Solutions Problem Set #1

Problem 1
Simon 2.1

(a) $Z = \int \frac{d\vec{p}}{(2\pi\hbar)^3} \int d\vec{x}\ e^{-\beta\left(\frac{p^2}{2m} + \frac{kx^2}{2}\right)}$

$= \frac{1}{(2\pi\hbar)^3} \int d\vec{p}\ e^{-\beta\frac{p^2}{2m}} \int d\vec{x}\ e^{-\beta\frac{kx^2}{2}}$

assume space is isotropic so $d\vec{p} = dp_x dp_y dp_z$
$= 4\pi p^2 dp$

same for $d\vec{x}$

$Z = \frac{1}{(2\pi\hbar)^3} (4\pi)^2 \int_0^\infty p^2 e^{-\frac{\beta p^2}{2m}} dp \int_0^\infty x^2 e^{-\beta\frac{kx^2}{2}} dx$

$\int_0^\infty x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{4 a^{3/2}}$

$\therefore Z = \frac{1}{(2\pi\hbar)^3} (4\pi)^2 \left(\frac{2m}{\beta}\right)^{3/2} \left(\frac{2}{\beta k}\right)^{3/2} \frac{\pi}{16}$

$Z = \left(\frac{m}{\beta^2 k \hbar^2}\right)^{3/2}$

spring constant

$\langle E \rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = -\left(\frac{\beta^2 k \hbar^2}{m}\right)^{3/2} \left(\frac{m}{k\hbar^2}\right)^{3/2} \frac{-3}{\beta^4}$

$= +\frac{3}{\beta} = 3 k_B T$

$C = \frac{\partial \langle E \rangle}{\partial T} = 3 k_B$ This is per atom
N_A atoms per mole

$C_{mole} = 3 N_A k_B = 3R!$

(B) $Z = \sum_j e^{-\beta E_j}$

now like Simon page 8 (This is just like in lecture)

$$Z = \sum_{n=0}^{\infty} e^{-\hbar\omega(n+1/2)\beta} \quad \text{for 1 oscillator in 1D}$$

$$= e^{-\frac{\beta\hbar\omega}{2}} \sum_{n=0}^{\infty} e^{-\hbar\omega n\beta} = e^{-\frac{\beta\hbar\omega}{2}}\left(1 + \sum_{n=1}^{\infty} e^{-\beta\hbar\omega n}\right)$$

recall that $\sum_{n=1}^{\infty} e^{-xn} = \frac{1}{e^x - 1}$

So $Z = e^{-\frac{\beta\hbar\omega}{2}}\left(1 + \frac{1}{e^{+\beta\hbar\omega} - 1}\right) = e^{-\frac{\beta\hbar\omega}{2}}\left[\frac{\cancel{1} + e^{\beta\hbar\omega} - \cancel{1}}{e^{\beta\hbar\omega} - 1}\right]$

$$Z = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}} \quad \text{for a single 1D oscillator}$$

for 3D $\quad Z_{3D} = Z_{1D}^3$

$$\langle E \rangle_{3D} = \frac{-1}{Z_{3D}}\frac{\partial Z_{3D}}{\partial \beta} = \frac{-1}{(Z_{1D})^3}\, 3(Z_{1D})^2\left[\frac{-\frac{\hbar\omega}{2} e^{-\frac{\beta\hbar\omega}{2}}}{1 - e^{-\beta\hbar\omega}}\right.$$

$$\left. -\frac{1}{(Z_{1D})^3}\, 3(Z_{1D})^2 \left[\frac{e^{-\frac{\beta\hbar\omega}{2}}(-1)(-1)e^{-\beta\hbar\omega}\,\hbar\omega(-1)}{(1 - e^{-\beta\hbar\omega})^2}\right]\right]$$

$$\langle E \rangle_{3D} = 3\left[\frac{\hbar\omega}{2} + \frac{\hbar\omega\, e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}}\right]$$

$$\langle E \rangle_{3D} = 3\left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}\right]$$

$$C = \frac{\partial T}{\partial \beta}\frac{\partial \langle E \rangle}{\partial \beta} = \left(\frac{-1}{k_B T^2}\right)\left(\frac{3\hbar\omega(-1)e^{\beta\hbar\omega}\hbar\omega}{(e^{\beta\hbar\omega} - 1)^2}\right)$$

$$\boxed{= 3k_B\left[\frac{\hbar\omega}{k_B T}\right]^2 \frac{e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega} - 1)^2}}$$

