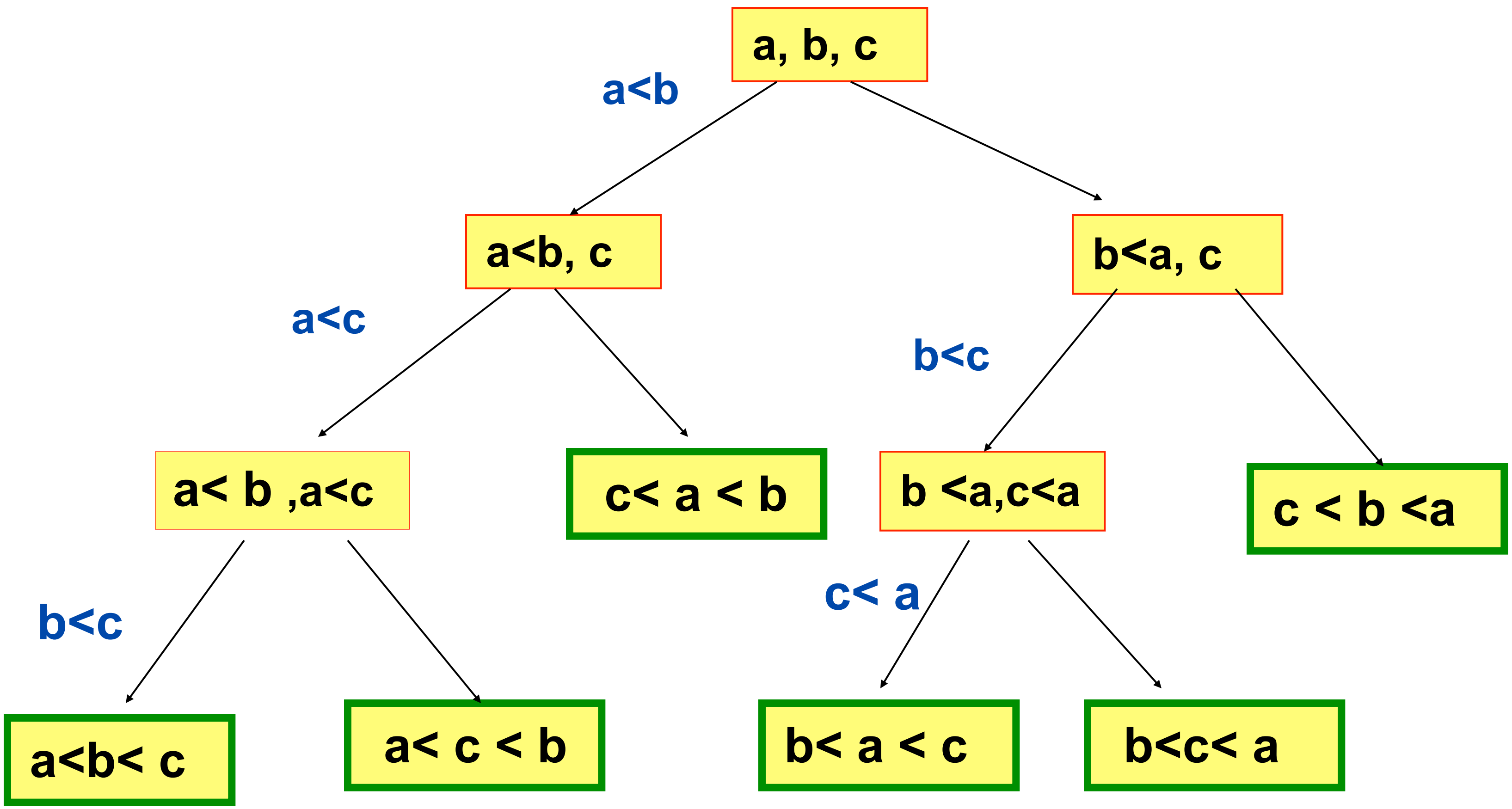
Searching and Sorting

- Searching
  - Linear O(N)
  - bisection O(log(N))
  - dictionary O(log(log(N
- Sorting
  - Insertion, bubble, selection $O(N^2)$ CRLS: 2.1
  - Merge, Quick, Heap O(N log (N)) CRLS: 2.2,
    Bin (Count), Radix, Bucket O(N) CLRS: 8
- Proof: $\Omega(N^2)$ near neighbor exchange
- Proof: $\Omega(N \log(N))$ Comparison search
- Median (or k quick selection) Problem CLRS: 9

Decision Tree

Proof of $\Omega(N\log(N))$



Binary decisions: 3! = 6 possible outcomes. Longest path: log(3!)

