Addendum to: Lossless trajectories of singularly passive systems

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1 Proof of Statement (1) of Theorem 1

A system with minimal input-state-output (i/s/o) representation

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \qquad \text{with } A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}, C \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}, D \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}} \tag{1}$$

is called passive if there exists a symmetric, positive-semidefinite $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that all system trajectories satisfy

$$\frac{d}{dt}\left(x(t)^T K x(t)\right) \leqslant 2u(t)^T y(t) \text{ for all } t \in \mathbb{R}. \tag{2}$$

The corresponding KYP LMI is:

$$\mathscr{L}(K) := \begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & -(D + D^T) \end{bmatrix} \leqslant 0. \tag{3}$$

The corresponding Extended Hamiltonian pencil is:

$$s\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathscr{E}} - \underbrace{\begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & D + D^T \end{bmatrix}}_{\mathscr{H}} \tag{4}$$

The output nulling representation is:

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}}_{\widehat{A}}\begin{bmatrix} x \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ C^T \end{bmatrix}}_{\widehat{B}}, \quad 0 = \underbrace{\begin{bmatrix} C & -B^T \end{bmatrix}}_{\widehat{C}}\begin{bmatrix} x \\ z \end{bmatrix} + (D + D^T)u. \tag{5}$$

Theorem 1.1 *Consider a regularly/singularly passive system Σ with a minimal i/s/o representation as given in equation* (1). *Let the corresponding EHP be as in equation* (4). *Assume Λ to be a Lambda-set of $\mathtt{det}(s\mathscr{E} - \mathscr{H})$ with cardinality $\mathtt{n_s}$. Define $\mathtt{n_f} := \mathtt{n} - \mathtt{n_s}$, $\widehat{A} := \left[\begin{smallmatrix} A & 0 \\ 0 & -A^T \end{smallmatrix}\right]$ and $\widehat{B} := \left[\begin{smallmatrix} B \\ C^T \end{smallmatrix}\right]$. Let $V_{1\Lambda}, V_{2\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$ and $V_{3\Lambda} \in \mathbb{R}^{1\times\mathtt{n_s}}$ be such that*

$$\begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & D + D^T \end{bmatrix}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ V_{3\Lambda} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ V_{3\Lambda} \end{bmatrix}\Gamma, \tag{6}$$

where $\Gamma \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$ and $\sigma(\Gamma) = \Lambda$. Define $V_\Lambda := \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n_s}}$ and
$W := \begin{bmatrix} \widehat{B} & \widehat{A}\widehat{B} & \ldots & \widehat{A}^{\mathtt{n_f}-1}\widehat{B} \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n_f}}$. Partition $\begin{bmatrix} V_\Lambda & W \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n}}$ as

$$\begin{bmatrix} V_\Lambda & W \end{bmatrix} =: \begin{bmatrix} X_{1\Lambda} \\ X_{2\Lambda} \end{bmatrix} \text{ where } X_{1\Lambda}, X_{2\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}. \tag{7}$$

Then, the following statements hold.

1. *$X_{1\Lambda}$ is invertible.*
2. *$K := X_{2\Lambda}X_{1\Lambda}^{-1}$ is symmetric.*
3. *K is a rank-minimizing solution of the LMI* (3).
4. *K is positive semi-definite, i.e., $K \geqslant 0$.*

W in Theorem 1.1 can be partitioned as follows:

$$W = \begin{bmatrix}\widehat{B} & \widehat{A}\widehat{B} & \cdots & \widehat{A}^{\mathtt{n_f}-1}\widehat{B}\end{bmatrix} = \begin{bmatrix} B & AB & \cdots & A^{\mathtt{n_f}-1}B \\ C^T & -(CA)^T & \cdots & (-1)^{\mathtt{n_f}-1}(CA^{\mathtt{n_f}-1})^T\end{bmatrix} =: \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}. \tag{8}$$

Define $V_{\mathtt{e}\Lambda} := \mathtt{col}(V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda})$. The following notational convention is required for the auxiliary results. Let $T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, nonsingular, be such that, under the similarity transformation induced by T, the system matrix A transforms to $A_{\mathtt{t}} := T^{-1}AT$. It is known that under this transformation, matrices b and c are transformed to $B_{\mathtt{t}} := T^{-1}B$ and $C_{\mathtt{t}} := CT$, respectively. We assume that $(A_{\mathtt{t}}, B_{\mathtt{t}}, C_{\mathtt{t}})$ is in the controller canonical form. Let $(\mathscr{E}_{\mathtt{t}}, \mathscr{H}_{\mathtt{t}})$ be the Hamiltonian matrix pair formed using the matrices $(A_{\mathtt{t}}, B_{\mathtt{t}}, C_{\mathtt{t}})$. Let $X_{1\Lambda}$ and $X_{1\Lambda_{\mathtt{t}}}$ be constructed as defined in Theorem 1.1 using Hamiltonian matrix pair (E,H) and $(\mathscr{E}_{\mathtt{t}}, \mathscr{H}_{\mathtt{t}})$, respectively.

