

SET THEORY

Set is a collection of well-defined and distinct objects.

POINTS TO KNOW

- $n(A \cup B)$ is the number of elements present in either of the sets A or B.
- $n(A \cap B)$ is the number of elements present in both the sets A and B.

FORMULAS

For two sets A and B

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

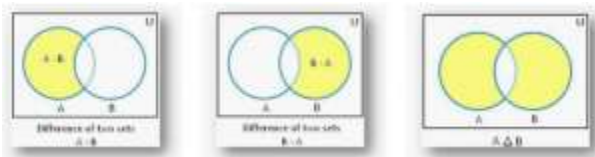
For three sets A, B and C

- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

SOME VENN DIAGRAM REPRESENTATIONS:

In above diagram set A is the subset of set B

-
-
-



REAL NUMBER

IN FACT

Nearly any number you can think of is a Real Number.

Real Numbers include:

- Whole Numbers (like 0, 1, 2, 3, 4, etc)
- Rational Numbers (like $\frac{3}{4}$, 0.125, 0.333..., 1.1, etc)
- Irrational Numbers (like π , $\sqrt{2}$, etc)

Real Numbers can also be positive, negative or zero.

So what is NOT a Real Number?

- Imaginary Numbers like $\sqrt{-1}$ (the square root of minus 1) are not Real Number
- Infinity is not a Real Number

This diagram may help you to understand better:

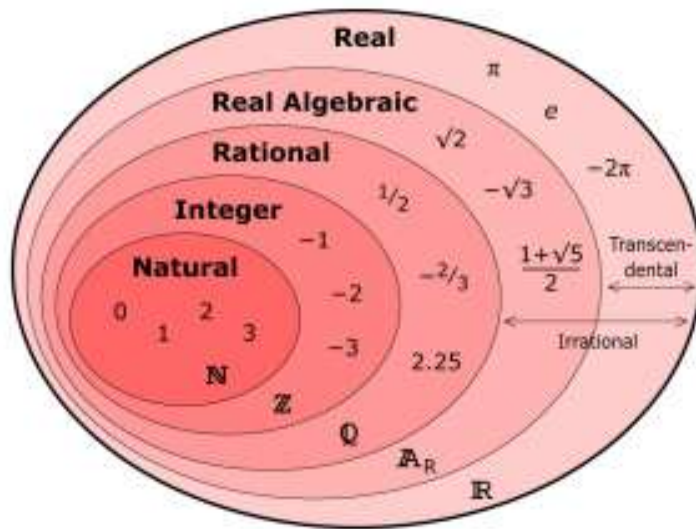


Fig: real number hierarchy

COMPLEX NUMBER

A complex number is any number which can be written as $a+ib$ where a and b are real numbers and $i=\sqrt{-1}$ is an imaginary number.

a is the real part of the complex number and b is the imaginary part of the complex number.

Example for a complex number: $9 + i2$

Properties of Cube Roots of Unity

(1) Cube Roots of Unity are in G.P.

(2) Each complex cube root of unity is the square of the other complex cube root of unity.

Example: $w = \frac{-1 + \sqrt{3}i}{2}$, $w^2 = \frac{-1 - \sqrt{3}i}{2}$

(3) $1 + w + w^2 = 0$

(4) Product of all cube roots of unity = 1
i.e., $w^3 = 1$

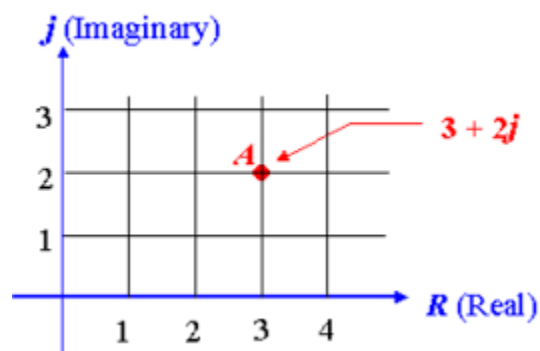
(5) $\frac{1}{w} = w^2$ and $\frac{1}{w^2} = w$

fig: complex number

POINTS TO KNOW

- If the complex number $a+ib=x+iy$, then $a=x$ and $b=y$
- Fourth Roots of Unity, $(1)^{1/4}$ are +1, -1, +i, -i

This may help you to identify real and imaginary numbers



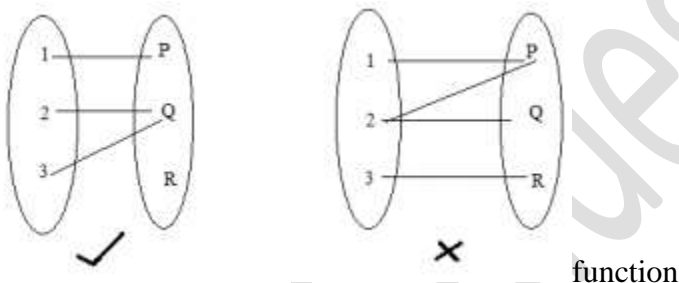
Relation:

A rule which associates each element of the set (A) with at least one element in set (B).

Function:

A function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.

Eg:



A Condition for a Function:

Set **A** and Set **B** should be non-empty.

In a function, a particular input is given to get a particular output. So, A function **f: A → B** denotes that f is a function from **A** to **B**, where **A** is a domain and **B** is a co-domain.

- For an element, **a**, which belongs to **A**, $a \in A$, a unique element **b**, $b \in B$ is there such that $(a, b) \in f$.

The unique element **b** to which **f** relates **a**, is denoted by **f(a)** and is called **f of a**, or the value of **f at a**, or **the image of a under f**.

- The *range* of **f** (image of **a** under **f**)
- It is the set of all values of **f(x)** taken together.
- **Range of f** = $\{ y \in Y \mid y = f(x), \text{ for some } x \text{ in } X \}$

A real-valued function has either **P** or any one of its subsets as its range. Further, if its domain is also either **P** or a subset of **P**, it is called a **real function**.

