EC 504 – Fall 2019 – Homework 4

Due Thursday, Nov 14, 2019 in the beginning of class. Coding problems submitted in the directory `/projectnb/alg504/yourname/HW4` on your SCC account by Friday Nov 15, 11:59PM.

Reading Assignment: CLRS Chapters 22, 23, 24 and appendix B.4

1. Consider the directed graph shown below in Fig. 1

   (a) (10 pts) Perform breadth first search (BFS), starting from node 1 and draw the tree on the figure and label each node by the depth of the node, $d(i)$, and its parent (or predecessor), $p(i)$ for $i = 1, 2, \cdots |E|$. Give a list of the order in which the nodes leave the queue. **Assume that arcs out of a node will be examined in order of increasing end node; i.e. break ties in order of arc selection by examining the arc with the smallest value of end node first.**

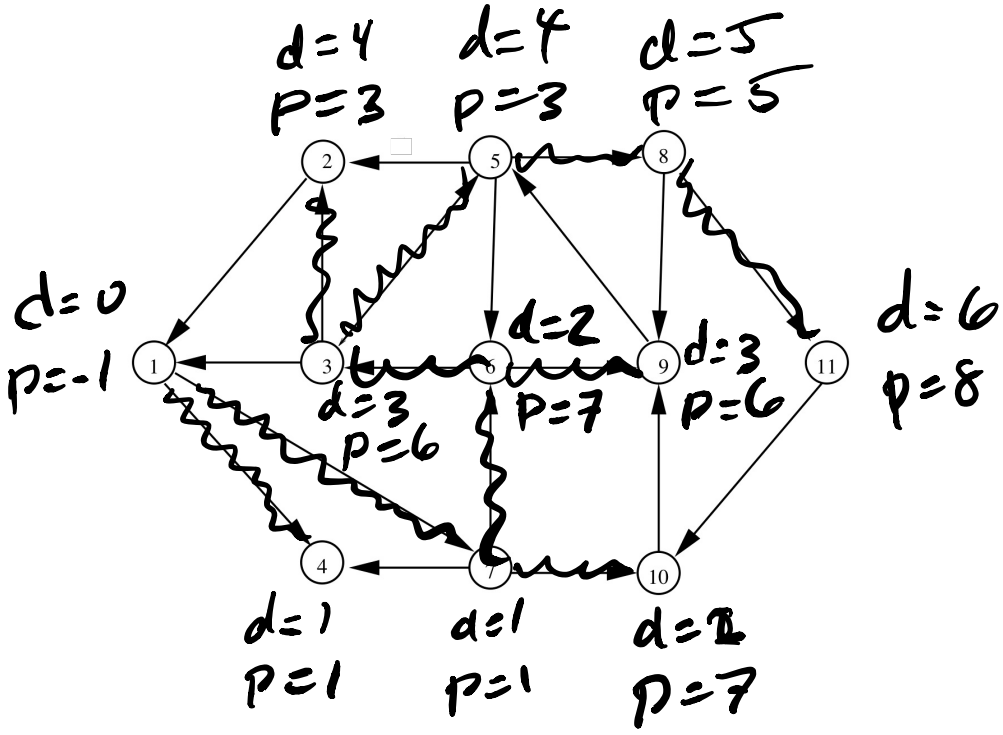


Figure 1

Order Exit Queue:

1 4 7 6 10 3 9 2 5 8 11

(b) (10 pts) For the same directed graph in Fig. 1a , use depth first search (DFS), starting from node 1. **Give two lists for the order in which nodes entered into the stack and the order they leave the stack**. (An extra figure is appended to this Homework below.) Draw the stack in enough detail to understand its function. **Again assume that arcs out of a node will be examined in order of increasing end node; i.e. break ties in order of arc selection by examining the arc with the smallest value of end node first.** Is the reverse order a topological sort? (Explain)

