Real Number

1.12 Introduction

A study of the order and regularity among numbers is an important feature of Mathematics. Discovery of pattern in numbers can help to predict future behavior through chains of reasoning. The real number system is one of the important concepts of Mathematics. We now give a brief discussion of the various types of numbers.

1.13 Natural Numbers

The most familiar numbers are the counting numbers or the natural numbers. They are 1, 2, 3, 4, ...

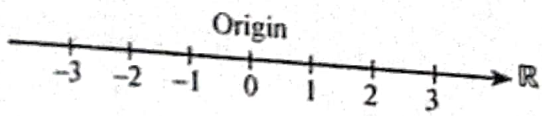
They are also called cardinal numbers or positive integers. Natural numbers are closed under the operation of addition and multiplication. The set of natural numbers is denoted by $\mathbb{N}$. So, $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. Note that 0 is not a natural number.

1.14 Integers

The set of integers contains all natural numbers, negatives of natural numbers and zero defined as,

$$\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

An integer is a whole number and does not have a fractional part.



We can associate the concepts of distance and measurement with numbers.

Take a point 'O' on the real line, on the right of O mark all positive integers with a certain unit of measurement. With the same unit, mark all negative integers on the left of O. Manipulation of integers by adding, subtracting, multiplying, dividing etc. gives rise to integers as well as fractions, decimals numbers.

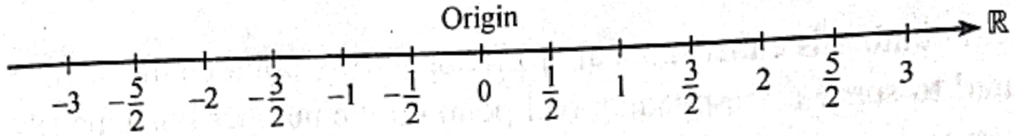
Integers are closed under the operation of addition, subtraction and multiplication. The set of integers is denoted by $\mathbb{I}$ or $\mathbb{Z}$.

1.15 Rational Numbers

A rational number is a ratio of two integers, positive, or negative, the denominator in the quotient being non–zero.

If x and y are integers and $y \neq 0$ then $\frac{x}{y}$ is a rational number.

All integers are rational numbers but all rational numbers are not integers. The fraction $\frac{8}{4}$ is a rational number. Being equal to 2, it is an integer but the fraction $\frac{1}{4}$ is not an integer.



A rational number $\frac{x}{y}$ can be represented by a point on the number line shown as a distance from the origin O. There is an infinite number of rational number on any line segment.

$\frac{1}{4} = 0.25, \frac{1}{3} = 0.333\ldots, \frac{1}{6} = 0.1666 \ldots$ are examples of rational numbers. A rational number can thus be expressed as a terminating decimal or as a repeating decimal. Note that $\frac{1}{3} = 0.333 \ldots$ can be written as $0.\overline{3}$ or $0.\dot{3}$.

Likewise 0.324324324 ... can be written as $0.\overline{324}$ etc.

Rational numbers are closed under the operation of addition, subtraction, multiplication and division (excluding division by zero). The set of rational numbers is denoted by $\mathbb{Q}$.

1.16 Irrational Numbers

Rational numbers do not constitute the set of all real numbers. A number, which cannot be expressed as the ratio of two integers, is called an irrational number. For example, $\sqrt{2} = 1.4142 \ldots$ is an irrational number as it cannot be expressed as the ratio of two integers and the process of finding the square root in decimal never terminates.

Other examples are $\sqrt[3]{7}$, $\sqrt[5]{2}$, π etc.

An irrational number does not have a rational co–ordinate on the number line. The co–ordinate for an irrational number exists but its distance from origin cannot be taken as the ratio of two integers. Numbers represented by non–terminating and non–repeating decimals are known as irrational numbers.

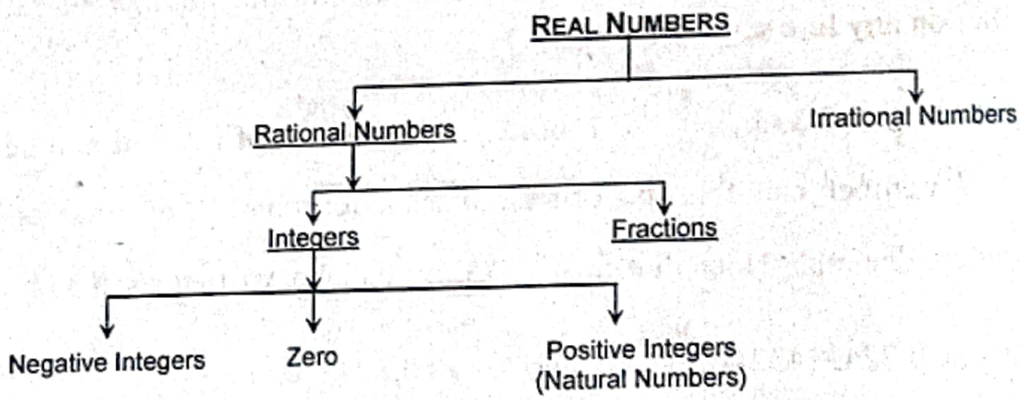
The set of irrational numbers is denoted by $\bar{\mathbb{Q}}$.

