	25. (b)	26. (d)	27. (b)	28. (b)	29. (b)	30. (a
31. (d)	32. (a)	33. (d)	34. (d)	35. (c)					

❑❑❑

Assignment Problem

Introduction

In real life, we have faced with the problem of allocating different personnel or workers to different jobs. Not everyone has the same ability to perform a given job. Different persons have different abilities to execute the same task and these different capabilities are expressed in terms of cost/profit/time involved in executing a given job. Therefore, we have to decide "how to assign different workers to different jobs" so that cost of performing such job is minimized. Assignment problem can be defined as the mathematical models, used to allocate the jobs to the workers as per their performance, ability, skill and knowledge. It follows the principle of right man at right place.

The basic and fundamental principle of this model is to allocate jobs to the workers on one to one basic to minimize the total cost and then to maximize the total revenue of the organization.

The main objective of this chapter is to equip the learner to deal with following situation: a) Assignments of different jobs to different workers/ machines on one to one basis where time or cost of performing such job is given. b) Assignment of different personnel to different location or service station with the objective to maximize sales/profit/consumer reaches. c) To deal with a situation where number of jobs to be assigned do not match with number of machines/workers. d) To deal with a situation where some jobs cannot be assigned to specific machines/workers.

The assignment problem is a special type of linear programming problem. We know that linear programming is an allocation technique to optimize a given objective. In linear programming we decide how to allocate limited resources over different activities so that, we maximize the profits or minimized the cost.

Similarly in assignment problem, assignees are being assigned to perform different tasks. For example, the assignees can be employees who need to be given work assignments, is a common application of assignment problem. However, assignees need not be people, they could be machines, vehicles, plants, time slots etc. to be assigned different task.

Assignment problem pertains to problem of assigning 'n' jobs to 'n' different machines. This model can be effectively used for any other problem in which 'n' items (or persons) are to be assigned to other 'n' items so that each one of the first group is assigned to one distinct item from the second group.

Characteristics of the Assignment Problem

From the general description of the assignment problem and the methods described for solving such problems, we can list out following characteristics of these problems:

1. The availability of resources such as workers, machines, projects managers, salesmen, jobs are finite in number.
2. The available resources can be assigned only one to one basis i.e. one job can be assigned to only one any worker.
3. The outcomes or results are expressed in terms of cost, time or profit.
4. All the persons are capable of handling any job.
5. The number of workers/ machines must be equal to number of jobs.

Mathematical Model of Assignment Problem

An assignment problem with 'n' workers and 'n' jobs can be presented in the following table. The cost or time taken by workers on various jobs is indicated by C_{ij} in the table.

Workers	Jobs				
	J_1	J_2	J_3	...	J_n
W_1	C_{11}	C_{12}	C_{13}	...	C_{1n}
W_2	C_{21}	C_{22}	C_{23}	...	C_{2n}
W_3	C_{31}	C_{32}	C_{33}	...	C_{3n}
.	.	.	.	.	.
W_n	C_{n1}	C_{n2}	C_{n3}	...	C_{nn}

Let X_{ij} denotes the assignment of the i^{th} worker to j^{th} job such that

X_{ij} = 1 if i^{th} worker is assigned to j^{th} job

= 0 if i^{th} worker is not assigned to j^{th} job

Under the assumption that number of workers is equal to the number of jobs, the mathematical model of the assignment problem can be stated as linear programming problem as

$$\text{Minimize total cost } (Z) = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}X_{ij}$$

Under the conditions,

$\sum_{j=1}^{n} X_{ij} = 1$ for all i = 1, 2, 3 .... n

$\sum_{i=1}^{n} X_{ij} = 1$ for all j = 1, 2, 3 .... n

$X_{ij} \geq 0$ for all i = 1, 2, 3 .... n and j = 1, 2, 3 .... n

Types of Assignment Problem

There are generally two types of assignment problem depending upon the number of resources and number of activities where resources have to be assigned.

1. **Balanced Assignment Problem:** An assignment problem is called a balanced assignment problem if number of resources is equal to the number of activities. For example,

   XYZ Company has three jobs to be done on three machines. Each job must be done on one and only one machine. The cost of each machine is given in the following table.

Job	Machine		
	P	Q	R
A	Rs. 18	Rs. 24	Rs. 28
B	Rs. 8	Rs. 13	Rs. 17
C	Rs. 10	Rs. 15	Rs. 25

This is a balanced assignment problem because number of jobs is equal to the number of machines.

2. **Unbalanced Assignment Problem:** An assignment problem is called an unbalanced assignment problem if number of resources is not equal to the number of activities. For example, XYZ Company has three jobs to be done on four machines. Each job must be done on one and only one machine. The cost of each machine is given in the following table.

Job	Machine			
	P	Q	R	S
A	Rs. 18	Rs. 24	Rs. 28	Rs. 38
B	Rs. 8	Rs. 13	Rs. 17	Rs. 27
C	Rs. 10	Rs. 15	Rs. 25	Rs. 35

This is an unbalanced assignment problem because number of jobs is not equal to the number of machines.

It should be noted that an assignment problem can be solved only when it is balanced. So the unbalanced assignment problem should be converted into balanced assignment problem before solving it by using dummy row (resource) if number of column (job) is greater than the number of row (resource) or by using dummy column (job) if number of row (resource) is greater than the number of column (job).

Solution to Assignment Problem

There are four methods of solving an assignment problem. They are described below in detail.

Simplex Method

Let us consider an example to illustrate the simlex method for solving assignment problem of assigning three salesmen to three different counters. A company has three salesmen S_1, S_2, and S_3 to assign them to three counters C_1, C_2, and C_3. The costs of operating them to different counters are given below.

Salesmen	Counters		
	C_1	C_2	C_3
S_1	9	7	10
S_2	18	11	19
S_3	13	14	12

Let X_{ij} denotes the assignment of the i^{th} salesman to j^{th} counter such that

X_{ij} = 1 if i^{th} salesman is assigned to j^{th} counter

= 0 if i^{th} salesman is not assigned to j^{th} counter for all i = 1, 2, 3 and j = 1, 2, 3.

The total cost of assigning is given by

$$\text{Minimize } Z = 9X_{11} + 7X_{12} + 10X_{13} + 18X_{21} + 11X_{22} + 19X_{23} + 13X_{31} + 14X_{32} + 12X_{33}$$

Under the conditions:

(i) Row Constraints

$X_{11} + X_{12} + X_{13} = 1$

$X_{21} + X_{22} + X_{23} = 1$

$X_{31} + X_{32} + X_{33} = 1$

(ii) Column Constraints

$X_{11} + X_{21} + X_{31} = 1$

$X_{12} + X_{22} + X_{32} = 1$

$X_{13} + X_{23} + X_{33} = 1$

and $X_{ij} \geq 0$ for all i = 1, 2, 3 and j = 1, 2, 3.

The above problem is a problem of linear programming having nine decision variable and six constraints which can be solved by using simplex method but it will be tedious task.

Transportation Method

An assignment problem also can be represented as transportation problem by taking demand and supply as 1 for each row and column. Let us illustrate by using the same above example.

Salesman \ Counter	C_1	C_2	C_3	Supply	I	II
S_1	9 [1]	7	10	1	2	
S_2	18	11 [1]	19	1	7 ←	
S_3	13	14	12 [1]	1	1	
Demand	1	1	1	3		
I	4	4	2			
II	4	-	2			

Here we see that the number of occupied cells = 3 and m + n − 1 = 3 + 3 − 1 = 5, so the problem is degenerate. To make it non degenerate, we have to assign two θ in two unoccupied cells. In case of 5×5 table, there will be need of four θ to make it non degenerate. Hence, if assignment problem is solved by transportation problem, the solution will be generally degenerate and it requires large number of θ for testing of optimality.

Enumeration Method

The above problem can be solved by using complete enumeration method. It requires the listing of all possible assignments. In 3 × 3 table, there will be 3! = 3 × 2 × 1= 6 assignments. They are

1. $S_1 - C_1$, $S_2 - C_2$ and $S_3 - C_3$
2. $S_1 - C_1$, $S_2 - C_3$ and $S_3 - C_2$
3. $S_1 - C_2$, $S_2 - C_1$ and $S_3 - C_3$
4. $S_1 - C_2$, $S_2 - C_3$ and $S_3 - C_1$
5. $S_1 - C_3$, $S_2 - C_1$ and $S_3 - C_2$
6. $S_1 - C_3$, $S_2 - C_2$ and $S_3 - C_1$

This method also becomes tedious work if there are large of number of resources and jobs to be assigned. For example, suppose there are 5 salesmen and 5 counters to be assigned then the number of possible assignment = 5! = 120 and its enumeration is very tedious job.

Trial and error works well enough for this problem, but suppose you had ten salespeople flying to ten cities. How many trials would this take? There are n! ways of assigning 'n' resources to 'n' tasks, which means that as 'n' gets large, we have too many trials to consider.

Hungarian Method

Out of these four methods, Hungarian Method developed by a Hungarian mathematician – D. Konig's is the most efficient and easy for solving an assignment problem. This method involves the following steps:

Step 1 ***Row Operation:*** Select the least element from each row and subtract this least element from each element of that row which yields the reduced cost matrix.

