Q2_main

2024-10-30

2. Monte Carlo Simulation for pricing Asian Call Option

2.1 Introduction

An Asian option is a type of exotic option where the payoff depends on the average price of the underlying asset over a certain period, rather than its price at a specific point in time. This average can be computed as either an arithmetic average or a geometric average. Asian options are particularly useful in markets where the underlying asset's price is highly volatile, as they reduce the impact of price manipulation or extreme price movements near expiration. For an Asian call or put option, the payoff formulas are listed as follows.

The payoff for a continuous arithmetic average Asian call or put option is:

$$\Phi(S) = \max\left(\frac{1}{T}\int_0^T S(t)dt - K, 0\right) \quad \text{or} \quad \Phi(S) = \max\left(K - \frac{1}{T}\int_0^T S(t)dt, 0\right).$$

The payoff for a continuous geometric average Asian call or put option is:

$$\Phi(S) = \max\left(e^{\frac{1}{T}\int_0^T \log S(t)dt} - K, 0\right) \quad \text{or} \quad \Phi(S) = \max\left(K - e^{\frac{1}{T}\int_0^T \log S(t)dt}, 0\right).$$

For discrete monitoring, the arithmetic average Asian call or put option has the following payoff:

$$\Phi(S) = \max\left(\frac{1}{m+1}\sum_{i=0}^{m} S\left(\frac{iT}{m}\right) - K, 0\right) \quad \text{or} \quad \Phi(S) = \max\left(K - \frac{1}{m+1}\sum_{i=0}^{m} S\left(\frac{iT}{m}\right), 0\right).$$

Similarly, for discrete monitoring, the geometric average Asian call or put option has the payoff:

$$\Phi(S) = \max\left(e^{\frac{1}{m+1}\sum_{i=0}^{m} \log S\left(\frac{iT}{m}\right)} - K, 0\right) \quad \text{or} \quad \Phi(S) = \max\left(K - e^{\frac{1}{m+1}\sum_{i=0}^{m} \log S\left(\frac{iT}{m}\right)}, 0\right).$$

Asian options are widely used in financial markets for energy and commodity trading. Their structure reduces the risk of price manipulation near expiration and smooths out price volatility through averaging.

In the following section, we will focus on three parts. The first is pricing an arithmetic average Asian call option using the Monte Carlo simulation method. Then, we will investigate how to enhance the efficiency of the Monte Carlo method by implementing variance reduction techniques: the Control Variate Method. Additionally, we will further explore variance reduction methods, such as combining control variates with stratified sampling and the brownian bridge.

2.2 Asian Option Pricing– Monte Carlo Method

Monte Carlo (MC) simulation is a widely used numerical method for pricing financial derivatives. The MC method involves generating a large number of simulated paths for the underlying asset price and computing the average payoff over these paths to estimate the option price. Its flexibility makes it suitable for pricing

Asian options, as it can handle the arithmetic averaging of the underlying prices, which does not have a closed-form solution.

The key steps in pricing an Asian call option using the Monte Carlo method are as follows:

Simulate Asset Price Paths: The underlying asset price S_t is modeled as a geometric Brownian motion (GBM) under the risk-neutral measure: $dS_t = (r - q)S_t\,dt + \sigma S_t\,dW_t$, where: - r: Risk-free interest rate, - q: Dividend yield, - σ: Volatility of the asset, - W_t: Standard Brownian motion.

We apply a logarithmic transformation on S_t, and use Ito's Lemma to solve the stochastic differential equation. Integrating the log-transformed SDE, we get the stock price at time t under risk-neutral probability: $\log(S_t) = \log(S_0) + (r - q - 0.5\sigma^2)t + \sigma W_t$. To Simulate stock prices, we utilize a vectorized approach in the logarithmic domain to ensure both computational efficiency and numerical stability. We begin by generating a matrix of standard normal random variables, $Z \sim \mathcal{N}(0, 1)$, representing N paths over m discrete time steps. We calculate the drift term and diffusion term separately. The drift term is added to account for deterministic growth. A cumulative sum is applied to the diffusion term across time steps to approximate W_t. This approach leverages matrix operations to compute all paths simultaneously, eliminating the need for iterative updates and enhancing performance for large-scale simulations. Furthermore, working in the logarithmic domain mitigates numerical instabilities that may arise from compounding errors in the stock price domain, particularly for long time horizons or high volatility.

