HW # 2 - Solutions

1. Velocities in the Free Electron Theory.

a) 3d

$$N = 2\,\frac{\frac{4}{3}\pi k_F^3}{\frac{(2\pi)^3}{V}}$$

$$\frac{N}{V} = \frac{k_F^3}{3\pi^2} \Rightarrow k_F = (3\pi^2 n)^{1/3}$$

$$\boxed{v_F = \frac{\hbar k_F}{m} = \frac{\hbar}{m}(3\pi^2 n)^{1/3}}$$

b) $j = \sigma E \qquad \sigma = \dfrac{ne^2\tau}{m}$

$$n e v_d = \sigma E$$

$$\boxed{v_d = \frac{\sigma E}{ne}} \qquad \tau = \frac{\lambda}{v_F} \Rightarrow \boxed{\sigma = \frac{ne^2\lambda}{m v_F}}$$

) Cu at 300 K $\quad E = 1V/m.$

$n = 8.45 \times 10^{28} \ m^{-3}.$

$\sigma = 5.9 \times 10^{7} \ \Omega^{-1} m^{-1}$ at 300 K

i) $v_d = \dfrac{(5.9 \times 10^{7})(1)}{(8.45 \times 10^{28})(1.6 \times 10^{-19} \ C)} = \dfrac{(5.9)}{(8.45)(1.6)} \cdot 10^{-2} \ m/s$

$$\boxed{v_d = 4.36 \times 10^{-3} \ m/sec.}$$

$v_F = \dfrac{\hbar}{m} (3\pi^2 n)^{1/3}.$

$= \dfrac{1.05 \times 10^{-34}}{9 \times 10^{-31}} \left(3\pi^2 \cdot 8.45 \times 10^{28}\right)^{1/3}.$

$= \dfrac{1.05}{9} \cdot 10^{-3} \left(3\pi^2 \cdot 8.45 \cdot 10^{1}\right) 10^{9}$

$= \dfrac{1.05}{9} \cdot 10^{6} \left\{2502\right\}^{1/3}. \qquad c = 3 \times 10^{8} \ m/s$

$$\boxed{v_F \sim 1.58 \times 10^{6} \ m/s} \sim \frac{1}{200} c$$

$$\frac{v_d}{v_F} = \frac{4.36 \times 10^{-3} \ m/s}{1.58 \times 10^{6} \ m/s} = 2.8 \times 10^{9} \ !$$

We note that v_d, the drift velocity, refers to the shift $\left(\frac{m v_d}{\hbar}\right)$ of the center of mass of the entire Fermi sphere and is indeed very small compared to the velocity of the most energetic electrons.

ii) $$\lambda = \frac{m v_F}{ne^2} \sigma$$

$$= \frac{(9 \times 10^{-31})(1.58 \times 10^{6})}{(8.45 \times 10^{28})(1.6 \times 10^{-19})^2} \quad (5.9 \times 10^{7})$$

$$= \frac{(9)(1.58)(5.9)}{(8.45)(1.6)^2} \quad \frac{10^{-18}}{10^{-10}}$$

$$\lambda = 3.9 \times 10^{-8} \ m$$

Mean atom spacing in Cu $\sim 1 \ \mathring{A} = 10^{-10}$ m

$\lambda \sim 400$ atomic spacings !

Another Review of the Free Electron Theory.

- Free Electron Model of a Metal

Here I assume the free electron model refers to the Sommerfeld theory.

A metal is viewed as a gas of noninteracting fermions with spin 1/2 in an infinite square well potential.

- Fermi energy E_F = chemical potential at $T=0$

$$E_F = \frac{\hbar^2 k_F^2}{2m}$$

Fermi temperature $T_F = \frac{E_F}{k_B}$.

- Even at temperatures $0 < T \ll T_F$, some electrons are thermally excited. These electrons transport both electricity and heat as evidenced by the Wiedemann-Franz ratio.