Step 2 ***Column Operation:*** In the reduced cost matrix, select the least element from each column and subtract it from each element of that column.

Step 3 ***Assignment:*** Draw the minimum number of lines that will cross all the zeroes in the reduced cost matrix. In order to draw minimum number of lines, check the number of zero in the first row of the reduced cost matrix after row and column operation. If there is single zero in the first row, make a box (☐) over it and draw a line through the column. If there are two or more zeros in the first row, leave the first row and check the second row. If there is single zero in the second row, make a box over it and draw a line through the column. If there are two or more zeros in the second row, leave the second row and check third row. Repeat these processes to all the rows and columns and draw lines accordingly. This operation should be repeated until all the zeros in the reduced matrix are covered. It should be noted that checking the number of zero in row or column and the drawing of line should be in perpendicular direction i.e. when we are checking zero in row we should draw a line through the column and while we are checking zero in column we should draw a line through the row.

Step 4 ***Test of Optimality:*** If the number of lines drawn is equal to the number of rows/columns, the assignment schedule is optimal and it has least operating cost. If the number of lines drawn is not equal to the number of rows, the assignment schedule is not optimal. If the assignment schedule is not optimal, then to get an optimal assignment schedule, select the minimum uncrossed element and subtract it from all the uncrossed elements and add it to all the elements which are at the intersection of two lines. The logic behind adding the minimum uncrossed element to the elements which are at the intersection of two lines is to keep such positions away from the assignment. The element which is crossed by a single line should remain as it is. Repeat this process until we get the optimal schedule.

Step 5 Finally calculate the minimum cost or maximum profit as per problem adding the elements of assigned positions.

Example 1 A company has three salesmen S_1, S_2, and S_3 to assign them to three counters C_1, C_2, and C_3. The costs of operating them to different counters are given below.

Salesmen	Counters		
	C_1	C_2	C_3
S_1	9	7	10
S_2	18	11	19
S_3	13	14	12

Find the optimal assignment schedule of salesmen to counters that minimizes the total cost.

Solution: Here, no. of rows = 3, no. of columns = 3

Since, no. of rows = no. of columns, So, given Assignment problem is balanced.

The given cost table is

Table 1

Salesmen	Counters			Row minimum
	C_1	C_2	C_3	
S_1	9	7	10	7
S_2	18	11	19	11
S_3	13	14	12	12

Row Operation: Select the minimum element from the first row. The minimum element among the elements 9, 7 and 10 of first row is 7 and subtract this minimum element (7) from each of 9, 7 and 10. Similarly, select the minimum element from second row. The minimum element among the elements 18, 11 and 19 of second row is 11 and subtract this minimum element (11) from each of 18, 11 and 19. Similarly, select the minimum element from third row. The minimum element among the elements 13, 14 and 12 of third row is 12 and subtract this minimum element (12) from each of 13, 14 and 12. The reduced cost table after row operation ($R_1 - 7$, $R_2 - 11$ and $R_3 - 12$) becomes,

Table 2

Salesmen	Counters		
	C_1	C_2	C_3
S_1	2	0	3
S_2	7	0	8
S_3	1	2	0
Column minimum	1	0	0

Column Operation: Select the minimum element from the first column. The minimum element among the elements 2, 7 and 1 of first column is 1 and subtract this minimum element (1) from each of 2, 7 and 1. Similarly, select the minimum element from second column. The minimum element among the elements 0, 0 and 2 of second column is 0 and subtract this minimum element (0) from each of 0, 0 and 2. Similarly, select the minimum element from third column. The minimum element among the elements 3, 8 and 0 of third column is 0 and subtract this minimum element (0) from each of 3, 8 and 0. The reduced cost table after column operation ($C_1 - 1$, $C_2 - 0$ and $C_3 - 0$) can be seen as Table 3.

Table 3

Salesmen	Counters		
	C_1	C_2	C_3
S_1	1	0	3
S_2	6	0	8
S_3	0	2	0

Table 4

Salesmen	Counters		
	C_1	C_2	C_3
S_1	1	$\boxed{0}$	3
S_2	6	0	8
S_3	0	2	0

Table 5

Salesmen	Counters		
	C_1	C_2	C_3
S_1	1	$\boxed{0}$	3
S_2	6	0	8
S_3	$\boxed{0}$	2	0

Drawing of Lines

For the assignment, Count the number of zero in the first row. Since, there is single zero in the first row, select it by a box over it and draw a line through second column. Then table will be as Table 4.

Again, count the number of zero in the second row where no zero is found. The elements crossed by a line are not considered. Then count the number of zeros in the third row where there are two zeros so leave the third.

Now count the number of zeros along column wise. Since, there is single zero in the first column select it by a box over it and a line through third row. The table can be seen as Table 5. Since there is no uncrossed zero remaining, so we should go for testing of optimality.

Testing of Optimality

Count the number of lines drawn. Since number of lines (2) ≠ number of row/column (3), the solution is not optimal. To get optimal solution, select the minimum element (1) among the uncrossed elements (1, 3, 6 and 8) and subtract it from uncrossed elements (1, 3, 6 and 8) and add to double crossed elements (2). The reduced cost table can be seen as Table 6.

Table 6

Salesmen	Counters		
	C_1	C_2	C_3
S_1	0	0	2
S_2	5	0	7
S_3	0	3	0

Table 7

Salesmen	Counters		
	C_1	C_2	C_3
S_1	0	0	2
S_2	5	$\boxed{0}$	7
S_3	0	3	0

Table 8

Salesmen	Counters		
	C_1	C_2	C_3
S_1	0	0	2
S_2	5	$\boxed{0}$	7
S_3	0	3	$\boxed{0}$

Table 9

Salesman	Counters		
	C_1	C_2	C_3
S_1	$\boxed{0}$	0	2
S_2	5	$\boxed{0}$	7
S_3	0	3	$\boxed{0}$

Count the no. of zero in the first row. Since, there are two zeros in the first row, leave it and move to the second row to count the no. of zero. Since, there is single zero in the second row, select it by a box over it and draw a line along second column. The table will be as Table 7.

Move to the third row where two zeros are present. So, leave third row and repeat this process along column wise. Count the number of zero in first column where two zeros are found, so leave it and move to third column. Since there is single zero in the third column, select it by a box over it and draw a line along third row. The table can be seen as Table 8.

Finally there is single zero is remaining in the first row, so select it by a box over it and draw a line along first column. Then the table can be seen as Table 9.

Since, number of lines (3) = number of row/column (3), the solution is optimal. Hence, the total minimum cost to the company for using different salesmen to different counters is

Salesmen	Counters	Cost
S_1	C_1	9
S_2	C_2	11
S_3	C_3	12
Total Minimum Cost		32

Unbalanced Assignment Problem

An assignment problem is called an unbalanced problem if the number of rows (resources) is not equal to the number of columns (jobs). While solving the unbalanced assignment problem, like *unbalanced transportation problem*, we convert the unbalanced assignment problem into balanced assignment problem by creating dummy row(s) (resource) or dummy column(s) (job) as per required. If the number of row exceeds the number of column we create a dummy column having zero cost in all the dummy cells. If the number of column exceeds the number of row, we create a dummy row having zero cost in all the dummy cells. Then, the ***Hungarian Method*** can be applied to the balanced cost table to find the optimal assignment.

Example 2 There are five work centers in a production shop and there are four workmen, who can be deployed on these work centers. The study carried out shows that time taken in hour by various workmen on these centers are much at variance and therefore a judicious selection and allocation of workmen is needed to get optimum. The table showing time taken by workmen is given as follows:

Work Centers	Workmen			
	A	B	C	D
P	10	9	8	12
Q	3	4	5	2
R	25	20	14	16
S	7	9	10	9
T	18	14	16	25

Work out the optimum allocation for various work centers to various workmen.

Solution:

Here, no. of rows = 5, no. of columns = 4

Since, no. of rows ≠ no. of columns. So, given Assignment problem is unbalanced.