Calculate Payoff: For each simulated path, compute the arithmetic average of the asset prices: $\bar{S}_{\text{arith}} = \frac{1}{m}\sum_{i=1}^{m} S_{t_i}$. Then, calculate the payoff of the option for each path as: $\text{Payoff}_i = \max(\bar{S}_{\text{arith}} - K, 0)$, where K is the strike price.

Discount Payoffs and Estimate Option Price: Discount the payoff to present value using the risk-free rate: $\text{Discounted Payoff}_i = e^{-rT} \cdot \text{Payoff}_i$. The Monte Carlo estimate of the Asian option price is the average of the discounted payoffs: $\text{Option Price} = \frac{1}{N}\sum_{i=1}^{N} \text{Discounted Payoff}_i$, where N is the number of simulated paths.

Measure Uncertainty: The standard error of the Monte Carlo estimate is given by: $\text{Standard Error} = \frac{\text{Standard Deviation of Payoffs}}{\sqrt{N}}$.

Below is the R implementation of calculating Asian call option prices using monte carlo method:

```r
price_asian_call_MC <- function(S0, K, T, r, q, sigma, m, N) {
  start_time <- proc.time()
  delta_t <- T / m
  sqrt_delta_t <- sqrt(delta_t)

  # Generate N x m standard normal random variables
  Z <- matrix(rnorm(N * m), nrow = N, ncol = m)

  # Simulate log prices and then transform to prices
  drift <- (r - q - 0.5 * sigma^2) * delta_t
  diffusion <- sigma * sqrt_delta_t * Z
  W <- t(apply(diffusion, 1, cumsum))
  log_S <- log(S0) + outer(rep(1, N), drift * (1:m)) + W
  S <- exp(log_S)

  # Calculate arithmetic average and payoffs
  S_bar <- rowMeans(S)
  payoffs <- pmax(S_bar - K, 0)
  discounted_payoffs <- exp(-r * T) * payoffs

  # Estimate option price
  option_price <- mean(discounted_payoffs)
```

```
  std_error <- sd(discounted_payoffs) / sqrt(N)

  # Calculate 95% confidence interval
  CI_lower <- option_price - 1.96 * std_error
  CI_upper <- option_price + 1.96 * std_error

  end_time <- proc.time()
  comp_time <- (end_time - start_time)[["elapsed"]]

  return(list(
    N = N,
    Option_Price = round(option_price, 2),
    Standard_Error = round(std_error, 5),
    Confidence_Interval = c(round(CI_lower, 2), round(CI_upper, 2)),
    Computation_Time_sec = round(comp_time, 4)
  ))
}
```