Radix Sort (IBM Card Sorter!)

- Represent integers in a[i] in base B: $n_0 + n_1 B + n_2 B^2 + \dots + n_p B^P$
- Sort into buckets by low digits first: n_0, then n_1, etc.

Queues: B= 10 Example: 64, 8, 216, 512, 27, 729, 0, 1, 343, 125

Bucket:	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
#1	0	1	512	343	64	125	216	27	8	729
	8 1 0	216 512	729 27 125		343		64			
#3	64 27 8 1 0	125	216	343		512		729		

$O(N\,P)$ where $B^P = N$ or $P = \log(N) / \log(B) = O(1)$

Searching: *"Why Sort at All?"*

- int a[0], a[1],a[2],a[3],.... a[m],.... a[2],a[N-1]

Three Algorithms:

- *Linear Search* ➔ O(N)
  (after Sorting)
- *Bisection Search* ➔ O(log(N)).
- *Dictionary Search* ➔ O(log[log[N]])

Bisection Search of Sorted List

- int a[0], a[1],a[2],a[3],.... a[m],.... a[N-2],a[N-1]

i j

```
i= 0; j= N-1; m = N/2
 while(b!=a[m] && i!=j ){
           if(b>a[m])    i = m+1;
           if(b<a[m])    j = m-1;
            m =   (j-i)/2 + i;}
if(b==a[m])) “found it” else “not found”
```

Choose mid point

$T(N) = T(N/2) + c_0 \quad \rightarrow \quad T(N)$ » $Log(N)$

Dictionary: Sorted and Uniform

- int a[0], a[1],a[2],a[3],.... a[m],.... a[2],a[N-1]

i j

Dictionary: Same code EXCEPT

estimate location of b

x = fractional distance $(0<x<1)$

x = (b-a[i])/(a[j] – a[i]) ;

m = x (j-i) + i ;

m

T(N) = T($N^{1/2}$) + c_0 ➔ T(N) » Log(Log(N))

$$N \rightarrow N^{\frac{1}{2}} \rightarrow N^{\frac{1}{4}} \rightarrow N^{\frac{1}{8}} \cdots \rightarrow N^{\frac{1}{2^n}} = 1 \quad \text{or} \quad n = log_2(log_2(N))$$

- *Extra Knowledge Helps: % Error » 1/$N^{1/2}$*

Insertion Sort --- *Deck of Cards*

- Insertion Sort(a[0:N-1]):
  for (i=1; i < n; i ++)
  	for (j = i; (j>0) && (a[j]<a[j-1]); j--)
  		swap a[j] and a[j-1] ;

Worst case $\Theta(N^2)$ number of "swaps" (i.e. time)

Outer loop trace for Insertion Sort

- a[0] a[1] a[2] a[3] a[4] a[5] a[6] a[7]
  (Swaps)
- 6 | 5 2 8 3 4 7 1 (1)
  5 ← → 6
- 5 6 | 2 8 3 4 7 1
  (2)
  2 ← → 6
  2 ← → 5
- 2 5 6 | 8 3 4 7 1 (0)
- 2 5 6 8 | 3 4 7 1 (3)
- 2 3 5 6 8 | 4 7 1 (3)
- 2 3 4 5 6 8 | 7 1 (1)
- 2 3 4 5 6 7 8 | 1 (7)

Bubble Sort --- *Sweep R to L*

- *<u>Bubble Sort(a[0:N-1])</u>:*
  for i=0 to n-1
  for j = n-1 to i + 1
  if a[j]<a[j-1] then
  swap a[i] and a[j]

Worst case $\Theta(N^2)$ swaps (time)

Outer loop trace for Bubble Sort

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	(Swaps)
6	5	2	8	3	4	7	1	(7)
1 |	6	5	2	8	3	4	7	(3)
1	2 |	6	5	3	8	4	7	(3)
1	2	3 |	6	5	4	8	7	(3)
1	2	3	4 |	6	5	7	8	(1)
1	2	3	4	5 |	6	7	8	(0)
1	2	3	4	5	6 |	7	8	(0)
1	2	3	4	5	6	7	8	(17)

- NOTE SAME # OF SWAPS? WHY?

