L29 04/11/2020

Let $P(s)$ and $\hat{K}(s)$ have a stabilizable & detectable realization as shown below:

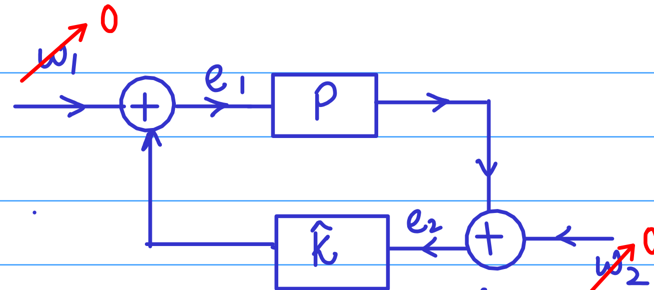

fig ①

$P(s)$ $\boxed{\begin{aligned}\dot{x} &= Ax + Be_1 \\ e_2 &= Cx + De_1\end{aligned}}$ $\hat{K}(s)$ $\boxed{\begin{aligned}\dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}e_2 \\ e_1 &= \hat{C}\hat{x} + \hat{D}e_2\end{aligned}}$

Definition: The system in fig ① is said to be internally stable if the states $(x, \hat{x})$ go to zero for all initial condition.

$$\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} \Rightarrow \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \underbrace{\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1}}_{\text{well-posedness}}\begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} \longrightarrow (1)$$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \underbrace{\left(\begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1}\begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}\right)}_{\tilde{A}}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

$$\underbrace{\mathrm{spec}(\tilde{A}) \subseteq \mathbb{C}^-}_{\text{Hurwitz (stable)}} \Longleftrightarrow \text{System is internally stable.}$$

e.g. $P(s) = 2/s+1$ $\hat{K}(s) = (s+3)/(s+1)$ $P(\infty) = 0$ $\hat{K}(\infty) = 1$ $\begin{bmatrix} I & -\hat{K}(\infty) \\ -P(\infty) & I \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$P = \left[\begin{array}{c|c} -1 & 2 \\ \hline 1 & 0 \end{array}\right]$ $\hat{K} = \left[\begin{array}{c|c} -1 & 2 \\ \hline 1 & 1 \end{array}\right]$

Interconnection is well posed

$\tilde{A} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ $\mathrm{spec}(\tilde{A}) = \{-\sqrt{5}, \sqrt{5}\}$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

$$\begin{aligned}\begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} &= e^{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}t}\begin{bmatrix} x(0) \\ \hat{x}(0) \end{bmatrix} \\ &= T e^{\begin{bmatrix} -\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}t}\underbrace{\boxed{T^{-1}\begin{bmatrix} x(0) \\ \hat{x}(0) \end{bmatrix}}}_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \\ &= T\begin{bmatrix} 0 \\ e^{\sqrt{5}t} \end{bmatrix} \leftarrow\end{aligned}$$

☐ The system in fig (i) with a stabilizable & detectable realization of P & $\hat{K}$ is internally stable if and only if $\tilde{A}$ is Hurwitz.

☐ The system in fig (i) is internally stable if and only if the transfer matrix from (w_1, w_2) to (e_1, e_2), i.e.

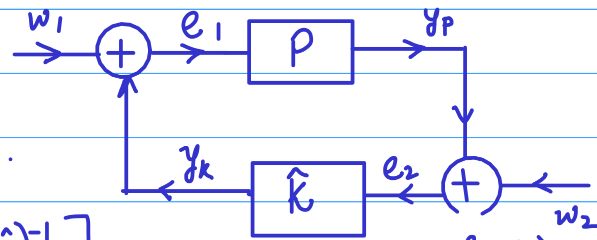

fig (ii)

$$\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} I+\hat{K}\underbrace{(I-P\hat{K})^{-1}}P & \hat{K}(I-P\hat{K})^{-1} \\ (I-P\hat{K})^{-1}P & (I-P\hat{K})^{-1} \end{bmatrix} \in RH_\infty$$

Proof:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \longrightarrow (I)$$

$$\begin{bmatrix} y_P \\ y_K \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & \hat{C} \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & \hat{D} \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \longrightarrow (II)$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} y_P \\ y_K \end{bmatrix} \longrightarrow (III)$$

(II) in (III)

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\left(\begin{bmatrix} C & 0 \\ 0 & \hat{C} \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & \hat{D} \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}\right)$$

$$\Rightarrow \begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1}\begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \longrightarrow (IV)$$

(IV) in (I)

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \overbrace{\left(\begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1}\begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}\right)}^{\tilde{A}}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \overbrace{\begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1}}^{\tilde{B}}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \overbrace{\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1}\begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}}^{\tilde{C}}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \overbrace{\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1}}^{\tilde{D}}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$\boxed{\tilde{D} + \tilde{C}(sI-\tilde{A})^{-1}\tilde{B}} = \begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1}$$

Only if: The system is internally stable.

(i) $(I - P(\infty)\hat{K}(\infty))$ is invertible due to well-posedness

$\Downarrow$

$\checkmark$ $(I - P\hat{K})$ is invertible $\longrightarrow$ real, rational & proper transfer matrix

(ii) Internal stability $\tilde{A}$ is Hurwitz. $\Rightarrow$ $\begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} \in RH_\infty$

If: Given $\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1} \in RH_\infty \Rightarrow (I - P\hat{K})^{-1}$ is proper $\Rightarrow (I - P(\infty)\hat{K}(\infty))^{-1}$ exists.

$\Updownarrow$

$\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}$ is invertible

$$\tilde{D} + \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}$$

$$= \begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1} + \begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} (sI - \tilde{A})^{-1} \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix} \begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1}$$

$\Downarrow$

$$\Rightarrow \underbrace{\begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} (sI - \tilde{A})^{-1} \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}}_{H(s)} \in RH_\infty \qquad \text{poles } H(s) \leftarrow \mathbb{C}^-$$

$\because$ (A, B, C) and $(\hat{A}, \hat{B}, \hat{C})$ are stabilizable & detectable

$$\ast \Rightarrow \left(\tilde{A}, \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}, \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}\right) \text{ is also stabilizable \& detectable}$$

$\text{spec}(\tilde{A}) \in \mathbb{C}^-$

$\tilde{A}$ is Hurwitz. $\Leftrightarrow$ System is internally stable

e.g. $P = \frac{s-1}{s+1} \in RH_\infty$ $\quad \hat{K} = -1/s-1 \notin RH_\infty$

$$\begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} \overset{\in RH_\infty}{(s+1)/s+2} & \overset{\notin RH_\infty}{(s+1)/(s-1)(s+2)} \\ \underset{RH_\infty}{(s-1)/s+2} & \underset{RH_\infty}{(s+1)/s+2} \end{bmatrix} \notin RH_\infty$$