The connection with Bose statistics is the factor $\frac{1}{e^{\beta\hbar\omega}-1} = n$

which for $\beta = 0$ $(kT \to \infty)$ where $e^x \approx 1 + x$

Bose $\Rightarrow \frac{1}{(1+\beta\hbar\omega - 1)} = \frac{k_B T}{\hbar\omega}$ increases

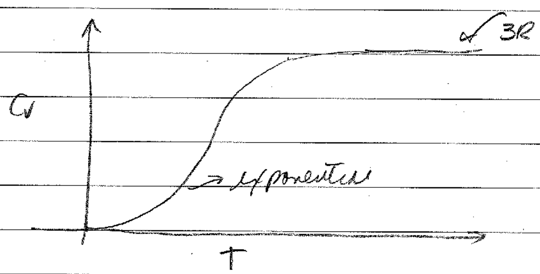
but for $\beta \to \infty$ $(T \to 0)$ $n \to 0$ for any state other than $\hbar\omega = 0$ state

At high temp $C_{T\to\infty} = 3k_B \left[\frac{\hbar\omega}{k_B T}\right]^2 \frac{(1)}{(1+\beta\hbar\omega - 1)^2}$

$$= 3k_B \left[\frac{\hbar\omega}{k_B T}\right]^2 \frac{1}{(\beta\hbar\omega)^2} = 3k_B$$ per oscillator

for a mole of oscillators $C = 3k_B N = 3R$!

Dulong Petit result as expected



Prob #2 Simon 2.2

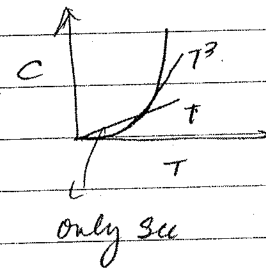
(Taken from lecture notes)

START HERE

Actual Density $C \sim T^3$

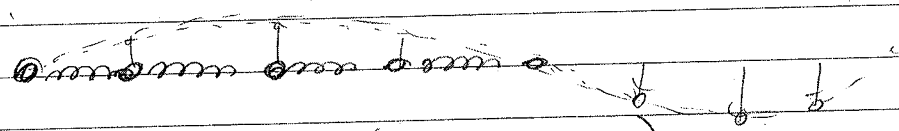
How can you tell. Plot C vs T^3 → get a straight line

→ What about electrons?



→ Debye for further refinement (put in knowledge of waves)

(A) → not independent oscillators



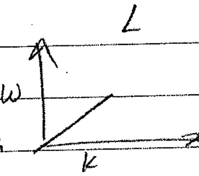
cyclical boundary conditions (periodic)

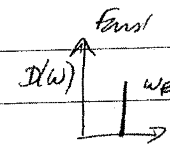
new use 2 things

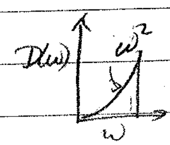
$$k_{min} = \frac{2\pi}{\lambda_{max}} = \frac{2\pi}{L} \qquad k = \frac{n2\pi}{L}$$

phase velocity

$\omega = vk$ energy → wavelength







(Important for problem set)

$$\sum_n \to \sum_k \xrightarrow{1D} \frac{L}{2\pi}\int_{-\infty}^{\infty} dk$$

Einstein

modes

but in 3D $\quad 3\sum_k^{3D} \to \frac{L^3}{(2\pi)^3}\int_{-\infty}^{\infty} d\vec{k} \xrightarrow{\text{isotropic}} 4\pi\int_0^{\infty} k^2\,dk$

