Part 1: Analytic Assignment

- Compute the gradient vector for a plane in 3d space.

$$z = f(x,y) = ax + by + c$$

$$\nabla f = \nabla(ax + by + c) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

- Compute the gradient vector for a hyperplane

$$z = f(\mathbf{x}) = f(x_1, x_2, \dots, x_N) = \sum_{i=1}^{N} a_i(x_i - b_i) + S = a_1x_1 + a_2x_2 + \dots + a_Nx_N + d$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_N} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

- Compute the partial derivative of the paraboloid function

$$z = f(x,y) = A(x - x_0)^2 + B(y - y_0)^2 + C$$
$$= A(x^2 - 2xx_0 + x_0^2) + B(y^2 - 2yy_0 + y_0^2) + C$$
$$= Ax^2 - 2Axx_0 + Ax_0^2 + By^2 - 2Byy_0 + By_0^2 + C$$

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(Ax^2 - 2Axx_0 + Ax_0^2 + By^2 - 2Byy_0 + By_0^2 + C)$$

$$= 2Ax - 2Ax_0$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(Ax^2 - 2Axx_0 + Ax_0^2 + By^2 - 2Byy_0 + By_0^2 + C$$

$$= 2By - 2By_0$$

- Compute quantities & specify shape

$x^T = [3 \ 1 \ 4]$ shape: $[1 \times 3]$ $\quad y^T = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$ shape $[3 \times 1]$

$B^T = \begin{bmatrix} 3 & 5 & 1 \\ 5 & 2 & 4 \end{bmatrix}$ shape $[2 \times 3]$

$\underset{3\times1}{x} \cdot \underset{3\times1}{y}$ → not defined $\quad \underset{[3\times1]}{x} \cdot \underset{[3\times1]}{y^T}$ → not defined

$\underset{[3\times1]}{x} \times \underset{[1\times3]}{y} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} [2 \ 5 \ 1] = \begin{bmatrix} 3\times2 & 3\times5 & 3\times1 \\ 1\times2 & 1\times5 & 1\times1 \\ 4\times2 & 4\times5 & 4\times1 \end{bmatrix} = \begin{bmatrix} 6 & 15 & 3 \\ 2 & 5 & 1 \\ 8 & 20 & 4 \end{bmatrix}$

shape $[3 \times 3]$

$\underset{[1\times3]}{y} \times \underset{[3\times1]}{x} = [2 \ 5 \ 1] \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = [2\times3 + 5\times1 + 1\times4] = [6+5+4] = [15]$

shape $[1 \times 1]$

$\underset{[3\times3]}{A} \times \underset{[3\times1]}{x} = \begin{bmatrix} 4 & 5 & 2 \\ 3 & 1 & 5 \\ 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4\times3 + 5\times1 + 2\times4 \\ 3\times3 + 1\times1 + 5\times4 \\ 6\times3 + 4\times1 + 3\times4 \end{bmatrix} = \begin{bmatrix} 12+5+8 \\ 9+1+20 \\ 18+4+12 \end{bmatrix} = \begin{bmatrix} 25 \\ 30 \\ 34 \end{bmatrix}$ shape $[3 \times 1]$

$\underset{[3\times3]}{A} \times \underset{[3\times2]}{B} = \begin{bmatrix} 4 & 5 & 2 \\ 3 & 1 & 5 \\ 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4\times3+5\times5+2\times1 & 4\times5+5\times2+2\times4 \\ 3\times3+1\times5+5\times1 & 3\times5+1\times2+5\times4 \\ 6\times3+4\times5+3\times1 & 6\times5+4\times2+3\times4 \end{bmatrix}$