REPRESENTATION OF FUNCTIONS

Functions are generally represented as $f(x)$

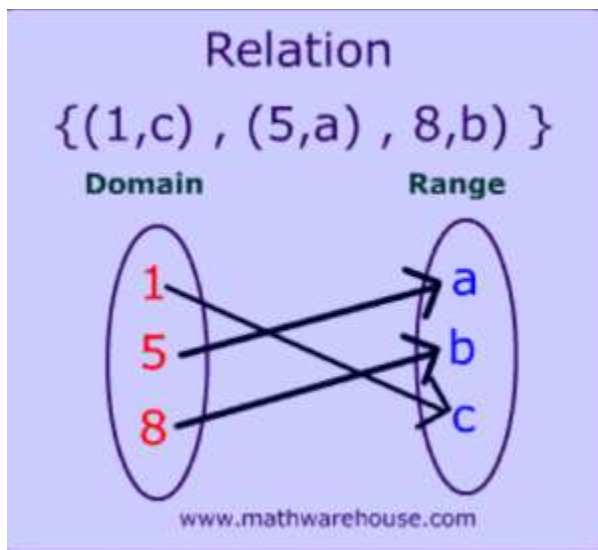
Let , $f(x)=x^3$

It is said as f of x is equal to x cube.

Functions can also be represented by $g()$, $t()$,... etc.

Domain & Range

The **domain** of a function $f(x)$ is the set of all values for which the function is defined, and the **range** of the function is the set of all values that ff takes.



domain and range

Here, Domain = $\{1,5,8\}$

Range = $\{a,b,c\}$

This will help you more:

Example

- The set "A" is the **Domain**,
- The set "B" is the **Codomain**,
- And the set of elements that get pointed to in B (the actual values produced by the function) are the **Range**, also called the Image.

And we have:

- Domain: {1, 2, 3, 4}
- Codomain: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
- Range: {3, 5, 7, 9}

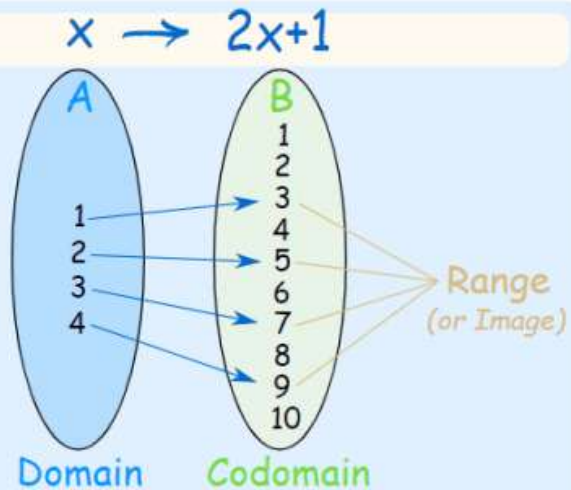


Fig: co-domain

Note: All ranges are co-domain but all co-domain are not range.

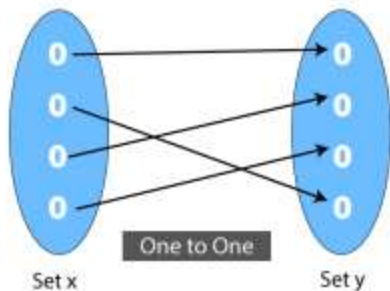
Types of functions

There are various types of functions in mathematics some of which are explained below in detail. The different functions types covered here are:

- One – one function (Injective function)
- Many – one function
- Onto – function (Surjective Function)
- Into – function
- Polynomial function
- Linear Function
- Quadratic Function
- Algebraic Functions
- Cubic Function
- Even and Odd Function
- Composite Function
- Constant Function
- Identity Function

ONE – ONE FUNCTION (INJECTIVE FUNCTION)

If each element in the domain of a function has a distinct image in the co-domain, the function is said to be one to one function.

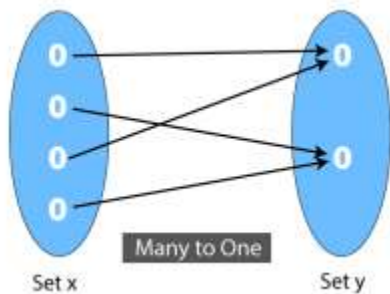


one-one function

For examples $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 3x + 5$ is one – one.

Many – one function

On the other hand, if there are at least two elements in the domain whose images are the same, the function is known as many to one function.



many-one function

ONTO – FUNCTION (SURJECTIVE FUNCTION)

A function is called an onto function if each element in the co-domain has at least one pre-image in the domain.

INTO – FUNCTION

If there exists at least one element in the co-domain which is not an image of any element in the domain then the function will be Into function.

LINEAR FUNCTION

All functions in the form of $ax + b$ where $a, b \in \mathbb{R}$ & $a \neq 0$ are called as linear functions. The graph will be a straight line. In other words, a linear polynomial function is a first-degree polynomial where the input needs to be multiplied by m and added to c . It can be expressed by $f(x) = mx + c$.

For example, $f(x) = 2x + 1$ at $x = 1$

$$f(1) = 2 \cdot 1 + 1 = 3$$

$$f(1) = 3$$

CUBIC FUNCTION

A cubic polynomial function is a polynomial of degree three and can be expressed as;

$F(x) = ax^3 + bx^2 + cx + d$ and a is not equal to zero.

QUADRATIC FUNCTION

All functions in the form of $y = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$, $a \neq 0$ will be known as Quadratic function.

ALGEBRAIC FUNCTIONS

A function that consists of a finite number of terms involving powers and roots of independent variable x and fundamental operations such as addition, subtraction, multiplication, and division is known as an algebraic equation.

Even and Odd Function

If $f(x) = f(-x)$ then the function will be even function & $f(x) = -f(-x)$ then the function will be odd function

Example 1:

$$f(x) = x^2 \sin x \quad f(-x) = -f(-x)$$

$$f(-x) = -x^2 \sin x$$

it is odd function.