Average Number of N(N-1)/4 swaps

- **Best Case:** sorted order → 0 swaps
- **Worst Case:** reverse order → N(N-1)/2 swaps

  since 1 + 2 + ...+N-1 = N(N-1)/2
- **Average Case:** Pair up each of the N! permutations with its reverse order → Every pair must swap in one or the other: Thus average is half of all swaps → (1/2) N(N-1)/2 q.e.d.

Selection Sort --- (Bubble only the index)

- *<u>Selection Sort(a[0:N-1])</u>:*

for *i=1 to n-2*
    { min = i
        for *j = n-1 to i + 1*
                if *a[j]<a[min]* ***then***
                        min = j;
            swap a[i] and a[min];
        }

worst case $\Theta(N)$ swaps + $\Theta(N^2)$ comparisons

Outer loop trace for Selection Sort

- a[0] a[1] a[2] a[3] a[4] a[5] a[6] a[7]

(Swaps)

- 6 5 2 8 3 4 7 1

(1)

1 ← → 6

- 1 5 2 8 3 4 7 6

(1)

- 2 ↔ 5
- 1 2 | 5 8 3 4 7 6

(1)

- 1 2 3 | 8 5 4 7 6

(1)

- 1 2 3 4 | 5 8 7 6

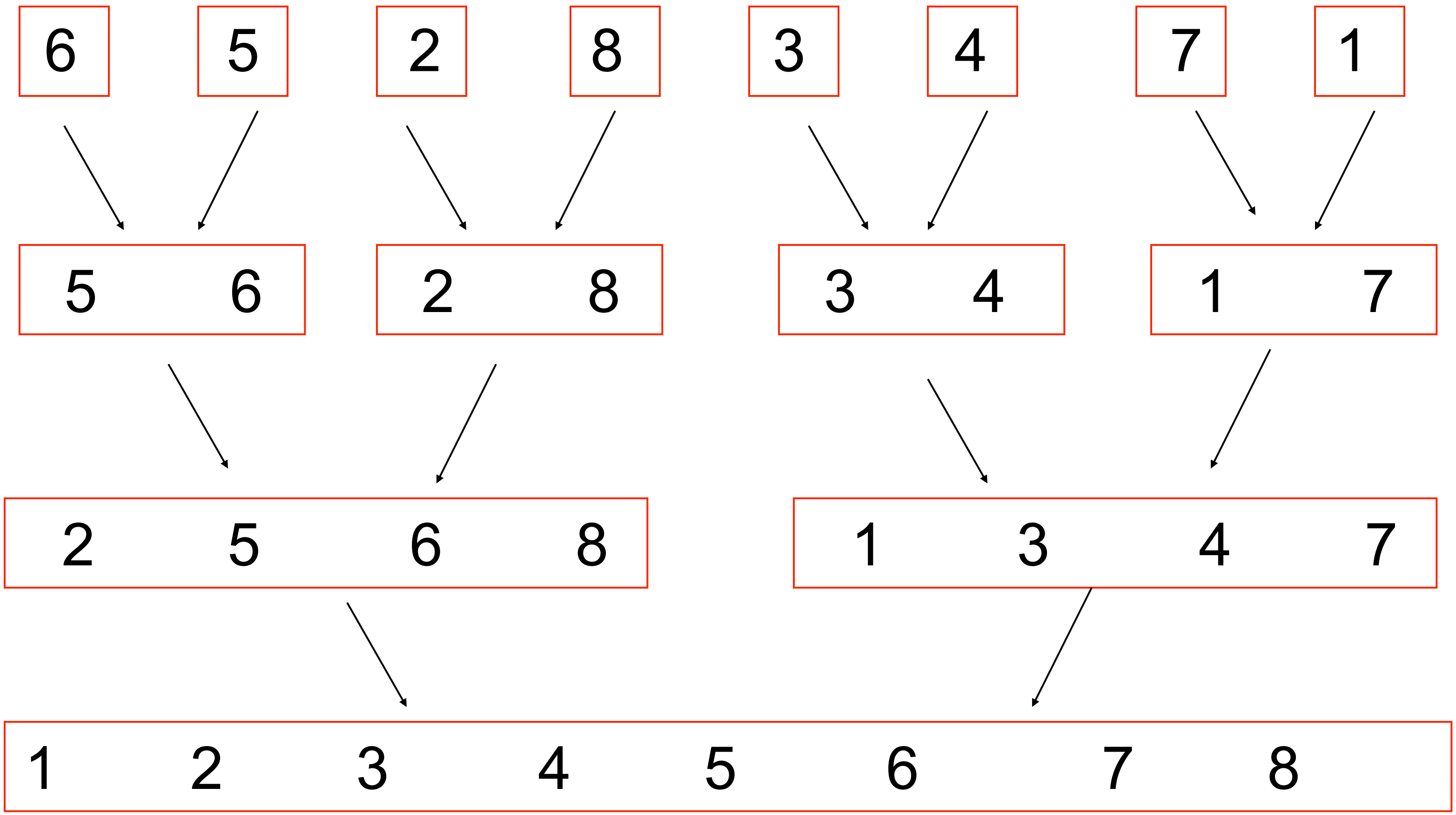
(0)