(a) d-dimensional sample w/ volume L^d

w/ N electrons.

$$N = \underset{\substack{\downarrow \\ \text{spin}}}{2} \frac{c_d \, k_F^d}{\left(\frac{2\pi}{L}\right)^d}$$

$$N = 2 \left(\frac{L}{2\pi}\right)^d c_d \, k_F^d$$

$$\Downarrow$$

$$k_F = \left(\frac{N}{2c_d}\right)^{1/d} \frac{2\pi}{L}$$

$$E_F = \frac{\hbar^2}{2m} \left(\frac{2\pi}{L}\right)^2 \left(\frac{N}{2c_d}\right)^{2/d}$$

$$= \frac{\hbar^2}{2mL^2} \left(\frac{N (2\pi)^d}{2c_d}\right)^{2/d}$$

$$= \frac{\hbar^2}{2mL^2} (N a_d)^{2/d} \qquad a_d = \frac{(2\pi)^d}{2c_d}$$

$d=1.$ $\quad c_d = 2$

$$a_d = \frac{\pi}{2}.$$

$d=2$ $\quad c_d = \pi.$

$$a_d = \frac{(2\pi)^2}{2\pi} = 2\pi$$

$d=3$ $\quad c_d = \frac{4}{3}\pi$

$$a_d = \frac{(2\pi)^3}{2\,\frac{4}{3}\pi} = 3\pi^2$$

(v). $g(\epsilon_F)$

We've just found that

$$\epsilon_F = \frac{\hbar^2}{2mL^2}(N a_d)^{2/d}$$

density of electrons N/V

$$= \frac{\hbar^2}{2m}\left(\frac{N}{L^d}\,a_d\right)^{2/d} = \frac{\hbar^2}{2m}\left(n\,a_d\right)^{2/d}$$

$$\ln E_F = \frac{2}{d} \ln n + \text{const}$$

We differentiate each side to obtain

$$\frac{dE_F}{E_F} = \frac{2}{d} \frac{dn}{n}$$

$$\Downarrow$$

$$\frac{dn}{dE_F} = g(E_F) = \frac{d}{2} \frac{n}{E_F} = \frac{d}{2L^d} \frac{N}{E_F}$$

$$\boxed{g(E_F) = \frac{Nd}{2L^d E_F}}$$

1-d $\quad n = N/L = 1/.8 \text{ nm} = 1.25 \times 10^{+9}$

$$E_F = \frac{\hbar^2}{2mL^2} (Na_1)^2$$

$$= \frac{\hbar^2}{2m} n^2 \left(\frac{\pi}{2}\right)^2$$

$$E_F = \frac{\left(1.05 \times 10^{-34}\right)^2}{2\left(9 \times 10^{-31}\right)} \left(1.25 \times 10^{9}\right)^2 \left(\frac{\pi}{2}\right)^2$$

$$= \frac{(1.05)^2 (1.25)^2 \pi^2}{(2)(9)(2)^2} \cdot \frac{10^{-68}\ 10^{18}}{10^{-31}}$$

$$\approx .236 \times 10^{-19}\ J$$

$$= 2.36 \times 10^{-20}\ J \ \frac{1\,eV}{1.6 \times 10^{-19}\ J} = .14\ eV$$

$$\boxed{E_F = .14\ eV}$$

$$T_F = \frac{E_F}{k_B}$$

$$= \frac{2.4 \times 10^{-20}\ J}{1.4 \times 10^{-23}\ J/K} = 1.7 \times 10^{3}\ K$$

$$\boxed{T_F = 1.7 \times 10^{3}\ K}$$

(c) $E = c|p|$

From (a) we have

$$k_F = \left(\frac{N}{2c_d}\right)^{1/d} \frac{2\pi}{L}$$

Now $E_F = c\left(\frac{\hbar k_F}{m}\right) = \frac{\hbar c}{m} \frac{2\pi}{L} \left(\frac{N}{2c_d}\right)^{1/d}$

$$E_F = \frac{2\pi\hbar c}{m} \left(\frac{N}{L^d\, 2c_d}\right)^{1/d} \qquad n = \frac{N}{L^d}$$

$$\boxed{E_F = \frac{2\pi\hbar c}{m} \left(\frac{n}{2c_d}\right)^{1/d}}$$

$$\ln E_F = \frac{1}{d} \ln n + \text{constant}$$

$$\frac{dE_F}{dE_F} = \frac{1}{d} \frac{dn}{n} \Rightarrow \frac{dn}{dE_F} = g(\varepsilon_F) = \frac{nd}{\varepsilon_F}$$

$$\boxed{g(E_F) = \frac{nd}{E_F}}$$

$d = 1$ $\quad C_d = 2.$

$$E_F = \frac{2\pi \hbar c}{m}\left(\frac{n}{2c_d}\right)^{1/d} = \frac{2\pi \hbar c}{m}\left(\frac{n}{4}\right)^{1}$$

$$\boxed{E_F = \frac{\pi \hbar c\, n}{2m}} \quad d=1$$

$$g(E_F) = \frac{nd}{\varepsilon_F} \Rightarrow \boxed{g(E_F) = \frac{n}{\varepsilon_F}} \quad d=1$$