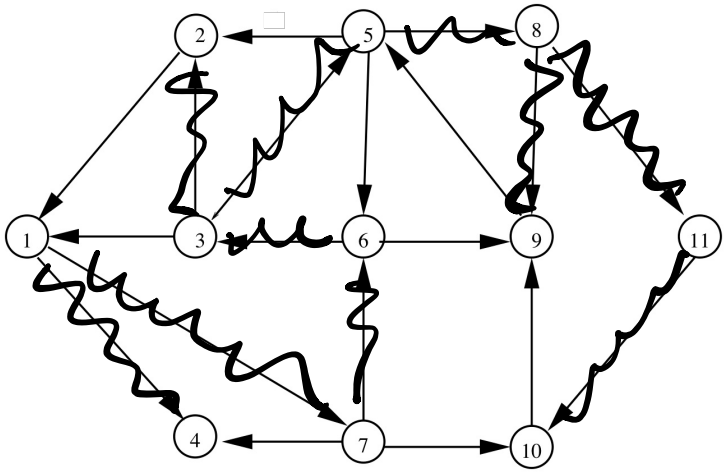


Figure 1a

push: 1 4 7 6 2 3 5 8 9 11 10

Pop: 4 2 9 10 11 8 5 3 6 7 1

The reverse order is NOT a topological sort because it is not a DAG: Has cycles!

2. (10 pts) Compute a minimum spanning tree for Fig. 2 below, and **evaluate the weight** of that spanning tree using Kruskal's algorithm. Draw the tree and list in order the edges that are added to arrive at the tree. (HINT: You may check this by using Prim's algorithm, which will get the same MST but of course the order of adding edges is very different.)

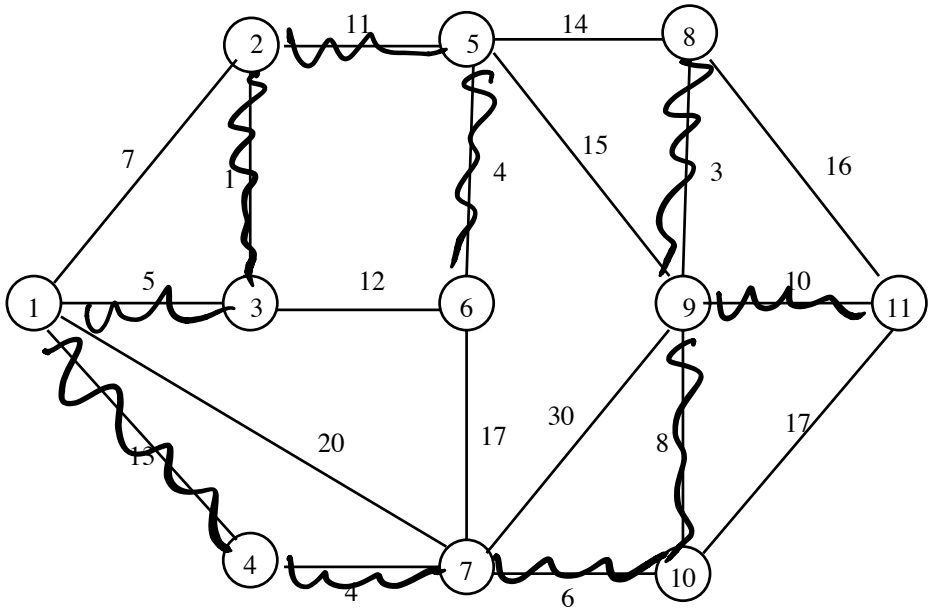


Figure 2:

Order of Edges added

1 3 4 4 5 6 8 10 11 13

either order

7 skipped 12 skipped

MST weight

= 1+3+4+4+5+6+8+10+11+13

= 65

3. (20 pts) This is a forward star representation for a directed graph with $|V| = 11$ vertices and $|E| = 16$ edges.

```
Vertex Number:   1   2   3   4   5   6   7   8   9  10  11  12
Array First:   { 1,  3,  4,  5,  7,  8, 12, 12, 14, 14, 15, 17 }

Edge Number:  1   2   3   4   5   6   7   8   9  10  11  12  13  14  15  16  17
Array Edge: { 2,  6,  6,  7,  3,  7,  8,  5,  8,  9, 10,  9, 11,  9,  9, 10, -1 }
```

(a) Draw the graph on the template in Fig. 3. (HINT: You may want to do part b first.)