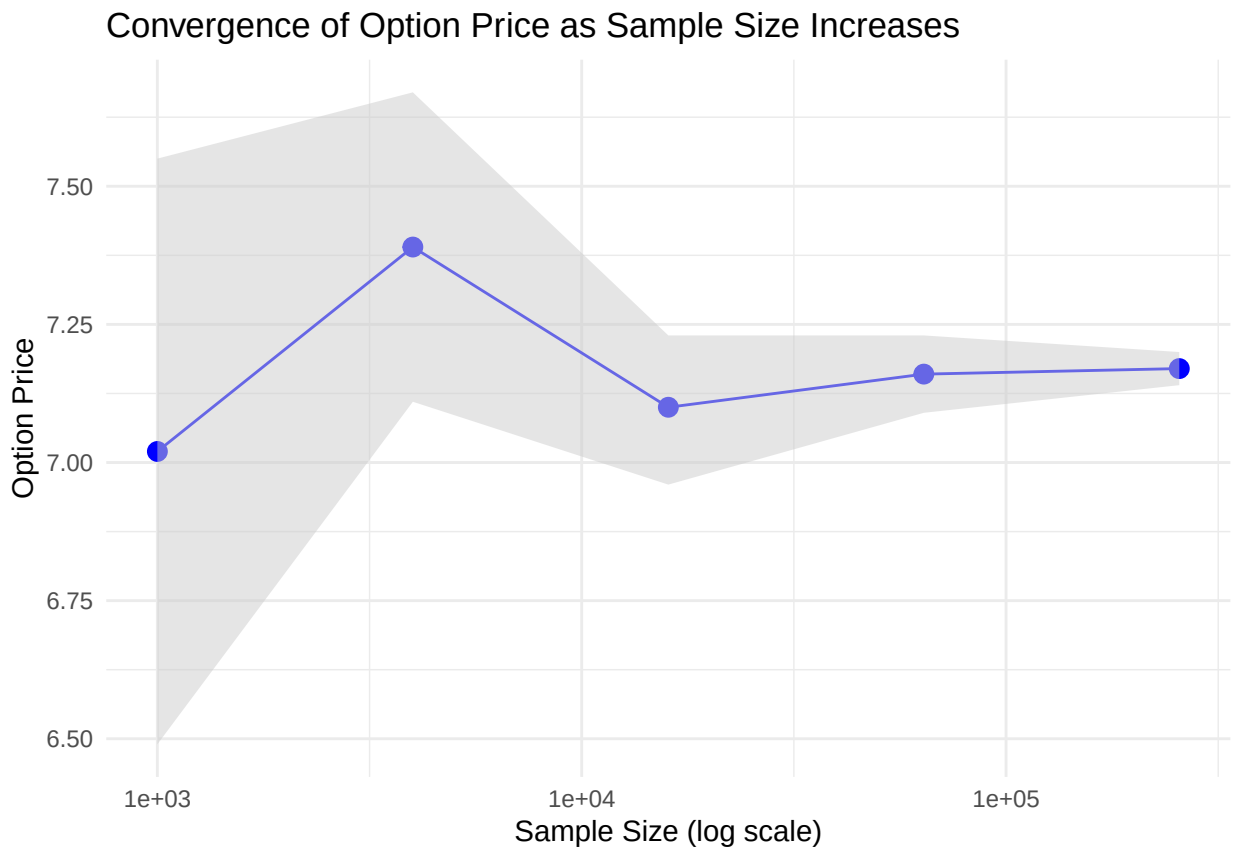
```
# Set parameters
S0 <- 100       # Initial asset price
K <- 100        # Strike price
T <- 1          # Time to maturity (in years)
r <- 0.10       # Risk-free rate
q <- 0          # Dividend yield
sigma <- 0.20   # Volatility
m <- 50         # Number of monitoring points

sample_sizes <- c(1000, 4000, 16000, 64000, 256000)

# Initialize a data frame to store results
standard_mc_results <- data.frame(
  Sample_Size = numeric(),
  Option_Price = numeric(),
  Standard_Error = numeric(),
  CI_Lower = numeric(),
  CI_Upper = numeric(),
  Computation_Time_sec = numeric(),
  stringsAsFactors = FALSE
)

# Set the seed once before the loop
set.seed(123)

# Loop through different sample sizes
for (N in sample_sizes) {
  result <- price_asian_call_MC(S0, K, T, r, q, sigma, m, N)
  standard_mc_results <- rbind(standard_mc_results, data.frame(
    Sample_Size = N,
    Option_Price = result$Option_Price,
    Standard_Error = result$Standard_Error,
    CI_Lower = result$Confidence_Interval[1],
    CI_Upper = result$Confidence_Interval[2],
    Computation_Time_sec = result$Computation_Time_sec
```

```
  ))
}

suppressWarnings(suppressPackageStartupMessages({
  library(ggplot2)
  library(knitr)
  library(dplyr)
}))
# Display results in a table
kable(standard_mc_results, caption = "Monte Carlo Simulation Results")
```