Lemma 1.2 *Consider a BIBO stable, SISO system Σ with transfer matrix $G(s)$ and a minimal i/s/o representation as given in equation* (1)*. Let $(s\mathscr{E} - \mathscr{H})$ with $\mathscr{E}$ and $\mathscr{H}$, as defined in equation* (4) *above, be the corresponding Hamiltonian pencil. Then,*

$$\mathtt{det}(s\mathscr{E} - \mathscr{H}) = -\mathtt{num}\,(G(s) + G(-s)).$$

In particular, $\sigma(E,H) = \mathtt{rootnum}\,(G(s) + G(-s))$.

Proof: Let $G(s) =: n(s)/d(s)$, where $n(s), d(s) \in \mathbb{R}[s]$ are coprime. The eigenvalues of (E,H) are given by the roots of the characteristic polynomial $\mathtt{det}(s\mathscr{E} - \mathscr{H})$. Using the procedure known as Schur complement, we get the following identity of rational functions:

$$\begin{aligned}
\mathtt{det}(s\mathscr{E} - \mathscr{H}) = & -\mathtt{det}\,(sI - A)\,\mathtt{det}\,(sI + A^T) \\
& \mathtt{det}\left((D + D^T) + \begin{bmatrix} C & -B^T \end{bmatrix}\begin{bmatrix} (sI-A)^{-1} & 0 \\ 0 & (sI+A^T)^{-1}\end{bmatrix}\begin{bmatrix} B \\ C^T \end{bmatrix}\right) \\
= & -\mathtt{det}\,(sI - A)\,\mathtt{det}\,(sI + A^T)\left[\left(D + C(sI-A)^{-1}B\right) + \left(D^T - B^T(sI + A^T)^{-1}C^T\right)\right] \\
= & -d(s)d(-s)\left(\frac{n(s)}{d(s)} + \frac{n(-s)}{d(-s)}\right) \\
= & -\left(n(s)d(-s) + n(-s)d(s)\right)
\end{aligned}$$

Since Σ is BIBO stable, from Lemma 1.3 it follows that $\mathtt{num}\,(G(s) + G(-s)) = n(s)d(-s) + n(-s)d(s)$. Therefore, $\mathtt{det}(s\mathscr{E} - \mathscr{H}) = -\mathtt{num}\,(G(s) + G(-s))$. Hence, we infer that $\sigma(\mathscr{E}, \mathscr{H}) = \mathtt{rootnum}\,(G(s) + G(-s))$. ■

The first lemma reveals an interesting fact about all BIBO stable SISO systems. It establishes that a BIBO stable SISO system never admits common spectral zeros and poles.

Lemma 1.3 *Consider a SISO system Σ with transfer function $G(s) := \frac{n(s)}{d(s)}$, where $n(s), d(s) \in \mathbb{R}[s]$ are coprime. Define $q(s) := n(s)d(-s) + d(s)n(-s)$. Let $\lambda \in \mathtt{roots}\,(d(s))$. Then, $\lambda \in \mathtt{roots}\,(q(s)) \cap \mathtt{roots}\,(d(s))$ if and only if $\lambda \in \mathtt{roots}\,(d(-s))$.*
In particular, if Σ is BIBO stable, then the following statements are true

(1) $\mathtt{roots}(q(s)) \cap \mathtt{roots}(d(s)) = \emptyset$.

(2) $\mathtt{num}\,(G(s) + G(-s)) = q(s)$.

Proof: **If:** Given $d(\lambda) = d(-\lambda) = 0$. Therefore, $q(\lambda) = n(\lambda)d(-\lambda) + n(-\lambda)d(\lambda) = 0$. Thus, $\lambda \in \mathtt{roots}\,(q(s)) \cap \mathtt{roots}\,(d(s))$.
Only if: Given $d(\lambda) = q(\lambda) = 0$. Thus, $q(\lambda) = n(\lambda)d(-\lambda) + n(-\lambda)d(\lambda) = n(\lambda)d(-\lambda) = 0$. Since $n(s)$ and $d(s)$ are coprime, $n(\lambda) \neq 0$. Therefore, $d(-\lambda) = 0$, i.e., $\lambda \in \mathtt{roots}\,(d(-s))$.
(1): Clearly, if $G(s)$ is BIBO stable then $\lambda \in \mathtt{roots}\,(d(s))$ implies that $\lambda \notin \mathtt{roots}\,(d(-s))$. Therefore, $\lambda \notin \mathtt{roots}\,(q(s)) \cap \mathtt{roots}\,(d(s))$. Since this is true for all roots of $d(s)$, we must have $\mathtt{roots}\,(q(s)) \cap \mathtt{roots}\,(d(s)) = \emptyset$.
(2): From Statement *(1)* it is clear that $q(s)$ and $d(s)$ are coprime. We claim that $q(s)$ and $d(-s)$ are coprime, as well. To the contrary, assume that $q(s)$ and $d(-s)$ are not coprime. Let $\lambda_1 \in \mathtt{roots}\,(q(s)) \cap \mathtt{roots}\,(d(-s))$. Then, $\lambda_1 \in \mathtt{roots}(d(-s)) \Rightarrow -\lambda_1 \in \mathtt{roots}(d(s))$. Further, $q(\lambda_1) = n(\lambda_1)d(-\lambda_1) + n(-\lambda_1)d(\lambda_1) = 0 \Rightarrow n(-\lambda_1)d(\lambda_1) = 0 \Rightarrow d(\lambda_1) = 0$. Therefore, $\lambda_1 \in \mathtt{roots}(d(s))$. However, since Σ is BIBO stable, $\pm\lambda_1 \in \mathtt{roots}(d(s))$ is not possible. Therefore, we must have $\mathtt{roots}\,(q(s)) \cap \mathtt{roots}\,(d(-s)) = \emptyset$. Thus, $q(s)$ and $d(-s)$ are coprime, as well. Therefore, $q(s)$ and $d(s)d(-s)$ are coprime. This implies that $\mathtt{num}\,(G(s) + G(-s)) = \mathtt{num}\left(\frac{q(s)}{d(s)d(-s)}\right) = q(s)$. ■