(B) so $\displaystyle \langle E\rangle_{mole} = \frac{3L^3}{(2\pi)^3}\, 4\pi \int k^2\,dk\,(\hbar\omega)\left(\frac{1}{2} + \frac{1}{e^{\hbar\omega\beta}-1}\right)$

switch to ω

$$\langle E\rangle = \frac{3L^3}{(2\pi)^3}\, 4\pi \int \frac{\omega^2}{v^3}\,d\omega\,\hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1}\right)$$

$$\langle E \rangle = \frac{12\pi L^3}{(2\pi)^3} \int_0^\infty \frac{\hbar\omega\, \omega^2\, d\omega}{v^3} \left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1} \right)$$

$nL^3 = N$

$\downarrow$ # atoms unit volm

$$\langle E \rangle = \frac{12\pi L^3}{(2\pi)^3 v^3} \int_0^\infty \hbar\omega\, \omega^2\, d\omega \left[\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1} \right]$$

$$\langle E \rangle = \int_0^\infty g(\omega)\, \hbar\omega \left[\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1} \right]$$

Density of states → # of levels ΔE

$$g(\omega) = \frac{12\pi L^3 \omega^2}{(2\pi)^3 v^3} = N \left[\frac{12\pi\, \omega^2}{(2\pi)^3 n v^3} \right] = N \frac{9\omega^2}{\omega_D^3}$$

$\omega_D^3 = 6\pi^2 n v^3$ ← sound velocity

$\omega_D = \sqrt[3]{6\pi^2 n}\; v$ Debye frequency

$$\langle E \rangle = \frac{9N\hbar}{\omega_D^3} \int_0^\infty d\omega \frac{\omega^3}{e^{\beta\hbar\omega} - 1} + \text{temp independent}$$

let $x = \beta\hbar\omega$ $\quad d\omega = \frac{dx}{\beta\hbar} \quad \omega^3 = \frac{x^3}{(\beta\hbar)^3}$

$$\langle E \rangle = \frac{9N\hbar}{\omega_D^3} \frac{1}{(\beta\hbar)^4} \int_0^\infty \frac{dx\, x^3}{e^x - 1} \to \pi^4/15$$

$$\langle E \rangle = 9N \frac{(k_B T)^4}{(\hbar\omega_D)^3} \frac{\pi^4}{15}$$

LOW T LIMIT

$$C = \frac{\partial E}{\partial T} = 9N \frac{4 k_B^4 T^3}{(\hbar\omega_D)^3} \frac{\pi^4}{15} \qquad T^3 \checkmark$$

can scale w/ ω_D → $\hbar\omega_D = k_B T_D$

but can't have T^3 french

fill parameter or is it?

So $$3N = \int_0^{\omega_{cutoff}} d\omega \; g(\omega)$$

$$3N = \int_0^{\omega_{cutoff}} d\omega \quad N \cdot \frac{9\omega^2}{\omega_D^3}$$

$$= \frac{N 9}{\omega_D^3} \frac{\omega^3}{3} \Big]_0^{\omega_{cutoff}}$$

$$3N = 3N \left[\frac{\omega_{cutoff}}{\omega_{Debye}} \right]^3 \qquad \omega_{cutoff} = \omega_D \checkmark$$

Limitations

ω_{cutoff}

$\omega = vk$ (linear)

still need to explain $C \sim T$

Put it all together

$$\langle E \rangle = \int_0^{\omega_{max}} g(\omega)\, \hbar\omega \left[\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1} \right] d\omega$$

when $T \to 0$ then $\frac{\hbar \omega_{max}}{k_B T} \to \infty$

so may as well have $\omega_{max} = \infty$

LOW T LIMIT AGAIN — then $\langle E \rangle \propto T^4$ and $C = \frac{\partial \langle E \rangle}{\partial T} \Rightarrow T^3$

↓

HIGH T LIMIT HERE — but when T high then keep $\omega_{max} = \omega_D$

$$\langle E \rangle = \int_0^{\omega_D} g(\omega)\, \hbar\omega \left[\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1} \right] d\omega$$