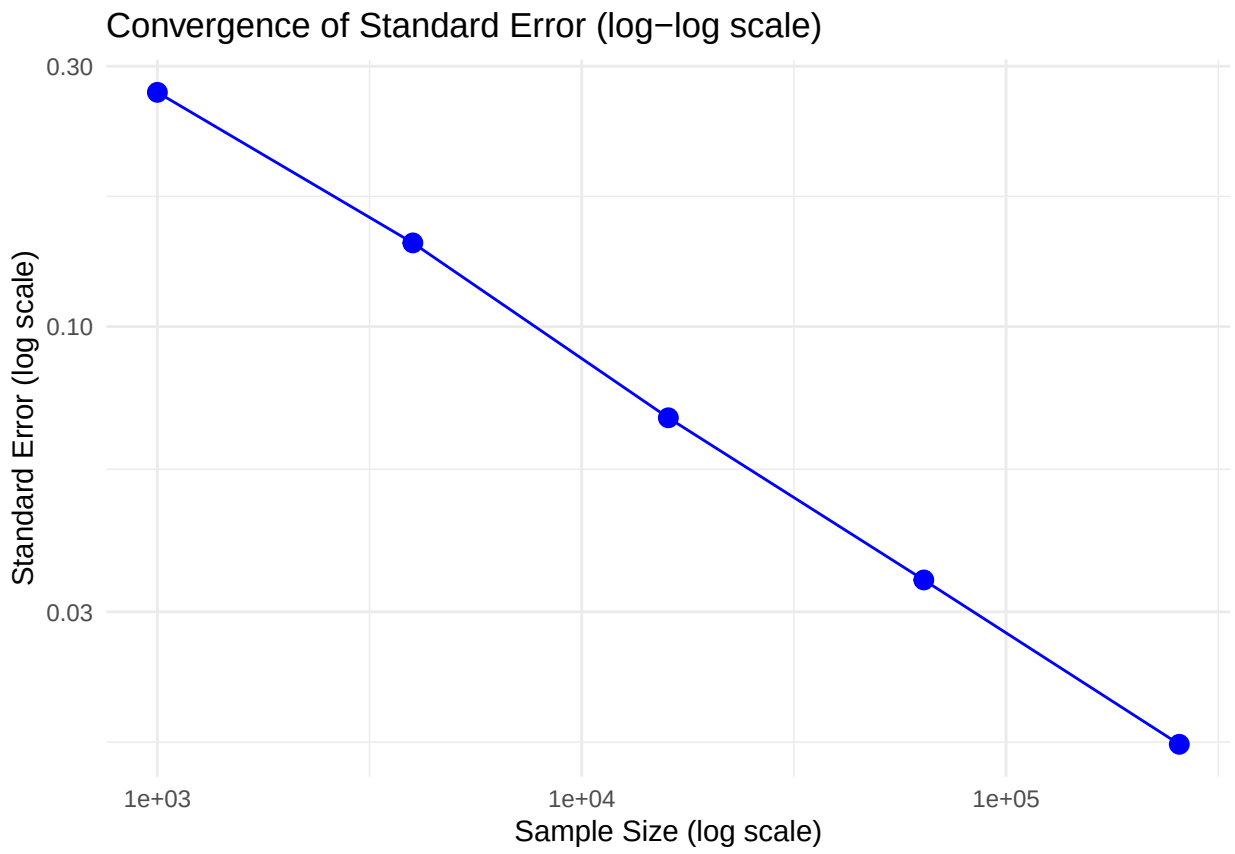
Table 1: Monte Carlo Simulation Results

Sample_Size	Option_Price	Standard_Error	CI_Lower	CI_Upper	Computation_Time_sec
1000	7.02	0.26869	6.49	7.55	0.00
4000	7.39	0.14233	7.11	7.67	0.06
16000	7.10	0.06805	6.96	7.23	0.14
64000	7.16	0.03433	7.09	7.23	0.56
256000	7.17	0.01716	7.14	7.20	2.31

```
# Plot the convergence of Option Price
ggplot(standard_mc_results, aes(x = Sample_Size, y = Option_Price)) +
  geom_line(color = "blue") +
  geom_point(size = 3, color = "blue") +
  geom_ribbon(aes(ymin = CI_Lower, ymax = CI_Upper), fill = "grey80", alpha = 0.5) +
  scale_x_log10() +
  labs(
    title = "Convergence of Option Price as Sample Size Increases",
    x = "Sample Size (log scale)",
    y = "Option Price"
  ) +
  theme_minimal()
```




```r
# Plot the convergence of Standard Error on log-log scale
ggplot(standard_mc_results, aes(x = Sample_Size, y = Standard_Error)) +
  geom_line(color = "blue") +
  geom_point(size = 3, color = "blue") +
  scale_x_log10() +
  scale_y_log10() +
  labs(
    title = "Convergence of Standard Error (log-log scale)",
    x = "Sample Size (log scale)",
    y = "Standard Error (log scale)"
  ) +
  theme_minimal()
```



Convergence of Option Price

The first plot highlights the behavior of the estimated option price as the sample size increases. For smaller sample sizes (e.g., $N = 1000$), the option price exhibits larger variability and a wider confidence interval, indicating less precision. As the sample size increases to $N = 256,000$, the option price converges to a stable value of approximately 7.17, with the 95% confidence interval narrowing significantly to a width of 0.06. This reflects the expected improvement in accuracy with larger sample sizes, consistent with Monte Carlo theory. The convergence is gradual, with the estimates stabilizing beyond $N = 64,000$.