1.17 Real Numbers

Rational and irrational numbers taken together constitute the real number system. The set of real numbers is denoted by $\mathbb{R}$.

Any real number is either rational or irrational. Any point on the number line is related to some corresponding real point on the number line uniquely and vice versa.

The classification of the set of all real numbers is as follows:



1.18 Properties of Real Numbers

The real numbers satisfy the following properties. Let a, b and c be the three real numbers.

1. **Additional Property**

   (i) $a + b$ is a real number. [Closure property]

   (ii) $a + b = b + a$. [Commutative property]

   (iii) $a + (b + c) = (a + b) + c$. [Associative property]

   (iv) $a + 0 = 0 + a = a$ for all a. [Existence of identity element]

   (v) $a + (-a) = (-a) + a = 0$ for all a. [Existence of inverse element]

2. **Multiplication Property**

   (i) ab is a real number. [Closure property]

   (ii) $ab = ba$. [Commutative property]

   (iii) $a(bc) = (ab)c$. [Associative property]

   (iv) $a \cdot 1 = 1 \cdot a = a$ for all a. [Existence of identity element]

   (v) $a \cdot a^{-1} = a^{-1} \cdot a = 1, a \neq 0$. [Existence of inverse element]

3. **Multiplication Over Addition Property (Distributive Property)**

   (i) $a(b + c) = ab + ac$. [Left distributive property]

   (ii) $(b + c)a = ba + ca$. [Right distributive property]

4. **Order Property**

   (i) Only one of the relations,

   $a = b, a > b, a < b$ is true. [Trichotomy property]

   (ii) $a > b$ and $b > c$ implies $a > c$. [Transitivity property]

   (iii) $a > b$ implies $a + c > b + c$. [Additivity property]

   (iv) $a > b$ implies $ca > cb$ if $c > 0$.

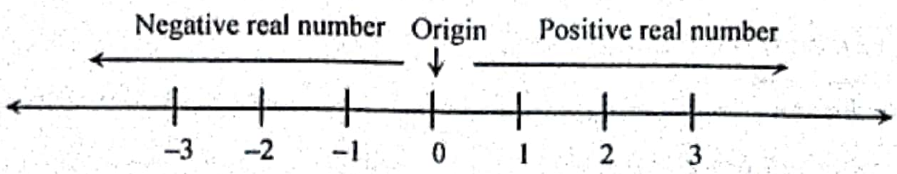
   $a < b$ implies $ca > cb$ if $c < 0$. [Multiplicativity property]

1.19 Representation of a Real Number in a Real Line

One of the most interesting characteristics of real numbers is that they can be identified with a point on a real line, called the real number line or the real axis. A real number can be identified with exactly one point on the real line and conversely. A point on the line can be identified with exactly one real number.

For this reason we often use the words "point" and "number" interchangeably. Thus the real numbers leave no gap on the real number line. The point representing the number 0 is called origin which is shown in the figure given below.

The numbers to the right of zero are positive and the numbers to the left of zero are negative. The real numbers are arranged on the number line in increasing order of magnitude of the numbers from left to right.



Example: Prove that $\sqrt{5}$ is an irrational number.

Solution

If possible, suppose $\sqrt{5}$ is a rational number.

Then, by definition, we can write

$$\sqrt{5} = \frac{p}{q},\ q \neq 0 \qquad \text{... (i)}$$

where p and q are integers and have no common factors.

Squaring both sides of equation (i), we get

$$5 = \frac{p^2}{q^2} \Rightarrow p^2 = 5q^2 \qquad \text{... (ii)}$$

$\therefore$ p^2 is multiple of 5 and hence p is also multiple of 5

Let $p = 5k$ where k is an integer.

Then equation (ii) can be written as $(5k)^2 = 5q^2$

or, $25k^2 = 5q^2$

or, $q^2 = 5k^2 \qquad \text{... (iii)}$

$\therefore$ q^2 is multiple of 5 and hence q is also multiple of 5. Thus, p and q both are multiple of 5 and hence have a common factor which contradicts our assumption.

Hence, $\sqrt{5}$ is an irrational number.

1.20 Inequality

Let a and b be any two real numbers. The possible relation between a and b are as follows: (i) $a > b$ (ii) $a < b$ (iii) $a = b$. The first two relations (i.e. $a > b$ and $a < b$) are known as inequalities and the last one (i.e. $a = b$) is known as equality or equation. For example $3 > 2$ since $3 - 2 = 1$ is positive, $2 < 4$ since $2 - 4 = -2$ is negative, $8 = 8$ since $8 - 8 = 0$.

The inequality $a > b$ means the difference $a - b$ is positive. Similarly, the inequality $a < b$ implies that the difference $a - b$ is negative.

1.21 Properties of Inequalities

The following are the properties of inequalities

Let a, b and c be three real numbers.