- 1 2 3 4 5 | 6 7 8

Merge Sort: Worst Case $\Theta(N\log(N))$

```
void mergesort(int a[ ], int l, int r)
  if (r > l) {
    m = (r+l)/2;
    mergesort(a, l, m);
    mergesort(a, m+1, r);
    for (i = l; i < m+1; I++)  b[i] = a[i];
    for (j = m; j < r; j++)    b[r+m-j] = a[j+1];   // reverse
    for (k = l; k <= r; k++)
            a[k] = (b[i] < b[j]) ? b[i++] : b[j--]; }
```

Outer loop trace for Merge Sort

- a[0] a[1] a[2] a[3] a[4] a[5] a[6] a[7]
- 

Lower Bound Theorem for Camparision Sort

Proof: Compute the maximum depth D of decision tree?

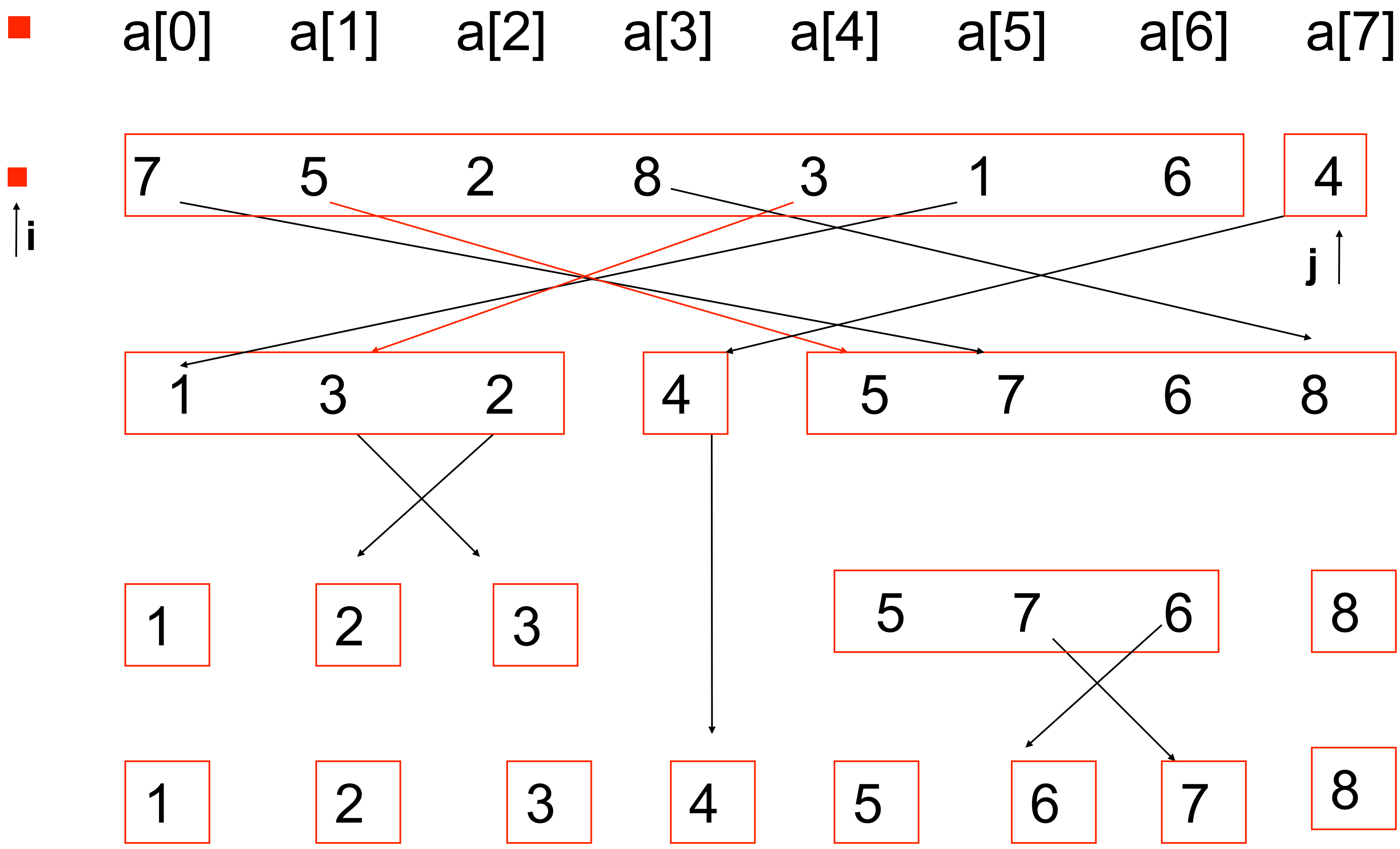
- Need N! leaves to get all possible outcomes of a sorting routine.
- Each level at most doubles: $1 ® 2 ® 4 ® 8 ® \bullet\bullet\bullet ® 2^D$
- Consequently for D levels: $N! \leq 2^D \Rightarrow D \geq log_2(N!)$