Convergence of Standard Error

The second plot, presented on a log-log scale, demonstrates the relationship between the standard error and the sample size. The convergence follows the theoretical rate of:

$$\text{Standard Error} \propto \frac{1}{\sqrt{N}}$$

This means that as the sample size increases by a factor of 4, the standard error decreases approximately by half. The straight line in the log-log plot shows that doubling the sample size results in a consistent proportional reduction in the standard error, making it predictable and reliable. For example: at $N = 1000$, the standard error is 0.26869. At $N = 4000$ (4 times $N = 1000$), the standard error decreases to approximately 0.14233. Similarly, at $N = 16,000$ (another 4x increase), the standard error further decreases to 0.06805.

2.3 Variance Reduction Techniques– Control Variate Method

In this section, we will introduce a variance reduction technique: control variate method. Suppose we want to estimate the expected value $\theta = \mathbb{E}[Y]$, where $Y = g(X)$ is a function of some random variable X. If we can find another random variable Z, for which the expected value $\mathbb{E}[Z]$ is known, we can construct alternative estimators for θ. For example,

The standard Monte Carlo estimator:

$$\hat{\theta} = Y$$

The control variate estimator:

$$\hat{\theta}_c = Y + c \cdot (Z - \mathbb{E}[Z]),$$

where c is a constant.

It can be shown that the control variate estimator $\hat{\theta}_c$ is unbiased, as:

$$\mathbb{E}[\hat{\theta}_c] = \mathbb{E}[Y] + c \cdot (\mathbb{E}[Z] - \mathbb{E}[Z]) = \mathbb{E}[Y] = \theta.$$

To minimize the variance of $\hat{\theta}_c$, we start with its variance formula:

$$\mathrm{Var}(\hat{\theta}_c) = \mathrm{Var}(Y) + c^2 \cdot \mathrm{Var}(Z) + 2c \cdot \mathrm{Cov}(Y, Z).$$

Treating this as a function of c, we differentiate with respect to c to obtain:

$$f'(c) = 2c \cdot \mathrm{Var}(Z) + 2 \cdot \mathrm{Cov}(Y, Z).$$

Setting $f'(c) = 0$, we solve for c to find the critical point:

$$c_{\text{opt}} = -\frac{\mathrm{Cov}(Y, Z)}{\mathrm{Var}(Z)}.$$

To confirm that this value minimizes the variance, we compute the second derivative:

$$f''(c) = 2 \cdot \mathrm{Var}(Z).$$

Since $\mathrm{Var}(Z) > 0$, $f(c)$ is convex, and c_{opt} is the minimizer.

This demonstrates that the control variate method reduces variance by leveraging the correlation between Y and Z, particularly when they are highly correlated.

In this practice, we will use geometric Asian Call as a control variant. This method is efficient, since first, the correlation between the arithmetic average and the geometric average is very high; second, the counterpart geometric average Asian option price has a closed-form solution. We use the following formula to calculate geometric Asian call option price, serving as the expected value of the control variant in the control variate estimator:

$$\begin{aligned}
\sigma_z^2 &= \sigma^2 \cdot \frac{(m+1)(2m+1)}{6m^2} \\
\mu &= \left(r - q - \frac{1}{2}\sigma^2\right) \cdot \frac{m+1}{2m} + \frac{1}{2}\sigma_z^2 \\
d_1 &= \frac{\ln\left(\frac{S_0}{K}\right) + \left(\mu + \frac{1}{2}\sigma_z^2\right) T}{\sigma_z \sqrt{T}} \\
d_2 &= \frac{\ln\left(\frac{S_0}{K}\right) + \left(\mu - \frac{1}{2}\sigma_z^2\right) T}{\sigma_z \sqrt{T}} \\
\text{Geometric Asian Call Price} &= e^{-rT}\left[S_0 \cdot e^{\mu T} \cdot N(d_1) - K \cdot N(d_2)\right]
\end{aligned}$$

Here: σ_z^2: Effective volatility of the geometric average. μ: Drift term adjusted for the geometric average. d_1, d_2: Terms used in the Black-Scholes framework for the option price. $N(d)$: Cumulative distribution function of the standard normal distribution.