$d = 2$ $\quad C_d = 4\pi$

$$E_F = \frac{2\pi \hbar c}{m}\left(\frac{n}{2c_d}\right)^{1/d} = \frac{2\pi \hbar c}{m}\left(\frac{n}{8\pi}\right)^{1/2}$$

$$\boxed{E_F = \frac{\hbar c}{m}\left(\frac{\pi n}{2}\right)^{1/2}} \quad d=2.$$

$$g(E_F) = \frac{nd}{\varepsilon_F} \Rightarrow \boxed{g(E_F) = \frac{2n}{\varepsilon_F}} \qquad d = 2.$$

$$d = 3 \qquad c_d = \frac{4}{3}\pi$$

$$E_F = \frac{2\pi\hbar c}{m}\left(\frac{n}{2c_d}\right)^{1/d} = \frac{2\pi\hbar c}{m}\left(\frac{3n}{8\pi}\right)^{1/3}.$$

$$\boxed{E_F = \left(\frac{2\pi\hbar c}{m}\right)\left(\frac{3n}{8\pi}\right)^{1/3}.} \qquad d = 3$$

$$g(\varepsilon_F) = \frac{nd}{\varepsilon_F} \Rightarrow \boxed{g(E_F) = \frac{3n}{\varepsilon_F}} \qquad d = 3.$$

7 a)

$$\Delta KE = \underbrace{\frac{m}{2}\left(\vec{v} - \frac{e\vec{E}t}{m}\right)^2}_{\substack{\text{kinetic energy}\\ \text{before second}\\ \text{collision}}} - \underbrace{\frac{m}{2}\left(\vec{v}\,'\right)^2}_{\substack{\text{kinetic energy}\\ \text{just after}\\ \text{second collision}}}$$

$$= \underbrace{\frac{m}{2}\left(v^2 - v'^2\right)}_{\substack{= 0\\ \text{(assumption of}\\ \text{Drude model)}}} + \frac{m}{2}\left(\frac{eEt}{m}\right)^2 - \underbrace{\frac{met}{m}\,\vec{E}\cdot\vec{v}}_{\substack{\vec{v}\text{ averaged over}\\ \text{spherically}\\ \text{symmetric}\\ \text{distribution} \Rightarrow 0}}$$

$\therefore$ Average energy loss $= \dfrac{(eEt)^2}{2m}$

τ = mean time between two collisions for a single electron

$$\frac{\text{Energy loss}}{\text{Electron collision}} = \frac{(eE\tau)^2}{2m}$$

For each collision $\Rightarrow$ average # electrons = 2.

$$\frac{\text{Average energy loss}}{\text{cm}^3 - \text{s}} = \underbrace{2}_{\text{\# electrons in collision}} \cdot \underbrace{\left(\frac{n}{\tau}\right)}_{\text{\# collisions/sec}} \left(\frac{(eE\tau)^2}{2m}\right)$$

$$= \underbrace{\frac{ne^2\tau}{m}}_{\sigma} E^2.$$

$$[P] = [E]/[T]$$

$$\frac{P}{LA} = (\sigma E)E = \frac{j^2}{\sigma} = \frac{(I/A)^2 RA}{L}$$

$j = I/A \quad \sigma = 1/\rho \quad \rho = \frac{RA}{L}$

$$\frac{P}{LA} = \frac{\left(I/A\right)^2 RA}{L}$$

$$\Downarrow$$

$$P = I^2 R$$ Joule heating

4. Fermi gases in astrophysics

a) N_e = # electrons

$$N_p \sim \frac{M_\odot}{m_p} = \frac{2 \times 10^{33}\ g}{1.7 \times 10^{-24}\ g} \sim 10^{57}.$$

protons in sun

Let us assume that there are roughly the same number of e's and p's

$$\Downarrow$$

$$\varepsilon_F = \frac{\hbar}{2m} (3\pi^2 n)^{2/3}$$

where

$$n = \frac{N_e}{V} \quad \text{where} \quad V = \frac{4}{3}\pi R_s^3$$

$$= \frac{4}{3}\pi (2 \times 10^9)^3.$$

$$\sim 3 \times 10^{28}\ cm^3.$$

$$n \sim \frac{10^{57}}{3 \times 10^{28}} \sim 3 \times 10^{28} \text{ electrons}/cm^3.$$