Only T dependence is in ↑ so keep only 2nd term for C

$$\langle E \rangle_{\text{Temp dep}} = \int_0^{\omega_D} g(\omega) \frac{\hbar\omega}{e^{\beta\hbar\omega}-1} d\omega$$

$e^x \simeq 1 + x$ for x small so $\frac{1}{e^{\beta\hbar\omega}-1} \to \frac{1}{\beta\hbar\omega}$

$$\langle E \rangle_{\text{temp dep}} = \int_0^{\omega_D} g(\omega) \frac{\cancel{\hbar\omega}}{\cancel{\hbar\omega}\,\beta} d\omega$$

$$= k_B T \int_0^{\omega_D} g(\omega)\, d\omega \quad \text{— } 3N$$

$$\langle E \rangle = 3N k_B T$$

$\boxed{C_V = 3N k_B}$ Dulong Petit

Problem 2.2 (B) and (C) (2.5)

Here is the Potassium iodide data plotted

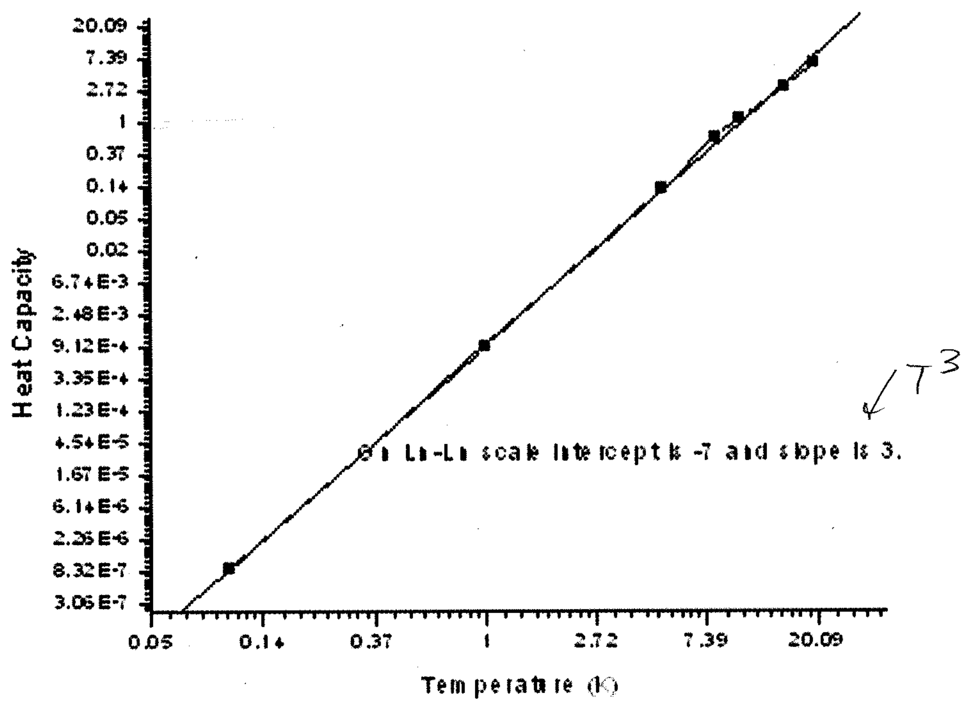
$\ln C$ vs $\ln T$

If $C = Nk_B \frac{T^3}{T_D^3} \frac{12\pi^4}{5}$

then $\ln C = \ln \frac{Nk_B 12\pi^4}{T_D^3 5} + 3 \ln T$

1

(B)



(C) Since slope is 3 this confirms the expected slope.

at $1K = T$ $C = 8.6 \times 10^{-4}$ J/K mole

$\therefore$ $T_D^3 = \frac{Nk_B (1K)^3 \frac{12\pi^4}{5}}{C \text{ at } 1K} = \frac{(6 \times 10^{23})(1.4 \times 10^{-23} \frac{J}{K})(12)\left(\frac{\pi^4}{5}\right)(K^3)}{8.6 \times 10^{-4} \text{ J/K mole}}$