Below is the R implementation of calculating the analytical price of Geometric Asian Call Option:

```r
# function to calculate the analytical price of Geometric Asian Call Option
price_geometric_asian_call <- function(S0, K, T, r, q, sigma, m){
  delta_t <- T/m
  sigma_sq <- sigma^2
  sigma_z_sq <- (sigma_sq) * (m + 1) * (2 * m + 1) / (6 * m^2)
  drift <-(r - q - 0.5*sigma_sq) * (m + 1) / (2 * m) + 0.5 * sigma_z_sq
  sigma_z <- sqrt(sigma_z_sq)
  d1 <- (log(S0/K) + (drift + 0.5 * sigma_z_sq) * T) / (sigma_z * sqrt(T))
  d2 <- (log(S0/K) + (drift - 0.5 * sigma_z_sq) * T) / (sigma_z * sqrt(T))
  geo_asian_price <- exp(-r*T) * (S0* exp(drift * T) * pnorm(d1) - K * pnorm(d2))
  return(geo_asian_price)
}
```

To implement this method, we first simulate the stock price Y using standard Monte Carlo techniques. Next, we compute the geometric average of the stock prices and take its logarithm to derive the geometric option price, denoted as Z. The closed-form solution for the geometric option price allows us to calculate its expected value, $\mathbb{E}[Z]$. Using this information, we construct the control variate estimator:

$\hat{\theta}_c = Y + c \cdot (Z - \mathbb{E}[Z])$, where $c_{\text{opt}} = -\frac{\text{Cov}(Y,Z)}{\text{Var}(Z)}$ is the optimal control coefficient. Below is the R implementation of the Monte Carlo control variate method:

```r
# function to price asian call option using geometric asian call price as a control variate
price_asian_call_MC_control_variate <- function(S0, K, T, r, q, sigma, m, N) {
  start_time <- proc.time()

  delta_t <- T / m
  sqrt_delta_t <- sqrt(delta_t)

  Z <- matrix(rnorm(N * m), nrow = N, ncol = m)

  drift <- (r - q - 0.5 * sigma^2) * delta_t
  diffusion <- sigma * sqrt_delta_t * Z
  W <- t(apply(diffusion, 1, cumsum))
  log_S <- log(S0) + outer(rep(1, N), (r - q - 0.5 * sigma^2) * (delta_t * (1:m))) + W
  S <- exp(log_S)

  S_bar_arith <- rowMeans(S)
  S_bar_geo <- exp(rowMeans(log(S)))

  payoffs_arith <- pmax(S_bar_arith - K, 0)

  # Calculate payoffs for geometric Asian call
  payoffs_geo <- pmax(S_bar_geo - K, 0)

  # Discount payoffs to present value
  discounted_payoffs_arith <- exp(-r * T) * payoffs_arith
  discounted_payoffs_geo <- exp(-r * T) * payoffs_geo

  # Calculate analytical price of Geometric Asian Call
  geo_asian_price <- price_geometric_asian_call(S0, K, T, r, q, sigma, m)

  # Calculate covariance and variance for Control Variate
  cov_xy <- cov(discounted_payoffs_arith, discounted_payoffs_geo)
  var_y <- var(discounted_payoffs_geo)
```

```
  # Optimal coefficient
  theta <- cov_xy / var_y

  # Calculate Control Variate estimator
  control_variate_estimator <- discounted_payoffs_arith - theta * (discounted_payoffs_geo - geo_asian_pr

  # Calculate option price using Control Variate
  option_price_cv <- mean(control_variate_estimator)

  std_error_cv <- sd(control_variate_estimator) / sqrt(N)

  CI_lower_cv <- option_price_cv - 1.96 * std_error_cv
  CI_upper_cv <- option_price_cv + 1.96 * std_error_cv

  end_time <- proc.time()
  comp_time <- (end_time - start_time)[["elapsed"]]

  return(list(
    N = N,
    Option_Price = round(option_price_cv, 2),
    Standard_Error = round(std_error_cv, 5),
    Confidence_Interval = c(round(CI_lower_cv, 2), round(CI_upper_cv, 2)),
    Computation_Time_sec = round(comp_time, 4)
  ))
}
```