Example 2:

$$f(x) = x^2 \Rightarrow f(x) = f(-x) \quad f(-x) = x^2 \text{ it is even function.}$$

Periodic Function

A function is said to be a periodic function if there exists a positive real numbers T such that $f(u - t) = f(x)$ for all $x \in \text{Domain}$.

For example $f(x) = \sin x$

$$f(x + 2\pi) = \sin(x + 2\pi) = \sin x \text{ fundamental}$$

then period of $\sin x$ is 2π

Composite Function

Let A, B, C' be three non-empty sets

Let $f: A \rightarrow B$ & $g: B \rightarrow C$ be two functions then $g \circ f: A \rightarrow C$. This function is called composition of f and g

$$\text{given } g \circ f(x) = g(f(x))$$

For example $f(x) = x^2$ & $g(x) = 2x$

$$f(g(x)) = f(2x) = (2x)^2 = 4x^2$$

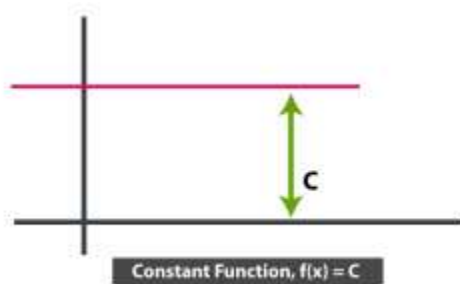
$$g(f(x)) = g(x^2) = 2x^2$$

Constant Function

The function $f : P \rightarrow P$ defined by $b = f(x) = D$, $a \in P$, where D is a constant $\in P$, is a constant function.

- Domain of $f = P$
- Range of $f = \{D\}$
- Graph type: A straight line which is parallel to the x-axis.

In simple words, the polynomial of 0th degree where $f(x) = f(0) = a_0 = c$. Regardless of the input, the output always results in constant value. The graph for this is a horizontal line.



Constant-Function

Identity Function

P = set of real numbers

The function $f : P \rightarrow P$ defined by $b = f(a) = a$ for each $a \in P$ is called the identity function.

- Domain of $f = P$
- Range of $f = P$
- Graph type: A straight line passing through the origin.

Vertical line test

Given the graph of a relation, there is a simple test for whether or not the relation is a function. This test is called the vertical line test. If it is possible to draw any vertical line (a line of constant x) which crosses the graph of the relation more than once, then the relation is not a function. If more than one intersection point exists, then the intersections correspond to multiple values of y for a single value of x (one-to-many).

- If any vertical line cuts the graph only once, then the relation is a function (one-to-one or many-to-one).
- The red vertical line cuts the circle twice and therefore the circle is not a function.
- The red vertical line only cuts the parabola once and therefore the parabola is a function.

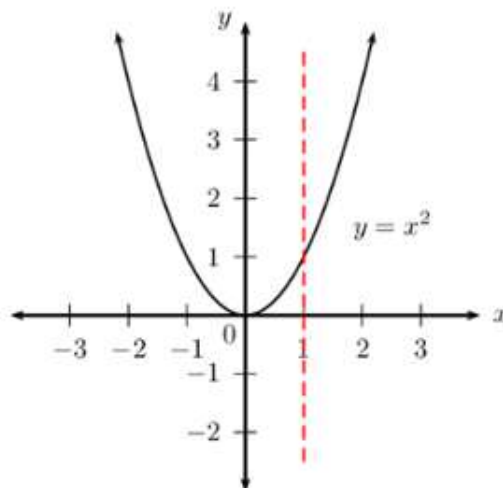
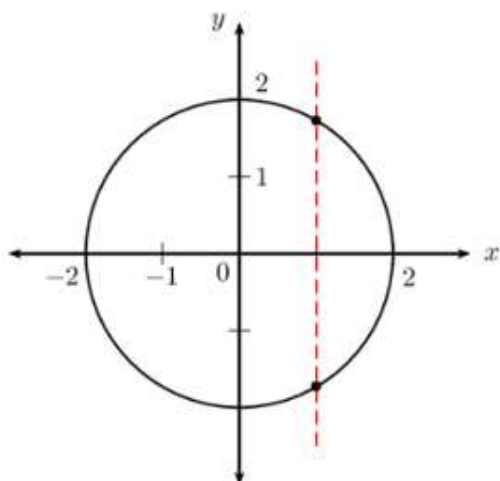
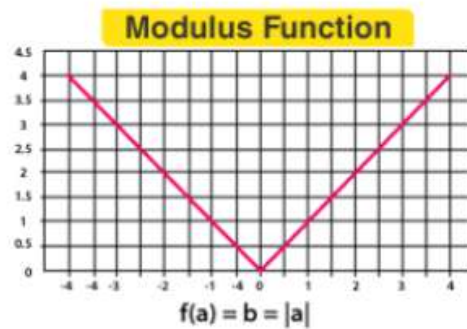
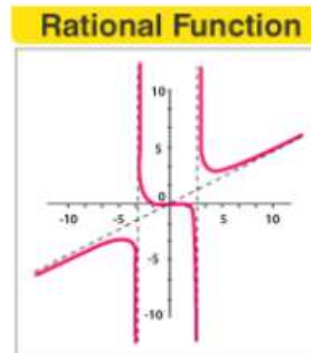
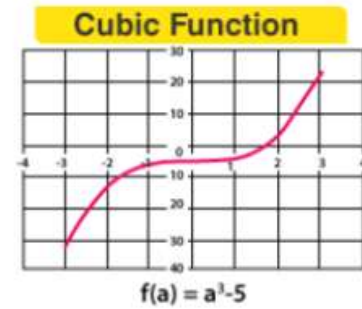
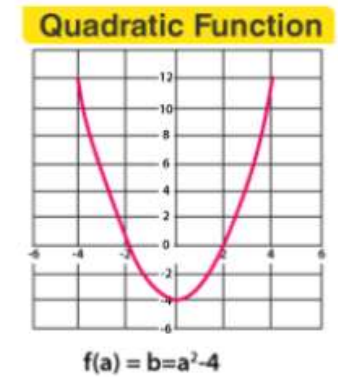
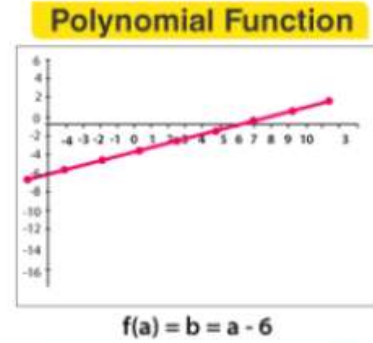
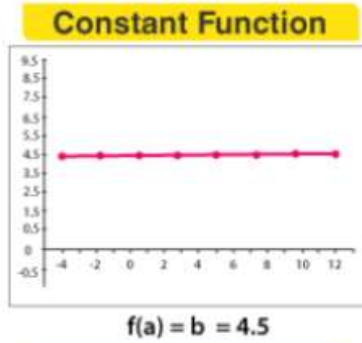
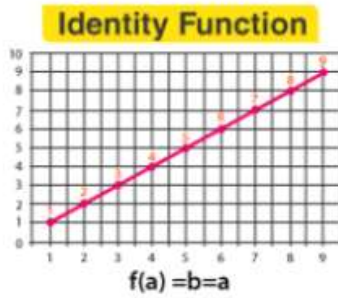