Lemma 1.4 *Consider a BIBO stable SISO system Σ with a minimal i/s/o representation as in equation* (1)*. Let $(\mathscr{E}, \mathscr{H})$ be the corresponding Hamiltonian pencil as defined in equation* (4)*. Then $\sigma(\mathscr{E}, \mathscr{H}) \cap \sigma(A) = \emptyset$.*

Proof: Define $G(s) := \frac{n(s)}{d(s)}$, where $n(s), d(s) \in \mathbb{R}[s]$ are coprime. Since Σ is BIBO stable, we have $\mathtt{num}\,(G(s) + G(-s)) = n(s)d(-s) + n(-s)d(s) =: q(s)$. For a BIBO stable system, $\sigma(\mathscr{E}, \mathscr{H}) = \mathtt{rootnum}\,(G(s) + G(-s)) = \mathtt{roots}\,(q(s))$. Since Σ is BIBO stable, from Lemma 1.3 we have $\mathtt{roots}\,(q(s)) \cap \mathtt{roots}\,(d(s)) = \emptyset$. Thus, $\sigma(\mathscr{E}, \mathscr{H}) \cap \sigma(A) = \emptyset$. ■

Since singularly passive SISO systems are BIBO stable, from Lemma 1.4 it is evident that such systems have no common poles and spectral zeros.

The next lemma establishes the relation between the eigenvalues of the Hamiltonian matrix pair $(\mathscr{E},\mathscr{H})$ constructed using (A,B,C) as given in equation (4), and the transformed Hamiltonian matrix pair $(\mathscr{E}_{\mathtt{t}},\mathscr{H}_{\mathtt{t}})$ constructed using $(A_{\mathtt{t}},B_{\mathtt{t}},C_{\mathtt{t}})$.

Lemma 1.5 *Consider a singularly passive SISO system Σ with a minimal i/s/o representation as in equation* (1). *Let the corresponding Hamiltonian matrix pair be $(\mathscr{E},\mathscr{H})$ as constructed in equation* (4). *Let a controller canonical form i/s/o representation of Σ be $\frac{d}{dt}x = A_{\mathtt{t}}x + B_{\mathtt{t}}u$ and $y = C_{\mathtt{t}}x$. Let the Hamiltonian matrix pair constructed using $(A_{\mathtt{t}},B_{\mathtt{t}},C_{\mathtt{t}})$ be $(\mathscr{E}_{\mathtt{t}},\mathscr{H}_{\mathtt{t}})$. Then,*

$$\sigma(\mathscr{E},\mathscr{H}) = \sigma(\mathscr{E}_{\mathtt{t}},\mathscr{H}_{\mathtt{t}}).$$

Further, let $X_{1\Lambda},X_{2\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ be constructed as defined in equation (7) *of Theorem 1.1 using system matrices (A,B,C) and Hamiltonian matrix pair $(\mathscr{E},\mathscr{H})$ corresponding to a Lambda-set Λ. Similarly, let $X_{1\Lambda\mathtt{t}},X_{2\Lambda\mathtt{t}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ be constructed using equation* (7)*, system matrices $(A_{\mathtt{t}},B_{\mathtt{t}},C_{\mathtt{t}})$ and the Hamiltonian matrix pair $(\mathscr{E}_{\mathtt{t}},\mathscr{H}_{\mathtt{t}})$ corresponding to a Lambda-set Λ. Then,*

$$X_{1\Lambda} \text{ is invertible if and only if } X_{1\Lambda\mathtt{t}} \text{ is invertible.}$$

Proof: Let $T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ be a nonsingular matrix such that $T^{-1}AT = A_{\mathtt{t}}, T^{-1}B = B_{\mathtt{t}}$, and $CT = C_{\mathtt{t}}$. Define $\widehat{A}_{\mathtt{t}} := \mathtt{diag}(A_{\mathtt{t}}, -A_{\mathtt{t}}^T)$, $\widehat{B}_{\mathtt{t}} = \mathtt{col}(B_{\mathtt{t}}, C_{\mathtt{t}}^T)$, and $\widehat{C}_{\mathtt{t}} := \begin{bmatrix} C_{\mathtt{t}} & -B_{\mathtt{t}} \end{bmatrix}$. Further, define $\widehat{T} := \mathtt{diag}(T, T^{-T})$ and $\widetilde{T} := \mathtt{diag}(\widehat{T}, I_{\mathtt{p}})$, where $T^{-T} := (T^{-1})^T$. Then, we have the following

$$\widetilde{T}^{-1}H\widetilde{T} = \begin{bmatrix} T & & \\ & T^{-T} & \\ & & I_{\mathtt{p}} \end{bmatrix}^{-1} \begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & 0 \end{bmatrix} \begin{bmatrix} T & & \\ & T^{-T} & \\ & & I_{\mathtt{p}} \end{bmatrix} = \begin{bmatrix} A_{\mathtt{t}} & 0 & B_{\mathtt{t}} \\ 0 & -A_{\mathtt{t}}^T & C_{\mathtt{t}}^T \\ C_{\mathtt{t}} & -B_{\mathtt{t}}^T & 0 \end{bmatrix} = \mathscr{H}_{\mathtt{t}},$$

$$\widetilde{T}^{-1}E\widetilde{T} = \begin{bmatrix} T & & \\ & T^{-T} & \\ & & I_{\mathtt{p}} \end{bmatrix}^{-1} \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T & & \\ & T^{-T} & \\ & & I_{\mathtt{p}} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathscr{E}_{\mathtt{t}}$$

