HW#3 Solution

1. (a) Like the derivations for 3D in the textbook.

in 1D we have

$$N_{1D} = 2\sum_k n_F(\beta(\epsilon(\vec{k})-\mu)) = 2\frac{L}{2\pi}\int d\vec{k}\; n_F(\beta(\epsilon(\vec{k})-\mu))$$

At $T=0$ K the Fermi distribution becomes a step function, then

$$N_{1D} = \frac{L}{\pi}\int d\vec{k}\;\theta(E_F-\epsilon(\vec{k})) = \frac{L}{\pi}\int^{|k|<k_F} dk = \frac{L}{\pi}k_F$$

Define the density of electrons per unit length as

$$n = \frac{N_{1D}}{L}$$

we have $k_F = n\pi$

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2m}$$

$$N_{1D} = L\int_0^{\infty} d\epsilon\, g(\epsilon)\, n_F(\beta(\epsilon-\mu)) \qquad \text{Eqn. 4.9}$$

$$= \frac{L}{\pi}\int_{1D} d\vec{k}\; n_F(\beta(\epsilon-\mu)) \qquad k=\sqrt{\frac{2m\epsilon}{\hbar^2}} \Rightarrow dk = \sqrt{\frac{m}{2\epsilon\hbar^2}}\,d\epsilon$$

$$\Rightarrow g(\epsilon)d\epsilon = \frac{1}{\pi}\sqrt{\frac{m}{2\epsilon\hbar^2}}\,d\epsilon \quad \propto \epsilon^{-\frac{1}{2}}$$

in 2D, likewise

$$N_{2D} = 2\left(\frac{L}{2\pi}\right)^2\int d\vec{k}\; n_F(\beta(\epsilon-\mu))$$

$$= 2\left(\frac{L}{2\pi}\right)^2\int^{|k|<k_F} d\vec{k} = 2\left(\frac{L}{2\pi}\right)^2 \pi k_F^2 = \frac{L^2}{2\pi}k_F^2$$

$$n = \frac{N}{L^2} = \frac{k_F^2}{2\pi} \Rightarrow k_F = \sqrt{2\pi n}$$

$$E_F = \frac{\hbar^2 n\pi}{m}$$

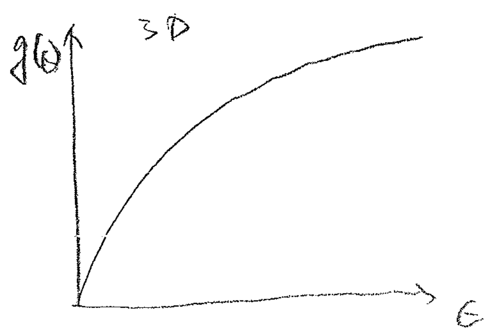
$$N_{2D}' = L^2 \int_0^\infty d\epsilon\, g(\epsilon)\, n_F(\beta(\epsilon-\mu))$$

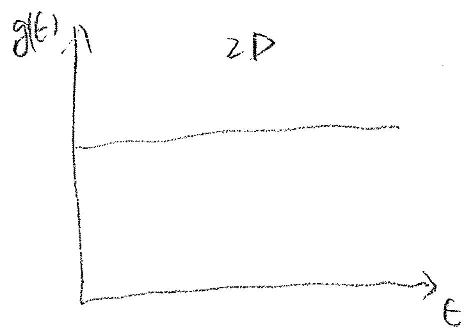
$$= 2\left(\frac{L}{2\pi}\right)^2 \int 2\pi k\, dk\, n_F(\beta(\epsilon-\mu))$$

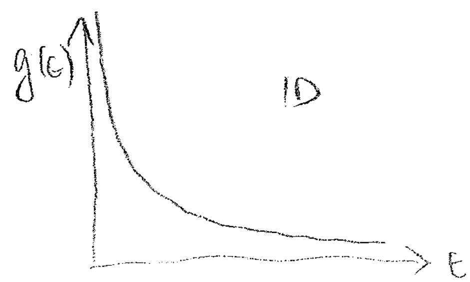
$$\Rightarrow g(\epsilon)d\epsilon = 2\left(\frac{1}{2\pi}\right)^2 2\pi \sqrt{\frac{2m\epsilon}{\hbar^2}} \sqrt{\frac{m}{2\epsilon\hbar^2}}\, d\epsilon$$

$$= \frac{m}{\pi\hbar^2} d\epsilon \quad \propto \epsilon^0$$

(b) in 3D $\quad g(\epsilon) = \frac{(2m)^{\frac{3}{2}}}{2\pi^2\hbar^3} \epsilon^{\frac{1}{2}}$