Fig: Vertical Test

This will help you more:



Sequence and Series

Sequence and Series

Arithmetic Progression (AP)

- An arithmetic progression is a sequence of numbers in which each term after the first is obtained by adding a constant 'd' to the preceding term. The constant d is called the common difference.
- An arithmetic progression is given by a, (a + d), (a + 2d), (a + 3d), ...

where a = the first term, d = the common difference

- If a, b, c are in AP then $b = (a + c)/2$
- n^{th} term of an arithmetic progression

$$t_n = a + (n - 1)d$$

where t_n = n^{th} term, a = the first term, d = common difference

- Number of terms of an arithmetic progression

$$n = (l - a)/d + 1$$

where n = number of terms, a = the first term, l = last term, d = common difference

-

FORMULAS

Number of terms of an arithmetic progression

$$n = \frac{(l - a)}{d} + 1$$

where n = number of terms, a = the first term, l = last term, d = common difference

no of term

Sum of first n terms in an arithmetic progression

$$S_n = \frac{n}{2} [2a + (n-1)d] = \frac{n}{2}(a + l)$$

where a = the first term,

d = common difference,

$l = t_n = n^{\text{th}} \text{ term} = a + (n-1)d$

the sum of first n terms

ADDITIONAL NOTES ON AP

To solve most of the problems related to AP, the terms can be conveniently taken as

3 terms: $(a - d), a, (a + d)$

4 terms: $(a - 3d), (a - d), (a + d), (a + 3d)$

5 terms: $(a - 2d), (a - d), a, (a + d), (a + 2d)$

Harmonic Progression(HP)

Harmonic Progression(HP)

Non-zero numbers $a_1, a_2, a_3, \dots, a_n$ are in Harmonic Progression(HP) if

$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \dots, \frac{1}{a_n}$ are in AP. Harmonic Progression is also known as harmonic sequence.

Examples

$\frac{1}{2}, \frac{1}{6}, \frac{1}{10}, \dots$ is a harmonic progression (HP)

Three non-zero numbers a, b, c will be in HP, if $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP

If a, $(a+d), (a+2d), \dots$ are in AP, n^{th} term of the AP = $a + (n-1)d$

Hence, if $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \dots$ are in HP, n^{th} term of the HP = $\frac{1}{a + (n-1)d}$

harmonic progression

If a, b, c are in HP, b is the Harmonic Mean(HM) between a and c

$$\text{In this case, } b = \frac{2ac}{a+c}$$

The Harmonic Mean(HM) between two numbers a and $b = \frac{2ab}{a+b}$

If $a, a_1, a_2 \dots a_n, b$ are in HP we can say that $a_1, a_2 \dots a_n$ are the n Harmonic Means between a and b .

$$\text{If } a, b, c \text{ are in HP, } \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

harmonic progression

Geometric Progression (GP)

Geometric Progression (GP) is a sequence of non-zero numbers in which the ratio of any term and its preceding term is always constant.

A geometric progression(GP) is given by a, ar, ar^2, ar^3, \dots
where a = the first term, r = the common ratio

Examples

1, 3, 9, 27, ... is a geometric progression(GP) with $a = 1$ and $r = 3$

2, 4, 8, 16, ... is a geometric progression(GP) with $a = 2$ and $r = 2$

$$\text{If } a, b, c \text{ are in GP, } b^2 = ac$$

geometric progression

n^{th} term of a geometric progression(GP)

$$t_n = ar^{n-1}$$

where $t_n = n^{\text{th}}$ term, $a =$ the first term, $r =$ common ratio, $n =$ number of terms

Sum of first n terms in a geometric progression(GP)

$$S_n = \begin{cases} \frac{a(r^n - 1)}{r - 1} & (\text{if } r > 1) \\ \frac{a(1 - r^n)}{1 - r} & (\text{if } r < 1) \end{cases}$$

where $a =$ the first term,
 $r =$ common ratio,
 $n =$ number of terms

sum of GP

Sum of an infinite geometric progression(GP)

$$S_{\infty} = \frac{a}{1 - r} \quad (\text{if } -1 < r < 1)$$

where $a =$ the first term, $r =$ common ratio

sum of infinite

Additional Notes on GP

To solve most of the problems related to GP, the terms of the GP can be conveniently taken as

3 terms: $\frac{a}{r}, a, ar$

5 terms: $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

If a, b, c are in GP, $\frac{a - b}{b - c} = \frac{a}{b}$

In a GP, product of terms equidistant from beginning and end will be constant.

geometric progression

Power Series : Important formulas

$$1 + 1 + 1 + \dots \text{ n terms} = \sum 1 = n$$

$$1 + 2 + 3 + \dots + n = \sum n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \sum n^3 = \frac{n^2(n+1)^2}{4} = \left[\frac{n(n+1)}{2} \right]^2$$