(i) If $a > b$ then $a + c > b + c$

(ii) If $a > b$, then $a - c > b - c$

(iii) If $a > b$, then $ca > cb$ if $c > 0$

(iv) If $a > b$, then $ca < cb$ if $c < 0$

(v) If $a < b$, then $a + c < b + c$

(vi) If $a < b$, then $a - c < b - c$

(vii) If $a < b$, then $ca < cb$ if $c > 0$

(viii) If $a < b$, then $ca > cb$ if $c < 0$

(ix) If $a > b$, then $-a < -b$

(x) If $a > 0$, then $-a < 0$.

Example: Solve: $6 + 4x \leq 12$.

Solution

$$6 + 4x \leq 12$$

Adding -6 on both sides

$$-6 + 6 + 4x \leq 12 - 6$$

or, $4x \leq 6$

Dividing both sides by 4

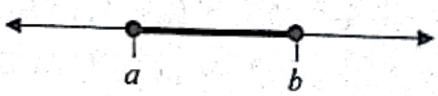
or, $\frac{4x}{4} \leq \frac{6}{4}$

$\therefore$ $x \leq \frac{3}{2}$.

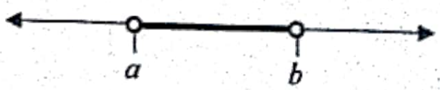
1.22 Intervals

Let 'a' and 'b' be any two points on the real line with $a < b$. The set of all points between a and b is called an interval. The points 'a' and 'b' are called the end points of the interval. An interval may or may not contain end points. There are four types of intervals. They are:

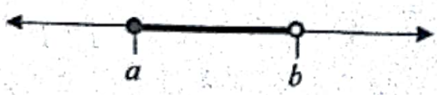
Closed interval : $[a, b] = \{x : a \leq x \leq b\}$



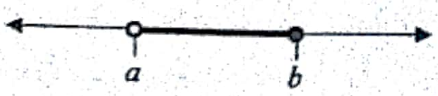
Open interval : $(a, b) = \{x : a < x < b\}$



Right open interval : $[a, b) = \{x : a \leq x < b\}$



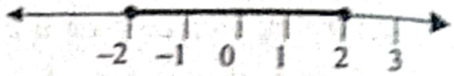
Left open interval : $(a, b] = \{x : a < x \leq b\}$



Some examples of intervals are as follows:

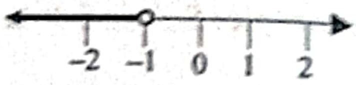
The figure alongside represents the half closed interval $[0, \infty)$. This is written as $[0, \infty) = \{x: 0 \leq x < \infty\}$.

The figure alongside represents the closed interval $-2 \leq x \leq 2$,



i.e., $[-2, 2] = \{x: -2 \leq x \leq 2\}$.

The figure alongside represents the half open interval $x < -1$,



i.e., $(-\infty, -1) = \{x: -\infty < x < -1\}$.

The real number line is represented by

$$\mathbb{R} = (-\infty, \infty) = \{x: -\infty < x < \infty\}.$$

Note: '( )' is used for the open interval, where end points are excluded and '[ ]' is used for the closed interval where the end points are included.

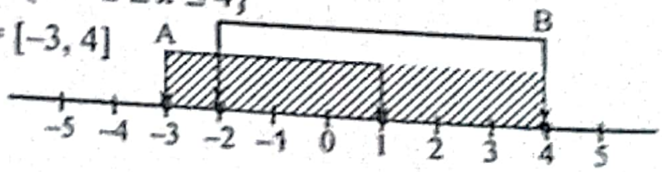
Example : Let $A = [-3, 1]$ and $B = [-2, 4]$. Find $A \cup B$, $A \cap B$, $A - B$ and $B - A$.

Solution

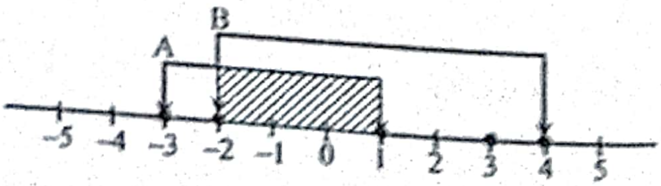
Here, $A = [-3, 1]$

$B = [-2, 4]$

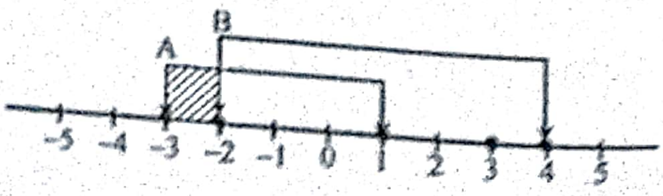
$$\begin{aligned} A \cup B &= [-3, 1] \cup [-2, 4] \\ &= \{x: -3 \leq x \leq 1\} \cup \{x: -2 \leq x \leq 4\} \\ &= \{x: -3 \leq x \leq 4\} = [-3, 4] \end{aligned}$$