Therefore, the Hamiltonian matrix pair $(\mathscr{E},\mathscr{H})$ and $(\mathscr{E}_{\mathtt{t}},\mathscr{H}_{\mathtt{t}})$ are equivalent[1]. By the property of equivalent matrix pencils, $\sigma(\mathscr{E},\mathscr{H}) = \sigma(\mathscr{E}_{\mathtt{t}},\mathscr{H}_{\mathtt{t}})$.

Define $V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda}$ as in Theorem 1.1 corresponding to a Lambda-set Λ of $\mathtt{det}(s\mathscr{E} - \mathscr{H})$. Similarly, Define $V_{1\Lambda\mathtt{t}}, V_{2\Lambda\mathtt{t}}, V_{3\Lambda\mathtt{t}}$ as in Theorem 1.1 corresponding to a Lambda-set Λ of $\mathtt{det}(s\mathscr{E}_{\mathtt{t}} - \mathscr{H}_{\mathtt{t}})$. Then, from equation (7) we have

$$\begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & 0 \end{bmatrix} \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ V_{3\Lambda} \end{bmatrix} = \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ 0 \end{bmatrix} \Gamma \text{ and } \begin{bmatrix} A_{\mathtt{t}} & 0 & B_{\mathtt{t}} \\ 0 & -A_{\mathtt{t}}^T & C_{\mathtt{t}}^T \\ C_{\mathtt{t}} & -B_{\mathtt{t}}^T & 0 \end{bmatrix} \begin{bmatrix} V_{1\Lambda\mathtt{t}} \\ V_{2\Lambda\mathtt{t}} \\ V_{3\Lambda\mathtt{t}} \end{bmatrix} = \begin{bmatrix} V_{1\Lambda\mathtt{t}} \\ V_{2\Lambda\mathtt{t}} \\ 0 \end{bmatrix} \Gamma, \text{ where } \sigma(\Gamma) = \Lambda.$$

Replacing (A,B,C) by $(TA_{\mathtt{t}}T^{-1}, TB_{\mathtt{t}}, C_{\mathtt{t}}T^{-1})$ in the above equation, we have

$$\begin{bmatrix} TA_{\mathtt{t}}T^{-1} & 0 & TB_{\mathtt{t}} \\ 0 & -(TA_{\mathtt{t}}T^{-1})^T & (C_{\mathtt{t}}T^{-1})^T \\ C_{\mathtt{t}}T^{-1} & -(TB_{\mathtt{t}})^T & 0 \end{bmatrix} \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ V_{3\Lambda} \end{bmatrix} = \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ 0 \end{bmatrix} \Gamma, \text{ where } \sigma(\Gamma) = \Lambda$$

$$\begin{bmatrix} A_{\mathtt{t}} & 0 & B_{\mathtt{t}} \\ 0 & -A_{\mathtt{t}}^T & C_{\mathtt{t}}^T \\ C_{\mathtt{t}} & -B_{\mathtt{t}}^T & 0 \end{bmatrix} \begin{bmatrix} T & & \\ & T^{-T} & \\ & & I_{\mathtt{p}} \end{bmatrix}^{-1} \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ V_{3\Lambda} \end{bmatrix} = \begin{bmatrix} T & & \\ & T^{-T} & \\ & & I_{\mathtt{p}} \end{bmatrix}^{-1} \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ 0 \end{bmatrix} \Gamma. \tag{9}$$

Therefore, from equation (9) it is clear that $\widehat{T}^{-1}\mathtt{col}(V_{1\Lambda}, V_{2\Lambda}) = \mathtt{col}(V_{1\Lambda\mathtt{t}}, V_{2\Lambda\mathtt{t}})$. Further, it is easy to verify that

$$W = \begin{bmatrix} \widehat{B} & \widehat{A}\widehat{B} & \cdots & \widehat{A}^{\mathtt{n_f}-1}\widehat{B} \end{bmatrix} = \widehat{T} \begin{bmatrix} \widehat{B}_{\mathtt{t}} & \widehat{A}_{\mathtt{t}}\widehat{B}_{\mathtt{t}} & \cdots & \widehat{A}_{\mathtt{t}}^{\mathtt{n_f}-1}\widehat{B}_{\mathtt{t}} \end{bmatrix}.$$

