Proof →

→ Consider the Bivariate Gaussian with $x1$ and $x2$ random variables with mean $= \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ and covariance matrix $= \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}$

with cofficient of correlation b/w $x1$ and $x2$ as

$$p = \frac{\sigma_{12}}{\sigma_1 \sigma_2} \quad . \quad \text{Here } \sigma_{12} = \sigma_{21}$$

→ The PDF function of the gaussian in terms $x1$ and $x2$ and p

$$f(x1, x2) = \frac{-1}{1-p^2}\left[\frac{(x1-\mu 1)^2}{\sigma_1^2} + \frac{(x2-\mu 2)^2}{\sigma_2^2} - \frac{2p\,(x1-\mu_1)\,(x2-\mu 2)}{\sigma_1 \sigma_2}\right] \Big/ 2\pi\sigma_1\sigma_2\sqrt{1-p^2}$$

→ Finding the conditional distribution of variable $x2$ when $x1 = x_1$

$$f(x2 / x1 = x1) = \frac{f(x1, x2)}{f(x1)}$$

$$f(x_1) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \frac{(x_1-\mu_1)^2}{\sigma_1^2} \right\}$$

$$f(x_2) = \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left\{ \frac{-1}{2} \frac{(x_2-\mu_2)^2}{\sigma_2^2} \right\}$$

Considering the marginalised pdf as given above

→ Computing $f(x2 / X1 = x1)$

$$f(x2 / X1 = x1) = \frac{f(x1, x2)}{f(x1)}$$

$$= \frac{\dfrac{\exp\left\{\dfrac{-1}{2(1-p^2)}\left[\dfrac{(x1-\mu1)^2}{\sigma_1^2} + \dfrac{(x2-\mu2)^2}{\sigma_2^2} - \dfrac{2p(x1-\mu1)(x2-\mu2)}{\sigma_1\sigma_2}\right]\right\}}{2\pi\sigma_1\sigma_2\sqrt{1-p^2}}}{\dfrac{\exp\left\{\dfrac{-1}{2}\dfrac{(x1-\mu1)^2}{\sigma_1^2}\right\}}{\sqrt{2\pi}\,\sigma_1}}$$

$$= \frac{\exp\left\{\dfrac{-1}{2(1-p^2)}\left[\dfrac{(x1-\mu_1)^2}{\sigma_1^2} + \dfrac{(x2-\mu_2)^2}{\sigma_2^2} - \dfrac{2p(x1-\mu_1)(x2-\mu2)}{\sigma_1\sigma_2}\right]\right\}}{\exp\left\{\dfrac{-1}{2}\dfrac{(x1-\mu_1)^2}{\sigma_1^2}\right\}} \times \frac{\sqrt{2\pi}\sigma_1}{2\pi\sigma_1\sigma_2\sqrt{1-p2}}$$

$$= \frac{\exp\left\{ \frac{-1}{2(1-p^2)} \left[\frac{(x1-\mu_1)^2}{\sigma_1^2} + \frac{(x2-\mu_2)^2}{\sigma_2^2} - \frac{2p(x1-\mu_1)(x2-\mu_2)}{\sigma_1\sigma_2} \right] + \frac{1}{2}\frac{(x1-\mu_1)^2}{\sigma_1^2} \right\}}{\sqrt{2\pi} \quad \sigma_2\sqrt{1-p^2}}$$

$$= \frac{\exp\left\{ \frac{-1}{2(1-p^2)\sigma_2^2} \left[\frac{\sigma_2^2}{\sigma_1^2}(x1-\mu_1)^2 + (x2-\mu_2)^2 - \frac{2p(x1-\mu_1)(x2-\mu_2)\sigma_2}{\sigma_1} - \frac{\sigma_2^2}{\sigma_1^2}(1-p^2)(x1-\mu_1)^2 \right] \right\}}{\sqrt{2\pi} \quad \sigma_2\sqrt{1-p^2}}$$