Good agreement w/ internet

$\boxed{T_D = 131 K}$ & $129 K$! $\leftarrow T_D^3 = 2.28 \times 10^6 K$

Problem 3 Simon 2.3

Assumptions

Periodic boundary conditions $K = \frac{2\pi n}{L}$

$K = \frac{2\pi}{L}(n_1, n_2)$ allowed

$$\sum_K \rightarrow \frac{L^2}{(2\pi)^2} \int d\vec{K}_{2D}$$

So we will have a 2D solid that has motion in 3D (2 in plane mode, 1 out of plane)

Assume all mode are linear ∴ $\omega = vK$

Assume all v are the same (probably a bad assumption in that out of plane mode will be different than in plane but lets leave it)

$$\langle E \rangle = 3 \sum_{\vec{K}(2D)} \hbar\omega(\vec{K}) \left[n(\beta\hbar\omega) + \tfrac{1}{2} \right]$$

3 modes (or polarization = 2 transverse + 1 longitudinal)

$$= 3 \int \frac{L^2}{(2\pi)^2} d\vec{K}_{2D} \; \hbar\omega(K) \left[n_B(\beta\hbar\omega_K) + \tfrac{1}{2} \right]$$

Circular symmetry $\rightarrow \int d\vec{k} \rightarrow \int_0^{K_{max}} 2\pi K\, dK$ and $K = \frac{\omega}{v}$, $dK = \frac{d\omega}{v}$

$$\langle E \rangle = 3 \frac{L^2}{(2\pi)^2} 2\pi \int_0^{\omega_{max}} \hbar\omega \frac{\omega}{v} \frac{d\omega}{v} \left[n(\beta\hbar\omega) + \tfrac{1}{2} \right]$$

$$\langle E \rangle = \frac{3}{(2\pi)} \frac{L^2 \hbar}{v^2} \int_0^{\omega_{max}} \omega^2 d\omega \left[n(\beta\hbar\omega) + \tfrac{1}{2} \right]$$

For $C = \frac{\partial E}{\partial T} = \frac{\partial \beta}{\partial T}\frac{\partial E}{\partial \beta}$ only this term counts

Problem 3.2

$$\langle E \rangle_{\text{Temp dep}} = \frac{3}{(2\pi)} \cdot \frac{L^2 \hbar}{v^2} \int_0^{\omega_{max}} \frac{\omega^2 \, d\omega}{e^{\beta\hbar\omega} - 1}$$

$$\frac{\partial \langle E \rangle}{\partial \beta} = \frac{3}{(2\pi)} \frac{L^2 \hbar}{v^2} \int_0^{\omega_{max}} \frac{\omega^2 \, d\omega}{(e^{\beta\hbar\omega} - 1)^2} (-1) \, e^{\beta\hbar\omega} \, \hbar\omega$$

$$\therefore \; C = \underbrace{\left(-\frac{1}{K_B T^2}\right)}_{\frac{\partial \beta}{\partial T}} \left(-\frac{3}{(2\pi)} \frac{L^2 \hbar^2}{v^2} \int_0^{\omega_{max}} \frac{\omega^3 \, d\omega \, e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega} - 1)^2}\right)$$

In the high Temp limit $T \to \infty$ so $\beta \to 0$

$$e^x_{x \to 0} = 1 + x$$

$$\therefore \; C = \frac{3}{K_B T^2 \, (2\pi)} \frac{L^2 \cancel{\hbar^2}}{v^2} \int_0^{\omega_{max}} \frac{\omega^{\cancel{3}} \, d\omega \, (1)}{(\beta\hbar\omega)^2}$$

$$= \frac{3}{(2\pi)} \frac{K_B L^2}{v^2} \left.\frac{\omega^2}{2}\right]_0^{\omega_{max}}$$

How to get ω_{max} in 2D

$$N = (L/2\pi)^2 (\pi K^2)$$ ← area of 2D circle

↑ # of modes of 1 polarization

$$g(\omega) = \frac{dN}{d\omega} = \left(\frac{L}{2\pi}\right)^2 \pi^2 K \, 2 \frac{dK}{d\omega}$$

but $\omega = vK \quad \therefore \; \frac{dk}{d\omega} = \frac{1}{v} \quad K = \frac{\omega}{v}$