Since, the number of column (workmen) is less than the number of row (work centers), the assignment problem is unbalanced. To make it balanced, we create a dummy column/workman (E). Since the dummy workman does not require any time, so the operating time is supposed to be zero for each dummy cell. The time table becomes

Table 1

Work Centers	Workmen				
	A	B	C	D	E
P	10	9	8	12	0
Q	3	4	5	2	0
R	25	20	14	16	0
S	7	9	10	9	0
T	18	14	16	25	0
Column Minimum	3	4	5	2	0

Since, the minimum element of each row is zero, so the row operation is not necessary to perform. The minimum elements 3, 4, 5, 2 and 0 of different are subtracted from the elements of the corresponding column. The reduced time table is

Table 2

Work Centers	Workmen				
	A	B	C	D	E
P	7	5	3	10	$\boxed{0}$
Q	$\boxed{0}$	0	0	0	0
R	22	16	9	14	0
S	4	5	5	7	0
T	15	10	11	23	0

Since, number of line (2) ≠ number of row/column (5), the solution is not optimal. To get optimal solution, the minimum element (3) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced time table becomes

Table 3

Work Centers	Workmen				
	A	B	C	D	E
P	4	2	$\boxed{0}$	7	0
Q	$\boxed{0}$	0	0	0	3
R	19	13	6	11	$\boxed{0}$
S	1	2	2	4	0
T	12	7	8	20	0

Since, number of line (3) ≠ number of row/column (5), the solution is not optimal. To get optimal solution, the minimum element (1) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced time table becomes

Table 4

Work Centers	Workmen				
	A	B	C	D	E
P	4	2	$\boxed{0}$	7	1
Q	0	$\boxed{0}$	0	0	4
R	18	12	5	10	$\boxed{0}$
S	$\boxed{0}$	1	1	3	0
T	11	6	7	19	0

Since, number of line (4) ≠ number of row/column (5), the solution is not optimal. To get optimal solution, the minimum element (1) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced time table becomes

Table 5

Work Centers	Workmen				
	A	B	C	D	E
P	4	1	$\boxed{0}$	6	1
Q	1	$\boxed{0}$	1	0	5
R	18	11	5	9	$\boxed{0}$
S	$\boxed{0}$	0	1	2	0
T	11	5	7	18	0

Since, number of line (4) ≠ number of row/column (5), the solution is not optimal. To get optimal solution, the minimum element (1) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced time table becomes

Table 6

Work Centers	Workmen				
	A	B	C	D	E
P	3	0	$\boxed{0}$	5	1
Q	1	0	2	$\boxed{0}$	6
R	17	10	5	8	$\boxed{0}$
S	$\boxed{0}$	0	2	2	1
T	10	4	7	17	0

Since number of line (4) ≠ number of row/column (5), the solution is not optimal. To get optimal solution, the minimum element (4) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced time table becomes

Table 7

Work Centers	Workmen				
	A	B	C	D	E
P	3	0	0	5	5
Q	1	0	2	0	10
R	13	6	1	4	0
S	0	0	2	2	5
T	6	0	3	13	0

Since number of lines (5) = number of row/column (5), the solution is optimal. Hence, the total time required for different workmen to different work centers is given by

Work Centers	Workmen	Time
P	C	8
Q	D	2
R	E	0
S	A	7
T	B	14
Total Minimum Time		31 hours

Maximization Assignment Problem

For maximization of assignment problem, first of all we convert the given profit matrix into *opportunity loss matrix* by subtracting each entry from the highest one. Then we proceed the same process proceeded in the minimization one. For calculation of final maximum value, we should consider the original table. If the assignment problem is unbalanced maximization problem, first of all make it balanced by creating dummy row(s) or column(s) as per required with profit zero for all the dummy cells and then convert it into opportunity loss matrix by subtracting each elements from highest element. Then we proceed the same process proceeded in the minimization one.

Example 3 A marketing manager has four salesmen and four sales districts. Considering the capabilities of the salesmen and nature of districts, the marketing manager estimates that sales per month in hundred of rupees for each salesmen in each district would be as follows:

Salesmen	Districts			
	A	B	C	D
P	32	38	40	28
Q	40	24	28	21
R	41	27	33	30
S	22	38	41	36

Solution: Here, no. of rows = 4, no. of columns = 4

Since, no. of rows = no. of columns, So, given Assignment problem is balanced.

The given table of sales of different salesmen in different districts is

Table 1

Salesmen	Districts			
	A	B	C	D
P	32	38	40	28
Q	40	24	28	21
R	41	27	33	30
S	22	38	41	36

Convert this table into opportunity loss table by subtracting each element from highest element (41). The opportunity loss table is

Table 2

Salesmen	Districts				Row Minimum
	A	B	C	D	
P	9	3	1	13	1
Q	1	17	13	20	1
R	0	14	8	11	0
S	19	3	0	5	0

The minimum elements 1, 1, 0 and 0 of different rows are subtracted from the elements of the corresponding row. The reduced opportunity loss table becomes

Table 3

Salesmen	Districts			
	A	B	C	D
P	8	2	0	12
Q	0	16	12	19
R	0	14	8	11
S	19	3	0	5
Column Minimum	0	2	0	5

The minimum elements 0, 2, 0 and 5 of different columns are subtracted from the elements of the corresponding column. The reduced opportunity loss table becomes

Table 4

Salesmen	Districts			
	A	B	C	D
P	8	0	0	7
Q	0	14	12	14
R	0	12	8	6
S	19	1	0	0

Since, number of line (3) ≠ number of row/column (4), the solution is not optimal. To get optimal solution, the minimum element (6) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced opportunity loss table becomes

Table 5

Salesmen	Districts			
	A	B	C	D
P	14	$\boxed{0}$	0	7
Q	$\boxed{0}$	8	6	8
R	0	6	2	$\boxed{0}$
S	25	1	$\boxed{0}$	0

Since, number of lines (4) = number of row/column (4), the solution is optimal. Hence, the total sales of different salesmen in different districts is

Jobs	Machines	Sales ('00)
P	B	38
Q	A	40
R	D	30
S	C	41
Total Maximum Sales		149

Constrained Assignment Problem

Sometimes it may happen that a particular resource (man, machine, and worker) cannot be assigned to perform a particular activity (job, counter). For example, suppose a particular worker say *Ram* does not have any about the computer work then we cannot assign him to the computer work. In such cases, the cost of performing that particular activity by a particular resource is either assumed to very large so as to prohibit the entry of this pair of resource-activity into final solution or neglect that particular cell cost by leaving it vacant by keeping das (-) throughout the solution. That is we do not perform any kind of operation in such vacant cells having das (-).

Example 4 Three workers are available to work with the machines and the respective cost in Rs associated with each worker-machine assignment is given below. The vacant places in table represent the prohibited case i.e. inappropriate place for assignment.

Workers	Machines			
	A	B	C	D
P	12	3	6	10
Q	4	11	-	5
R	-	2	10	9

Solution: Here, no. of rows = 3, no. of columns = 4

Since, no. of rows ≠ no. of columns, So, given Assignment problem is unbalanced.

Since, the number of row is less than the number of column, the assignment problem is unbalanced. To make it balanced, we create a dummy row/worker (S) having cell cost to all dummy cells. Then the balanced cost table is

Table 1

Workers	Machines				Row Minimum
	A	B	C	D	
P	12	3	6	10	3
Q	4	11	-	5	4
R	-	2	10	9	2
S	0	0	0	0	0

The minimum elements 3, 4, 2 and 0 of the different rows are subtracted from the elements of the corresponding row. The reduced cost table becomes

Table 2

Workers	Machines			
	A	B	C	D
P	9	$\boxed{0}$	3	7
Q	$\boxed{0}$	7	-	1
R	-	0	8	7
S	0	0	$\boxed{0}$	0

Since, the minimum element of each column is zero, so the column operation is not necessary. Since the number of lines (3) ≠ the number of row/column (4), the solution is not optimal. To get optimal solution, the minimum element (1) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced cost table becomes

Table 3

Workers	Machines			
	A	B	C	D
P	9	$\boxed{0}$	2	6
Q	$\boxed{0}$	7	-	0
R	-	0	7	6
S	1	1	$\boxed{0}$	0

Since, number of lines (3) ≠ number of row/column (4), the solution is not optimal. To get optimal solution, the minimum element (2) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced cost table becomes

Table 4

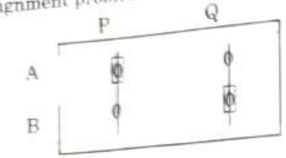
Workers	Machines			
	A	B	C	D
P	7	0	$\boxed{0}$	4
Q	$\boxed{0}$	9	-	0
R	-	$\boxed{0}$	5	4
S	1	3	0	$\boxed{0}$

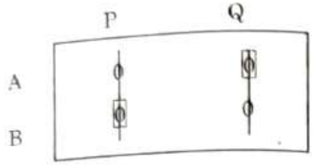
Since, number of lines (4) = number of row/column (4), the solution is optimal. Hence, the total operating /assigning cost is given by

Workers	Machines	Costs
P	C	6
Q	A	4
R	B	2
S	D	0
Total Minimum Cost		12

Alternative Solution to an Assignment Problem

In the process of making assignments, it has been stated earlier that we select a row/column with only a single zero to make as assignment. However, a situation may arise where in the various rows and columns, where assignments are yet to be done, have two or more zeros. In such cases, we get an alternative assignment solution without changing the total cost. The situation of an alternative solution to an assignment problem may be as





The first solution is A – P and B – Q. The alternative solution can be obtained by interchanging the assignment of A and B among themselves i.e. the alternative solution is A – Q and B – P.

Example 5 There are three jobs P, Q and R to be completed on four machines A, B, C and D. The costs of performing the different jobs on different machines are given below. Assign the jobs to different machines to minimize the total cost of performing the jobs on machines.

Jobs	Machines			
	A	B	C	D
P	90	120	140	160
Q	40	65	85	95
R	50	75	95	110

Solution: Here, no. of rows = 3, no. of columns = 4

Since, no. of rows = no. of columns, So, given Assignment problem is balanced.