$$\Rightarrow T(N) = \Omega(D) = \Omega(\log_2(N!)) = \Omega(N \log_2(N))$$

$$Information = \log_2(N!) \cong N \log_2(N)$$

Number of bits to encode any (initial) state is information (- Entropy)

Outer loop trace for Quick Sort (i moves before j)



Quick Sort: original Hoare partition

```
void quicksort(int a[ ], int l, int r)
  if (r > l){
      v = a[r]; i = l-1; j = r;
      for (;;){  while (a[++i] < v);       // move first i to right
                 while (a[--j] > v) ;      // then mover j left
                  if (i >= j) break  ;
                 swap(&a[i],  &a[j]); }
       swap(&a[i], &a[r]);                 // move pivot in to center
       quicksort(a, l, i-1);
       quicksort(a, i+1, r);
    }
```

Quick Sort: My Implementation

```
void quicksort(int a[], int l, int r)
 {  int i, j, vl
   if (r > l)
    {   v = a[r]; i = l-1; j = r;
      for (;;)
       {   while (a[++i] < v && i >= j ) ;  // Move i first from left
          while (a[--j] >= v && i >= j) ;  // Move j second from right
          if (i >= j) break;
          swap(&a[i], &a[j]);   } //  swap root to divide left and right sublist
      swap(&a[i], &a[r]);
      quicksort(a, l, i-1);     quicksort(a, i+1, r);
    }  }
```

Quick Sort: CLRS:7 Implementation

```
void quicksort(int a[ ], int l, int r)
  if (r > l){
      v = a[r]; i = l-1;
      for (int j= l ; j<r ; j++){
          if (a[j] <= v) {
               i +=1 ;                          // move first i to right
               swap(&a[i],  &a[j])  }
        swap(&a[i], &a[r]);                     // move pivot in to center
        quicksort(a, l, i-1);
        quicksort(a, i+1, r);
    }
```

Very cute: See CLRS page 172 Figure 7.1

- Worst Case (choose smallest as pivot!):
  - T(N) = T(N-1) + c N ➔ T(N) =O(N^2)
- Best Case:(choose median as pivot)
  - T(N) = 2 T(N/2) + cN ➔ T(N) = O(N log(N))
- Average Case:
  - T(N) = 2[T(0) + T(1) + …+ T(N-1)]/N + c N
    - ➔ T(N) = O(N log (N))
    - ➔ Using Calculus if you are lazy! (x = N)

$$xT(x) \simeq 2\int_0^x dy T(y) + cx^2 \quad x\frac{dT(x)}{dx} = T(x) + 2cx \Rightarrow T(x) = 2cx\log(x)$$

Can do better-- Worst Case O(N) !

