Problem 1

Using the following equations in Simon's Solid State Basics:

Equation 4.3:

$$N = 2\sum_k n_F(\beta(\epsilon(\mathbf{k}) - \mu)) = 2\frac{V}{(2\pi)^3}\int \mathbf{dk} n_F(\beta(\epsilon(\mathbf{k}) - \mu))$$

Equation 4.6:

$$N = 2\frac{V}{(2\pi)^3}(\frac{4}{3}\pi k_F^3)$$

Equation 4.7:

$$E_F = \frac{\hbar^2(3\pi^2 n)^{\frac{2}{3}}}{2m}$$

Equation 4.7:

$$g(\epsilon)d\epsilon = \frac{(2m)^{\frac{3}{2}}}{2\pi^2\hbar^3}\epsilon^{\frac{1}{2}}d\epsilon$$

a) Provide the 1D and 2D analogues of these equations.

b) Sketch the Density of States in 1D, 2D, and 3D.

Solution

a)

Equation 4.3:

N $\equiv$ The total number of electrons within the system.
There are 2 possible spin states for electrons (fermions), Therefore we have a prefector of 2.
Due to the spacing between points in k-space ($\frac{2\pi}{L}$), the sum over all **k** is replaced by an integral multiplied by the spacing:

1D:

$$\sum_k \rightarrow \frac{L}{2\pi}\int \mathbf{dk}$$

2D:

$$\sum_k \rightarrow \frac{L^2}{(2\pi)^2}\int \mathbf{dk} = \frac{A}{(2\pi)^3}\int \mathbf{dk}$$

3D:

$$\sum_k \rightarrow \frac{L^3}{(2\pi)^3}\int \mathbf{dk} = \frac{V}{(2\pi)^3}\int \mathbf{dk}$$

Therefore, the total number of electrons for each dimensional system is given by:

1D:

$$\boxed{N_{1D} = 2\frac{L}{2\pi}\int \mathbf{dk} n_F(\beta(\epsilon(\mathbf{k}) - \mu))}$$

2D:

$$\boxed{N_{2D} = 2\frac{A}{(2\pi)^2}\int \mathbf{dk} n_F(\beta(\epsilon(\mathbf{k}) - \mu))}$$

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Equation 4.6:

Solving the integral provides the solution of a sphere (for 3D) of radius k_F.
Likewise, a circle (2D) and a line (1D) can be used with a radius or length of k_F.

1D:

$$N = 2\frac{L}{2\pi}(k_F) = \frac{L}{\pi}k_F$$

$$\Rightarrow \boxed{N_{1D} = \frac{L}{\pi}k_F}$$

2D:

$$N = 2\frac{A}{(2\pi)^2}(\pi k_F^2) = \frac{A}{2\pi}k_F^2$$

$$\Rightarrow \boxed{N_{2D} = \frac{A}{2\pi}k_F^2}$$

Equation 4.7:

The Fermi energy (E_F) is then found by substituting the electron density $n = \frac{N}{V}$, solving for k_F and using Equation 4.4 ($E_F = \frac{\hbar^2 k_F^2}{2m}$):

1D:

$$N/L = n = \frac{k_{F,1D}}{\pi} \Rightarrow k_{F,1D} = n\pi$$

$$\therefore \boxed{E_{F,1D} = \frac{\hbar^2 n^2 \pi^2}{2m}}$$

2D:

$$N/A = n = \frac{\frac{2}{(2\pi)^2}\pi k_{F,2D}^2}{\pi} \Rightarrow k_{F,2D}^2 = \frac{n(2\pi)^2}{2\pi} = 2\pi n$$

$$\Rightarrow k_{F,2D} = \sqrt{2n\pi}$$

$$\therefore \boxed{E_{F,2D} = \frac{\hbar^2 n\pi}{m}}$$

Equation 4.10:

1D:

$$g_{1D}(\epsilon)d\epsilon = \frac{2}{2\pi}(1)dk$$

$$\Rightarrow g_{1D}(\epsilon)d\epsilon = \frac{dk}{\pi}$$

$$\because k = \sqrt{\frac{2\epsilon m}{\hbar^2}} \Rightarrow dk = \sqrt{\frac{m}{2\epsilon\hbar^2}}d\epsilon$$

$$\therefore \boxed{g_{1D}(\epsilon) = \sqrt{\frac{m}{2\pi^2\epsilon\hbar^2}}}$$

$$\because E_{F,1D} = \frac{\hbar^2 n^2\pi^2}{2m}$$

$$\Rightarrow \sqrt{\frac{m}{2\epsilon\hbar^2}} = \sqrt{\frac{1}{4E_{F,1D}\epsilon}}n\pi$$

$$\therefore \boxed{g_{1D}(\epsilon) = n\sqrt{\frac{1}{4\epsilon E_{F,1D}}}}$$

2D:

$$g_{2D}(\epsilon)d\epsilon = \frac{2}{(2\pi)^2}(2\pi k)dk$$

$$\because k = \sqrt{\frac{2\epsilon m}{\hbar^2}} \Rightarrow dk = \sqrt{\frac{m}{2\epsilon\hbar^2}}d\epsilon$$

$$\Rightarrow g_{2D}(\epsilon)d\epsilon = \frac{2}{(2\pi)^2}(2\pi\sqrt{\frac{2\epsilon m}{\hbar^2}})\sqrt{\frac{m}{2\epsilon\hbar^2}}d\epsilon$$

$$\Rightarrow g_{2D}(\epsilon)d\epsilon = \frac{1}{\pi}\sqrt{\frac{2\epsilon m^2}{2\epsilon\hbar^4}}d\epsilon = \frac{m}{\pi\hbar^2}d\epsilon$$

$$\Rightarrow \boxed{g_{2D}(\epsilon) = \frac{m}{\pi\hbar^2}}$$

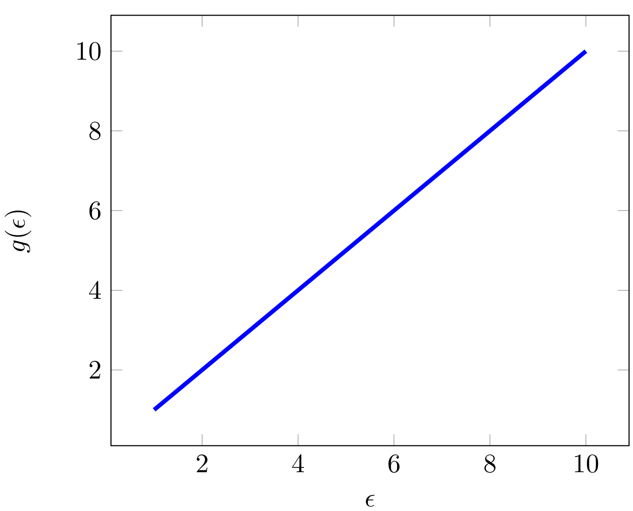
$$\because E_{F,2D} = \frac{\hbar^2 n\pi}{m}$$

$$\Rightarrow \frac{m}{\pi\hbar^2} = \frac{n}{E_{F,2D}}$$

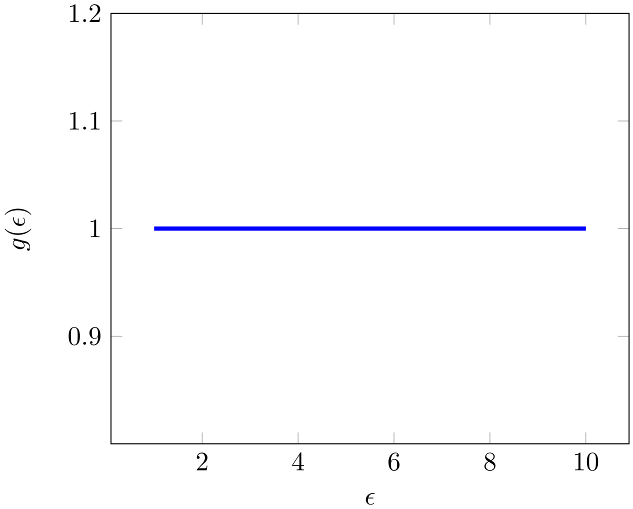
$$\therefore \boxed{g_{2D}(\epsilon) = \frac{n}{E_{F,2D}}}$$

b) Sketch the Density of States in 1D, 2D, and 3D.

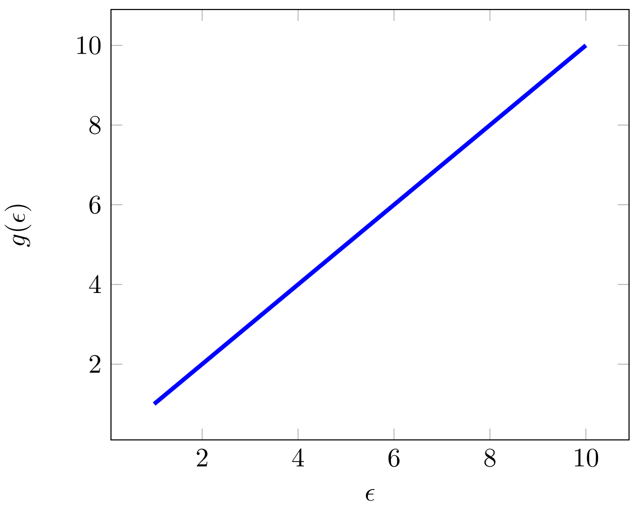
1D:



2D:


1.2
1.1
1
0.9
$g(\epsilon)$
2
4
6
8
10
ϵ

3D:


10
8
6
4
2
$g(\epsilon)$
2
4
6
8
10
ϵ