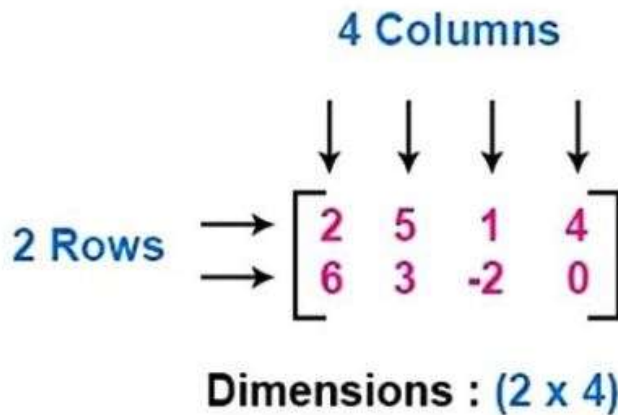
power series

Matrix and Determinants

MATRIX

A **matrix** is simply a set of numbers arranged in a rectangular table.

Determine the order of matrix



Determine-the-order-of-matrix

Types of Matrices

There are several types of matrices, but the most commonly used are:

- Rows Matrix
- Columns Matrix
- Rectangular Matrix
- Square Matrix
- Diagonal Matrix
- Scalar Matrix
- Identity Matrix
- Triangular Matrix
- Null or Zero Matrix
- Transpose of a Matrix

Row Matrix:

A matrix is said to be a row matrix if it has only one row.

$$A = [1 \quad 2 \quad 3]$$

row matrix

Column Matrix:

A matrix is said to be a column matrix if it has only one column.

$$B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Rectangular Matrix:

A matrix is said to be rectangular if the number of rows is not equal to the number of columns.

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 2 \end{bmatrix}$$

rectangular matrix

Square Matrix:

A matrix is said to be square if the number of rows is equal to the number of columns.

$$B = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 2 & 4 \\ 1 & 9 & 6 \end{bmatrix}$$

square matrix

Diagonal Matrix:

A square matrix is said to be diagonal if at least one element of principal diagonal is non-zero and all the other elements are zero.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

diagonal matrix

Scalar Matrix:

A diagonal matrix is said to be scalar if all of its diagonal elements are the same.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

scalar matrix

Identity or Unit Matrix:

A diagonal matrix is said to be identity if all of its diagonal elements are equal to one, denoted by I.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

identity matrix

Triangular Matrix:

A square matrix is said to be triangular if all of its elements above the principal diagonal are zero (**lower triangular matrix**) or all of its elements below the principal diagonal are zero (**upper triangular matrix**).

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix} \text{ } \backslash \text{ } [(LowerTriangularMatrix) \backslash \text{ } [B = \begin{bmatrix} 5 & 8 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

traingular

Null or Zero Matrix:

A matrix is said to be a null or zero matrix if all of its elements are equal to zero. It is denoted by O.

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

zero matrix

Transpose of a Matrix:

Suppose A is a given matrix, then the matrix obtained by interchanging its rows into columns is called the transpose of A. It is denoted by A^t .

e.g.

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 8 & 4 & 6 \end{bmatrix}$$

then

$$A^t = \begin{bmatrix} 1 & 8 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

transpose matrix

Determinants of a matrix

The determinant of a matrix is a **special number** that can be calculated from a square matrix.

For a 2x2 Matrix

For a 2x2 matrix (2 rows and 2 columns):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is:

$$|A| = ad - bc$$

"The determinant of A equals a times d minus b times c"

determinant

$$B = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

$$\begin{aligned} |B| &= 4 \times 8 - 6 \times 3 \\ &= 32 - 18 \\ &= 14 \end{aligned}$$

For a 3x3 Matrix

For a 3x3 matrix (3 rows and 3 columns):

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is:

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

"The determinant of A equals ... etc"

It may look complicated, but **there is a pattern**:

$$\left[\begin{array}{c|cc} a & & \\ \hline & e & f \\ & h & i \end{array} \right] - \left[\begin{array}{c|cc} & b & \\ \hline d & & \\ g & & \end{array} \right] + \left[\begin{array}{c|cc} & & c \\ \hline d & e & \\ g & h & \end{array} \right]$$

To work out the determinant of a **3x3** matrix:

- Multiply **a** by the **determinant of the 2x2 matrix** that is **not in a's** row or column.
- Likewise for **b**, and for **c**
- Sum them up, but remember the minus in front of the **b**



ADJOINT OF A MATRIX

Let $A=[a_{ij}]$ be a square matrix of order n . The adjoint of matrix A is the transpose of the cofactor matrix of A . It is denoted by $\text{adj } A$. An adjoint matrix is also called an adjoint

Example:

Find the adjoint of the matrix.

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

matrix.

eg of the cofactor matrix

To find the adjoint of a matrix, first find the cofactor matrix of the given matrix. Then find the transpose of the cofactor matrix.

$$\text{Cofactor of } 3 = A_{11} = \begin{vmatrix} -2 & 0 \\ 2 & -1 \end{vmatrix} = 2$$

$$\text{Cofactor of } 1 = A_{12} = -\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = 2$$

$$\text{Cofactor of } -1 = A_{13} = \begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} = 6$$

$$\text{Cofactor of } 2 = A_{21} = -\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1$$

$$\text{Cofactor of } -2 = A_{22} = \begin{vmatrix} 3 & -1 \\ 1 & -1 \end{vmatrix} = -2$$

$$\text{Cofactor of } 0 = A_{23} = -\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = -5$$

$$\text{Cofactor of } 1 = A_{31} = \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -2$$

$$\text{Cofactor of } 2 = A_{32} = -\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = -2$$

$$\text{Cofactor of } -1 = A_{33} = \begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} = -8$$

$$\text{The cofactor matrix of } A \text{ is } [A_{ij}] = \begin{bmatrix} 2 & 2 & 6 \\ -1 & -2 & -5 \\ -2 & -2 & -8 \end{bmatrix}$$

Now find the transpose of A_{ij} .

$$\begin{aligned} \text{adj } A &= (A_{ij})^T \\ &= \begin{bmatrix} 2 & -1 & -2 \\ 2 & -2 & -2 \\ 6 & -5 & -8 \end{bmatrix} \end{aligned}$$

Inverse of a matrix

The inverse of A is A^{-1} only when:

$$A \times A^{-1} = A^{-1} \times A = I$$

Sometimes there is no inverse at all.