So $$g(\omega) = \left(\frac{L}{2\pi}\right)^2 \pi \, 2 \left[\frac{\omega}{v^2}\right] = \frac{L^2}{2\pi} \frac{\omega}{v^2}$$

Problem 3.3

So $\quad N = \int_0^{w_{max}} g(w)\,dw$

$$N = \int_0^{w_{max}} \frac{L^2 w}{2\pi v^2}\,dw = \frac{L^2 w_{max}^2}{4\pi v^2}$$

$$w_{max}^2 = \frac{N}{L^2}\frac{4\pi}{v^2} \rightarrow w_{max\,2D} = \sqrt{\frac{N\,4\pi v^2}{L^2}}$$

So going back to $C = \frac{3}{2\pi} k_B \frac{L^2}{v^2} \left.\frac{w^2}{2}\right]_0^{w_{max}}$

$$= \frac{3}{\cancel{2\pi}} k_B \frac{\cancel{L^2}}{\cancel{v^2}} \frac{N}{\cancel{L^2}} \cancel{4\pi v^2} \frac{1}{\cancel{2}} = 3k_B N$$

So we get the same high temp result as in 3D because we assumed the oscillators could move in 3D. If the out of plane motion were restricted then we would have only 2 polarizations (transverse and longitudinal and the $3k_BN \Rightarrow 2k_BN$

Now for the low Temp limit

$$C = \frac{3}{k_B T^2 (2\pi)} \frac{L^2 \hbar^2}{v^2} \int_0^{w_{max}} \frac{w^3\,dw\,e^{\beta\hbar w}}{(e^{\beta\hbar w}-1)^2}$$

let $x = \beta\hbar w$

$$C = \frac{3}{k_B T^2} \frac{1}{2\pi} \frac{L^2}{v^2} \hbar^2 \int_0^{x_{max}} \frac{x^3\,dx\,e^x}{(e^x-1)^2} \frac{1}{(\beta\hbar)^4}$$

as $T \to 0$ $\beta \to \infty$ $x_{max} \to \infty$

Problem 3.4

$$C = \frac{3}{k_B T^2} \frac{1}{2\pi} \frac{L^2 \hbar^3}{v^2} \frac{1}{(\beta \hbar)^4} \int_0^\infty \frac{x^3 dx\, e^x}{(e^x-1)^2}$$

$$\int_0^\infty \frac{x^3 e^x dx}{(e^x-1)^2} = 6\,\zeta(3) \cong 7.2$$

↑ zeta function

$$C = \frac{(3)(7.2)}{\cancel{k_B T^2}} \frac{1}{2\pi} \frac{L^2 \cancel{\hbar^2}}{v^2} k_B \left(\frac{k_B T}{\hbar}\right)^{\cancel{4}\,2}$$

$n = 2$

$$\boxed{C = \frac{(3)(7.2)}{2\pi} \frac{L^2}{\hbar^2 v^2} k_B^3 T^2}$$

check units $= \dfrac{\cancel{\text{length}^2}}{\text{energy}^2 \cdot \cancel{\text{sec}^2}\ \dfrac{\cancel{\text{length}^2}}{\cancel{\text{sec}^2}}} \dfrac{\text{energy}^3\ \cancel{{}^oK^2}}{{}^oK^{\cancel{3}}}$

$= \dfrac{\text{energy}}{{}^oK}$ ✓

Problem 4

Highest and Lowest Debye Temp

$$\Theta_D = \frac{\hbar \omega_D}{k_B} = \underbrace{\frac{\hbar}{k_B}\left[6\pi^2\right]^{1/3}}_{\text{Constants}} n^{1/3} v$$

$n^{1/3} \rightarrow$ [material number density]$^{1/3}$
$v \rightarrow$ velocity of sound

so Θ_D high when number density is high
close atomic bonds
and when speed of sound is high (so when the material is hard.)