Defining $V_\Lambda := \mathtt{col}(V_{1\Lambda}, V_{2\Lambda})$ and $V_{\Lambda_{\mathtt{t}}} := \mathtt{col}(V_{1\Lambda\mathtt{t}}, V_{2\Lambda\mathtt{t}})$, we therefore have

$$\begin{bmatrix} X_{1\Lambda} \\ X_{2\Lambda} \end{bmatrix} = \begin{bmatrix} V_\Lambda & \widehat{B} & \widehat{A}\widehat{B} & \cdots & \widehat{A}^{\mathtt{n_f}-1}\widehat{B} \end{bmatrix} = \widehat{T} \begin{bmatrix} V_{\Lambda_{\mathtt{t}}} & \widehat{B}_{\mathtt{t}} & \widehat{A}_{\mathtt{t}}\widehat{B}_{\mathtt{t}} & \cdots & \widehat{A}_{\mathtt{t}}^{\mathtt{n_f}-1}\widehat{B}_{\mathtt{t}} \end{bmatrix} = \begin{bmatrix} T & \\ & T^{-T} \end{bmatrix} \begin{bmatrix} X_{1\Lambda\mathtt{t}} \\ X_{2\Lambda\mathtt{t}} \end{bmatrix}.$$

Thus, we have $X_{1\Lambda} = TX_{1\Lambda\mathtt{t}}$. Since T is nonsingular, $X_{1\Lambda}$ is nonsingular if and only if $X_{1\Lambda\mathtt{t}}$ is nonsingular. ■

Note that in the proof of Lemma 1.5 we have not used the fact that $A_{\mathtt{t}}$, $B_{\mathtt{t}}$, and $C_{\mathtt{t}}$ are in controller canonical form. This indicates that the lemma holds true for any change in basis of the state-space $\mathbb{R}^{\mathtt{n}}$ and hence, the title of the lemma: $X_{1\Lambda}$ is invariant under change of basis on the state-space.

The next lemma shows the existence and the structure of the eigenvectors corresponding to the spectral zeros of a singularly passive SISO system. The structure of the eigenvectors is crucially used in the proof of Statement *(1)* of Theorem 1.1.

[1] Two matrix pairs (A_1,A_2) and (B_1,B_2) are equivalent if there exist nonsingular matrices P and Q such that $P(sA_1 - A_2)Q = (sB_1 - B_2)$. Note that $\mathtt{det}(sB_1 - B_2) = \mathtt{det}(P)\mathtt{det}(Q)\mathtt{det}(sA_1 - A_2)$. Therefore, characteristic polynomials of (A_1,A_2) and (B_1,B_2) are the same (up to to scaling), i.e., $\sigma(A_1,A_2) = \sigma(B_1,B_2)$.

Lemma 1.6 *Consider a singularly passive SISO system Σ of order $\mathtt{n_s}$ with a minimal i/s/o representation as in equation* (1) *and with the Hamiltonian pencil as defined in equation* (4)*. Let (A,B,C) be in the controller canonical form. Assume $\lambda \in \mathbb{C}$ is an eigenvalue of $(\mathscr{E},\mathscr{H})$ with algebraic multiplicity $\mathtt{m}$. Let $J_\lambda := \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix} \in \mathbb{C}^{\mathtt{m}\times\mathtt{m}}$ be the complex Jordan block of size $\mathtt{m}$. Then, there exists $S,P \in \mathbb{C}^{\mathtt{n}\times\mathtt{m}}$ and $Q \in \mathbb{C}^{1\times\mathtt{m}}$ such that*

$$H \begin{bmatrix} S \\ P \\ Q \end{bmatrix} = E \begin{bmatrix} S \\ P \\ Q \end{bmatrix} J_\lambda, \text{ i.e., } \begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & 0 \end{bmatrix} \begin{bmatrix} S \\ P \\ Q \end{bmatrix} = \begin{bmatrix} S \\ P \\ 0 \end{bmatrix} J_\lambda. \tag{10}$$

Proof: Let the characteristic polynomial of A be $\mathscr{X}_A(s) := \mathtt{det}(sI_\mathtt{n} - A)$. Construct

$$Q := \begin{bmatrix} \mathscr{X}_A(\lambda) & \mathscr{X}_A^{(1)}(\lambda) & \mathscr{X}_A^{(2)}(\lambda) & \cdots & \mathscr{X}_A^{(\mathtt{m}-1)}(\lambda) \end{bmatrix} \tag{11}$$

where $\mathscr{X}_A^{(i)}(\lambda) := \frac{d^i}{ds^i}\left(\mathscr{X}_A(s)\right)|_{s=\lambda}$. We need to find S,P such that $AS+BQ = SJ_\lambda$, $-A^TP+C^TQ = PJ_\lambda$ and $CS - B^TP = 0$. Note that the equation $AS+BQ = SJ_\lambda$, after re-arrangement reduces to

$$-AS + SJ_\lambda = BQ, \tag{12}$$

which is a Sylvester equation in the unknown S. By construction, we know that λ is the eigenvalue of J_λ, i.e., $\lambda \in \sigma(\mathscr{E},\mathscr{H})$. Owing to the fact that $\sigma(\mathscr{E},\mathscr{H})$ has a reflection symmetry with respect to the imaginary axis, we get $-\lambda \in \sigma(\mathscr{E},\mathscr{H})$. Since Σ is singularly passive, and equation (1) is a minimal i/s/o representation of Σ, the system matrix A must be Hurwitz. Therefore, by Lemma 1.3, $-\lambda \notin \sigma(A)$. Hence $\sigma(J_\lambda) \cap \sigma(-A) = \emptyset$. Therefore, there exists a unique S that satisfies equation (12). It can be verified that this unique S for Q defined in equation (11) is given by the following Vandermonde matrix:

$$S := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \lambda & 1 & \cdots & 0 \\ \lambda^2 & 2\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{\mathtt{n}-1} & \binom{\mathtt{n}-1}{1}\lambda^{\mathtt{n}-2} & \cdots & \binom{\mathtt{n}-1}{\mathtt{m}-1}\lambda^{\mathtt{n}-\mathtt{m}} \end{bmatrix} \in \mathbb{C}^{\mathtt{n}\times\mathtt{m}}. \tag{13}$$