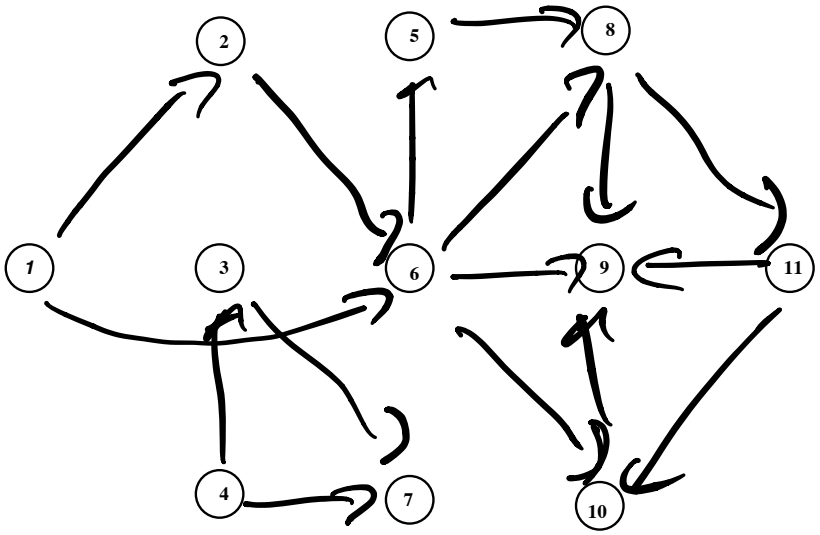
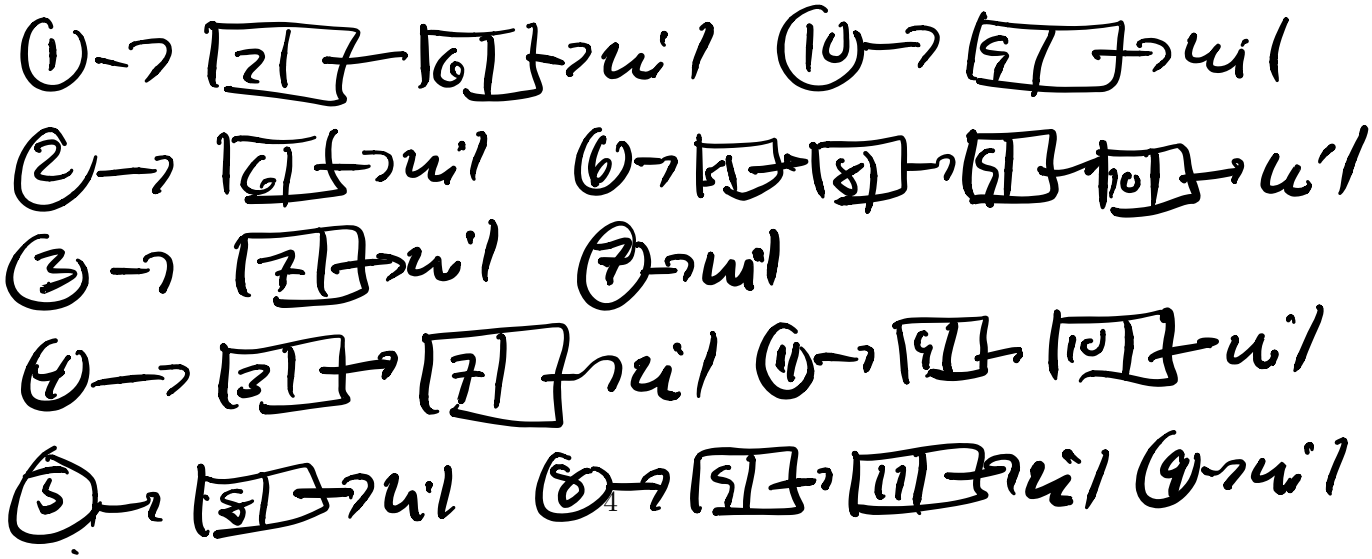


Figure 3:

(b) Represent this graph as an adjacency list.