Formula:

Multiplication of a matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

$$(1, 2, 3) \cdot (8, 10, 12) = 1 \times 8 + 2 \times 10 + 3 \times 12 = 64$$

We can do the same thing for the **2nd row** and **1st column**:

$$(4, 5, 6) \cdot (7, 9, 11) = 4 \times 7 + 5 \times 9 + 6 \times 11 = 139$$

And for the **2nd row** and **2nd column**:

$$(4, 5, 6) \cdot (8, 10, 12) = 4 \times 8 + 5 \times 10 + 6 \times 12 = 154$$

And we get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix} \quad \checkmark$$

DONE!

matrix multiplication

Analytical Geometry:

ANALYTICAL GEOMETRY

Analytic Geometry is a branch of algebra that is used to model geometric objects – points, (straight) lines, and circles being the most basic of these. Analytic geometry is a great invention of Descartes and Fermat.

In plane analytic geometry, points are defined as ordered pairs of numbers, say, (x, y), while the straight lines are in turn defined as the sets of points that satisfy linear

equations, see the excellent expositions by D. Pedoe or D. Brannan et al. From the view of analytic geometry, geometric axioms are derivable theorems. For example, for any two distinct points (x_1, y_1) and (x_2, y_2) , there is a single line $ax + by + c = 0$ that passes through these points. Its coefficients a, b, c can be found (up to a constant factor) from the linear system of two equations

$$\begin{aligned} ax_1 + by_1 + c &= 0 \\ ax_2 + by_2 + c &= 0, \end{aligned}$$

1. [Conic Section](#)
2. [Vector in space](#)

1. Conic Sections:

Circle

A circle is the set of all points in a plane, which are at a fixed distance from a fixed point in the plane. The fixed point is called the centre of the circle and the distance from centre to any point on the circle is called the radius of the circle.

The equation of a circle with radius r having centre (h, k) is given by $(x - h)^2 + (y - k)^2 = r^2$.

The general equation of the circle is given by $x^2 + y^2 + 2gx + 2fy + c = 0$, where, g, f and c are constants.

- The centre of the circle is $(-g, -f)$.
- The radius of the circle is $r = \sqrt{g^2 + f^2 - c}$

The general equation of the circle passing through origin is $x^2 + y^2 + 2gx + 2fy = 0$.

The parametric equation of the circle $x^2 + y^2 = r^2$ are given by $x = r \cos \theta, y = r \sin \theta$, where θ is the parametre and the parametric equation of the circle $(x - h)^2 + (y - k)^2 = r^2$ are given by $x = h + r \cos \theta, y = k + r \sin \theta$.

Note: The general equation of the circle involves three constants which implies that at least three conditions are required to determine a circle uniquely.

Parabola

A parabola is the set of points P whose distances from a fixed point F in the plane are equal to their distance from a fixed line l in the plane. The fixed point F is called focus and the fixed line l is the directrix of the parabola.

Main Facts About the Parabola

Forms of parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Axis of parabola	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Directrix of parabola	$x = -a$	$x = a$	$y = -a$	$y = a$
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$
Focal length	$ x + a $	$ x - a $	$ y + a $	$ y - a $

Ellipse

An ellipse is the set of all points in a plane such that the sum of whose distances from two fixed points is constant.

or

An ellipse is the set of all points in the plane whose distances from a fixed point in the plane bears a constant ratio, less than to their distance from a fixed point in the plane. The fixed point is called focus, the fixed line a directrix and the constant ratio(e) the

eccentricity of the ellipse. We have two standard forms of ellipse i.e.

- (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and
(ii) $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ in both cases $a > b$ and
 $b^2 = a^2(1 - e^2), e < 1$

Main Facts about the Ellipse

Forms of the ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b$
Equation of major axis	$y = 0$	$x = 0$
Length of major axis	$2a$	$2a$
Equation of minor axis	$x = 0$	$y = 0$
Length of minor axis	$2b$	$2b$
Equation of directrices	$x = \pm a/e$	$y = \pm a/e$
Vertex of ellipse	$(\pm a, 0)$	$(0, \pm a)$
Focus of ellipse	$(\pm ae, 0)$	$(0, \pm ae)$
Length of latusrectum	$2b^2/a$	$2b^2/a$

Hyperbola

A hyperbola is the locus of a point in a plane which moves in such a way that the ratio of its distance from a fixed point in the same plane to its distance from a fixed line is always constant which is always greater than unity. The fixed point is called the focus, the fixed line is called the directrix and the constant ratio, generally denoted by e , is known as the eccentricity of the hyperbola.

We have two standard forms of hyperbola i.e.

- (i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and (ii) $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Main Facts About Hyperbola

Forms of the hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Coordinates of centre	(0, 0)	(0, 0)
Coordinates of vertices	($\pm a$, 0)	(0, $\pm a$)
Coordinates of foci	($\pm ae$, 0)	(0, $\pm ae$)
Length of transverse axis	2a	2a
Length of conjugate axis	2b	2b
Equation of directrices	$x = \pm a/e$	$y = \pm a/e$
Eccentricity	$e = \sqrt{\frac{a^2 + b^2}{a^2}}$	$e = \sqrt{\frac{a^2 + b^2}{a^2}}$
Length of latusrectum	$2b^2/a$	$2b^2/a$
Equation of transverse axis	$y = 0$	$x = 0$
Equation of conjugate axis	$x = 0$	$y = 0$

2. Vector in Space

Formula for Distance Between Two Points:

The formula for the distance, dd , between two points whose coordinates are (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This is called the **Distance Formula**.