Note that the i-th column of S, i.e., s_i can also be found using the following formula

$$s_i = \sum_{\ell=1}^{i} (-1)^{\ell+1}(\lambda I - A)^{-\ell} B \left(\mathscr{X}_A^{(i-\ell)}(\lambda)\right), \text{ for } i \in \{1,2,\dots,\mathtt{m}\}. \tag{14}$$

In equation (14), we have used the fact that $\lambda \notin \sigma(A)$ because $\lambda \in \sigma(\mathscr{E},\mathscr{H})$ (see Lemma 1.4). Similarly, the equation involving P, i.e., $-A^TP + C^TQ = PJ_\lambda$ can be transformed to the Sylvester equation $A^TP + PJ_\lambda = c^TQ$ in the unknown P. Arguing like before, this Sylvester equation can be shown to admit a unique solution because $\sigma(J_\lambda) \cap \sigma(A^T) = \emptyset$. Like before, the i-th column of P, say p_i, can be found using the following formula:

$$p_i = \sum_{\ell=1}^{i} (-1)^{\ell+1}(\lambda I + A^T)^{-\ell} c^T \left(\mathscr{X}_A^{(i-\ell)}(\lambda)\right), \text{ for } i \in \{1,2,\dots,\mathtt{m}\}. \tag{15}$$

We have used the fact that $\lambda \in \sigma(\mathscr{E},\mathscr{H})$ implies $-\lambda \in \sigma(\mathscr{E},\mathscr{H})$ and therefore by Lemma 1.4, $-\lambda \notin \sigma(A) \implies \lambda \notin \sigma(-A^T)$. In order to show that S, P, thus constructed, satisfies $cS - b^TP = 0$, we note that the i-th column of $cS - b^TP$ is given by

$$\begin{aligned} [CS - B^TP]_i &= \sum_{\ell=1}^{i} (-1)^{\ell+1} \left(c(\lambda I - A)^{-\ell}B - B^T(sI + A^T)^{-\ell}C^T\right)\left(\mathscr{X}_A^{(i-\ell)}(\lambda)\right) \\ &= \sum_{\ell=1}^{i} \left(\frac{d^{(\ell-1)}}{ds^{(\ell-1)}}\left(G(s) + G(-s)\right)\Big|_{s=\lambda}\right)\left(\mathscr{X}_A^{(i-\ell)}(\lambda)\right). \end{aligned} \tag{16}$$

Since λ has algebraic multiplicity $\mathtt{m}$, and $\mathtt{det}(s\mathscr{E} - \mathscr{H}) = \mathtt{num}(G(s)+G(-s))$ (Lemma 1.2), we have $\frac{d^{(\ell-1)}}{ds^{(\ell-1)}}\left(G(s)+G(-s)\right)\Big|_{s=\lambda} = 0$, for $\ell \in \{1,2,\cdots,\mathtt{m}\}$. Therefore, the ride hand side of equation (16) evaluates to zero for all $\ell \in \{1,2,\cdots,\mathtt{m}\}$; we thus infer that $CS - B^TP = 0$. ■

1.1 Proof of Statement (1) of Theorem 1.1

Statement (1): We prove this in two-steps.

Step 1 (Construction of $V_{e\Lambda} \in \mathbb{R}^{(2\mathtt{n}+1)\times\mathtt{n_s}}$ that satisfies eqn. (4)*):* Lemma 1.5 implies that (A,B,C) can be assumed to be in controller canonical form without loss of generality. From the definition of Lambda-sets we know that if $\lambda \in \Lambda$ then, $\bar{\lambda} \in \Lambda$. Thus, without loss of generality, we assume that there are α number of complex-conjugate pairs in Λ and β number of real elements in Λ such that each distinct element λ_i in Λ has an algebraic multiplicity $\mathtt{m}_{\lambda_i}$. Thus for a Lambda-set with cardinality $\mathtt{n_s}$, we have $\sum_{i=1}^{2\alpha+\beta} \mathtt{m}_{\lambda_i} = \mathtt{n_s}$.