$$\Downarrow$$

$$\varepsilon_F \sim \frac{(10^{-27})^2}{2(9 \times 10^{-28})} \left\{ 3\pi^2 (3 \times 10^{-28}) \right\}^{2/3}$$

$$\sim \frac{1}{2} (10^{-27})(10^{20}) \sim 5 \times 10^{-6} \text{ ergs}$$

$$\varepsilon_F \sim 5 \times 10^{-6} \text{ ergs} \times \frac{1 \text{ eV}}{1.6 \times 10^{-12} \text{ ergs}}$$

$$\Downarrow$$

$$\boxed{\varepsilon_F = 3.6 \times 10^{4} \text{ eV}}$$

b) k_F is not affected by relativity

In 3d we determine k_F

$$\Downarrow$$

$$N = \frac{2 \frac{4}{3} \pi k_F^3}{(2\pi)^3 / V} \Rightarrow k_F \sim \left(\frac{N}{V} \right)^{1/3}$$

Then in the relativistic limit

$$\Downarrow$$

$$\boxed{\varepsilon_F = \hbar k_F c \sim \hbar c \left(\frac{N}{V}\right)^{1/3}.}$$

c) Now $\tilde{R}_S = 10$ km $= 10^6$ cm

$(R_S = 2 \times 10^9$ cm$)$

$$n \sim 3 \times 10^{28} \frac{e}{cm^3} \times \frac{(2 \times 10^9)^3}{10^{18}}$$

$$\sim 2.4 \times 10^{38} \ e/cm^3$$

$$\varepsilon_F \sim \hbar c n^{1/3} \sim (10^{-27})(3 \times 10^{10})(10^{13})$$

$$\sim 2 \times 10^{-4} \text{ erg} \cdot \frac{1 \text{ eV}}{1.6 \times 10^{-12} \text{ erg}}$$

$$\Downarrow$$

$$\boxed{\varepsilon_F \sim 10^8 \text{ eV}}$$ relativistic

$(m_e/c^2 \sim .51 \times 10^6$ eV$)$.

Liquid He^3.

$$\varepsilon_F = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}.$$

$$\rho = .081 \ g/cm^3.$$

$$\frac{\#\ \text{moles}}{cm^3} = \frac{1}{3}\ 81 \times 10^{-3}$$

$$= 27 \times 10^{-3} = 2.7 \times 10^{-2} \ \frac{\text{moles}}{cm^3}.$$

$$n_H = \underset{\text{of atoms}}{\text{concentration}} = 2.7 \times 10^{-2} \frac{\text{moles}}{cm^3}$$

$$\times \ 6 \times 10^{23} \ \text{atoms}/\text{mole}$$

$$= 1.6 \times 10^{22} \ \text{atoms}/cm^3.$$

$$m_H = (3)\ m_p = (3)(1.6 \times 10^{-24}\ g)$$

$$\sim 5 \times 10^{-24}\ g.$$

$$\varepsilon_F \sim \frac{(10^{-27})^2}{2.5 \times 10^{-24}} \left\{ 3 \cdot \pi^2 \, (1.6 \times 10^{22}) \right\}^{2/3}.$$

$$\sim \frac{10^{-54}}{10^{-23}} \left\{ (3\pi^2)(16)(10^{21}) \right\}^{2/3}.$$

$$\sim \underbrace{10^{-31}}\left[[30][16] \right]^{2/3} \cdot 10^{14}$$

$$\boxed{\varepsilon_F \sim 6 \times 10^{-16} \text{ erg}}$$

$$\boxed{T_F = \frac{\varepsilon_F}{k_B} \sim \frac{6 \times 10^{-16} \text{ erg}}{1.4 \times 10^{-16} \text{ erg/K}} \sim 4.29 \text{ K}}$$

- Challenges associated w/ observing quantum degeneracy in Fermi gases of cold atoms and how this was achieved (from Jin, "A Fermi Gas of Atoms").
  - Cooling and equilibration of gases at low temperature achieved through hands-on "s-wave" collisions - not possible for fermions, due to Pauli exclusion principle, which makes it challenging to achieve low temperatures in equilibrium
  - Successful cooling of gases of fermionic atoms using s-wave collisions of mixtures of atoms that are in different states
  - Jin's group used atoms in two distinct spin states
  - Hulet's group used mixture of isotopes.
  - Evaporative cooling used for these mixtures to achieve quantum degeneracy via collisions.