(c) Is this graph a DAG? Yes. No cycle!



4. An undirected bipartite graph $G(V, E)$ is one where its nodes can be partitioned into two disjoint sets $V = V_1 \cup V_2$, such that that every edge $e \in E$ is an arc $\{v_1, v_2\}$, where $v_1 \in V_1$ and $v_2 \in V_2$. Note that, in a bipartite graph, the length of every cycle must be an even number.

   Define a pseudo code for an algorithm to determine whether an undirected graph is bipartite with worst case complexity $O(m)$, where m is the number of edges in the graph.

   Solution: We will do a variation of breadth-first search.

   (a) Mark all nodes as parity 0. Set all parents of nodes to null. Initialize *Do* queue to empty, *Done* queue to empty.

   (b) While there are nodes of parity 0, select one such node n, set its parity to 1, and perform a breadth first search as follows by adding n to a *Do* queue, first in, first out.

   - While there are nodes in *Do*, remove a node k from *Do*.
   - For each edge $\{k, j\}$ in E such that parent$(j) \neq k$,
     - If parity(j) == parity(k), the graph is not bipartite, so exit and declare not bipartite..
     - else set parity(j) = -1*parity(k). If j is not in *Do* or *Done*, add j to *Do*.
   - Add node k to *Done*.

   (c) end while loop

   (d) If you get here, the graph is bipartite.

   The complexity of this algorithm is similar to the complexity of breadth-first search, which is $O(m)$, where m is the number of edges. Each edge may be examined at most twice, once by each node in the edge.

5. While Euler's theorem gives a characterization of planar graphs in terms of numbers of vertices, edges and faces, it is hard to establish whether a graph is planar or not if it is difficult to count faces. There are a couple of other properties of simple, connected planar graphs that derive from Euler's theorem:

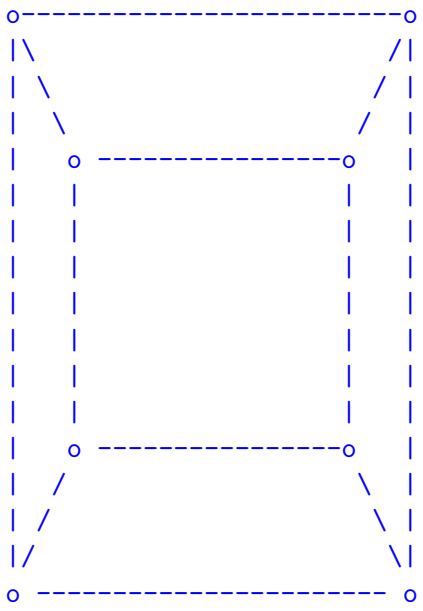
   - A simple, connected planar graph with $n \geq 3$ vertices and e edges must satisfy $e \leq 3n - 6$
   - A simple, connected planar graph with $n \geq 3$ vertices, e edges and no cycles of length 3 must satisfy $e \leq 2n - 4$

   A popular architecture for parallel computers is a hypercube. A hypercube of dimension k, denoted by Q_k, has 2^k nodes, and each node is connected to k other nodes. The nodes can be embedded into a k-dimensional boolean vector, and nodes are connected to other nodes that differ along one of its coordinates. Thus, Q_2 has nodes $(0, 0), (0, 1), (1, 0), (1, 1)$, and has 4 edges. The node (0,0) is connected to (0,1) and (1,0). Q_3 has nodes $(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1)$. Node (1,1,1) is connected to nodes (0,1,1), (1,0,1) and (1,1,0). Note that the number of edges in a hypercube of dimension k is $k * 2^{k-1}$, since each node has k edges, and we divide by 2 so as not to count the number of arcs twice. Other important facts about hypercubes is that every hypercube is a bipartite graph.