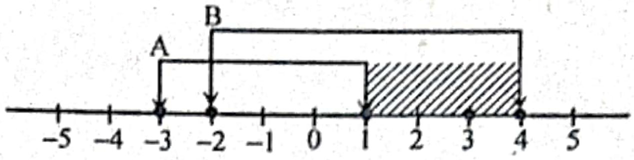
$$\begin{aligned} A \cap B &= [-3, 1] \cap [-2, 4] \\ &= \{x: -3 \leq x \leq 1\} \cap \{x: -2 \leq x \leq 4\} \\ &= \{x: -2 \leq x \leq 1\} \\ &= [-2, 1] \end{aligned}$$



$$\begin{aligned} A - B &= [-3, 1] - [-2, 4] \\ &= \{x: -3 \leq x \leq 1\} - \{x: -2 \leq x \leq 4\} \\ &= \{x: -3 \leq x < -2\} \\ &= [-3, -2) \end{aligned}$$



$$
\begin{aligned}
B - A &= [-2, 4] - [-3, 1] \\
&= \{x : -2 \le x \le 4\} - \{x : -3 \le x \le 1\} \\
&= \{x : 1 < x \le 4\} \\
&= (1, 4].
\end{aligned}
$$

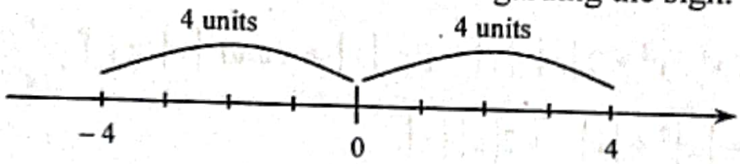


1.23 Absolute (Modulus) Value

The absolute value or modulus of a real number x, denoted by $|x|$, is defined as $|x| = \begin{cases} x \text{ for } x \ge 0 \\ -x \text{ for } x < 0 \end{cases}$

Thus, it is clear that the absolute value of a real number is always non–negative i.e. $|x| \ge 0$. So, $|4| = 4$, $|-4| = -(-4) = 4$, $|0| = 0$.

Geometrically, the absolute value of a real number is its distance from the origin. It is the magnitude of the number disregarding the sign.



1.24 Some Properties of Absolute Values

1. For any two real numbers x and y,

   (i) $x \le |x|$ and $-x \le |x|$

   (ii) $|x\,y| = |x|\ |y|$

   (iii) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ where $y \neq 0$

   (iv) $|x + y| \le |x| + |y|$ [Triangle inequality]

   (v) $|x - y| \ge |x| - |y|$

   (vi) $|x - y| \le |x| + |y|$.

Proof

(i) Obvious

(ii) $$
\begin{aligned}
|xy|^2 &= (xy)^2 \\
&= x^2y^2 \\
&= |x|^2\,|y|^2 \\
&= (|x|\,|y|)^2
\end{aligned}
$$

$\therefore\ |xy| = |x|\,|y|$.

(iii) $\left|\frac{x}{y}\right|^2 = \left(\frac{x}{y}\right)^2$

$= \frac{x^2}{y^2}$

$= \frac{|x|^2}{|y|^2}$

$= \left(\frac{|x|}{|y|}\right)^2$

$\therefore \quad \left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$

(iv) $|x+y|^2 = (x+y)^2$

$|x+y|^2 = x^2 + y^2 + 2xy$

$|x+y|^2 \leq x^2 + y^2 + 2\,|x|\,|y|$ $\quad [\because\ |x| \geq x \text{ and } |y| \geq y\,]$

$= |x|^2 + |y|^2 + 2|x|\,|y|$

$= [\,|x| + |y|\,]^2$

$\therefore \quad |x+y| \leq |x| + |y|.$

(v) $|x| = |x - y + y|$

or, $|x| \leq |x-y| + |y|$ [Using Triangle inequality]

or, $|x| - |y| \leq |x-y|$

$\therefore \quad |x-y| \geq |x| - |y|.$

(vi) $|x-y| = |x + (-y)| \leq |x| + |-y|$ [Using Triangle inequality]

$= |x| + |y|$

$\therefore \quad |x-y| \leq |x| + |y|.$

2. If a be any positive real number and $x \in R$, prove that $|x| < a \Leftrightarrow -a < x < a$.

Proof

For all $x \in R$, $x \leq |x|$

Given $|x| < a$

$\therefore \quad x \leq |x| < a$

$\Rightarrow \quad x < a \qquad \ldots \text{(i)}$

Again, for all $x \in R$, $-x \leq |x|$

Given $|x| < a$

$\therefore \quad -x \leq |x| < a$

$\Rightarrow \quad -x < a$

$\Rightarrow \quad x > -a$

$\Rightarrow \quad -a < x \qquad \ldots \text{(ii)}$

Combining (i) and (ii) $-a < x < a$.

Conversely, let $-a < x < a$

At first, $x < a$

If $x \geq 0$, then $|x| = x$

$\therefore \quad |x| = x < a \qquad \ldots \text{(iii)}$

Again, $-a < x$

$\Rightarrow \quad a > -x$

$\Rightarrow \quad -x < a$

But for $x < 0$, $|x| = -x$

$\therefore \quad |x| = -x < a \qquad \ldots \text{(iv)}$

Hence, from (iii) and (iv), for all $x \in R$, $|x| <$ a.

