Generalized Inequalities

This report is literature survey of various generalized convexity properties and inequalities formulated from them as a result. Most of these inequalties are derived from various properties of convex functions defined by **Maclaurin Series**.

The following **Lemma** is important is important to study the convexity and monotonicity of many of the following power series and inequalities.

Lemma 1: For 0 < R $\leq \infty$, let *f(x)* = $\sum_{n=0}^{\infty} a_n x^n$ and *g(x)* = $\sum_{n=0}^{\infty} b_n x^n$ be two real power series converging on the interval *(-R, R)*. If the sequence $\frac{a_n}{b_n}$ is increasing(decreasing), and $b_n > 0$ for all n, then the function

$$h(x) = \frac{f(x)}{g(x)} = \frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n}$$

is also increasing(decreasing) on *(0,R)*. Also the function

$$g(x)f\prime(x) - f(x)g\prime(x)$$

has positive *Maclaurin coefficients.*

Definitions: 1) A function $M : (0, \infty) \times (0, \infty) - > (0, \infty)$ is called a *Mean function* if,

a) *M(x,y) = M(y,x)*
b) *M(x,x) = x*
c) *x < M(x,y) < y, whenever x < y*
d) *M(ax,ay) = aM(x,y) for all a > 0*

2) Let $f : I \to (0, \infty)$ be continuous, where I is a subinterval of $(0, \infty)$. Let M and N be any two Mean functions. We say f if *MN-convex(concave)* is $f(M(x, y)) \leq (\geq) N(f(x), f(y))$ for all *x,y* $\in$ *I.*

The following inequalities are concerned with the Mean functions, *M,N = A,G,H*, where A is the Arithmetic Mean, *A(x,y) = (x+y)/2*, G is the Geometric Mean,

$G(x,y) = \sqrt{xy}$ and H is the Harmonic Mean, $H(x,y) = \frac{1}{A(1/x,1/y)}$.

Theorem 1: Let I be an open subinterval of $(0, \infty)$ and let $f : I \to (0, \infty)$ be continuous. In parts (4-9), $I = (0,b)$, $0 < b < \infty$. We have the following:

a) *f is AA-convex(concave) if and only if f is convex(concave).*

b) *f is AG-convex(concave) if and only if log(f) is convex(concave).*

c) *f is AH-convex(concave) if and only if 1/f is concave(convex).*

d) *f is GA-convex(concave) on I if and only if* $f(be^{-t})$ *is convex(concave) on* $(0,\infty)$.

e) *f is GG-convex(concave) on I if and only if* $log(f(be^{-t}))$ *is convex(concave) on* $(0,\infty)$.

f) *f is GH-convex(concave) on I if and only if* $1/f(be^{-t})$ *is concave(convex) on* $(0,\infty)$.

g) *f is HA-convex(concave) on I if and only if f(1/x) is convex(concave) on* $(1/b,\infty)$.

h) *f is HG-convex(concave) on I if and only if log(f(1/x)) is convex(concave) on* $(1/b,\infty)$.

i) *f is HH-convex(concave) on I if and only if 1/f(1/x) is concave(convex) on* $(1/b,\infty)$.

Proof:

1) This is the definition of convex functions, put $\theta = \frac{1}{2}$ in the convex inequaltiy.

2) From the definitions

$$f(A(x,y)) \leq (\geq) G(f(x), f(y))$$

$$\iff f((x+y)/2) \leq (\geq) \sqrt{f(x)f(y)}$$

Taking log,

$$\iff log(f((x+y)/2)) \leq (\geq) \tfrac{1}{2}(log(f(x)) + log(f(y))),$$

hence the result follows.

3) From the definitions

$$f(A(x,y)) \leq (\geq) H(f(x), f(y))$$

$$\iff f((x+y)/2) \leq (\geq) 2/(1/f(x) + 1/f(y))$$

cross-multiplying (since f is positive),

$$\iff 1/(f((x+y)/2)) \geq (\leq) \tfrac{1}{2}(1/(f(x)) + 1/(f(y))),$$

hence the result follows.

