4.2) a) Show $v_F = \frac{\hbar k_F}{m} = \frac{\hbar}{m}(3\pi^2 n)^{1/3}$

essentially show $k_F = (3\pi^2 n)^{1/3}$.

$$N = 2\int g(k)\, d^3\vec{k} \qquad \text{(factor of 2 from spin degeneracy)}$$

integrating density of states over Fermi sphere gives # of electrons N.

$$N = 2\int_0^{2\pi}\int_0^{\pi}\int_0^{k_F} \left(\frac{L}{2\pi}\right)^3 k^2\, dk \sin\theta\, d\theta\, d\phi$$

$$N = \frac{2V}{8\pi^3}\frac{4}{3}\pi k_F^3 \rightsquigarrow \frac{N}{V} = \frac{1}{3\pi^2}k_F^3 \rightsquigarrow \boxed{k_F = (3\pi^2 n)^{1/3}}$$

b.) Recall $\vec{J} = -ne\vec{v}_d = \sigma\vec{E}$

$$\therefore \boxed{v_d = \left|\frac{\sigma E}{ne}\right|}$$

Recall from the classical eqn of motion $\dot{\vec{p}} = -e\vec{E} - \vec{p}/\tau$ scattering term

we arrived at the following steady state expr. for σ (eqn 3.2)

$$\sigma = \frac{e^2\tau n}{m}$$

We can rewrite τ in terms of mean free path λ as

$$\lambda = v_F\tau$$

We use v_F above and not v_d because v_d is the net effect of having these fast moving electrons bouncing around and scattering/relaxing on $\sim 10^{-14}$s time scales. But λ is really talking about how far one of these electrons moves in that short amount of time τ, not the distance it drifts in the metal over long time scales.

$$\sigma = \frac{e^2 \frac{\lambda}{v_F} n}{m} = \boxed{\frac{ne^2\lambda}{mv_F}}$$

c) i. Using $E = 1\ V/m$ & $n = 8.45\times10^{28}\ m^{-3}$ I get (note this is e-density and not mass density so we're all good.)

$$v_F = \frac{1.05\times10^{-34}\ Js}{9.11\times10^{-31}\ kg}\left(3\pi^2 \times 8.45\times10^{28}\ m^{-3}\right)^{1/3}$$

$$= \boxed{1.56\times10^{6}\ m/s} \simeq 0.005c$$

$$v_d = \frac{(5.9\times10^{7}\ \Omega^{-1}m^{-1})(1\ V/m)}{(1.602\times10^{-19}\ C)(8.45\times10^{28}\ m^{-3})} = \boxed{4.36\times10^{-3}\ m/s} = 4.36\ mm/s$$

So scattering REALLY attenuates net charge movement rate in a metal.

ii $$\lambda = \frac{m v_F \sigma}{ne^2} = \frac{(9.11\times10^{-31}\ kg)(1.56\times10^{6}\ m/s)(5.9\times10^{7}\ \Omega^{-1}m^{-1})}{(8.45\times10^{28}\ m^{-3})(1.602\times10^{-19})^2}$$

$$= 3.87\times10^{-8}\ m = \boxed{387\ \mathring{A}}$$

Internet tells me lattice constant in copper is ~3.61 Å (FCC lattice so mean spacing is probably even smaller but same order of mag).

Roughly speaking a conduction electron moves $\sim10^2$ atomic distances before scattering.

4.3) a & b & c) Read the book

d.) ▸ The linear term comes from the electrons and the T^3 term comes from phonons (remember $C_{ph} \sim T^d$ from hw #1)

▸ A variety of ways to do this, one is to use the coefficient γ along w/ book derivations of the heat capacity due to elec, which depends on $g(E_F)$ to solve for E_F

↳ $C_{elec} = \frac{\pi^2}{3} k_B^2 g(E_F) T V$

$$\text{Now } \frac{\pi^2}{3} k_B^2 g(E_F) V = \frac{\pi^2}{3} k_B^2 \frac{3nV}{2E_F} \cancel{\left(\frac{E_F}{E_F}\right)^{1/2}}^{1}$$

$$= \frac{\pi^2}{2} k_B^2 \frac{N}{E_F}$$

$$= \frac{\pi^2}{2} k_B^2 \frac{1}{E_F} n_m N_A$$

n_m ← # Moles.

$$\text{So } \gamma = \frac{\pi^2}{2} k_B^2 \frac{N_A}{E_F} = 2.08 \times 10^{-3} \frac{J}{mol \cdot K^2}$$

$$E_F = \frac{\pi^2}{2} \frac{k_B^2 N_A}{2.08 \times 10^{-3}\, J/mol\,K^2} = 2.72 \times 10^{-19}\, J = \boxed{1.70\ eV}$$

4.7) Can use eqn. 4.8 in text

a) $E = V\int_0^\infty d\epsilon\, \epsilon\, g(\epsilon)\, n_F(\beta(\epsilon-\epsilon_F)) = V\int_0^\infty d\epsilon\, \epsilon \frac{3n}{2E_F}\left(\frac{\epsilon}{E_F}\right)^{1/2} \frac{1}{e^{\beta(\epsilon-E_F)}+1}$