WORKED OUT EXAMPLES

Example 1. Find the value of $|-16| + |3| + |-5|$.

Solution

$|-16| + |3| + |-5|$

$= -(-16) + 3 + \{-(-5)\}$

$= 16 + 3 + 5 = 24.$

Example 2. If $x = -4$ and $y = 3$ then verify that

(i) $|x - y| \geq |x| - |y|$ (ii) $|xy| = |x| \cdot |y|$

(iii) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ (iv) $|x + y| \leq |x| + |y|$.

Solution

(i) Now,

$$|x|-|y| = |-4|-|3| = -(-4)-3 = 4-3 = 1$$

and $$|x-y| = |-4-3| = |-7| = -(-7) = 7$$

$\therefore \quad |x-y| > |x|-|y|.$

(ii) $$|xy| = |(-4)\times 3| = |-12| = 12$$

$$|x|\cdot|y| = |-4|\ |3| = 4\times 3 = 12$$

$\therefore\ |xy| = |x|\cdot|y|.$

(iii) $$\frac{x}{y} = \frac{-4}{3}$$

$$\therefore \quad \left|\frac{x}{y}\right| = \left|\frac{-4}{3}\right| = \frac{4}{3}$$

And $$\frac{|x|}{|y|} = \frac{|-4|}{|3|} = \frac{4}{3}$$

$$\therefore \quad \left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

(iv) $$|x+y| = |-4+3| = |-1| = 1$$

$$|x|+|y| = |-4|+|3| = 4+3 = 7.$$

$\therefore \quad |x+y| < |x|+|y|.$

Example 3. Solve the equation $|3x+5| = 7$.

Solution

Let $3x+5$ be positive.

Then $3x+5 = 7$

or, $3x = 7-5$

or, $3x = 2$

or, $x = \frac{2}{3}$

Again, let $3x + 5$ be negative.

Then $-(3x + 5) = 7$

or, $3x + 5 = -7$

or, $3x = -7 - 5$

or, $3x = -12$

or, $x = -\frac{12}{3}$

$= -4$

$\therefore x = \frac{2}{3}, -4.$

Example 4. Rewrite the following without absolute sign

(i) $|x - 1| < 6$ (ii) $|3x - 2| \leq 7.$

Solution

(i) $|x - 1| < 6$

$\Rightarrow -6 < x - 1 < 6$ [By definition]

Adding 1 to each side.

$\Rightarrow -6 + 1 < x - 1 + 1 < 6 + 1$

$\Rightarrow -5 < x < 7.$

(ii) $|3x - 2| \leq 7$

$\Rightarrow -7 \leq 3x - 2 \leq 7$

Adding 2 to each side.

$\Rightarrow -7 + 2 \leq 3x + 2 - 2 \leq 7 + 2$

$\Rightarrow -5 \leq 3x \leq 9$

Dividing each side by 3.

$\Rightarrow -\frac{5}{3} \leq x \leq 3.$

Example 5. Solve the following inequalities

(i) $|4x - 3| < 6$ (ii) $|3x - 15| \leq \frac{3}{2}.$

Solution

(i) Here, $|4x - 3| < 6$

$\Rightarrow -6 < 4x - 3 < 6$

$\Rightarrow -6 + 3 < 4x - 3 + 3 < 6 + 3$

$\Rightarrow -3 < 4x < 9$

$\Rightarrow -\frac{3}{4} < \frac{4x}{4} < \frac{9}{4}$

$\Rightarrow \frac{-3}{4} < x < \frac{9}{4}.$

(ii) Here

$$|3x-15| \le \frac{3}{2}$$

$$\Rightarrow \quad \frac{-3}{2} \le 3x-15 \le \frac{3}{2}$$

$$\Rightarrow \quad \frac{-3}{2}+15 \le 3x-15+15 \le \frac{3}{2}+15$$

$$\Rightarrow \quad \frac{27}{2} \le 3x \le \frac{33}{2}$$

$$\Rightarrow \quad \frac{27}{2\times 3} \le \frac{3x}{3} \le \frac{33}{2\times 3}$$

$$\Rightarrow \quad \frac{9}{2} \le x \le \frac{11}{2}.$$

Example 6. Rewrite so that x is alone between the inequality sign $-7<-2x+3<5$

Solution

$$-7<-2x+3<5$$

Adding -3 to each side of the given inequality, we get

$$-7-3<-2x+3-3<5-3$$

$$\Rightarrow \quad -10<-2x<2$$

Dividing each side by -2, we have

$$\frac{-10}{-2} > \frac{-2x}{-2} > \frac{2}{-2} \qquad [\because a<b \Rightarrow ca>cb \text{ if } c<0]$$