Now, we associate a matrix $S_{\lambda_i} \in \mathbb{C}^{\mathtt{n}\times \mathtt{m}_{\lambda_i}}$ with each distinct element $\lambda_i \in \Lambda$. These matrices S_{λ_i} have a structure as defined in equation (13) of the proof of Lemma 1.6, i.e.,

$$S_{\lambda_i} := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \lambda_i & 1 & \cdots & 0 \\ \lambda_i^2 & 2\lambda_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_i^{\mathtt{n}-1} & \binom{\mathtt{n}-1}{1}\lambda_i^{\mathtt{n}-2} & \cdots & \binom{\mathtt{n}-1}{\mathtt{m}_{\lambda_i}-1}\lambda_i^{\mathtt{n}-\mathtt{m}_{\lambda_i}} \end{bmatrix} \tag{17}$$

Note that since E, H are real matrices, the algebraic multiplicities of $\lambda_i \in \sigma(\mathscr{E},\mathscr{H})$ and $\bar{\lambda}_i \in \sigma(\mathscr{E},\mathscr{H})$ are the same. Further, from the structure of S_{λ_i} it is evident that $S_{\bar{\lambda}_i} = \bar{S}_{\lambda_i}$, where $\bar{S}_{\lambda_i}$ is the complex-conjugate matrix of S_{λ_i}. Now we define a matrix $V_{1\Lambda}^{\mathbb{C}}$ as follows:

$$V_{1\Lambda}^{\mathbb{C}} := \begin{bmatrix} S_{\lambda_1} & \bar{S}_{\lambda_1} & \cdots & S_{\lambda_\alpha} & \bar{S}_{\lambda_\alpha} & S_{\lambda_{2\alpha+1}} & \cdots & S_{\lambda_{2\alpha+\beta}} \end{bmatrix} \in \mathbb{C}^{\mathtt{n}\times\mathtt{n_s}} \tag{18}$$

$V_{1\Lambda}^{\mathbb{C}}$ in equation (18) is constructed such that the matrices S_{λ_i} and $\bar{S}_{\lambda_i}$ corresponding to each of the complex-conjugate pairs in Λ are appended consecutively and this is followed by the matrices associated with the real elements in Λ. Using Lemma 1.6, we infer that corresponding to $V_{1\Lambda}^{\mathbb{C}}$ in equation (18) there exists $V_{2\Lambda}^{\mathbb{C}} \in \mathbb{C}^{\mathtt{n}\times\mathtt{n_s}}$ and $V_{3\Lambda}^{\mathbb{C}} \in \mathbb{C}^{1\times\mathtt{n_s}}$ such that $V_{\mathrm{e}\Lambda}^{\mathbb{C}} := \mathtt{col}(V_{1\Lambda}^{\mathbb{C}}, V_{2\Lambda}^{\mathbb{C}}, V_{3\Lambda}^{\mathbb{C}})$ satisfies $HV_{\mathrm{e}\Lambda}^{\mathbb{C}} = EV_{\mathrm{e}\Lambda}^{\mathbb{C}}J^{\mathbb{C}}$, where $J^{\mathbb{C}} \in \mathbb{C}^{\mathtt{n_s}\times\mathtt{n_s}}$ is a block diagonal matrix with each block being a complex Jordan block and $\sigma(J^{\mathbb{C}}) = \Lambda$. Now we construct a matrix $V_{1\Lambda}$ such that

$$V_{1\Lambda} := \begin{bmatrix} \mathtt{Re}(S_{\lambda_1}) & \mathtt{Im}(S_{\lambda_1}) & \cdots & \mathtt{Re}(S_{\lambda_\alpha}) & \mathtt{Im}(S_{\lambda_\alpha}) & S_{\lambda_{2\alpha+1}} & \cdots & S_{\lambda_{2\alpha+\beta}} \end{bmatrix} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_s}},$$

where $\mathtt{Re}(S_{\lambda_i})$ and $\mathtt{Im}(S_{\lambda_i})$ denotes the real-part and imaginary-part of the matrix S_{λ_i}, respectively. It can be verified that there exists a nonsingular matrix $L \in \mathbb{C}^{\mathtt{n_s}\times\mathtt{n_s}}$ such that $V_{1\Lambda}^{\mathbb{C}}L = V_{1\Lambda}$, where $V_{1\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$. Using this nonsingular matrix L, we now define $V_{\mathrm{e}\Lambda} := V_{\mathrm{e}\Lambda}^{\mathbb{C}}L$. It is easy to verify that $V_{\mathrm{e}\Lambda} \in \mathbb{R}^{(2\mathtt{n}+1)\times\mathtt{n_s}}$. Thus, $HV_{\mathrm{e}\Lambda}^{\mathbb{C}} = EV_{\mathrm{e}\Lambda}^{\mathbb{C}}J^{\mathbb{C}} \implies HV_{\mathrm{e}\Lambda} = EV_{\mathrm{e}\Lambda}\Gamma$, where $\Gamma := L^{-1}J^{\mathbb{C}}L \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$. Importantly, $\sigma(\Gamma) = \sigma(J^{\mathbb{C}}) = \Lambda$, albeit unlike the matrix $J^{\mathbb{C}}$, matrix Γ is in real Jordan form. This shows the existence of a matrix $V_{\mathrm{e}\Lambda}$ that satisfies equation (4).

Step 2 ($X_{1\Lambda}$ is nonsingular): Conforming to the partition of $V_{\mathrm{e}\Lambda}$ in equation (4), we partition $V_{\mathrm{e}\Lambda}$ as $V_{\mathrm{e}\Lambda} := \mathtt{col}\,(V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda})$ and define $V_\Lambda := \mathtt{col}\,(V_{1\Lambda}, V_{2\Lambda})$, where $V_{1\Lambda}, V_{2\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$ and $V_{3\Lambda} \in \mathbb{R}^{1\times\mathtt{n_s}}$. Similarly, partition W, defined in the statement of the theorem, as follows: $W := \mathtt{col}\,(W_1, W_2)$, where $W_1, W_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_f}}$. Recall from the theorem that $\begin{bmatrix} V_\Lambda & W \end{bmatrix} = \mathtt{col}\,(X_{1\Lambda}, X_{2\Lambda})$, where $X_{1\Lambda}, X_{2\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Therefore, $X_{1\Lambda} = \begin{bmatrix} V_{1\Lambda} & W_1 \end{bmatrix}$. Recall that $V_{1\Lambda}^{\mathbb{C}}L = V_{1\Lambda}$. We crucially use this in the next step of the proof.