Since, the number of rows (jobs) is less than the number of columns (machines), the assignment problem is unbalanced. To make it balanced, we create the dummy row/job (S). Since, the dummy job does not require any cost, so the operating cost is supposed to be zero for each dummy cell. The cost table becomes

Table 1

Jobs	Machines				Row Minimum
	A	B	C	D	
P	90	120	140	160	90
Q	40	65	85	95	40
R	50	75	95	110	50
S	0	0	0	0	0

The minimum elements 90, 40, 50 and 0 of different rows are subtracted from the elements of the corresponding row. The reduced cost table is

Table 2

Jobs	Machines			
	A	B	C	D
P	$\boxed{0}$	30	50	70
Q	0	25	45	55
R	0	25	45	60
S	0	$\boxed{0}$	0	0

Since, the minimum element of each column is zero, so column operation is not necessary to perform.

Since, number of line (2) ≠ number of row/column (4), the solution is not optimal. To get optimal solution, the minimum element (25) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced cost table becomes

Table 3

Jobs	Machines			
	A	B	C	D
P	$\boxed{0}$	5	25	45
Q	0	$\boxed{0}$	20	30
R	0	0	20	35
S	25	0	$\boxed{0}$	0

Since, number of line (3) ≠ number of row/column (4), the solution is not optimal. To get optimal solution, the minimum element (20) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced cost table becomes

Table 4

Jobs	Machines			
	A	B	C	D
P	$\boxed{0}$	5	5	25
Q	0	$\boxed{0}$	0	10
R	0	0	$\boxed{0}$	15
S	45	20	0	$\boxed{0}$

Since, number of lines (4) = number of row/column (4), the solution is optimal. Hence the total cost to the company for performing the different jobs on different machines is

Jobs	Machines	Cost
P	A	90
Q	B	65
R	C	95
S	D	0
Total Minimum Cost		250

Alternative Solution: The alternative solution of the problem can be obtained by interchanging the assignment Q and R. In the solution, Q and R have assigned to B and C respectively. Then the alternative solution can be obtained by assigning Q to C and R to B.

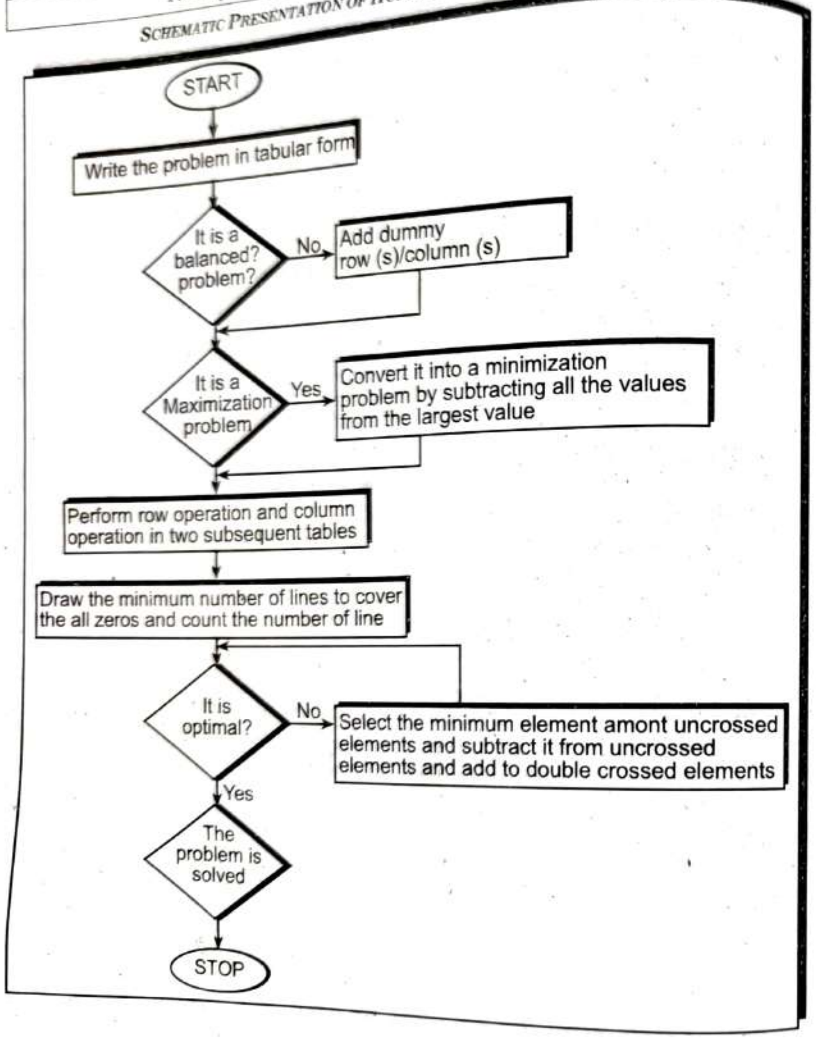
Table 4

Jobs	Machines			
	A	B	C	D
P	$\boxed{0}$	5	5	25
Q	0	0	$\boxed{0}$	10
R	0	$\boxed{0}$	0	15
S	0	0	0	$\boxed{0}$

Since, number of lines (4) = number of row/column (4), the solution is optimal. Hence the total cost to the company for performing the different jobs on different machines is

Jobs	Machines	Cost
P	A	90
Q	C	85
R	B	75
S	D	0
Total Minimum Cost		250

SCHEMATIC PRESENTATION OF HUNGARIAN ASSIGNMENT METHOD:



Workedout Examples

Example 6 The ABC Company has three jobs to be done on three machines. Each job must be done on one and only one machine. The cost of each job on each machine is given in the following table.

Cost information:

Jobs	Machine		
	X	Y	Z
A	4	6	8
B	2	3	4
C	4	8	5

Give the job assignments, which will minimize cost.

Solution: Here, no. of rows = 3, no. of columns = 3

Since, no. of rows = no. of columns, So, given Assignment problem is balanced.

The given cost table is

Table 1

Jobs	Machines			Row Minimum
	X	Y	Z	
A	4	6	8	4
B	2	3	4	2
C	4	8	5	4

The minimum elements 4, 2 and 4 are subtracted from the elements of the corresponding row. The reduced cost table becomes

Table 2

Jobs	Machines		
	X	Y	Z
A	0	2	4
B	0	1	2
C	0	4	1
Column Minimum	0	1	1

The minimum elements 0, 1 and 1 are subtracted from the elements of the corresponding column. The reduced cost table becomes

Table 3

Jobs	Machines		
	X	Y	Z
A	$\boxed{0}$	1	3
B	0	$\boxed{0}$	1
C	0	3	$\boxed{0}$

Since, number of lines (3) = number of row/column (3), the solution is optimal. Hence, the total cost to the company for performing the different jobs on different machines is

Jobs	Machines	Cost
A	X	4
B	Y	3
C	Z	5
Total Minimum Cost		Rs 12

Example 7 Four children in a household were assigned three different household chores to be done. The children are motivated to get pocket money for the job. Assign the jobs to the children in such a way that their pocket money income is maximum.

Children	Clean the house	Wash clothes	Cook dinner
Ram	1	4	5
Laxman	2	3	3
Bharat	3	3	3
Shatrudhan	5	1	2

Solution: Here, no. of rows = 4, no. of columns = 3

Since, no. of rows ≠ no. of columns, So, given Assignment problem is unbalanced.

Since, the number of column (Work) is less than the number of row (children), the assignment problem is unbalanced. To make it balanced, we create the dummy column/work (dummy work). Since the dummy work does not require any cost, so the operating cost is supposed to be zero for each dummy cell. The income table becomes

Table 1

Children	Clean the house	Wash clothes	Cook dinner	Dummy
Ram	1	4	5	0
Laxman	2	3	3	0
Bharat	3	3	3	0
Shatrudhan	5	1	2	0

Convert this table into opportunity loss table by subtracting each element from highest element (5). The opportunity loss table is

Table 2

Children	Clean the house	Wash clothes	Cook dinner	Dummy	Row Min.
Ram	4	1	0	5	0
Laxman	3	2	2	5	2
Bharat	2	2	2	5	2
Shatrudhan	0	4	3	5	0

The minimum elements 0, 2, 2 and 0 are subtracted from the elements of the corresponding row. The reduced opportunity loss table becomes

Table 3

Children	Clean the house	Wash clothes	Cook dinner	Dummy
Ram	4	1	0	5
Laxman	1	0	0	3
Bharat	0	0	0	3
Shatrudhan	0	4	3	5
Column Min.	0	0	0	3

The minimum elements 0, 0, 0 and 3 are subtracted from the elements of the corresponding column. The reduced opportunity loss table becomes

Table 4

Children	Clean the house	Wash clothes	Cook dinner	Dummy
Ram	4	1	$\boxed{0}$	2
Laxman	1	$\boxed{0}$	0	0
Bharat	0	0	0	$\boxed{0}$
Shatrudhan	$\boxed{0}$	4	3	2

Since, number of lines (4) = number of row/column (4), the solution is optimal. Hence, the total income of the four children from different work is

Children	Work	Income
Ram	Cook Dinner	5
Laxman	Wash Clothes	3
Bharat	Dummy	0
Shatrudha	Clean the house	5
Total Maximum Income		Rs 13

Example 8 From the following find the optimal assignment so that the profit is maximized on the basis of one machine to one job.