At T=0 n_F is a step function

$= V\int_0^{E_F} d\epsilon\, \epsilon \frac{3n}{2E_F}\left(\frac{\epsilon}{E_F}\right)^{1/2}$

$= \frac{3N}{2E_F^{3/2}}\int_0^{E_F} d\epsilon\, \epsilon^{3/2} = \frac{3N}{\cancel{2}E_F^{3/2}}\frac{\cancel{2}}{5}E_F^{5/2} = \boxed{\frac{3}{5}NE_F}$

b) $P = -\frac{\partial E}{\partial V} = -\frac{3}{5}N\frac{\partial}{\partial V}E_F = -\frac{3}{5}N\frac{\partial}{\partial V}\left(\frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3}\right)$ ~> see eqn. 4.7

$= -\frac{\cancel{3}}{5}N\frac{\hbar^2}{\cancel{2}m}\frac{\cancel{2}}{\cancel{3}}\left(3\pi^2\frac{N}{V}\right)^{-1/3}\left(-\frac{3\pi^2 N}{V^2}\right)$

$= \frac{\hbar^2}{5m}\left(\frac{N}{V}\right)^{5/3}\left(3\pi^2\right)^{2/3} = \frac{2}{5}\frac{N}{V}\left(\frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3}\right)$

$\boxed{= \frac{2}{5}nE_F}$

$B = -V\frac{\partial P}{\partial V} = -V\frac{2}{5}\frac{\partial}{\partial V}\left(\frac{N}{V}\left(\frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3}\right)\right) = -V\frac{2}{5}\left(3\pi^2\right)^{2/3}\frac{\hbar^2}{2m}\frac{\partial}{\partial V}\left(\frac{N}{V}\right)^{5/3}$

$= -V\frac{2}{5}\left(3\pi^2\right)^{2/3}\frac{\hbar^2}{2m}\frac{5}{3}\left(\frac{N}{V}\right)^{2/3}\left(-\frac{N}{V^2}\right)$

$= \frac{2}{\cancel{5}}\left(3\pi^2\right)^{2/3}\frac{\hbar^2}{2m}\frac{\cancel{5}}{3}\left(\frac{N}{V}\right)^{5/3} = \boxed{\frac{2}{3}nE_F}$

c) $B_{Na} = \frac{2}{3}n_{Na}\left(\frac{\hbar^2}{2m}\left(3\pi^2 n_{Na}\right)^{2/3}\right)$ = plug & chug $= 8.42\times10^9$ Pa = $\boxed{8.42\text{ GPa}}$

overestimate

$B_K = 2.88\times10^9$ Pa = $\boxed{2.88\text{ GPa}}$

underestimate

Different periods in table, perhaps core electrons are contributing to difference?

Estimate pressure needed to squeeze electrons.

$$B = V\frac{\partial P}{\partial V} \approx V\frac{\Delta P}{\Delta V}$$

$$\rightarrow B = \frac{\cancel{V}}{0.01\cancel{V}}\Delta P$$

$$\Delta P = 0.01B$$

$$\Delta P_{Na} = 0.01(8.42GPa) = \boxed{84.2MPa}$$

6.6

a) $$H_1 = \chi\left(\frac{1}{|\bar{R}|} + \frac{1}{|\bar{R}-\bar{r}_1+\bar{r}_2|} + \frac{1}{|\bar{R}-\bar{r}_1|} + \frac{1}{|\bar{R}+\bar{r}_2|}\right) \quad w/ \quad \chi = \frac{e^2}{4\pi\epsilon_0}$$

$$\frac{1}{|\bar{R}-\bar{r}_1|} = \frac{1}{\sqrt{(R-x_1)^2+y_1^2+z_1^2}} = \frac{1}{\sqrt{R^2-2Rx_1+r_1^2}}$$

$$= \frac{1}{R\sqrt{1-\frac{2x_1}{R}+r_1^2/R^2}}$$

Expand for large R

$$= \frac{1}{R}\left(1-\frac{1}{2}\left(\frac{-2x_1}{R}+r_1^2/R^2\right)+\frac{3}{8}\left(\frac{-2x_1}{R}+r_1^2/R_2\right)^2+\ldots\right)$$

Repeat for each term, cancellation, lots of algebra.

Recall field of dipole is

$$E_{dip} = \frac{1}{4\pi\epsilon_0}\left(\frac{3(\bar{p}_1\cdot\hat{r})\hat{r}-\bar{p}_1}{r^3}\right)$$