$= \begin{bmatrix} 12+25+2 & 20+10+8 \\ 9+5+5 & 15+2+20 \\ 18+20+3 & 30+8+12 \end{bmatrix} = \begin{bmatrix} 39 & 38 \\ 19 & 37 \\ 41 & 50 \end{bmatrix}$

shape $[3 \times 2]$

B.reshape(1,6) $= [3 \ 5 \ 5 \ 2 \ 1 \ 4]$ shape $[1 \times 6]$

- Use Calculus to derive expression for single variable linear reg

$y = mx + b$ $\quad L(m\ b) = \sum_{i=1}^{N} (\hat{y}_i - M(\hat{x}_i, m, b))^2$

$L(m, b) = \sum_{i=1}^{N} (\hat{y}_i - m\hat{x}_i - b)^2$

$= \dfrac{cov(x, y)}{var(x)} \rightarrow b = \bar{y} - \dfrac{cov(x,y)}{var(x)} \bar{x}$

$$L(m, b) = \sum_{i=1}^{N} (\hat{y}_i - m\hat{x}_i - b)^2$$

$$\Rightarrow \frac{\partial L}{\partial b} = 2\sum_{i=1}^{N} (-\hat{y}_i + m\hat{x}_i + b) = 2\left(-\sum_{i=1}^{n} \hat{y}_i + m\sum_{i=1}^{n} \hat{x}_i + n \cdot b\right) = 0$$

$$\Rightarrow m\sum_{i=1}^{n} \hat{x}_i - \sum_{i=1}^{N} \hat{y}_i + bn = 0$$

$$\Rightarrow bn = \sum_{i=1}^{N} \hat{y}_i - m\sum_{i=1}^{n} \hat{x}_i$$

$$\Rightarrow b = \overline{y} - m\overline{x}$$

$$\frac{\partial L}{\partial m} = \sum_{i=1}^{N} 2\hat{x}_i(-\hat{y}_i + m\hat{x}_i + b) = \sum_{i=1}^{N} 2(-\hat{x}_i\hat{y}_i + m\hat{x}_i^2 + b\hat{x}_i) = 0$$

$$\Rightarrow \sum_{i=1}^{N} 2(-\hat{x}_i\hat{y}_i + m\hat{x}_i^2 + (\overline{y} - m\overline{x})\hat{x}_i) = 0$$

$$\Rightarrow \sum_{i=1}^{N} (-\hat{x}_i\hat{y}_i + m\hat{x}_i^2 + \overline{y}\hat{x}_i - m\overline{x}\hat{x}_i = 0$$

$$\Rightarrow \sum_{i=1}^{N} (\overline{y}\hat{x}_i - \hat{y}_i\hat{x}_i) + \sum_{i=1}^{N} (m\hat{x}_i^2 - m\overline{x}\hat{x}_i) = 0$$

$$\Rightarrow \sum_{i=1}^{N} (\overline{y}\hat{x}_i - \hat{y}_i\hat{x}_i) + m\sum_{i=1}^{N} (\hat{x}_i^2 - \overline{x}\hat{x}_i) = 0$$

$$m\sum_{i=1}^{N} (\hat{x}_i^2 - \overline{x}\hat{x}_i) = \sum_{i=1}^{N} (\hat{y}_i\hat{x}_i - \overline{y}\hat{x}_i)$$

$$\Rightarrow m = \frac{\sum_{i=1}^{N} (\hat{y}_i\hat{x}_i - \overline{y}\hat{x}_i)}{\sum_{i=1}^{N} (\hat{x}_i^2 - \overline{x}\hat{x}_i)} = \frac{\sum_{i=1}^{N} (\hat{x}_i\hat{y}_i) - n\overline{y}\,\overline{x}}{\sum_{i=1}^{N} (\hat{x}_i^2) - n\overline{x}^2}$$

$$\sum_{i=1}^{n} (\overline{x}^2 + \hat{x}_i\overline{x}) = 0 \quad \& \quad \sum_{i=1}^{N} (\overline{x}\overline{y} - \hat{y}_i\overline{x}) = 0$$

$$\Rightarrow m = \frac{\sum_{i=1}^{n} (\hat{x}_i\hat{y}_i - \hat{x}_i\overline{y}) + \sum_{i=1}^{n} (\overline{x}\overline{y} - \hat{y}_i\overline{x})}{\sum_{i=1}^{n} (\hat{x}_i^2 - \hat{x}_i\overline{x}) + \sum_{i=1}^{N} (\overline{x}^2 - \overline{x}\hat{x}_i)} = \frac{\frac{1}{n}\sum_{i=1}^{n} (\hat{x}_i - \overline{x})(\hat{y}_i - \overline{y})}{\frac{1}{n}\sum_{i=1}^{n} (\hat{x}_i - \overline{x})^2}$$