To find the distance between two points given in 3-D plane, we can apply the 3D distance formula, given as,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Unit Vector Formula

As vectors have both magnitude (Value) and direction, they are shown with an arrow \vec{a} , and it denotes a unit vector. If we want to find the unit vector of any vector, we divide it by the vector's magnitude. Usually, the coordinates of x, y, z are used to represent any vector.

A vector can be represented in two ways:

1. $\vec{a} = (x, y, z)$ using the brackets.

2. $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

The formula for the magnitude of a vector is:

$$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$$

Unit Vector = Vector/Vector's magnitude

The above is a unit vector formula.

How to find the unit vector?

To find a unit vector with the same direction as a given vector, simply divide the vector by its magnitude. For example, consider a vector $v = (1, 4)$ which has a magnitude of $|v|$. If we divide each component of vector v by $|v|$ to get the unit vector \hat{v} which is in the same direction as v .

How to represent vector in a bracket format?

$$\hat{a} = \frac{a}{|a|} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

How to represent vector in a unit vector component format?

$$\hat{a} = \frac{a}{|a|} = \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \cdot \hat{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \cdot \hat{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \cdot \hat{k}$$

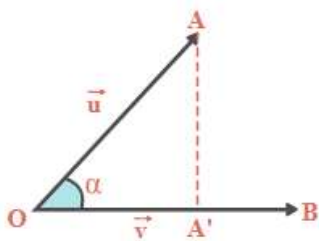
Where x, y, z represent the value of the vector along the x-axis, y-axis, z-axis respectively and \hat{a} is a unit vector, \vec{a} is a vector, $|\vec{a}|$ is the magnitude of the vector, \hat{i} , \hat{j} , \hat{k} are the directed unit vectors along the (x, y, z) axis respectively.

Null Vector: A null vector is a vector that has magnitude equal to zero and is directionless. It is the resultant of two or more equal vectors that are acting opposite to each other.

A most common example of null vector is pulling a rope from both the end with equal forces at opposite direction.

Geometric interpretation of the scalar product

The product of two non zero vectors is equal to the magnitude of one of them times the projection of the other onto it.



In the picture, OA' is the projection of the vector \vec{u} on \vec{v} . If we observe the OAA' triangle and apply the cosinus definition, we have:

$$\cos(\alpha) = \frac{OA'}{|\vec{u}|} \Rightarrow |OA'| = |\vec{u}| \cdot \cos(\alpha)$$

Finally, applying to the scalar product formula what we just have found:

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\alpha) = |\vec{u}|\text{proj}_{\vec{v}}(\vec{u})$$

Example

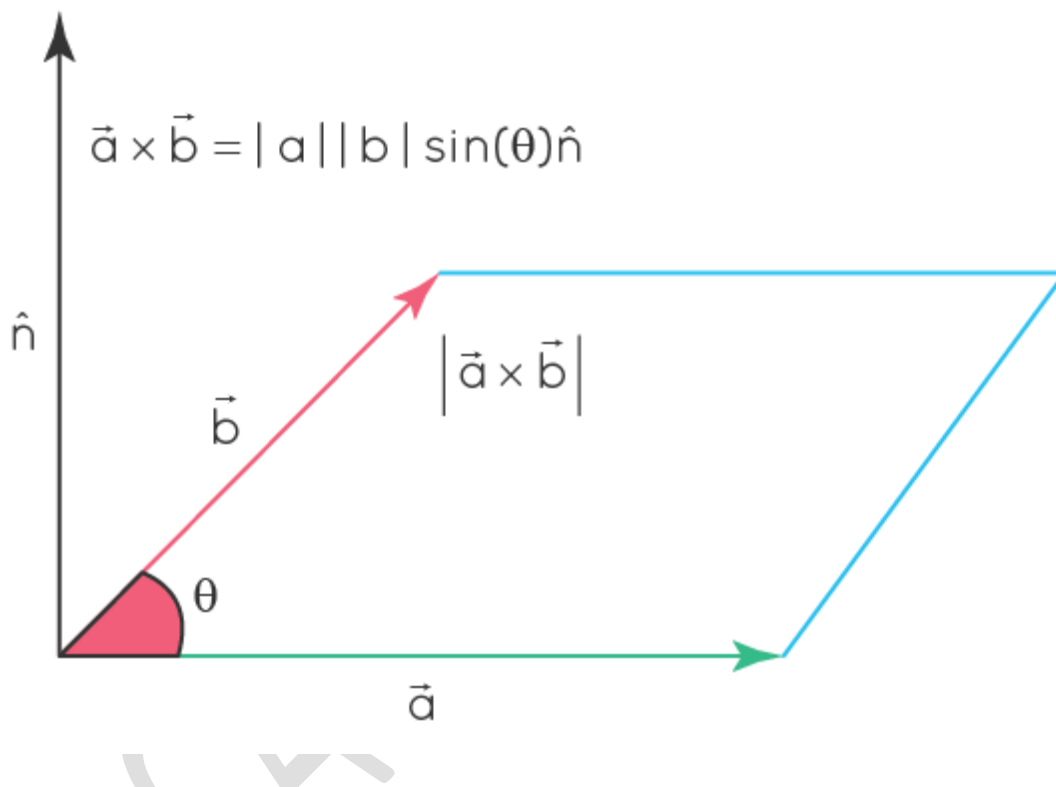
Find the projection of the vector $\vec{u} = (2, 3)$ on $\vec{v} = (-1, 4)$.

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \frac{2 \cdot (-1) + 3 \cdot 4}{\sqrt{(-1)^2 + 4^2}} = \frac{10}{\sqrt{17}}$$

Cross Product of Two Vectors Meaning

Use the image shown below and observe the angles between the vectors \vec{a} and \vec{c} and the angles between the vectors \vec{b} and \vec{c} .

$$a \times b = |a| |b| \sin \theta.$$



- The angle between \vec{a} and \vec{c} is always 90° . i.e., \vec{a} and \vec{c} are orthogonal vectors.
- The angle between \vec{b} and \vec{c} is always 90° . i.e., \vec{b} and \vec{c} are orthogonal vectors.
- We can position \vec{a} and \vec{b} parallel to each other or at an angle of 0° , making the resultant vector a zero vector.
- To get the greatest magnitude, the original vectors must be perpendicular (angle of 90°) so that the cross product of the two vectors will be maximum.

