#5

a) $$S = \sum_{n=0}^{N-1} x^n$$

$$S \cdot x = \sum_{n=0}^{N-1} x^{n+1}$$

$$Sx - S = x^N - 1$$

$$\Rightarrow \quad S(x-1) = x^N - 1$$

$$S = \frac{x^N - 1}{x - 1} = \frac{1 - x^N}{1 - x}$$

letting $x \to \exp(-2\pi i k/N)$

$$S = \sum_{j=0}^{N-1} \left[\exp\left(-2\pi i \frac{k}{N}\right)\right]^j = \frac{1 - \exp(-2\pi i k)}{1 - \exp\left(-2\pi i \frac{k}{N}\right)}$$

b)

$$\lim_{k \to 0} S = \frac{\frac{d}{dk}\left(1 - \exp(-2\pi i k)\right)}{d/dk\left(1 - \exp\left(-2\pi i \, k/N\right)\right)}$$

$$= \lim_{k \to 0} \frac{2\pi i \exp(-2\pi i k)}{\left(\frac{2\pi i}{N}\right) \exp\left(-2\pi i \frac{k}{N}\right)} = N$$

letting k be an integer

$$\sum_{x=0}^{N-1} \exp(-2\pi i\, kx/N) = \frac{1-\exp(-2\pi i k)}{1-\exp(-2\pi i\, k/N)} = \frac{A}{B}$$

$A = 1-1=0$ for any integer k.

if $k \nmid N$, $\exp(-2\pi i\, k/N) \neq 1$

and so $B \neq 0$ hence $\sum_{x=0}^{N-1} \exp\left(-2\pi i \frac{kx}{N}\right) = 0$

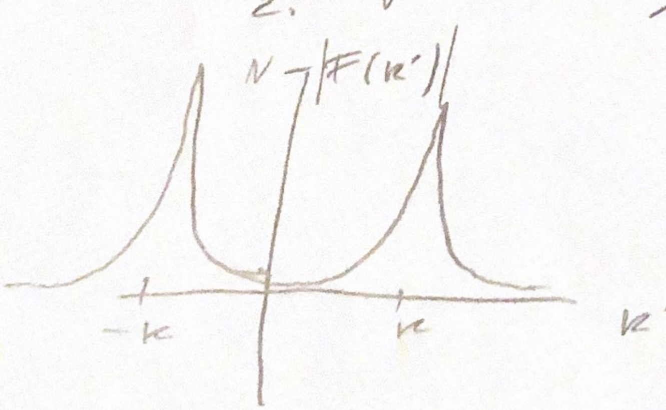
$\forall\, k \in \mathbb{N}$ s.t. $k \nmid N$

c) $f(x) = \sin\left(2\pi \frac{kx}{N}\right)$

$$F(k') = \sum_{x=0}^{N-1} \exp\left(-2\pi i \frac{k'x}{N}\right) \sin\left(2\pi \frac{kx}{N}\right)$$

$$= \frac{1}{2i} \sum_{x=0}^{N-1} \exp\left(-2\pi i \frac{x}{N}(k'-k)\right) - \exp\left(-\frac{2\pi i x}{N}(k'+k)\right)$$

$= \frac{\pm N}{2i}$ if $k' = \pm k$, $F(k') = 0$ otherwise



$$F(k') = \frac{1}{2i}\left[\frac{1-\exp(-2\pi i (k'-k))}{1-\exp(-2\pi i (k'-k)/N)} - \frac{1-\exp(-2\pi i (k'+k))}{1-\exp\left(-2\pi i \frac{(k'+k)}{N}\right)}\right]$$

#5 e) We note that

for $w(x) = 0.5\left(1 - \cos\left(2\pi x/N\right)\right)$

$$\hat{w}(k) = \frac{N}{4}\delta(k) - \frac{N}{2}\left(\delta(k-1) + \delta(N+1)\right)$$

Noting that $k \pm 1 \to k \pm 1 \bmod k$

then

$$FFT\left(f(x) \cdot w(x)\right) = \hat{f}(k)\,\hat{w}(k)$$

$$= \frac{N}{4}\hat{f}(k) - \frac{N}{2}\left[\hat{f}(k-1) + \hat{f}(k+1)\right]$$