So $$H \sim -(\vec{p}_2\cdot\vec{E}) = \frac{1}{4\pi\epsilon_0}\left(\frac{-3(\bar{p}_1\cdot\hat{r})(\bar{p}_2\cdot\hat{r})+\bar{p}_2\cdot\bar{p}_1}{r^3}\right)$$

$$= \frac{e^2}{4\pi\epsilon_0}\left(\frac{-3x_1x_2+x_1x_2+y_1y_2+z_1z_2}{r^3}\right)$$

since $\hat{r}=\hat{x}$

$$= \frac{e^2}{4\pi\epsilon_0}\frac{1}{r^3}\left(z_1z_2+y_1y_2-2x_1x_2\right)$$

b) The first order correction would be

$$\Delta E^{(1)} = \langle 1,0,0;1,0,0| H_1 |1,0,0;1,0,0\rangle$$

- However H_1 is odd in each in each atom's position so it's odd under inversion operator $\hat{\pi}$. Note that G.S. is spherically symmetric and <u>even</u> under inversion. Generally, for each atom,

$$\widetilde{|1,0,0\rangle} = \pi|1,0,0\rangle$$

↑ inverted state

So if I take the expectation value of, say, x_1 in the inverted state, I should get same thing as normal state

$$\widetilde{\langle 1,0,0|} x_1 \widetilde{|1,0,0\rangle} = \langle 1,0,0| x_1 |1,0,0\rangle$$

However x_1 as an operator is odd under inversion, meaning

$$\widetilde{\langle 1,0,0|} x_1 \widetilde{|1,0,0\rangle} = \langle 1,0,0|(\pi^{\dagger} x_1 \pi)|1,0,0\rangle$$

$$= -\langle 1,0,0| x_1 |1,0,0\rangle = \langle 1,0,0| x_1 |1,0,0\rangle$$

The only way this can be true is if the expec. value is zero.

The first non zero term will come from expec values of form

$$|\langle 1,0,0;1,0,0| H_1 |n_1,l_1,m_1;n_2,l_2,m_2\rangle|^2 \sim \left|\frac{1}{R^3}\right|^2 \sim \frac{1}{R^6}$$

- The denominator of δE contains $E_{0,0} - E_{n_1,n_2}$ which is always negative. Since the modulus squared in the numerator is always positive, each term in the sum for δE is negative. The perturbation <u>drops</u> the energy implying an attractive force.

c) $\cancel{\delta E = \sum_{\substack{n_1 n_2 \\ l_1 l_2 \\ m_1 m_2}} |\langle 1,0,0;1,0,0|H_1|}$

$$\delta E = \sum_{\substack{n_1 n_2 \\ l_1 l_2 \\ m_1 m_2}} \frac{|\langle 1,0,0;1,0,0|H_1|n_1 l_1 m_1; n_2 l_2 m_2\rangle|^2}{E_{00} - E_{n_1,n_2}}$$

The numerator is zero for the same reason as previous problem.
To get finite expectation value from

$$\langle 1,0,0|X|n, l, m\rangle$$

l must be 1. This is impossible unless $n \geq 2$.

Upper bound.

$$\delta E_> = \sum_{\substack{n_1 n_2 \\ l_1 l_2 \\ m_1 m_2}} \frac{\langle 1,0,0;1,0,0|H_1|n_1 l_1 m_1; n_2 l_2 m_2\rangle\langle n_1 l_1 m_1; n_2 l_2 m_2|H_1|1,0,0,1,0,0\rangle}{E_{00} - E_{22}}$$

$$= \frac{\langle 1,0,0;1,0,0|H_1^2|1,0,0;1,0,0\rangle}{E_0 - E_{22}}$$

only the even terms in H_1 (ie. $x_1^2 x_2^2$ and not $z_1 z_2 y_1 y_2$) contribute

$$= \frac{\left(e^2/4\pi\epsilon_0 R^3\right)^2}{E_0 - E_{22}}\left(a_0^4 + a_0^4 + 4a_0^4\right) = \frac{1}{\frac{-e^2}{8\pi\epsilon_0 a_0}\left(\left(\frac{1}{1^2}+\frac{1}{1^2}\right) - \left(\frac{1}{2^2}+\frac{1}{2^2}\right)\right)}$$

$$\times\left(\frac{e^2}{4\pi\epsilon_0 R^3}\right)^2 (6a_0^4)$$

$$= -\frac{8\pi\epsilon_0 a_0}{\cancel{e^2}} \frac{e^{\cancel{4}\,2}}{16\pi^2\epsilon_0^2 R^6}(6a_0^4) \times \frac{2}{3} = -2\frac{e^2 a_0^5}{\pi\epsilon_0 R^6}$$

So $|\delta E_>| = 2\dfrac{e^2 a_0^5}{\pi\epsilon_0 R^6}$ $\rightsquigarrow$ Not sure why book doesn't simplify $\frac{8}{4} \to 2$?

$|\delta E_<|$ is almost exactly same except sum in denominator changes slightly.

$$|\delta E_<| = \left| \frac{-8\pi\epsilon_0 a_0}{e^2} \frac{e^4}{16\pi^2\epsilon_0^2 R^6} 6a_0^4 \times \frac{1}{2} \right|$$

$$= \frac{3}{2} \frac{e^2 a_0^5}{\pi\epsilon_0 R^6} \rightsquigarrow \text{again } \frac{6}{4} = \frac{3}{2}$$

Thus $|\delta E_<| \leq |\delta E| \leq |\delta E_>|$

$$\frac{3}{2} \frac{e^2 a_0^5}{\pi\epsilon_0 R^6} \leq |\delta E| \leq 2 \frac{e^2 a_0^5}{\pi\epsilon_0 R^6}$$