4) Substituting $x = be^{-r} and x = be^{-s}$,

$$f(G(x,y)) \leq (\geq) A(f(x) + f(y))$$

Putting the values of x, y and from the definition of GM,

$$\iff f(be^{-(r+s)/2}) \leq (\geq) \tfrac{1}{2}(f(be^{-r}) + f(be^{-s})),$$

hence the result follows.

5) Substituting $x = be^{-r} and x = be^{-s}$,

$$f(G(x,y)) \leq (\geq) G(f(x), f(y))$$

Taking log, putting the values of x, y and from the definition of GM,

$$\iff log(f(be^{-(r+s)/2})) \leq (\geq) \tfrac{1}{2}(log(f(be^{-r})) + log(f(be^{-s}))),$$

hence the result follows.

6) Substituting $x = be^{-r} and x = be^{-s}$,

$$f(G(x,y)) \leq (\geq) H(f(x), f(y))$$

Cross-multiplying, putting the values of x, y and from the definition of GM and HM,

$$\iff 1/f(be^{-(r+s)/2}) \geq (\leq) \tfrac{1}{2}(1/f(be^{-r}) + 1/f(be^{-s})),$$

hence the result follows.

7) Let *g(x) = f(1/x)*, and let *x,y* $\in$ *(1/b,∞)*, so that *1/x,1/y* $\in (0,b)$. Then *f is HA-convex(concave) on (0,b) if and only if,*

$$\iff f(\tfrac{2}{x+y}) \leq (\geq) \tfrac{1}{2}(f(1/x) + f(1/y))$$

Substituting g for f,

$$\iff g(\tfrac{x+y}{2}) \leq (\geq) \tfrac{1}{2}(g(x) + g(y))$$

hence the result follows.

8) Let *g(x) = log(f(1/x))*, and let *x,y* $\in$ *(1/b,∞)*, so that *1/x,1/y* $\in (0,b)$. Then *f is HG-convex(concave) on (0,b) if and only if,*

$$f(\tfrac{2}{x+y}) \leq (\geq) \sqrt{f(1/x)f(1/y)}$$

Taking log (monotonicity is preserved),

$$\iff log(f(\tfrac{2}{x+y})) \leq (\geq) \tfrac{1}{2}(log(f(1/x)) + log(f(1/y)))$$

Substituting g for f,

$$\iff g(\tfrac{x+y}{2}) \le (\ge)\tfrac{1}{2}(g(x)+g(y))$$

hence the result follows.

9) Let *g(x) = 1/f(1/x)*, and let *x,y* $\in$ *(1/b,∞)*, so that *1/x,1/y* $\in (0,b)$. Then *f is HA-convex(concave) on (0,b) if and only if,*

$$f(\tfrac{2}{x+y}) \le (\ge)2/1/f(1/x)+1/f(1/y)$$

Cross-multiplying,

$$\iff 1/f(\tfrac{2}{x+y}) \ge (\le)\tfrac{1}{2}(1/f(1/x)+1/f(1/y))$$

$$\iff g(\tfrac{x+y}{2}) \ge (\le)\tfrac{1}{2}(g(x)+g(y))$$

hence the result follows.

The following results are direct consequences of above Theorem:

(Just take derivates of *f(x)* assuming convexity(concavity))

Corollary 1: Let I be an open subinterval of $(0,\infty$ and let $f: I \to (0,\infty)$ be differentiable. In parts (4-9), *I = (0,b), 0 < b < ∞*. We have the following:

a) *f is AA-convex(concave) if and only if $f\prime(x)$ is increasing(decreasing).*

b) *f is AG-convex(concave) if and only if $f\prime(x)/f(x)$ is increasing(decreasing).*

c) *f is AH-convex(concave) if and only if $f\prime(x)/f(x)^2$ is increasing(decreasing).*

d) *f is GA-convex(concave) on I if and only if $xf\prime(x)$ is increasing(decreasing) on (0,∞).*