- *Approximate Media Selector for pivot v:*
  - *Partition in 5 rows of N/5*
  - *Sort each column*
  - *Find (Exact) Medium of Middle list!*

- *Result there are (3/5)(1/2)N elements smaller than pivot v*

- *K-th find is O(N) --- Double recursion!*
  - *Sort of N/5 col O(N)*
  - *Find media of T(N/5)*
  - *Find k-th in T(7N/10) at worst*

- $T(N) < C0 * (N/5) + T(N/5) + T(7N/10)$
  - *Try solution:* $T = C\ N$
  - $C(N - N/5 - 7N/10) = C\ N/10 < C0\ N/5$
  - $C < 2\ C0$

Median Finding: Quick Select

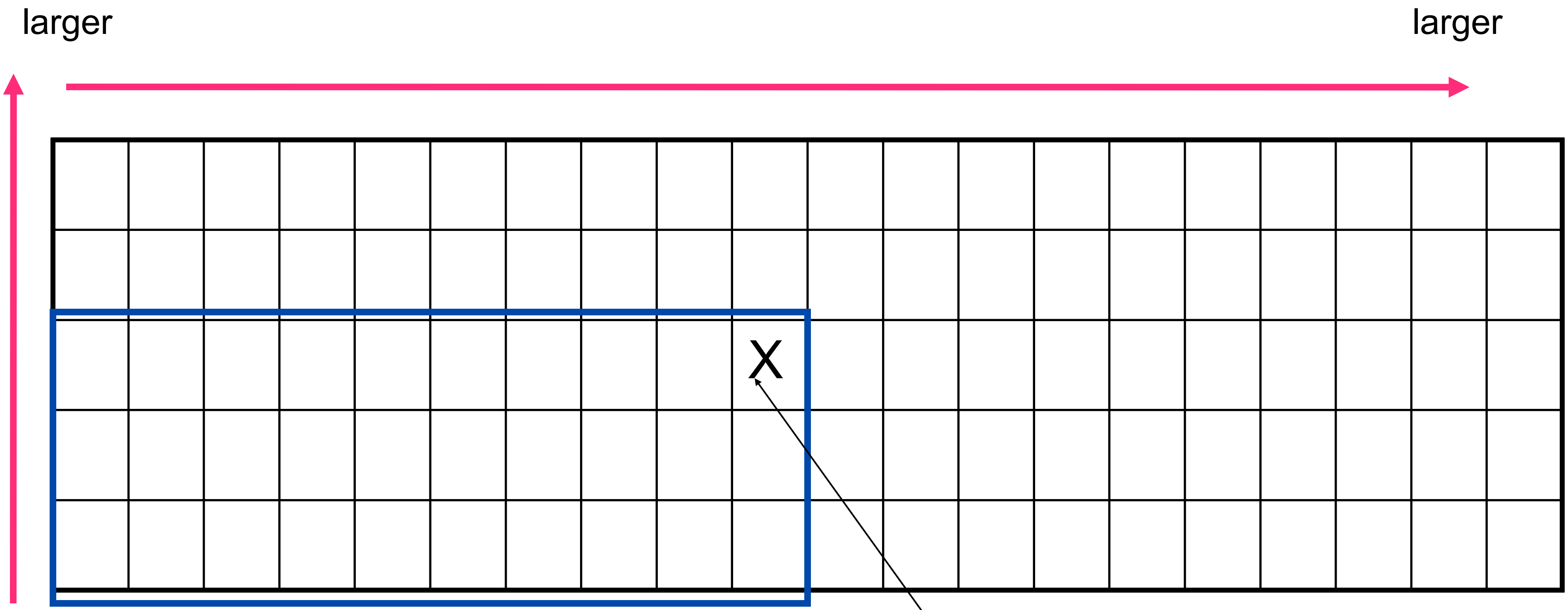
- Median is the element a[m] so that half is less/equal
- Generalize to finding k-th smallest in set S
- Quick(S,k): |S| = size of S
  - If |S| = 1, the k = 1 in S
  - Pick pivot v 2 S & Partition S – {v} into S_L & S_H
    - If k< $|S_L|$ + 1 then k-th 2 S_L: Quick(S_L,k)
    - If k = $|S_L|$ + 1 k-th is v : exit
    - If k > $|S_L|$ + 1 then k-th 2 S_R: Quick(S_R,k- $|S_L|$-1)

Now: T(N) = O(N) is average performance

T(N) = [T(0) + T(1) + … T(N-1)]/N + c N

$$xT(x) \simeq \int_0^x dy T(y) + cx^2 \quad x\frac{dT(x)}{dx} = 2cx \Rightarrow T(x) = 2cx$$

5 row of N/5 Columns



Exact Medium of Middle Row

3/5 * 1/2 N Boxes smaller than X