In order to prove the invertibility of $X_{1\Lambda}$, we partition $X_{1\Lambda}$ as $X_{1\Lambda} := \begin{bmatrix} V_{11} & W_{11} \\ V_{12} & W_{12} \end{bmatrix}$, where $V_{11} \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$, $V_{12} \in \mathbb{R}^{\mathtt{n_f}\times\mathtt{n_s}}$, $W_{11} \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_f}}$ and $W_{12} \in \mathbb{R}^{\mathtt{n_f}\times\mathtt{n_f}}$. Conforming to this partition, we partition $V_{1\Lambda}^{\mathbb{C}}$, as well: $V_{1\Lambda}^{\mathbb{C}} = \begin{bmatrix} V_{11}^{\mathbb{C}} \\ V_{12}^{\mathbb{C}} \end{bmatrix}$, where $V_{11}^{\mathbb{C}} \in \mathbb{C}^{\mathtt{n_s}\times\mathtt{n_s}}$, $V_{12}^{\mathbb{C}} \in \mathbb{C}^{\mathtt{n_f}\times\mathtt{n_s}}$. Since $V_{1\Lambda}^{\mathbb{C}}L = V_{1\Lambda}$, clearly $V_{11} = V_{11}^{\mathbb{C}}L$. From the structure of $V_{1\Lambda}^{\mathbb{C}}$ shown in equation (18), it is evident that $V_{11}^{\mathbb{C}} \in \mathbb{C}^{\mathtt{n_s}\times\mathtt{n_s}}$ is a Vandermonde matrix of the form:

$$V_{11}^{\mathbb{C}} = \begin{bmatrix} 1 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 \\ \lambda_1 & \cdots & 0 & \lambda_2 & \cdots & \lambda_{\mathtt{k}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{\mathtt{n_s}-1} & \cdots & \binom{\mathtt{n_s}-1}{\mathtt{m}_{\lambda_1}-1}\lambda_1^{\mathtt{n_s}-\mathtt{m}_{\lambda_1}} & \lambda_2^{\mathtt{n_s}-1} & \cdots & \lambda_{\mathtt{k}}^{\mathtt{n_s}-1} & \cdots & \binom{\mathtt{n_s}-1}{\mathtt{m}_{\lambda_{2\alpha+\beta}}-1}\lambda_{\mathtt{k}}^{\mathtt{n_s}-\mathtt{m}_{\lambda_{2\alpha+\beta}}} \end{bmatrix}.$$

Since $V_{11}^{\mathbb{C}}$ is a Vandermonde matrix with $2\alpha+\beta$ distinct λ_is such that their multiplicities add up to the size of the matrix, $V_{11}^{\mathbb{C}}$ must be invertible. Thus, V_{11} is the product of two nonsingular matrices $V_{11}^{\mathbb{C}}$ and L. Therefore, V_{11} is nonsingular, as well. Now, we concentrate on the structure of W_1. First, recall that

$$W = \begin{bmatrix} \widehat{B} & \widehat{A}\widehat{B} & \cdots & \widehat{A}^{\mathtt{n_f}-1}\widehat{B} \end{bmatrix} = \begin{bmatrix} b & Ab & \cdots & A^{\mathtt{n_f}-1}b \\ c^T & -(cA)^T & \cdots & (-1)^{\mathtt{n_f}-1}(cA^{\mathtt{n_f}-1})^T \end{bmatrix}. \tag{19}$$

Hence, $W_1 = \begin{bmatrix} b & Ab & \cdots & A^{\mathtt{n_f}-1}b \end{bmatrix} = \begin{bmatrix} W_{11} \\ W_{12} \end{bmatrix}$. Since (A,b,c) is in the controller canonical form, W_{11} is a zero matrix, i.e., $W_1 := \begin{bmatrix} 0 \\ W_{12} \end{bmatrix} \in \mathbb{C}^{\mathtt{n}\times\mathtt{n_f}}$, where $W_{12} \in \mathbb{R}^{\mathtt{n_f}\times\mathtt{n_f}}$ has the following structure $W_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & \star \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & \star & \star \\ 1 & \star & \cdots & \star & \star \end{bmatrix}$ with $\star$ denoting possibly nonzero entries. Clearly, W_{12} is nonsingular. Thus, $X_{1\Lambda}$ has the following structure

$$X_{1\Lambda} = \begin{bmatrix} V_{1\Lambda} & W_1 \end{bmatrix} =: \begin{bmatrix} V_{11} & 0 \\ V_{12} & W_{12} \end{bmatrix}, \tag{20}$$

Thus, $X_{1\Lambda}$ is a block lower-triangular matrix with the diagonal blocks being V_{11} and W_{12}. Since V_{11} and W_{12} are nonsingular matrices, $X_{1\Lambda}$ is nonsingular. $\square$