Cross Product Formula

Cross product formula between any two vectors gives the area between those vectors. The cross product formula gives the magnitude of the resultant vector which is the area of the parallelogram that is spanned by the two vectors.

Cross Product Formula

Vector Cross Product Formula

$$\vec{A} \times \vec{B} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$$

$$\vec{A} \times \vec{B} = i (a_2 b_3 - a_3 b_2) + j (a_1 b_3 - a_3 b_1) + k (a_1 b_2 - a_2 b_1)$$

Consider two

vectors $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ and $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$. Let θ be the angle formed

between \vec{a} and \vec{b} and \hat{n} is the unit vector perpendicular to the plane containing both \vec{a} and \vec{b} . The cross product of the two vectors is given by the formula:

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin(\theta) \hat{n}$$

Where

- $|\vec{a}|$ is the magnitude of the vector \vec{a} or the length of \vec{a} ,
- $|\vec{b}|$ is the magnitude of \vec{b}
-
-

- If the vector \vec{b} or the length of \vec{b} .

Let us assume that \vec{a} and \vec{b} are two vectors, such that $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then by using determinants, we could find the cross product and write the result as the cross product formula using matrix notation.

Permutation and Combination

What's the Difference?

- When the order doesn't matter, it is a **Combination**.
- When the order **does** matter it is a **Permutation**.

In other words:

A Permutation is an **ordered** Combination.

PERMUTATIONS

There are basically two types of permutation:

- **Repetition is Allowed:** It could be "333".
- **No Repetition:** for example, the first three people in a running race. You can't be first *and*

1. PERMUTATIONS WITH REPETITION

When a thing has n different types ... we have n choices each time!

For example: choosing **3** of those things, the permutations are:

$n \times n \times n$ (n multiplied 3 times)



Example: in the lock above, there are 10 numbers to choose from (0,1,2,3,4,5,6,7,8,9) and we choose 3 of them:

$10 \times 10 \times 10$ (3 times) = $10^3 = 1,000$ permutations

So, the formula is simply:

n^r

where n is the number of things to choose from,
and we choose r of them,
repetition is allowed,
and order matters.

2. PERMUTATIONS WITHOUT REPETITION

In this case, we have to **reduce** the number of available choices each time.

Example: what order could 16 pool balls be in?



After choosing, say, number "14" we can't choose it again.

So, our first choice has 16 possibilities, and our next choice has 15 possibilities, then 14, 13, 12, 11, ... etc. And the total permutations are:

$$16 \times 15 \times 14 \times 13 \times \dots = 20,922,789,888,000$$

But maybe we don't want to choose them all, **just 3** of them, and that is then:

$$16 \times 15 \times 14 = 3,360$$

In other words, there are 3,360 different ways that 3 pool balls could be arranged out of 16 balls.

But how do we write that mathematically? Answer: we use the "**factorial function**"

The **factorial function** (symbol: $!$) just means to multiply a series of descending natural numbers. Examples:

$$\downarrow 7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5,040$$

$$\downarrow 1! = 1$$

So, when we want to select **all** of the pool balls the permutations are:
 $16! = 20,922,789,888,000$

But when we want to select just 3 we don't want to multiply after 14. How do we do that? Simple, we divide by $13!$

$$16 \times 15 \times 14 \times 13 \times 12 \dots 13 \times 12 \dots = \underline{16 \times 15 \times 14}$$

Hint:

The $13 \times 12 \times \dots$ etc gets "cancelled out", leaving only $16 \times 15 \times 14$.

The formula is written:

$$n!(n - r)!$$

where n is the number of things to choose from,
and we choose r of them,
no repetitions,
order matters.

This will help you :

Example: Our "order of 3 out of 16 pool balls example" is:

$$\frac{16!}{(16-3)!} = \frac{16!}{13!} = \frac{20,922,789,888,000}{6,227,020,800} = 3,360$$

(which is just the same as: $16 \times 15 \times 14 = 3,360$)

Notation

Instead of writing the whole formula, people use different notations such as these:

$$P(n, r) = {}^nP_r = {}_nP_r = \frac{n!}{(n - r)!}$$

Example: $P(10,2) = 90$



Combinations

There are also two types of combinations (remember the order does **not** matter now):

- **Repetition is Allowed:** such as coins in your pocket (5,5,5,10,10)
- **No Repetition:** such as lottery numbers (2,14,15,27,30,33)

Notation

As well as the "big parentheses", people also use these notations:

$$C(n, r) = {}^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

(It is often called "n choose r" (such as "16 choose 3")

And is also known as the [Binomial Coefficient](#).)

1. Combinations with Repetition

$$\binom{r+n-1}{r} = \frac{(r+n-1)!}{r!(n-1)!}$$

where n is the number of things to choose from,
and we choose r of them
repetition allowed,
order doesn't matter.

Example:

Let us say there are five flavors of ice-cream: **banana, chocolate, lemon, strawberry and vanilla.**

We can have three scoops. How many variations will there be?

So, here $n=5$ and $r=3$, using formula

$$\frac{(3+5-1)!}{3!(5-1)!} = \frac{7!}{3! \times 4!} = \frac{5040}{6 \times 24} = 35$$

There are 35 ways of having 3 scoops from five flavors of ice-cream.

2. Combinations without Repetition

Example: Pool Balls (without order)

So, our pool ball example (now without order) is:

$$\begin{aligned} 16! \underline{3!} (16-3)! &= 16! \underline{3!} \times 13! \\ &= 20,922,789,888,000 \underline{6} \times 6,227,020,800 \\ &= 560 \end{aligned}$$

Or we could do it this way:

$$16 \times 15 \times 14 \underline{3} \times 2 \times 1 = 3360 \underline{6} = 560$$

The End