2.(a) From the text book we know in 3D

$$k_F = (3\pi^2 n)^{1/3}$$

thus $$v_F = \frac{\hbar k_F}{m} = \frac{\hbar}{m}(3\pi^2 n)^{\frac{1}{3}}$$

(b) we know $\vec{j} = \sigma \vec{E}$ and $\vec{j} = ne\vec{v}_D$

$$\Rightarrow v_D = \left|\frac{\sigma E}{ne}\right|$$

(c) For copper at 300K, $n = 8.45\times10^{28}\ m^{-3}$, $\sigma = 5.9\times10^{7}\ \Omega^{-1}m^{-1}$

then $$v_F = \frac{\hbar}{m}(3\pi^2 n)^{\frac{1}{3}}$$

$$= \frac{1.05\times10^{-34}\ m^2\cdot kg/s}{9.11\times10^{-31}\ kg}(3\pi^2\times 8.45\times10^{28}\ m^{-3})^{\frac{1}{3}}$$

$$= 1.56\times10^{6}\ m/s$$

$$v_d = \frac{\sigma E}{ne} = \frac{5.9\times10^{7}\ \Omega^{-1}m^{-1}\times 1\ V m^{-1}}{8.45\times10^{28}\ m^{-3}\times 1.6\times10^{-19}\ C}$$

$$= 4.36\times10^{-3}\ m/s$$

v_d and v_F are of 9 order of magnitude different, which can tell us the electrons move at a very high Fermi velocity randomly. In the current direction, the drift velocity induced by the $\vec{E}$ field

(C) since $\sigma = \frac{ne^2\tau}{m}$ and $\tau = \frac{\lambda}{V_F}$

then the mean free path for Cu is

$$\lambda = \frac{m\sigma V_F}{ne^2}$$

$$= \frac{9.11 \times 10^{-31}\,kg \times 5.9 \times 10^7\,\Omega^{-1} m^{-1} \times 1.56 \times 10^6\,m/s}{8.45 \times 10^{28}\,m^{-3} \times (1.6 \times 10^{-9}\,C)^2}$$

$$= 3.89 \times 10^{-8}\,m$$

The space betwean Cu atoms is about 10^{-10} m, compare to the mean free path $\sim 10^{-8}$ m, which could tell the Cu is a very good conductor.

3. For this problem, we assume $T = 0\,K$.

(a) $$E = V\int_0^\infty d\epsilon \cdot \epsilon\, g(\epsilon)\, n_F(\beta(\epsilon-\mu))$$

$$= V\int_0^{E_F} d\epsilon\, \epsilon\, g(\epsilon)$$

$$= V\int_0^{E_F} d\epsilon\, \epsilon\, \frac{(2m)^{\frac{3}{2}}}{2\pi^2\hbar^3}\epsilon^{\frac{1}{2}}$$

$$= \frac{2}{5}\frac{N}{n}\frac{(2m)^{\frac{3}{2}}}{2\pi^2\hbar^3}E_F^{\frac{5}{2}}$$

$$= \frac{3}{5}N E_F \qquad \text{where } E_F = \frac{\hbar^2(3\pi^2 n)^{2/3}}{2m}$$

(b) $$P = -\frac{\partial E}{\partial V} = -\frac{\partial}{\partial V}\left[\frac{3}{5}N\frac{\hbar^2\left(3\pi^2\frac{N}{V}\right)^{\frac{2}{3}}}{2m}\right]$$

$$= -\frac{3}{5}N\frac{\hbar^2}{2m}(3\pi^2 N)^{\frac{2}{3}}\left(-\frac{2}{3}V^{-\frac{5}{3}}\right)$$

$$= \frac{\hbar^2}{5m}(3\pi^2)^{\frac{2}{3}}\left(\frac{N}{V}\right)^{\frac{5}{3}}$$

$$B = -V\frac{\partial P}{\partial V} = -V\frac{\hbar^2}{5m}(3\pi^2)^{\frac{2}{3}}N^{\frac{5}{3}}\frac{\partial}{\partial V}\left(V^{-\frac{5}{3}}\right)$$

$$= \frac{\hbar^2}{3m}(3\pi^2)^{\frac{2}{3}}\left(\frac{N}{V}\right)^{\frac{5}{3}}$$

(c) $B = \frac{\hbar^2}{3m}(3\pi^2)^{\frac{2}{3}}\, n^{\frac{5}{3}}$

Then for Sodium, $n = 2.53\times10^{22}\ cm^{-3}$, while it's monovalent

$$B_s = \frac{(1.05\times10^{-34}\ m^2\cdot kg/s)^2}{3\times9.11\times10^{-31}\ kg}(3\pi^2)^{\frac{2}{3}}\times(2.53\times10^{28}\ m^{-3})^{\frac{5}{3}}$$