(a) Using the above facts, verify that Q_3 is a planar graph.

Solution: In a hypercube, all cycles have even length. The hypercube Q_3 has 8 nodes, and 12 edges. This satisfies $12 \leq 16 - 4 = 12$.

Another way of seeing this is that we can draw Q_3 as a planar graph, as follows:



(b) Using the above facts, show that Q_4 cannot be a planar graph.

Solution: For Q_4, there are 16 nodes, and there are 32 edges. Thus, $32 > 2(16) - 4$ so Q_4 cannot be a planar graph.

6. Suppose you have computed a minimum spanning tree T for a graph $G(V, E)$. Assume you now add a new vertex n and undirected arcs $E_n = \{\{n, v_i\}$, for some $v_i \in V\}$, with new weights w_{nv_i}. Provide the pseudocode for an algorithm to find the minimum weight spanning tree in the augmented graph $G_a(V \cup \{n\}, E \cup E_n)$. Estimate the complexity of this algorithm.

Solution:

Easy algorithm. Consider the subgraph consisting of the minimum spanning tree T plus the extra arcs E_n. Note that the total number of arcs in this graph is $O(n)$. One can show , by Kruskal's algorithm, that a minimum spanning tree in $G_a = (V \cup \{n\}, E \cup E_n)$ can be found with only the arcs in $T \cup E_n$. Finding a minimum spanning tree in $G_b = (V \cup \{n\}, T \cup E_n)$ can now be done in $O(n \log n)$ with either Kruskal's algorithm or Prim's algorithm.

7. (20 pts) An undirected graph G(V,E) is defined interm of two arrays: `First[0:Vsize]` and `Edge[0:Esize]` that label the Vertice(nodes) from $0, 1, \cdots, Vsize - 1$ and Edges (arcs) from $0, 1, \cdots, Esize - 1$ respectively. The last "fake" vertex (`Frist[Vsize] = Esize`) point to the null "fake" edge with "null" value (`Edge[Esize] = -1`), See Exercise 3 above for an small example.

   Implement your algorithm as a C/C++ function that counts the number of connected components. On GitHub there is the main file that reads input and writes output the result. You only write the required functions. A code `makeGraph.cpp` has been posted that generates "random" undirected graphs, which you use for you amusement if you want to test your algorithm small or large graphs of varying sparsity.

   Your connected component function needs to call **over and over again** a `Grow(...)` function using either BFS (or DFS) until you have visited **all** the nodes in the graph. It is preferable to use BFS as a more efficient way to grow each connected componet from its start node. The functions for BFS have been included. The number of time you call `Grow(..)` is the number of connected components. All you need to return is one extra a global array `Found[0:Vsize]` initialize to -1 for not found and then set to something else when each node is visited so your next call to search `Grow(..)` goes only to nodes that have been left out.

   The task is to first implement connected components with BFS and a Queue and then to add to the code another function doing DFS using a stack.

   Write the function:

```
int find_connected_components_BFS()
Stack * createStack()
Push(..)
Pop(..)
int find_connected_compoents_DFS()
```

   Find the number of time and connected components using BFS and then add the find DFS method so you can compare the time of execution and show that the number of connected components are the same of course!

   Do not make any changes to the `infield` reading format. so that it easy for the grader to run you code against a set of input files to test the code correctness. When you implement the DFS version the `outfile` writing format in `main()` You are given as set of input graphs and few solutions.

   Analysis For extra credit if you pick the label found as a distinct number inside each connected component (e.g. $1, 2, \cdots,$ NumberConnectCompoents), you can learn a lot more – like how many live in the largest most popular cluster etc.

   Let's discuss more extension class and how to graph resutls. Might lead to a fun project as well.

```
More fun on conected components
Google look at the curious
Regular graphs
Cluster finding variations etc
```