Machines	Profit in 'Rs.' Jobs A	B	C	D
X	1800	2400	2800	3200
Y	800	1300	1700	1900
Z	1000	1500	1900	2200

Solution: Here, no. of rows = 3, no. of columns = 4

Since, no. of rows ≠ no. of columns, So, given Assignment problem is unbalanced.

Since, the number of row (machine) is less than the number of column (job), the assignment problem is unbalanced. To make it balanced, we create the dummy row/machine (W). Since the dummy machine does not give any profit, so the profit is supposed to be zero for each dummy cell. The profit table becomes

Table 1

Machines	Profit in 'Rs.' Jobs A	B	C	D
X	1800	2400	2800	3200
Y	800	1300	1700	1900
Z	1000	1500	1900	2200
W	0	0	0	0

Convert the profit table into opportunity loss table by subtracting each element from highest element (3200). The opportunity loss table is

Table 2

Machine	A	B	C	D	Row Min.
X	1400	800	400	0	0
Y	2400	1900	1500	1300	1300
Z	2200	1700	1300	1000	1000
W	3200	3200	3200	3200	3200

The minimum elements 0, 1300, 1000 and 3200 are subtracted from the elements of the corresponding row. The reduced opportunity loss table becomes

Table 3

Machine	A	B	C	D
X	1400	800	400	0
Y	1100	600	200	0
Z	1200	700	300	0
W	0	0	0	0

Since the minimum element of each column is zero, so the column operation is not necessary.

Table 4

Machine	A	B	C	D
X	1400	800	400	$\boxed{0}$
Y	1100	600	200	0
Z	1200	700	300	0
W	$\boxed{0}$	0	0	0

Since, number of line (2) ≠ number of row/column (4), the solution is not optimal. To get optimal solution, the minimum element (200) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced opportunity loss table becomes

Table 5

Machine	A	B	C	D
X	1200	600	200	$\boxed{0}$
Y	900	400	$\boxed{0}$	0
Z	1000	500	100	0
W	$\boxed{0}$	0	0	200

Since, number of line (3) ≠ number of row/column (4), the solution is not optimal. To get optimal solution, the minimum element (400) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced opportunity loss table becomes

Table 6

Machine	A	B	C	D
X	800	200	200	$\boxed{0}$
Y	500	$\boxed{0}$	0	0
Z	600	100	100	0
W	$\boxed{0}$	0	400	600

...e, number of line (3) ≠ number of row/column (4), the solution is not optimal. To get optimal ...ion, the minimum element (100) among the uncrossed elements is subtracted from all the ...ssed elements and added to all the double crossed elements. The reduced opportunity loss table ...es

Table 7

Machine	A	B	C	D
X	700	100	100	$\boxed{0}$
Y	500	$\boxed{0}$	0	100
Z	500	0	$\boxed{0}$	0
W	$\boxed{0}$	0	400	700

Since, number of lines (4) = number of row/column (4), the solution is optimal. Hence the total profit is given by

Machine	Job	Profit
X	D	3200
Y	B	1300
Z	C	1900
W	A	0
Total Maximum Profit		6400

Example 9 A department has four subordinates and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. The estimates of the profit in rupees that each man would earn is given in the matrix below. How should the tasks be allocated, one to each man, so as to maximize the total earnings?

Subordinates	Task			
	1	2	3	4
1	5	40	20	5
2	25	35	30	25
3	15	25	20	10
4	15	5	30	15

Solution: Here, no. of rows = 4, no. of columns = 4

Since, no. of rows = no. of columns, So, given Assignment problem is balanced.

The profit table is

Table 1

Subordinates	Task			
	1	2	3	4
1	5	40	20	5
2	25	35	30	25
3	15	25	20	10
4	15	5	30	15

Convert the profit table into opportunity loss table by subtracting each element from highest element (40). The opportunity loss table is

Table 2

Subordinates	Task 1	2	3	4	Row Min.
1	35	0	20	35	0
2	15	5	10	15	5
3	25	15	20	30	15
4	25	35	10	25	10

The minimum elements 0, 5, 15 and 10 are subtracted from the elements of the corresponding row. The reduced opportunity loss table becomes

Table 3

Subordinates	Task 1	2	3	4
1	35	0	20	35
2	10	0	5	10
3	10	0	5	15
4	15	25	0	15
Column Min.	10	0	0	10

The minimum elements 10, 0, 0 and 10 are subtracted from the elements of the corresponding column. The reduced opportunity loss table becomes

Table 4

Subordinates	Task 1	2	3	4
1	25	$\boxed{0}$	20	25
2	0	0	5	$\boxed{0}$
3	$\boxed{0}$	0	5	5
4	5	25	$\boxed{0}$	5

Since, number of lines (4) = number of row/column (4), the solution is optimal. Hence, the total profit is given by

Subordinates	Tasks	Profit
1	2	40
2	4	25
3	1	15
4	3	30
Total Maximum Profit		110

Example 10 A solicitors firm employs typist on hourly basis. There are five typists for service and their speeds and charges are different. According to earlier understanding, only one job is given to one typist and the typist is paid for full hour even if he works for a fraction of an hour. Find the least cost allocation for the following data:

Job	P	Q	R	S	T
No. of pages	199	175	145	298	178

Typists	No. of pages typed per hour	Rate per hour
A	12	5
B	14	6
C	8	3
D	10	4
E	11	4

Solution:

According to understanding between typists and firm, a typist is paid of full hour even if he works for an fraction of an hour. For example, if typist takes 12.15 hour to complete a job then he is paid for 13 hours at the rate of his charge. The cost required for different typists to different jobs can be calculated by using the formulas,

$$\text{Time} = \frac{\text{Number of pages in a job}}{\text{Speed of the typist}} \text{ in full hour}$$

$$\text{Cost} = \text{Time in full hour} \times \text{Charge per hour of that typist}$$

For example, if typist A is engaged in job P having 199 pages, then time required to him for completing the job P = 199/12 = 16.58 hours which means he is paid for 17 hours at the rate of Rs. 5 per hour. So the cost required is 17 × 5 = Rs. 85. Similarly if typist A is engaged in job Q having 175 pages, then time required to him for completing the job Q = 175/12 = 14.58 hours which means he is paid for 15 hours at the rate of Rs. 5 per hour. So the cost required is 15 × 5 = Rs. 75 and so on. The cost table is

Here, no. of rows = 5, no. of columns = 5

Since, no. of rows = no. of columns, So, given Assignment problem is balanced.

Table 1

Typists	Jobs P	Q	R	S	T	Row Min.
A	85	75	65	125	75	65
B	90	78	66	132	78	66
C	75	66	57	114	69	57
D	80	72	60	120	72	60
E	76	64	56	112	68	56

The minimum elements 65, 66, 57, 60 and 56 are subtracted from the elements of the corresponding row. The reduced cost table becomes

Table 2

Typists	Jobs P	Q	R	S	T
A	20	10	0	60	10
B	24	12	0	66	12
C	18	9	0	57	12
D	20	12	0	60	12
E	20	8	0	56	12
Column Min.	18	8	0	56	10

The minimum elements 18, 8, 0, 56 and 10 are subtracted from the elements of the corresponding column. The reduced cost table becomes

Table 3

Typists	Jobs				
	P	Q	R	S	T
A	2	2	0	4	0
B	6	4	0	10	2
C	0	1	0	1	2
D	2	4	0	4	2
E	2	0	0	0	2

Since, number of line (4) ≠ number of row/column (5), the solution is not optimal. To get optimal solution, the minimum element (1) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced cost table becomes

Table 4

Typists	Jobs				
	P	Q	R	S	T
A	3	2	1	4	0
B	6	3	0	9	1
C	0	0	0	0	1
D	2	3	0	3	1
E	3	0	1	0	2

Since, number of line (4) ≠ number of row/column (5), the solution is not optimal. To get optimal solution, the minimum element (2) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced cost table becomes

Table 5

Typists	Jobs				
	P	Q	R	S	T
A	1	0	1	2	0
B	4	1	0	7	1
C	0	0	2	0	3
D	0	1	0	1	1
E	3	0	3	0	4

Since, number of lines (5) = number of row/column (5), the solution is optimal. Hence, the total cost is given by

Typists	Jobs	Cost
A	T	75
B	R	66
C	Q	66
D	P	80
E	S	112
Total Minimum Cost		Rs. 399

Alternative Solution: The alternative solution is

Table 5

Typists	Jobs				
	P	Q	R	S	T
A	1	0	1	2	0
B	4	1	0	7	1
C	0	0	2	0	3
D	0	1	0	1	1
E	3	0	3	0	4

Since, number of lines (5) = number of row/column (5), the solution is optimal. Hence the total cost is given by

Typists	Jobs	Cost
A	T	75
B	R	66
C	S	114
D	P	80
E	Q	64
Total Minimum Cost		Rs. 399

Example 11 A city corporation has decided to carry out road repairs on main four arteries of the city. The government has agreed to make a special grant of 50 lakhs towards the cost with a condition that the repair is done at the lowest cost and quickest time. If the condition warrant, a supplémentary token grant will also be favorably. The corporation has floated tenders and five contractors have sent in their bids. In order to expedite work, one road will be awarded to only one contractor.