$= 8.42\times10^{9}\ kg\cdot m^{-1}\cdot s^{-2}$

$= 8.42\ GPa$ $>$ (measured value $6.3\ GPa$)

For Potassium, $n = 1.33\times10^{22}\ cm^{-3}$, monovalent

in the same way

$B_p = 2.88\ GPa$ $<$ (measured value $3.1\ GPa$)

4. (a) From Eqn. 51 in kittle

$$\begin{cases} m\left(\frac{d}{dt}+\frac{1}{\tau}\right)v_x = -e\left(E_x+\frac{B}{c}v_y\right) \\ m\left(\frac{d}{dt}+\frac{1}{\tau}\right)v_y = -e\left(E_y-\frac{B}{c}v_x\right) \end{cases}$$

let v_x, v_y vary as $e^{-i\omega t}$, we have

$$\begin{cases} i\omega v_x = \frac{e}{m}E_x + \omega_c v_y \\ i\omega v_y = \frac{e}{m}E_y - \omega_c v_x \end{cases}$$

$$\Rightarrow \begin{cases} v_x = \frac{e}{m}\frac{i\omega E_x+\omega_c E_y}{\omega_c^2-\omega^2} \simeq -\frac{e}{m}\frac{i\omega E_x+\omega_c E_y}{\omega^2} \\ v_y = \frac{e}{m}\frac{i\omega E_y-\omega_c E_x}{\omega_c^2-\omega^2} \approx -\frac{e}{m}\frac{i\omega E_y-\omega_c E_x}{\omega^2} \end{cases} \quad (\omega \gg \omega_c)$$

Since $\vec{J} = -ne\vec{v}$, we have

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \frac{ne^2}{m}\begin{pmatrix} \frac{i}{\omega} & \frac{\omega_c}{\omega^2} \\ -\frac{\omega_c}{\omega^2} & \frac{i}{\omega} \end{pmatrix}\cdot\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$= \begin{pmatrix} \frac{i\omega_p^2}{4\pi\omega} & \frac{\omega_c\omega_p^2}{4\pi\omega^2} \\ -\frac{\omega_c\omega_p^2}{4\pi\omega^2} & \frac{i\omega_p^2}{4\pi\omega} \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix}, \quad \omega_p^2 = \frac{4\pi ne^2}{m}$$

while $\vec{J} = \overleftrightarrow{\sigma}\vec{E}$, we know

$$\sigma_{xx} = \sigma_{yy} = \frac{i\omega_p^2}{4\pi\omega}, \quad \sigma_{xy} = -\sigma_{yx} = \frac{\omega_c\omega_p^2}{4\pi\omega^2}$$

(b) From $\overleftrightarrow{\varepsilon} = 1 + i\frac{4\pi}{\omega}\overleftrightarrow{\sigma}$, we have

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} 1-\frac{\omega_p^2}{\omega^2} & \frac{i\omega_c\omega_p^2}{\omega^3} \\ -i\frac{\omega_c\omega_p^2}{\omega^3} & 1-\frac{\omega_p^2}{\omega^2} \end{pmatrix}, \quad \begin{matrix} \varepsilon_{xx} = \varepsilon_{yy} \\ \varepsilon_{xy} = -\varepsilon_{yx} \end{matrix}$$

From the Maxwell's eqn. let $E = E_0 e^{i(kz-\omega t)}$

$$c^2\nabla^2 E = \varepsilon\frac{\partial^2 E}{\partial t^2}$$

$$\Rightarrow c^2(-k^2)E = \varepsilon\omega^2 E$$

$$\Rightarrow \begin{pmatrix} \varepsilon_{xx}\omega^2 - c^2k^2 & \varepsilon_{xy}\omega^2 \\ \varepsilon_{yx}\omega^2 & \varepsilon_{yy}\omega^2 - c^2k^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix} = 0$$

The determinant is 0

$$\Rightarrow (\varepsilon_{xx}\omega^2 - c^2k^2)^2 + (\varepsilon_{xy}\omega^2)^2$$

$$\left[\left(1-\frac{\omega_p^2}{\omega^2}\right)\omega^2 - c^2k^2\right]^2 - \left(\frac{\omega_c\omega_p^2}{\omega^3}\omega^2\right)^2 = 0$$

$$\Rightarrow c^2k^2 = \omega^2 - \omega_p^2 \pm \frac{\omega_c\omega_p^2}{\omega}$$