$$= \frac{\exp\left\{ \frac{-1}{2(1-p^2)\sigma_2^2} \left[p^2\frac{\sigma_2^2}{\sigma_1^2}(x1-\mu_1)^2 + (x2-\mu_2)^2 - 2p\frac{\sigma_2}{\sigma_1}(x1-\mu_1)(x2-\mu_2) \right] \right\}}{\sqrt{2\pi} \quad \sigma_2\sqrt{1-p^2}}$$

$$= \exp\left\{ \frac{-1}{2(1-p^2)\sigma_2^2} \left[p^2\frac{\sigma_2^2}{\sigma_1^2}(x1-\mu_1)^2 + x^2 - 2\mu_2 x2 + \mu_2^2 - 2x2\left(p\frac{\sigma_2}{\sigma_1}(x1-\mu_1)\right) + \right.\right.$$

$$\left.\left. 2\mu_2\left(p\frac{\sigma_2}{\sigma_1}(x2-\mu_2)\right)\right]\right\} \Big/ \sqrt{2\pi}\,\sigma_2\,\sqrt{1-p^2}$$

Using the identity,

$$a^2 - 2ab + b^2 - 2ac + 2bc + c^2 = (a - b - c)^2$$

$$= \frac{\exp\left\{\frac{-1}{2(1-p^2)\sigma_2^2}\left[x2 - \mu 2 - p\frac{\sigma_2}{\sigma_1}(x1 - \mu_1)\right]^2\right\}}{\sqrt{2\pi}\ \sigma_2\sqrt{(1-p^2)}} = \frac{\exp\left\{\frac{-1}{2\sigma_*^2}\left[x2 - \mu_*\right]^2\right\}}{\sqrt{2\pi}\ \sigma_*}$$

→ This resembles the Univariate gaussian pdf with

$$\mu_* = \mu_2 + p\left(\frac{\sigma_2}{\sigma_1}\right)(x1 - \mu_1)$$

and

$$\sigma_*^2 = (1-p^2)\sigma_2$$

→ Computing $f(x1 \mid x2 = x2)$

$$f(x1 \mid x2 = x2) = \frac{f(x1, x2)}{f(x2)}$$

$$= \frac{\dfrac{\exp\left\{\dfrac{-1}{2(1-p^2)}\left[\dfrac{(x1-\mu1)^2}{\sigma_1^2} + \dfrac{(x2-\mu2)^2}{\sigma_2^2} - \dfrac{2p(x1-\mu1)(x2-\mu2)}{\sigma_1\sigma_2}\right]\right\}}{2\pi\sigma_1\sigma_2\sqrt{1-p^2}}}{\dfrac{\exp\left\{\dfrac{-1}{2}\dfrac{(x2-\mu2)^2}{\sigma_2^2}\right\}}{\sqrt{2\pi}\,\sigma_2}}$$

$$= \frac{\exp\left\{\dfrac{-1}{2(1-p^2)\sigma_1^2}\left[\dfrac{\sigma_1^2}{\sigma_2^2}(x2-\mu2)^2 + (x1-\mu_1)^2 - 2p\dfrac{\sigma_1}{\sigma_2}(x1-\mu1)(x2-\mu2) - \dfrac{\sigma_1^2}{\sigma_2^2}(1-p^2)(x2-\mu2)^2\right]\right\}}{\sqrt{2\pi}\,\sigma_1\sqrt{1-p^2}}$$

$$= \frac{\exp\left\{\frac{-1}{2(1-p^2)\sigma_1^2}\left[x1 - \mu_1 - p\frac{\sigma_1}{\sigma_2}(x2-\mu_2)\right]^2\right.}{\sqrt{2\pi}\ \sigma_1\sqrt{1-p^2}}$$

This resembles the uni variable gassium with random variable $x1$ and mean $\mu^* = \mu_1 + p\frac{\sigma_1}{\sigma_2}(x2-\mu_2)$

and variance $= \sigma_1^2(1-p^2)$