Cost of repair in lakh.

Contractor	Road			
	R_1	R_2	R_3	R_4
A	9	14	19	15
B	7	17	20	19
C	9	18	21	18
D	10	12	18	19
E	10	15	21	16

(a) Find the best way of assigning the repair work to the contractors to minimize the cost.

(b) If it is necessary to seek supplementary grant, what should be the amount sought?

(c) Which of the five contractors will be unsuccessful in his bid?

Solution: Here, no. of rows = 5, no. of columns = 4

Since, no. of rows ≠ no. of columns, So, given Assignment problem is unbalanced.

(a) Since, the number of column (road) is less than the number of row (contractor), the assignment problem is unbalanced. To make it balanced, we create the dummy column/road (R_5). Since the dummy road does not require any cost, so the cost is supposed to be zero for each dummy cell. The cost table becomes

Table 1

Contractor	Road				
	R_1	R_2	R_3	R_4	R_5
A	9	14	19	15	0
B	7	17	20	19	0
C	9	18	21	18	0
D	10	12	18	19	0
E	10	15	21	16	0
Column Min	7	12	18	15	0

Since, the minimum element of each row is zero, so the row operation is not necessary. The minimum elements 7, 12, 18, 15 and 0 are subtracted from the elements of the corresponding column. The reduced cost table becomes

Table 2

Contractor	Road				
	R_1	R_2	R_3	R_4	R_5
A	2	2	1	$\boxed{0}$	0
B	$\boxed{0}$	5	2	4	0
C	2	6	3	3	$\boxed{0}$
D	3	$\boxed{0}$	0	4	0
E	3	3	3	1	0

Since, number of line (4) ≠ number of row/column (5), the solution is not optimal. To get optimal solution, the minimum element (1) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced cost table becomes

Table 3

Contractor	Road				
	R_1	R_2	R_3	R_4	R_5
A	2	2	1	$\boxed{0}$	1
B	$\boxed{0}$	5	2	4	1
C	1	5	2	2	$\boxed{0}$
D	3	$\boxed{0}$	0	4	1
E	2	2	2	0	0

Since, number of line (4) ≠ number of row/column (5), the solution is not optimal. To get optimal solution, the minimum element (1) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced cost table becomes

Table 4

Contractor	Road				
	R_1	R_2	R_3	R_4	R_5
A	2	1	$\boxed{0}$	0	1
B	$\boxed{0}$	4	1	4	1
C	1	4	1	2	$\boxed{0}$
D	4	$\boxed{0}$	0	5	2
E	2	1	1	$\boxed{0}$	0

Since, number of lines (5) = number of row/column (5), the solution is optimal. Hence, the total cost is given by

Contractor	Road	Cost
A	R_3	19
B	R_1	7
C	R_5	0
D	R_2	12
E	R_4	16
Total Minimum Cost		Rs. 54

(b) The minimum cost of road construction according to bid applied there is Rs. 54 lakhs but the available fund is Rs. 50 lakhs only, so the supplementary grant of Rs. (54 – 50) = Rs. 4 lakhs should be sought.

(c) The contractor C is unsuccessful on his bid because he has been assigned to dummy road (R_5).

Example 12 An airlines company has drawn up a new flight schedule involving five flights. To assist in allocating five pilots to the flights, it has asked them to state their preference scores by giving each flight a number out of 10. The higher the number, greater is the preference. Certain of these flights are unsuitable to some pilots owing to domestic reasons. These have been marked with a (×).

Pilot	Flight number				
	1	2	3	4	5
A	8	2	×	5	4
B	10	9	2	8	4
C	5	4	9	6	×
D	3	6	2	8	7
E	5	6	10	4	3

What should be the allocation of the pilots to flights in order to meet as many preferences as possible?

Solution: Here, no. of rows = 5, no. of columns = 5

Since, no. of rows = no. of columns, So, given Assignment problem is balanced.

Since, the problem is to maximize the total preference score, in order to apply the Hungarian method to solve this assignment problem, the opportunity loss/cost matrix is required. Convert this table into corresponding opportunity loss/cost table by subtracting each element from highest element (10). The loss table is

Table 1

Pilot	Flight number					Row Min.
	1	2	3	4	5	
A	2	8	×	5	6	2
B	0	1	8	2	6	0
C	5	6	1	4	×	1
D	7	4	8	2	3	2
E	5	4	0	6	7	0

The minimum elements 2, 0, 1, 2 and 0 are subtracted from the elements of corresponding row. The reduced opportunity loss table becomes

Table 2

Pilot	Flight number				
	1	2	3	4	5
A	0	6	×	3	4
B	0	1	8	2	6
C	4	5	0	3	×
D	5	2	6	0	1
E	5	4	0	6	7
Column Min	0	1	0	0	1

The minimum elements 0, 1, 0, 0 and 1 are subtracted from the elements of corresponding column. The reduced opportunity loss table becomes

Table 3

Pilot	Flight number				
	1	2	3	4	5
A	$\boxed{0}$	5	×	3	3
B	0	$\boxed{0}$	8	2	5
C	4	4	$\boxed{0}$	3	×
D	5	1	6	$\boxed{0}$	0
E	5	3	0	6	6

Since, number of line (4) ≠ number of row/column (5), the solution is not optimal. To get optimal solution, the minimum element (2) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced opportunity loss table becomes

Table 4

Pilot	Flight number				
	1	2	3	4	5
A	$\boxed{0}$	5	×	1	1
B	0	$\boxed{0}$	8	0	3
C	4	4	$\boxed{0}$	1	×
D	7	3	8	$\boxed{0}$	0
E	5	3	0	4	4

Since, number of line (4) ≠ number of row/column (5), the solution is not optimal. To get optimal solution, the minimum element (1) among the uncrossed elements is subtracted from all the uncrossed elements and added to the double crossed elements. The reduced loss table becomes.

Table 5

Pilot	Flight number				
	1	2	3	4	5
A	$\boxed{0}$	4	×	0	0
B	1	$\boxed{0}$	9	0	3
C	4	3	0	$\boxed{0}$	×
D	8	3	9	0	$\boxed{0}$
E	5	2	$\boxed{0}$	3	3

Since, number of lines (5) = number of row/column (5), the solution is optimal. Hence, the total preference score is given by

Pilot	Flight Number	Preference Score
A	1	4
B	2	9
C	4	6
D	5	7
E	3	10
Total Maximum Score		10

Example 13 The captain of a cricket team has to allot five middle batting positions to five batsmen. The average runs scored by each batsman at these positions are as follows:

Batsmen	Batting Position				
	1	2	3	4	5
A	40	40	35	25	50
B	42	30	16	25	27
C	50	48	40	60	50
D	20	19	20	18	25
E	58	60	59	55	53

(i) Find the assignment of batsmen to positions, which maximizes the number of runs.

(ii) If another batsman 'F' with the following average runs in batting positions as given below:

Batting Position	1	2	3	4	5
Average Run	45	52	38	50	49

Should he be included to play? If yes, who will be replaced by him?

Solution: Here, no. of rows = 5, no. of columns = 5

Since, no. of rows = no. of columns, So, given Assignment problem is balanced.

(i) Since, the problem is to maximize the total runs scored by batsmen, in order to apply the Hungarian method to solve this assignment problem, the opportunity loss/cost matrix is required. Convert this table into corresponding opportunity loss/cost table by subtracting each element from highest element (60). The opportunity loss table is

Table 1

Batsmen	Batting Positions					Row Min.
	1	2	3	4	5	
A	20	20	25	35	10	10
B	18	30	44	35	33	18
C	10	12	20	0	10	0
D	40	41	40	42	35	35
E	2	0	1	5	7	0

The minimum elements 10, 18, 0, 35 and 0 are subtracted from the elements of corresponding row. The reduced opportunity loss table becomes

Table 2

Batsmen	Batting Positions				
	1	2	3	4	5
A	10	10	15	25	0
B	0	12	26	17	15
C	10	12	20	0	10
D	5	6	5	7	0
E	2	0	1	5	7
Column Min.	0	0	1	0	0

The minimum elements 0, 0, 1, 0 and 0 are subtracted from the elements of corresponding column. The reduced opportunity loss table becomes

Table 3

Batsmen	Batting Positions				
	1	2	3	4	5
A	10	10	14	25	$\boxed{0}$
B	$\boxed{0}$	12	25	17	15
C	10	12	19	$\boxed{0}$	10
D	5	6	4	7	0
E	2	$\boxed{0}$	0	5	7

Since, number of line (4) ≠ number of row/column (5), the solution is not optimal. To get optimal solution, the minimum element (4) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced opportunity loss table becomes

Table 4

Batsmen	Batting Positions				
	1	2	3	4	5
A	10	6	10	25	$\boxed{0}$
B	$\boxed{0}$	8	21	17	15
C	10	8	15	$\boxed{0}$	10
D	5	2	$\boxed{0}$	7	0
E	6	$\boxed{0}$	0	9	11

Since, number of lines (5) = number of row/column (5), the solution is optimal. Hence, the total runs scored by the batsmen is given by

Batsmen	A	B	C	D	E	Total maximum run scored
Batting positions	5	1	4	3	2	
Runs scored	50	42	60	20	60	232

(ii) The table of average run scored after including the average runs of batsman F becomes

Table 1

Batsmen	Batting Position				
	1	2	3	4	5
A	40	40	35	25	50
B	42	30	16	25	27
C	50	48	40	60	50
D	20	19	20	18	25
E	58	60	59	55	53
F	45	52	38	50	49

Here, no. of rows = 6, no. of columns = 5

Since, no. of rows ≠ no. of columns, So, given Assignment problem is unbalanced.