See attachments from Kittel Intro to Solid State Physics

Elements w/ highest atomic density	C	$17.6 \times 10^{22}/cm^3$
	B	12.1 "
	Ni	9.4 "
	Co	8.7 "
Element w/ lowest atomic density	K	$1.4 \times 10^{22}/cm^3$
	Rb	1.15 "
	Cs	0.905 "
	Ba	1.6 "

See attachment from book Understanding the Properties of Matter by de Podesta

Problem 4.2

Elements w/ large speeds of sound	Carbon	18350 m/s
	Be	12890 "
	Cr	6600
	Ti	6130
" w/ low speeds of sound	Pb	2160
	Cd	2780
	Au	3240
	Pt	3260

So we would anticipate that Θ_D is high for C (diamond) and maybe Be or B and should be low for Cs and maybe Pb or Rb.

lets look them up (see attachment from Kittel)

highest	Carbon	$\Theta_D = 2230$
	Be	$= 1440$
lowest	Cs	~~Θ_D~~ $= 38$
	Rb	$= 56$

So our predictions were pretty good.

Problem 5 (Simon 2.5)

Estimate Θ_D for diamond

$\Theta_D = \frac{\hbar \omega_D}{K_B}$ where $\omega_D = \left[6\pi^2 n\right]^{1/3} v$

from Previous problem $n = 17.6 \times 10^{22}/cm^3$
$v = 18350\ m/s$

$\omega_D = \left[6\pi^2\ 17.6 \times 10^{22}/cm^3\right]^{1/3}\ 1.835 \times 10^6\ cm/s$
$\omega_D = 3.93 \times 10^{14}/s$

$\Theta_D = \frac{\overset{\hbar}{(6.58 \times 10^{-16}\ eV \cdot s)}\ \overset{\omega_D}{(3.93 \times 10^{14}/s)}}{(8.6 \times 10^{-5}\ eV/K)}$

$\Theta_D = 3.02 \times 10^3\ {}^\circ K$

From previous problem value is about $2230\ {}^\circ K$
so we have over estimated

probably the speed of sound goes down as the material gets hotter as it starts to soften so we don't have as high a Θ_D also it might expand so $n \downarrow$ as $T \uparrow$

Grad Students only.

Problem 6 Simon 2.6

(A) If $v_t \neq v_\ell$ then

$$\langle E \rangle = 2 \sum_{k \text{ transverse}}' \hbar \omega_k \left[n(\beta \hbar \omega) + 1/2 \right] + \sum_{k \text{ longitudinal}}' \hbar \omega_k \left[n(\beta \hbar \omega) + 1/2 \right]$$

$$= 2 \frac{L^3}{(2\pi)^3} \int d\vec{k}\ \hbar\omega \left[n(\beta\hbar\omega) + 1/2 \right]$$
transverse

$$+ \frac{L^3}{(2\pi)^3} \int d\vec{k}\ \hbar\omega \left[n(\beta\hbar\omega) + 1/2 \right]$$
longitudinal

let $\omega_{transverse} = v_t k$ and
$\omega_{longitudinal} = v_\ell k$

Then $$\langle E \rangle = \frac{2L^3}{(2\pi)^3} \int 4\pi \frac{\omega^2 d\omega}{v_t^3} \left(n(\beta\hbar\omega) + 1/2 \right) + \frac{L^3}{(2\pi)^3} \int 4\pi \frac{\omega^2 d\omega}{v_\ell^3}\, n(\beta\hbar\omega + 1/2)$$

$$= \left[\frac{2}{v_t^3} + \frac{1}{v_\ell^3} \right] \frac{L^3}{(2\pi)^3} \int 4\pi \omega^2 d\omega \left(n(\beta(\hbar\omega)) + 1/2 \right)$$

So this modifies $g(\omega)$

from $$\frac{L^3}{(2\pi)^3} \frac{(4)(3)\pi\omega^2}{v^3} \rightarrow \frac{L^3}{(2\pi)^3} 4\pi\omega^2 \left[\frac{2}{v_t^3} + \frac{1}{v_\ell^3} \right]$$

So that ω_0^3 now is equal to

$$\omega_D^3 = 6\pi^2 n \left[\frac{2}{v_t^3} + \frac{1}{v_\ell^3} \right]^{-1} \left(\frac{1}{3} \right)$$ need this to get right behav when $v_t = v_\ell$

Temp dependence of C remains the same but slope of $C = \alpha T^3$ ← slope is different.

