MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

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1998-2018]

The optimal cost of the singular LQR problem and fast/slow subspaces of the Hamiltonian system

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CSC

Majority of the work completed at Indian Institute of Technology Bombay (IIT Bombay)

Problem statement

Consider a system with state-space dynamics

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \text{ where } A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}.$$

Infinite-horizon linear quadratic regulator (LQR) problem

For every initial condition $x_0 \in \mathbb{R}^{\mathtt{n}}$, find an input $u(t)$ (from admissible input space) that minimizes the functional

$$J\left(x_0, u(t)\right) := \int_0^{\infty} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \text{ where } \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geqslant 0.$$

$\mathfrak{C}^{\mathtt{m}}_{\mathtt{imp}} := \{f = f_{\mathtt{reg}} + f_{\mathtt{imp}} \mid f_{\mathtt{reg}} \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{m}})|_{\mathbb{R}_+}, f_{\mathtt{imp}} = \sum_{i=0}^{k} a_i \delta^{(i)}(t), a_i \in \mathbb{R}^{\mathtt{m}}, k \in \mathbb{N}\}.$

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For $R > 0$ (Regular case)

$$A^T K + KA + Q - (KB + S)R^{-1}(B^T K + S^T) = 0 \qquad u(t) = -R^{-1}(B^T K_{\mathtt{max}} + S^T)x(t)$$

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For $R \geqslant 0$, $\mathtt{det}(R) = 0$ (Singular/degenerate case)

$\cancel{A^TK + KA + Q - (KB + S)R^{-1}(B^TK + S^T) = 0}$ $\cancel{u(t) = -R^{-1}(B^TK_{\mathtt{max}} + S^T)x(t)}$

The LQR LMI

- Find the maximal rank-minimizing solution of the LQR LMI:

$$\begin{bmatrix} A^T K + KA + Q & KB + S \\ B^T K + S^T & R \end{bmatrix} \geqslant 0.$$

- For regular case: Schur-complement with respect to R gives the Algebraic Riccati Inequality (ARI)

$$A^T K + KA + Q - (KB + S)R^{-1}(B^T K + S^T) \geqslant 0.$$

- Solutions of the Algebraic Riccati Equation (ARE) are rank-miniziming solutions of the LQR LMI.

An algorithm

Theorem (Finding maximal rank-minimizing solution of an LQR LMI)

- *Define* $E := \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $H := \begin{bmatrix} A & 0 & b \\ -Q & -A^T & 0 \\ 0 & b^T & 0 \end{bmatrix}$. *(Hamiltonian pencil)*
- *Assume* $\sigma(E, H) \cap j\mathbb{R} = \emptyset$.
- *Define* $\mathtt{n_s} := \mathtt{degdet}(sE - H)/2$ *and* $\mathtt{n_f} := \mathtt{n} - \mathtt{n_s}$.

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- *Find eigenbasis of* (E, H) *corresponding to* $\sigma(E, H) \cap \mathbb{C}_-$.
  Define $V_1, V_2 \in \mathbb{R}^{\mathtt{n} \times \mathtt{n_s}}$ *and* $V_3 \in \mathbb{R}^{1 \times \mathtt{n_s}}$ *such that*
  $H \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = E \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \Gamma$, *where* $\sigma(\Gamma) = \sigma(E, H) \cap \mathbb{C}_-$.

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- $W = \begin{bmatrix} \widehat{b} & \widehat{A}\widehat{b} & \dots & \widehat{A}^{\mathtt{n_f}-1}\widehat{b} \end{bmatrix} \in \mathbb{R}^{\mathtt{n} \times \mathtt{n_f}}$, *where* $\widehat{A} := \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix}$, $\widehat{b} := \begin{bmatrix} b \\ 0 \end{bmatrix}$.

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- $X := \begin{bmatrix} V & W \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, *where* $V := \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ *and* $X_1, X_2 \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}$.

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  $X_2 X_1^{-1}$ *is the maximal rank-minimizing solution of the LQR LMI.*

PD-controllers

Theorem (Almost all singular LQR problems solvable using PD-controllers)

- *Define*

$$F_{\mathtt{p}} := \begin{bmatrix} V_3 & f_0 & f_1 & f_2 & \cdots & f_{\mathtt{n_f}-1} \end{bmatrix} X_1^{-1}$$
$$F_{\mathtt{d}} := \begin{bmatrix} 0_{1,\mathtt{n_s}} & 1 & -f_0 & -f_1 & \cdots & -f_{\mathtt{n_f}-2} \end{bmatrix} X_1^{-1},$$

such that $\mathtt{det}\left(s(I_{\mathtt{n}} - bF_{\mathtt{d}}) - (A + bF_{\mathtt{p}})\right) \neq 0$, $f_i \in \mathbb{R}$.

- *Then, the optimal input* $u(t)$ *is given by:*

$$u(t) = F_{\mathtt{p}}x(t) + F_{\mathtt{d}}\frac{d}{dt}x(t).$$

For the regular case: $F_{\mathtt{d}} = 0$. Hence, $u(t) = F_{\mathtt{p}}x(t)$ – as expected.

For details see: Bhawal and Pal, IEEE Control Systems Letters, 2019.

Primary idea

- Corresponding to certain special initial conditions compute the trajectories of the Hamiltonian system (a differential algebraic system):

$$\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E} \frac{d}{dt}\begin{bmatrix} x(t) \\ z(t) \\ u(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & b \\ -Q & -A^T & 0 \\ 0 & b^T & 0 \end{bmatrix}}_{H} \begin{bmatrix} x(t) \\ z(t) \\ u(t) \end{bmatrix}.$$

- Special initial conditions are:
  $\mathtt{img}\,(V)$ - Trajectories of the form $ce^{\lambda t}$ - Slow part.
  $\mathtt{img}(W)$ - Trajectories contain $\delta(t)$ and its derivatives - Fast part.

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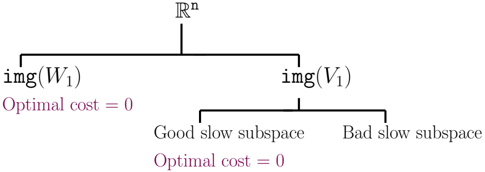
$$V =: \left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right], V_1, V_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$$

$$W =: \left[\begin{smallmatrix} W_1 \\ W_2 \end{smallmatrix}\right], W_1, W_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_f}}$$



Thank you