e) *f is GG-convex(concave) on I if and only if $xf\prime(x)$/f(x) is increasing(decreasing) on (0,∞).*

f) *f is GH-convex(concave) on I if and only if $xf\prime(x)/f(x)^2$ is increasing(decreasing) on (0,∞).*

g) *f is HA-convex(concave) on I if and only if $x^2f\prime(x)$ is increasing(decreasing) on (1/b,∞).*

h) *f is HG-convex(concave) on I if and only if $x^2f\prime(x)/f(x)$ is increasing(decreasing) on (1/b,∞).*

i) *f is HH-convex(concave) on I if and only if $x^2f\prime(x)/f(x)^2$ is increasing(decreasing) on (1/b,∞).*

Notations: If $f(x)=\sum_{n=0}^{\infty} a_n x^n$ *and* $g(x)=\sum_{n=0}^{\infty} b_n x^n$ are two power series, where $b_n > 0$ for all n. $T_n = T_n(f(x),g(x)) = a_n/b_n$. Also F = *F(a,b;c;x)* is used to denote the *Gaussian Hypergeometric function*

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \tfrac{(a,n)(b,n)}{(c,n)n!} x^n, |x|<1,$$

where (a,n) denotes the product *a(a+1)(a+2).....(a+n-1)* where $n \ge 1$, and (a,0) = 1 if a $\neq$ 0. The expression (0,0) is not defined. For x $\in (0,1)$,

$x\prime isusedtodenote\sqrt{1-x^2}$.

Theorem 2: *Let f(x)* $= \sum_{n=0}^{\infty} a_n x^n$, *where* $a_n > 0$ *for n = 0,1,2... be convergent on (-R,R),* $0 < R < \infty$. *Then the following results hold.*

1) *f is AA-convex and GG-convex on (0,R).*
2) *If the sequence* $(n+1)a_{n+1}/a_n$ *is increasing(decreasing), then f is AG-convex(concave) on (0,R). In particular,*

$$f(\tfrac{x+y}{2}) \leq (\geq)\sqrt{f(x)f(y)}$$

for all x,y $\in$ *(0,R), with equaltiy if and only if x=y.*
3) *Let* $b_n = \sum_{k=0}^{n} a_k a_{n-k}$. *If the sequence* $(n+1)a_{n+1}/b_n$ *is increasing(decreasing), then f is AH-convex(concave) on (0,R).*
4) *Let* $b_n = \sum_{k=0}^{n} a_k a_{n-k}$. *If the sequence* na_n/b_n *is increasing(decreasing), then f is GH-convex(concave) on (0,R).*
5) *If the sequence* $R(n+1)a_{n+1}/a_n - n$ *is increasing(decreasing), then the function* $(R-x)f\prime(x)/f(x)$ *is increasing(decreasing) on (0,R), so that the function* $log(f(R(1-e^{-t})))$ *is convex(concave) on* $(0,\infty)$. *In partcular,*

$$f(R-\sqrt{(R-x)(R-y)}) \leq (\geq)\sqrt{f(x)f(y)}$$

for all x,y $\in$ *(0,R), with equaltiy if and only if x=y.*
6) *If the sequence* na_nR^n *is increasing(decreasing), then the function* $(R-x)f\prime(x)$ *is increasing(decreasing) on (0,R), so that the function* $f(R(1-e^{-t}))$ *is convex(concave) as a function of t on (0,*∞*).*
In particular,

$$f(R-\sqrt{(R-x)(R-y)}) \leq (\geq)\tfrac{f(x)+f(y)}{2}$$

for all x,y $\in$ *(0,R), with equaltiy if and only if x=y.*
7) *If the sequence* na_nR^n *is increasing and if also the sequence* $n!a_nR^n/(1/2,n)$ *is decreasing, then the function 1/f(x) is concave on (0,R). In particular,*

$$f(\tfrac{x+y}{2}) \leq \tfrac{2f(x)f(y)}{f(x)+f(y)}$$

for all x,y $\in$ *(0,R), with equaltiy if and only if x=y.*

Proof: 1) Since all $a_n > \mathbf{0}$, any $f\prime^n$ is increasing function (i.e, any n^{th} derivative of f is incresing).