$$\Rightarrow \quad 5>x>-1$$

$$\therefore \quad -1<x<5.$$

Example 7. Rewrite the following using absolute value sign

(i) $-3<x<5$ (ii) $-1<x<7$ (iii) $-3 \le x \le 8.$

Solution

(i) $-3<x<5$

Adding -1 to each side

$$\Rightarrow \quad -3-1<x-1<5-1$$

$$\Rightarrow \quad -4<x-1<4$$

$$\therefore \quad |x-1|<4. \qquad [\because -a<x<a \Rightarrow |x|<a]$$

(ii) $-1<x<7$

Adding -3 to each side, we get

$$\Rightarrow \quad -1-3<x-3<7-3$$

$$\Rightarrow \quad -4<x-3<4$$

$$\therefore \quad |x-3|<4. \qquad [\because -a<x<a \Rightarrow |x|<a]$$

(iii) $-3 \le x \le 8$

Adding $-\frac{5}{2}$ to each side, we get

$\Rightarrow \quad -3-\frac{5}{2} \le x-\frac{5}{2} \le 8-\frac{5}{2}$

$\Rightarrow \quad \frac{-11}{2} \le x-\frac{5}{2} \le \frac{11}{2}$

$\Rightarrow \quad \frac{-11}{2} \le \frac{2x-5}{2} \le \frac{11}{2}$

$\Rightarrow \quad -11 \le 2x-5 \le 11$

$\therefore \quad |2x-5| \le 11.$ $\quad [\because -a \le x \le a \Rightarrow |x| \le a]$

Alternative Method:

$-3 \le x \le 8$

Multiplying each side by 2

$-6 \le 2x \le 16$

Again, adding -5 to each side

$-11 \le 2x-5 \le 11$

$\therefore \quad |2x-5| \le 11.$

Note: To write $a \le x \le b$ using absolute value sign we add $-\left(\frac{a+b}{2}\right)$ to each side.

Example 8. Rewrite the following relation without using absolute value sign $|2x-1| \le 5$. Also, draw the graph of the inequality..

Solution

$|2x-1| \le 5$

or, $\quad -5 \le 2x-1 \le 5$ $\quad [\because |x| \le a \Leftrightarrow -a \le x \le a]$

or, $\quad -5+1 \le 2x-1+1 \le 5+1$

or, $\quad -4 \le 2x \le 6$

or, $\quad \frac{-4}{2} \le \frac{2x}{2} \le \frac{6}{2}$

$\therefore \quad -2 \le x \le 3.$

The graph is as follows.

Example 9. Represent the solution set of x of $-2 \le 3x+4 \le 10$ in interval form.

Solution

Given, $-2 \le 3x+4 \le 10$

Subtracting 4 from each side, we get

$-2-4 \le 3x+4-4 \le 10-4$

or, $\quad -6 \le 3x \le 6$

Dividing each side by 3

$-2 \le x \le 2$

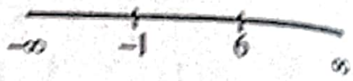
In interval form, this can be written as $[-2, 2]$.

Example 10. Solve the inequality: $6+5x-x^2 \geq 0$.

Solution

The corresponding equation of given inequality is

$$6+5x-x^2=0$$



or, $(6-x)(x+1)=0$

$\therefore$ $x=-1, x=6$

These two points divide the whole real line into 3 sub-intervals $(-\infty,-1)$, $(-1, 6$ and $(6, \infty)$.

Intervals	Sign of		
	$(6-x)$	$(x+1)$	$(6-x)(x+1)$
$(-\infty,-1)$	+ ve	− ve	− ve
$(-1, 6)$	+ ve	+ ve	+ ve
$(6, \infty)$	− ve	+ ve	− ve

Also, at $x=-1$, and $x=6$, $6+5x-x^2=0$

The possible interval is $(-1, 6) \cup \{-1, 6\} = [-1, 6]$.

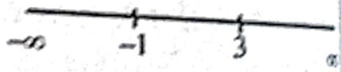
Example 11. Solve: $x^2-2x-3>0$.

Solution

The corresponding equation of given inequality is

$$x^2-2x-3=0$$

or, $x^2-3x+x-3=0$



or, $x(x-3)+1(x-3)=0$

or, $(x+1)(x-3)=0$

$\therefore$ $x=-1, 3$

Now, these two points divide the real line into 3 sub-intervals $(-\infty,-1)$, $(-1, 3$ and $(3, \infty)$.

Intervals	Sign of		
	$(x+1)$	$(x-3)$	$(x+1)(x-3)$
$(-\infty,-1)$	−ve	− ve	+ ve
$(-1, 3)$	+ ve	− ve	− ve
$(3, \infty)$	+ ve	+ ve	+ ve

$\therefore$ The possible intervals is $(-\infty,-1) \cup (3, \infty)$.

Example 12. Solve the following inequalities and graph their solution sets on th real line

(i) $2x-1<x+3$ (ii) $\dfrac{6}{x-1} \geq 5$.

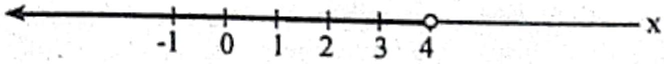
Solution

(i) $2x-1<x+3$

or, $2x-x<3+1$

or, $x<4$

The solution set is the interval $(-\infty, 4)$. The graph is,



(ii) $\frac{6}{x-1} \geq 5$

The inequality $\frac{6}{x-1} \geq 5$ will be hold if $x > 1$, because if it not so $\frac{6}{x-1}$ will be undefined or negative.