Since, the number of column (batting position) is less than the number of row (batsmen), the assignment problem is unbalanced. To make it balanced, we create the dummy column/batting position (6) with each element zero in the dummy cells. The table becomes

Table 2

Batsmen	Batting Position					
	1	2	3	4	5	6
A	40	40	35	25	50	0
B	42	30	16	25	27	0
C	50	48	40	60	50	0
D	20	19	20	18	25	0
E	58	60	59	55	53	0
F	45	52	38	50	49	0

Convert this table into corresponding opportunity loss/cost table by subtracting each element from highest element (60). The loss/ regret table becomes

Table 3

Batsmen	Batting Positions						Row Min.
	1	2	3	4	5	6	
A	20	20	25	35	10	60	10
B	18	30	44	35	33	60	18
C	10	12	20	0	10	60	0
D	40	41	40	42	35	60	35
E	2	0	1	5	7	60	0
F	15	8	22	10	11	60	8

The minimum elements 10, 18, 0, 35, 0 and 8 are subtracted from the elements of corresponding row. The reduced regret table becomes

Table 4

Batsmen	Batting Positions					
	1	2	3	4	5	6
A	10	10	15	25	0	50
B	0	12	26	17	15	42
C	10	12	20	0	10	60
D	5	6	5	7	0	25
E	2	0	1	5	7	60
F	7	0	14	2	3	52
Column Min.	0	0	1	0	0	25

The minimum elements 0, 0, 1, 0, 0 and ... are subtracted from ... corresponding column...
The reduced regret table becomes

Table 5

Batsmen	Batting Positions					
	1	2	3	4	5	6
A	10	10	14	25	0	25
B	0	12	25	17	15	17
C	10	12	19	0	10	36
D	5	6	4	7	0	0
E	3	0	0	5	7	35
F	7	0	13	2	3	27

Since, number of lines (6) = number of row/column (6), the solution is optimal. Hence the total runs scored by the batsmen is given by

Batsmen	Batting positions	Runs scored
A	5	50
B	1	42
C	4	60
D	6	0
E	3	59
F	2	52
Total Maximum Run Scored		263

The batsman F should be included in playing team and the batsman D should be replaced by batsman F.

Example 14 A company has four territories and four salesmen available for assignment. The territories are not equally rich in their sales potential. It is estimated that a typical salesman operating in each territory would bring the following sales:

Territory	:	I	II	III	IV
Annual Sales	:	60,000	50,000	40,000	30,000

The four salesmen are also considered to differ in ability. It is estimated that working under the same conditions their yearly sales would be proportionately as follows:

Salesman	:	A	B	C	D
Proportion	:	7	5	5	4

Assign the salesmen to different territories in order to maximize the total annual sales. Consider D as reference salesman.

Solution:

We can consider any salesman as reference. At the moment let's consider the salesman D as the reference salesman. Then the index of the salesmen will be

Salesman	:	A	B	C	D
Proportion	:	7	5	5	4
Index	:	7 ÷ 4 =1.75	5 ÷ 4 = 1.25	5 ÷ 4 = 1.25	4 ÷ 4 = 1

The sales ('000) for each salesman to be assigned to each territory are given below in the table.

Table 1

Salesmen	Territory			
	I	II	III	IV
A	60×1.75=105	50×1.75=87.5	40×1.75=70	30×1.75=52.5
B	60×1.25=75	50×1.25=62.5	40×1.25=50	30×1.25=37.5
C	60×1.25=75	50×1.25=62.5	40×1.25=50	30×1.25=37.5
D	60×1=60	50 × 1=50	40×1 =40	30×1=30

Here, no. of rows = 4, no. of columns = 4

Since, no. of rows = no. of columns. So, given Assignment problem is balanced.

Since, the problem is to maximize the total annual sales by salesmen, in order to apply the Hungarian method to solve this assignment problem, the opportunity loss/cost/regret matrix is required. Convert this table into corresponding opportunity regret table by subtracting each element from highest element (105). The regret table is

Table 2

Salesmen	Territory				Row Min.
	I	II	III	IV	
A	0	17.5	35	52.5	0
B	30	42.5	55	67.5	30
C	30	42.5	55	67.5	30
D	45	55	65	75	45

The minimum elements 0, 30, 30 and 45 are subtracted from the elements of corresponding row, the regret table becomes

Table 3

Salesmen	Territory			
	I	II	III	IV
A	0	17.5	35	52.5
B	0	12.5	25	37.5
C	0	12.5	25	37.5
D	0	10	20	30
Column Min.	0	10	20	30

The minimum elements 0, 10, 20 and 30 are subtracted from the elements of corresponding column, the regret table becomes

Table 4

Salesmen	Territory			
	I	II	III	IV
A	0	7.5	15	22.5
B	0	2.5	5	7.5
C	0	2.5	5	7.5
D	0	0	0	0

Since, number of line (2) ≠ number of row/column (4), the solution is not optimal. To get optimal solution, the minimum element (2.5) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced regret table becomes

Table 5

Salesmen	Territory I	II	III	IV
A	$\boxed{0}$	5.5	12.5	20
B	0	$\boxed{0}$	2.5	5
C	0	0	2.5	5
D	2.5	0	$\boxed{0}$	0

Since, number of line (3) ≠ number of row/column (4), the solution is not optimal. To get optimal solution, the minimum element (2.5) among the uncrossed elements is subtracted from all the uncrossed elements and added to all the double crossed elements. The reduced regret table becomes

Table 6

Salesmen	Territory I	II	III	IV
A	$\boxed{0}$	5.5	10	17.5
B	0	$\boxed{0}$	0	2.5
C	0	0	$\boxed{0}$	2.5
D	5	2.5	0	$\boxed{0}$

Since, number of lines (4) = number of row/column (4), the solution is optimal. Hence, the total sales revenue by the salesmen is given by

Salesmen	A	B	C	D	Total Maximum Sales
Territory	I	II	III	IV	
Sales in (000)	105	62.5	50	30	247.5

Alternative Solution:

The alternative solution is

Table 6

Salesmen	Territory I	II	III	IV
A	$\boxed{0}$	5.5	10	17.5
B	0	0	$\boxed{0}$	2.5
C	0	$\boxed{0}$	0	2.5
D	5	2.5	0	$\boxed{0}$

Since, number of lines (4) = number of row/column (4), the solution is optimal. Hence, the total sales revenue by the salesmen is given by

Salesmen	Territory	Sales in (000)
A	I	105
B	III	50
C	II	62.5
D	IV	30
Total Maximum Sales		247.5

Example 15 Kathmandu Metropolitan is putting up bids for four used motorbikes company. The Metropolitan allows individuals to make bids on all four motorbikes company but will accept only one bid per individual. Four individuals have made the following bids (in thousands Rs).

Individuals	Motorbike Company: Honda	Hero	Bajaj	Yamaha
A	100	90	110	90
B	110	100	95	95
C	105	95	90	105
D	115	100	95	100

Assign the individuals to different motorbike company in order to maximize the revenue.

Solution: Here, no. of rows = 4, no. of columns = 4

Since, no. of rows = no. of columns, So, given Assignment problem is balanced.

Since, the problem is to maximize the total revenue, in order to apply the Hungarian method to solve this assignment problem, the opportunity loss/cost/regret matrix is required. Convert this table into corresponding opportunity regret table by subtracting each element from highest element (115). The regret table is

Table 1

Individuals	Motorbike Company: Honda	Hero	Baja	Yamaha	Row Min.
A	15	25	5	25	5
B	5	15	20	20	5
C	10	20	25	10	10
D	0	15	20	15	0

The minimum elements 5, 5, 10 and 0 are subtracted from the elements of the corresponding row, the regret table becomes

Table 2

Individuals	Motorbike Company: Honda	Hero	Baja	Yamaha
A	10	20	0	20
B	0	10	15	15
C	0	10	15	0
D	0	15	20	15
Column Min.	0	10	0	0

The minimum elements 0, 10, 0 and 0 are subtracted from the elements of the corresponding column, the regret table becomes

Table 3

Individuals	Motorbike Company: Honda	Hero	Baja	Yamaha
A	10	10	$\boxed{0}$	20
B	0	$\boxed{0}$	15	15
C	0	0	15	$\boxed{0}$
D	$\boxed{0}$	5	20	15

7. What is an unbalanced assignment problem? How is the Hungarian Assignment Method applied in respect of such a problem?

8. Can there be multiple optimal solutions to an assignment problem? How would you identify, if possible, the existence of multiple solutions in the Hungarian assignment method?