2) We have $T_n(f\prime(x), f(x)) = (n+1)a_{n+1}/a_n$, which is increasing(decreasing). By **Corollary 1(b) and Lemma 1** , the assertion follows.
3) Since $T_n(f\prime(x), f(x)^2) = (n+1)a_{n+1}/b_n$. By **Corollary 1(c) and Lemma 1** , the assertion follows.
4) Since $T_n(xf\prime(x), f(x)^2) = na_n/b_n$. By **Corollary 1(f) and Lemma 1** , the assertion follows.
5) Taking derivative w.r.t t

$$\tfrac{d}{dt}log(f(R(1-e^{-t}))) = Re^{-t}\tfrac{f\prime(R(1-e^{-t}))}{f(R(1-e^{-t}))} = (R-x)\tfrac{f\prime(x)}{f(x)},$$

where $x = R(1 - e^{-t})$. Then

$$T_n((R-x)f\prime(x), f(x)) = R(n+1)a_{n+1}/a_n - n,$$

which is increasing(decreasing), so the assertion follows from **Lemma 1**.

6) Taking derivative w.r.t t

$\frac{d}{dt}f(R(1-e^{-t})) = Re^{-t}f\prime(R(1-e^{-t})) = (R-x)f\prime(x) = \frac{f\prime(x)}{1/(R-x)}$,

where $x = R(1 - e^{-t})$. Then

$$T_n(f\prime(x), 1/(R-x)) = (n+1)a_{n+1}R^{n+1},$$

which is increasing(decreasing), so the assertion follows from **Lemma 1**.

7) First,

$$\frac{d}{dx}\frac{1}{f(x)} = \frac{-f\prime(x)}{f(x)^2}$$

Now $(R-x)f\prime(x) = \frac{f\prime(x)}{1/(R-x)}$, so that

$$T_n(f\prime(x), 1/(R-x)) = (n+1)a_{n+1}R^{n+1},$$

which is increasing by hypothesis. Hence, $(R-x)f\prime(x)$ is increasing on (0,R), by **Lemma 1**.
Since

$$\sqrt{R-x}f(x) = \frac{f(x)}{(R-x)^{-1/2}},$$

we have

$$T_n(f(x), (R-x)^{-1/2}) = \frac{n!a_nR^{n+1/2}}{(1/2,n)},$$

which is decreasing by hypothesis. Hence, $\sqrt{R-x}f(x)$ is also decreasing on (0,R), by **Lemma 1**. Dividing $(R-x)f\prime(x)$ by square of $\sqrt{R-x}f(x)$, we see that $\frac{d}{dx}\frac{1}{f(x)}$ is decreasing in x on (0,R), proving the assertion.

References:

1) J. Aczl, A generalization of the notion of convex functions, Norske Vid. Selsk. Forhd., Trondhjem 19 (24) (1947) 8790.
2) R. Balasubramanian, S. Ponnusamy, M. Vuorinen, Functional inequalities for the quotients of hypergeometric functions, J. Math. Anal. Appl. 218 (1998) 256268.
3) Baricz, Functional inequalities involving special functions, J. Math. Anal. Appl. 319 (2006) 450459.
4) P.S. Bullen, D.S. Mitrinovic, P.M. Vasic, Means and Their Inequalities, Reidel, Dordrecht, 1988.
5) C.P. Niculescu, Convexity according to means, Math. Inequal. Appl. 6 (2003) 571579.
6) E. Rainville, Special Functions, Macmillan, New York, 1960.
7) A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series, vol. 3: More Special Functions, Gordon and Breach, New York, 1988 (trans. from the Russian by G.G. Gould), Errata: Math. Comp. 65 (1996) 13801384.