Therefore the inequality will be preserved if we multiply both sides by $(x-1)$.

Multiply both sides by $x - 1$.

$$6 \geq 5(x-1)$$

or, $6 \geq 5x - 5$

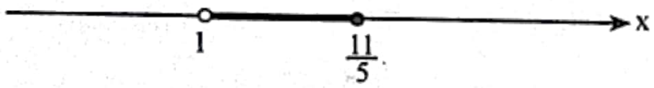
or, $6 + 5 \geq 5x$

or, $\frac{11}{5} \geq x$.

or, $x \leq \frac{11}{5}$

Hence, combining $x > 1$ and $x \leq \frac{11}{5}$, we get $1 < x \leq \frac{11}{5}$.

Thus the solution set is the half open interval $\left(1, \frac{11}{5}\right]$.



Example 13. Solve $|x-5| < 9$ and graph the solution set on the real line.

Solution

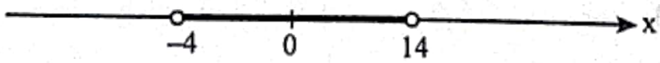
$$|x-5| < 9$$

or, $-9 < x - 5 < 9$ $\quad$ $[\because |x| < a \Leftrightarrow -a < x < a]$

or, $-9 + 5 < x < 9 + 5$ $\quad$ [Adding 5 to each side]

or, $-4 < x < 14$

The solution set is the open interval $(-4, 14)$.



Example 14. Solve the inequality $\left|5 - \frac{2}{x}\right| < 1$ and graph the solution set on the real line.

Solution

$$\left|5 - \frac{2}{x}\right| < 1$$

or, $-1 < 5 - \frac{2}{x} < 1$

or, $-1 - 5 < 5 - \frac{2}{x} - 5 < 1 - 5$

or, $-6 < -\frac{2}{x} < -4$

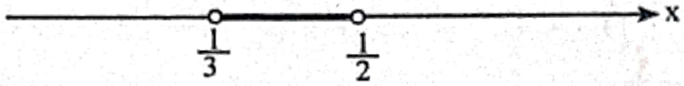
Multiplying each side by $-\frac{1}{2}$

$$3 > \frac{1}{x} > 2.$$

***Note:** Multiplying by negative number reverses the inequality.*

Taking reciprocals, we get, $\frac{1}{3} < x < \frac{1}{2}$ [Taking reciprocals in an inequality which both sides are positive also reverses the inequality sign]

Thus the solution set is $\left(\frac{1}{3}, \frac{1}{2}\right)$.



Example 15. Solve the inequality and graph the solution set

(i) $|2x - 3| \le 1$ (ii) $|2x - 3| \ge 1$.

Solution:

(i) $|2x - 3| \le 1$

or, $-1 \le 2x - 3 \le 1$

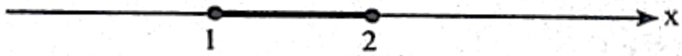
or, $-1 + 3 \le 2x - 3 + 3 \le 1 + 3$

or, $2 \le 2x \le 4$

or, $\frac{2}{2} \le \frac{2x}{2} \le \frac{4}{2}$

$\therefore$ $1 \le x \le 2$

The solution set is the closed set [1, 2].



(ii) If $2x - 3$ is positive, then

$2x - 3 \ge 1$

or, $2x \ge 1 + 3$

or, $2x \ge 4$

or, $x \ge 2$

i.e. $x \in [2, \infty)$

Again, if $2x - 3$ is negative, then

$-(2x - 3) \ge 1$

or, $2x - 3 \le -1$

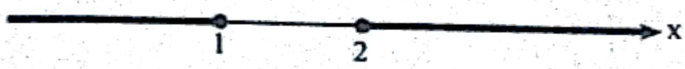
or, $2x \le 3 - 1$

or, $2x \le 2$

or, $x \le 1$

i.e. $x \in (-\infty, 1]$

Thus the solution set is $(-\infty, 1] \cup [2, \infty)$.



Example 16. Prove that $\sqrt{2}$ is an irrational number.

Solution:

If possible, suppose $\sqrt{2}$ is a rational number.

Then by definition, we can write

$$\sqrt{2} = \frac{p}{q}, q \neq 0 \qquad \dots \text{(i)}$$

where p and q are integers and have no common factors.

Squaring both sides of (i), we get

$$2 = \frac{p^2}{q^2}$$

or, $p^2 = 2q^2$ ... (ii)

$\therefore$ p^2 is even and hence p is even.

Let $p = 2k$, where k is an integer.

Then, equation (ii) can be written as

$$(2k)^2 = 2q^2$$

or, $4k^2 = 2q^2$

or, $q^2 = 2k^2$ ... (iii)

This shows that q^2 is even and hence q is even. Thus, p and q both are even and hence have a common factor namely 2, which contradicts our assumption.

Hence, $\sqrt{2}$ is an irrational number.