Problem 6.2

(B) If instead $V_x \neq V_y \neq V_z$
then lets backtrack to page 9.

$$E = \hbar\omega\left[[n_x + 1/2] + [n_y + 1/2] + [n_z + 1/2]\right]$$

but lets keep them separate for a while

Such that

$$\langle E \rangle = \sum_{k_x}' \sum_{k_y}' \sum_{k_z}' \hbar\omega_{(k_x, k_y, k_z)} \; n(\beta\hbar\omega_{k_x, k_y, k_z} + 1/2)$$

$$\sum_{k_x}' \Rightarrow \int_{-\infty}^{\infty} \frac{L}{2\pi} \, dk_x$$

So $\quad \omega = \alpha|k| \quad$ where $|k| = \sqrt{k_x^2 + k_y^2 + k_z^2}$

So $\quad \omega^2 = V_x^2 k_x^2 + V_y^2 k_y^2 + V_z^2 k_z^2$

So $\quad \alpha = \dfrac{|k|}{\sqrt{\omega^2}} = \dfrac{\sqrt{k_x^2 + k_y^2 + k_z^2}}{\sqrt{V_x^2 k_x^2 + V_y^2 k_y^2 + V_z^2 k_z^2}}$

then $\quad \langle E \rangle = 3 \dfrac{L^3}{(2\pi)^3} \displaystyle\int d\vec{k} \; \hbar|\omega| \left(n(\beta\hbar\omega) + 1/2\right)$

where $\quad d\vec{k} = dk_x dk_y dk_z$

So we need to solve the integral (see attached)

Prob 6.3

Assume $w = \sqrt{v_x^2 k_x^2 + v_y^2 k_y^2 + v_z^2 k_z}$

let $v_x = a$, $v_y = b$, $v_z = c$ and $k_x = x$ $k_y = y$ $k_z = z$

```
In[1]:= Integrate[Sqrt[a^2 x^2 + b^2 y^2 + c^2 z^2] / (Exp[d x Sqrt[a^2 x^2 + b^2 y^2 + c^2 z^2]] - 1),
     {x, -∞, ∞}, {y, -∞, ∞}, {z, -∞, ∞}]
```

$d = \beta\hbar$

```
Out[1]= ConditionalExpression[4 π^5 / (15 Sqrt[a^2 b^2] d^4 Abs[c]), Re[d] > 0]

In[3]:= (* Lets say that one of the velocities was huge
    (i.e. vx>>>vy and vz, then the energy associated with any kx>
     0 would be enormous and for most values of T except very high,
     the mode would be unoccupied . That would be the same as if we didn't integrate
      over kx (it would be held at kx=0). Then our integral changes as below.*)
    Integrate[Sqrt[b^2 y^2 + c^2 z^2] / (Exp[d x Sqrt[b^2 y^2 + c^2 z^2]] - 1), {y, -∞, ∞}, {z, -∞, ∞}]

Out[3]= ConditionalExpression[4 π Zeta[3] / (Sqrt[b^2 c^2] d^3), Re[d] > 0]
```

Low Temperature limit:

If $v_x \neq v_y \neq v_z$ but all are within a reasonable range of one another then

$$\langle E\rangle = \frac{3L^3}{(2\pi)^3}\,\frac{\hbar\, 4\pi^5}{15\, v_x v_y v_z\, (\beta\hbar)^4} \propto T^4$$

So $C = \frac{\partial E}{\partial T} \sim T^3$

If $v_x \gg v_y$ and $v_x \gg v_z$

then

$$\langle E\rangle \sim 1/d^3 \quad \text{and} \sim \frac{1}{v_y v_z} T^3$$

So $C = \frac{\partial\langle E\rangle}{\partial T} \sim T^2$