Practical Questions

1. The three technicians have to be assigned three machines. Each technician can operate any machine but one technician can be used in one machine only. The operating cost in rupees required to different technician to different machines are given below:

Technician	Machine		
	A	B	C
P	400	600	800
Q	200	300	400
R	400	800	500

Make the assignment of technicians to the machines in order to minimize the operating cost.

2. The cost in rupees required for using four machines to four plants is given. There is no restriction in using the machines in different plant except one machine to one plant. Suggest optimal assignment of machines to plants.

Machines	Plant			
	1	2	3	4
P	220	125	120	105
Q	115	150	140	70
R	90	165	45	155
S	60	150	105	75

3. Find the optimal assignment for the following cost matrix.

Salesman	Areas			
	A_1	A_2	A_3	A_4
P	11	17	8	16
Q	9	7	12	10
R	13	16	15	12
S	14	15	12	11

4. You are given the information relating to the Bike Company and bikers for a racing club. The assignment costs are given. Find the optimal assignment.

Bikers	Bike company			
	Yamaha	Honda	Hero	Bajaj
Ram	2200	3400	1600	3200
Laxman	1800	1400	2400	2000
Bharat	2600	3200	3000	2400
Hari	2800	3000	2400	2200

5. Three customers in a certain sales territory have requested technical assistance. Three technicians are available for assignment with the distance in km from each technician to each customer being as follows:

Technician	Customer		
	A	B	C
P	10	12	15
Q	20	25	18
R	17	16	13

If it costs Rs. 2 per km for travel, find the assignment of technicians to customers that will result in a minimum travel cost.

6. Three customers in a certain sales territory have requested technical assistance. Three technicians are available for assignment with the distance in km from each technician to each customer being as follows:

Technician	Customer		
	A	B	C
P	10	12	15
Q	20	25	18
R	17	16	13

If the costs for technician P, Q and R are Rs. 2, Rs. 1 and Rs. 3 respectively, find the assignment of technicians to customers that will result in a minimum travel cost.

7. A company has four major tasks for which four performers are appointed. The cost in thousand rupees is given in the matrix below:

Performers	Tasks			
	P	Q	R	S
W	12	21	32	10
X	16	22	28	12
Y	10	24	30	15
Z	9	26	34	16

Find the assignment which minimizes the total cost.

8. A city corporation has decided to carry out road repairs on main four arteries of the city. The government has agreed to make a special grant of 50 lakhs with a condition that the repair is done at the lowest cost and quickest time. If the condition warrant, a supplementary token grant will also be favorable. The corporation has floated tenders and five contractors have sent in their bids. In order to expedite work, one road will be awarded to only one contractor.

Cost of repair in lakh.

Contractor	Road			
	R_1	R_2	R_3	R_4
A	9	14	19	15
B	7	17	20	19
C	9	18	21	18
D	10	12	18	19
E	10	15	21	16

(a) Find the best way of assigning repair work to the contractors that will minimize the cost.

(b) If it is necessary to seek supplementary grant, what should be the amount sought?

(c) Which of the five contractors will be unsuccessful in his bid?

9. A company solicits bids on each of four projects from five contractors. Only one project may be assigned to any contractor. The bids received in thousands of rupees are given in the table given below. Contractor D feels unable to carry out project 3 and, therefore, submits no bid for project 3. Find the optimal assignment of the projects to contractors such that total achievable cost will be minimized.

Project	Contractor				
	A	B	C	D	E
1	18	25	22	26	25
2	26	29	26	27	24
3	28	31	30	-	31
4	26	28	27	26	29

10. Mr. X, the Campus Chief of Nepal Commerce Campus, has the problem of assigning the faculties of operation management to different sections such that the performance of the faculties becomes the maximum. One faculty cannot be used in more than one section. The performance of the different faculties in different sections is given below:

Faculties	Sections			
	A	B	C	D
P	65	78	83	60
Q	85	52	59	44
R	83	56	69	64
S	49	80	85	84

Find the optimal assignment of the faculties to different sections.

11. A company has four districts to sell its products and four salesmen A, B, C and D for it. The districts sale – record of each salesman is as given in the table. Determine the areas allocation so as to make the maximum sales.

Salesman	District			
	Kathmandu	Lalitpur	Bhaktapur	Gorkha
A	4200	3500	2800	2100
B	3000	2500	2000	1500
C	3000	2500	2000	1500
D	2400	2000	1600	1200

12. From the following find the optimal assignment so that the profit is maximized on the basis of one machine to one job.

	Profit in 'Rs.' Jobs			
Machines	A	B	C	D
X	18	24	28	32
Y	8	13	17	19
Z	10	15	19	22

13. In the modification of a plant layout of a factory, four new machines A, B, C and D are to be installed in five vacant plants P, Q, R, S and T. Because of limited space, machine B can not be installed at plant R and machine C cannot be installed at plant P. The profit (Rs. 000) contribution by different machines at different plants are given below:

Machines	Vacant Plant				
	P	Q	R	S	T
A	89	91	95	90	91
B	92	89	-	90	89
C	-	91	94	91	87
D	94	88	92	87	88

Find the optimal assignment schedule that would maximize the profit.

14. A production manager wants to assign one of the five new methods to each of the four operations. The following table summarizes the weekly output in units:

Operations	Methods				
	P	Q	R	S	T
A	4	6	11	16	9
B	5	8	16	19	9
C	9	13	21	21	13
D	6	7	9	11	7

If production cost per unit is Rs. 10 and selling price per unit is Rs. 35, find the maximum profit per month.

15. Nepal UniLever Company has four products and four salesmen as well. The selling quantity (in units) per day per salesman for each product is given below in the table along with per unit profit of products.

Salesmen	Products			
	A	B	C	D
1	600	100	900	1500
2	900	1500	1000	1500
3	360	600	600	1000
4	600	1000	900	900
Profit per unit (Rs)	2	3	4	5

Find optimum assignment in order to maximize the total profit.

16. The captain of a cricket team has to allot five middle batting positions to five batsmen. The average runs scored by each batsman at these positions are as follows:

Batsmen	Batting Position				
	1	2	3	4	5
A	40	40	35	25	50
B	42	30	16	25	27
C	50	48	40	60	50
D	20	19	20	18	25
E	58	60	59	55	53

Find the assignment of batsmen to positions, which maximizes the number of runs. Who are the opening batsmen?

17. A firm is in the thought of introducing three products shampoo, conditioner and hair oil in its three plants A, B and C. Only a single product is decided to be introduced in each of the plants. The unit cost of producing its product, the selling price and quantity produced are given in the table below. What assignment will maximize the total profit of the company?

Product	Plant						Quantity
	Production Cost			Selling Price			
	A	B	C	A	B	C	
Shampoo	8	12	–	15	18	–	2000
Conditioner	10	6	4	18	16	10	2000
Hair Oil	7	6	6	12	10	8	10000

ANSWER

1. (P, A), (Q, B), (R, C). Total Minimum Cost = Rs. 1200
2. (P, 2), (Q, 4), (R, 3), (S, 1) and Total cost = Rs. 300
3. (P, A_3), (Q, A_2), (R, A_1), (S, A_4) and Total cost = Rs. 39
4. (Ram, Hero), (Laxman, Honda), (Bharat, Yamaha), (Hari, Bajaj) and Total cost = Rs. 7800
5. (P, A), (Q, C), (R, B) and Total travel cost = Rs. 88
6. (P, B), (Q, A), (R, C) and Total travel cost = Rs. 83
7. {(W, S), (X, Q), (Y, R) and (Z, P)} or {(W, S), (X, R), (Y, Q) and (Z, P)}.The minimum cost = Rs 71,000
8. (a) (A, R_3), (B, R_1), (C, R_5), (D, R_2) and (E, R_4). The total cost = 54 lakhs (b) 4 lakhs (c) contractor C
9. (1, A), (2, E), (3, C), (4, D) and (5, B). Total Minimum cost = Rs. 98,000
10. (P, B), (Q, A), (R, C) and (S, D). The max. performance = 316
11. {(A, Kathmandu), (B, Bhaktapur), (C, Lalitpur) (D, Gorkha)} or {(A, Kathmandu), (B, Lalitpur), (C, Bharktapur) (D, Gorkha)} and Max Sales = 9900
12. {(X, D), (Y, B), (Z, C), (Dummy, A)} or {(X, D), (Y, C), (Z, B), (Dummy, A)}.Total Profit = Rs. 64
13. (A, R), (B, S), (C, Q), (D, P) and (Dummy, T). The maximum profit = Rs. 370('000)
14. (A, T), (B, S), (C, R), (D, Q), (Dummy, P) and Total monthly profit = Rs. (56 × 25 × 4 = 5600)
15. (1, D), (2, B), (3, A), (4, C) and the maximum profit = Rs. 16,320
16. (A, 5), (B, 1), (C, 4), (D, 3), (E, 2) and Total runs scored = 232. The opening batsmen are B and E
17. (Shampoo, B), (Conditioner, C), (Hair Oil, A) and Total Profit = Rs. 74,000.

Answers of Multiple Choice Questions

1. (d)	2. (d)	3. (a)	4. (b)	5. (a)	6. (b)	7. (a)	8. (c)	9. (c)	10. (d)
11. (c)	12. (b)	13. (a)							

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