EXERCISE – 1 B

1. (a) If $-10 < 5x + 10 < 5$, prove that $-4 < x < -1$.
   (b) If $-5 < 7x + 9 < 30$, prove that $-2 < x < 3$.
   (c) If $-8 \leq 2x + 2 < -2$, prove that $-5 \leq x < -2$.
   (d) If $0 \leq 3x + 9 \leq 27$, prove that $-3 \leq x \leq 6$.
2. Solve the inequalities.
   (a) $-2x > 4$
   (b) $5x - 3 \leq 7 - 3x$
   (c) $2x - \frac{1}{2} \geq 7x + \frac{7}{6}$
   (d) $\frac{4}{5}(x - 2) < \frac{1}{3}(x - 6)$
   (e) $-\frac{x+5}{2} \leq \frac{12+3x}{4}$.
3. If A = [– 2, 1) and B = (– 1, 3] then find
   (a) $A \cup B$
   (b) $A \cap B$
   (c) $A - B$
   (d) $B - A$.

4. Express $\frac{1}{9}$ as a repeating decimal, using a bar to indicate the repeating digits. What are the representations of $\frac{2}{9}, \frac{7}{9}$ and $\frac{8}{9}$?

5. If $x = -3, y = 5$, verify that:

(a) $|x+y| \leq |x| + |y|$ (b) $|x| - |y| \leq |x-y|$

(c) $|xy| = |x|\,|y|$ (d) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.

6. Solve:

(a) $|x| = 3$ (b) $|2x+5| = 4$

(c) $|8-3x| = \frac{9}{2}$.

7. (i) Rewrite the following without using modulus sign.

(a) $|x+2| \leq 1$ (b) $|2x-3| \leq 1$.

(ii) Solve the inequalities expressing the solution sets as intervals unions of intervals. Also, graph each solution set on the real line.

(a) $|x| < 2$ (b) $|x-2| < 6$

(c) $|3x-7| < 4$ (d) $\left|3 - \frac{1}{y}\right| < \frac{1}{2}$

(e) $|3-5x| \leq 2$ (f) $|1-x| > 1$

(g) $\left|\frac{x+1}{2}\right| \geq 1$.

8. Rewrite the following by using the modulus sign.

(a) $-3 < x < 3$ (b) $-3 < x < 9$

(c) $-4 \leq x \leq 1$ (d) $-3 \leq x \leq -2$

(e) $-5 \leq x \leq -2$.

9. Solve:

(a) $x^2 < 4$ (b) $(x-1)^2 < 4$

(c) $x^2 - x < 0$ (d) $x^2 - x - 2 \geq 0$.

10. Prove that $\sqrt{3}$ is an irrational number.

Answers

2. (a) $x < -2$ (b) $x \leq \frac{5}{4}$ (c) $x \leq -\frac{1}{3}$

(d) $x < -\frac{6}{7}$ (e) $x \geq -\frac{2}{5}$

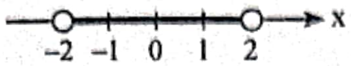
3. (a) $[-2, 3]$ (b) $(-1, 1)$
(c) $[-2, -1]$ (d) $[1, 3]$

4. $0.\overline{1}, 0.\overline{2}, 0.\overline{7}, 0.\overline{8}$

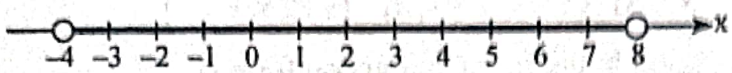
6. (a) ± 3 (b) $-\frac{1}{2}, -\frac{9}{2}$ (c) $\frac{7}{6}, \frac{25}{6}$

7. (i) (a) $-3 \le x \le -1$ (b) $1 \le x \le 2$

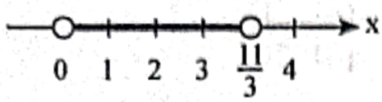
(ii) (a) $(-2, 2)$;



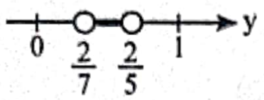
(b) $(-4, 8)$;



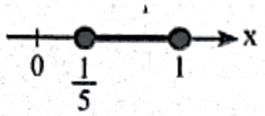
(c) $\left(1, \frac{11}{3}\right)$;



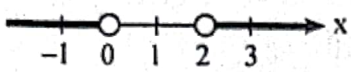
(d) $\left(\frac{2}{7}, \frac{2}{5}\right)$;



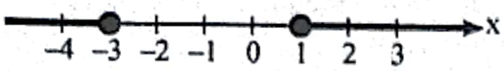
(e) $\left[\frac{1}{5}, 1\right]$;



(f) $(-\infty, 0) \cup (2, \infty)$;



(g) $(-\infty, -3] \cup [1, \infty)$;



8. (a) $|x| < 3$ (b) $|x-3| < 6$
(c) $|2x+3| \le 5$ (d) $|2x+5| \le 1$
(e) $|2x+7| \le 3$.

9. (a) $x \in (-2, 2)$ (b) $x \in (-1, 3)$
(c) $x \in (0, 1)$ (d) $x \in (-\infty, -1] \cup [2, \infty)$.