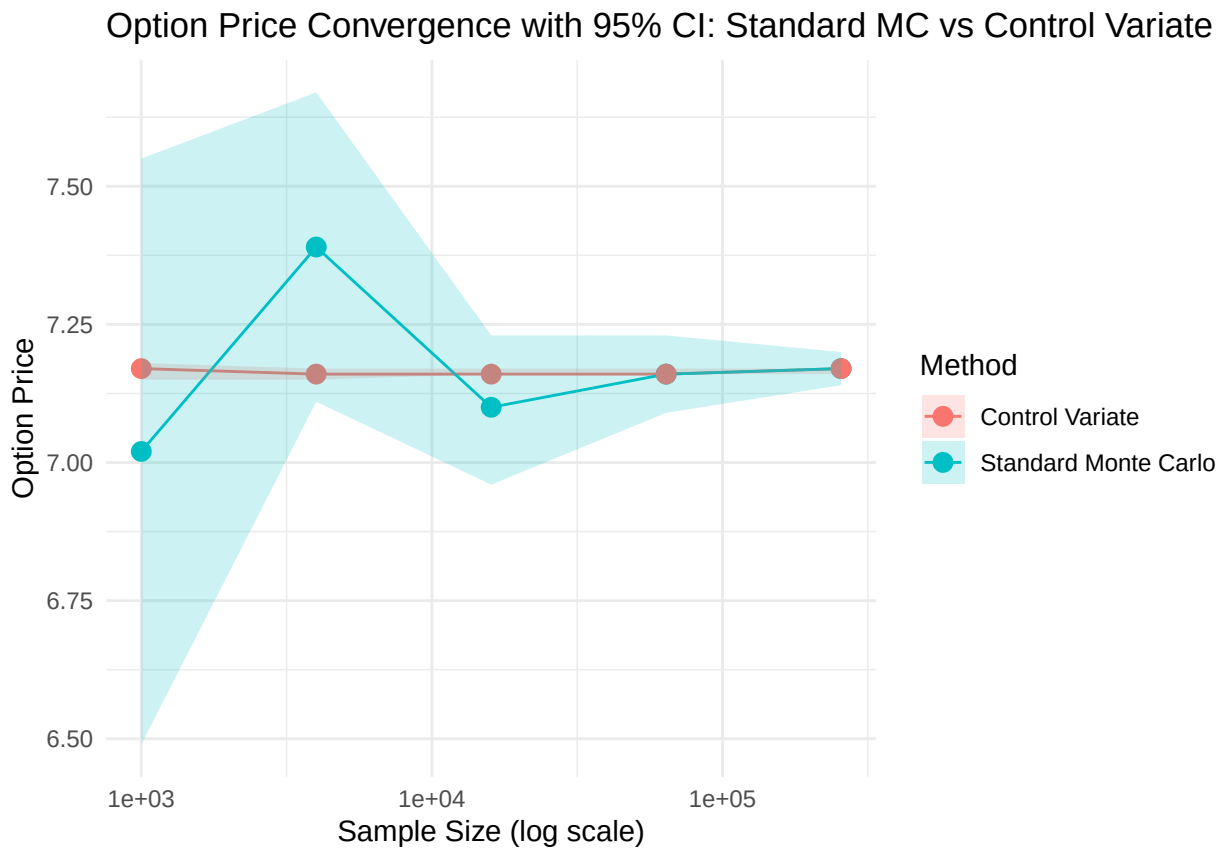
```
# Set parameters
S0 <- 100      # Initial asset price
K <- 100       # Strike price
T <- 1         # Time to maturity (in years)
r <- 0.10      # Risk-free rate
q <- 0         # Dividend yield
sigma <- 0.20  # Volatility
m <- 50        # Number of monitoring points

sample_sizes <- c(1000, 4000, 16000, 64000, 256000)
# Initialize the results data frame
results_cv <- data.frame(
  Sample_Size = numeric(),
  Option_Price = numeric(),
  Standard_Error = numeric(),
  CI_Lower = numeric(),
  CI_Upper = numeric(),
  Computation_Time_sec = numeric(),
  stringsAsFactors = FALSE
)
# Remove or comment out the next line to prevent printing the empty data frame
# results_cv

for (N in sample_sizes) {
  # Monte Carlo with Control Variate
  res_cv <- price_asian_call_MC_control_variate(S0, K, T, r, q, sigma, m, N)
  results_cv <- rbind(results_cv, data.frame(
```

```
    Sample_Size = N,
    Option_Price = res_cv$Option_Price,
    Standard_Error = res_cv$Standard_Error,
    CI_Lower = res_cv$Confidence_Interval[1],
    CI_Upper = res_cv$Confidence_Interval[2],
    Computation_Time_sec = res_cv$Computation_Time_sec
  ))
}

# Display the final results using kable
kable(results_cv, caption = "Monte Carlo with Control Variate Simulation Results")
```

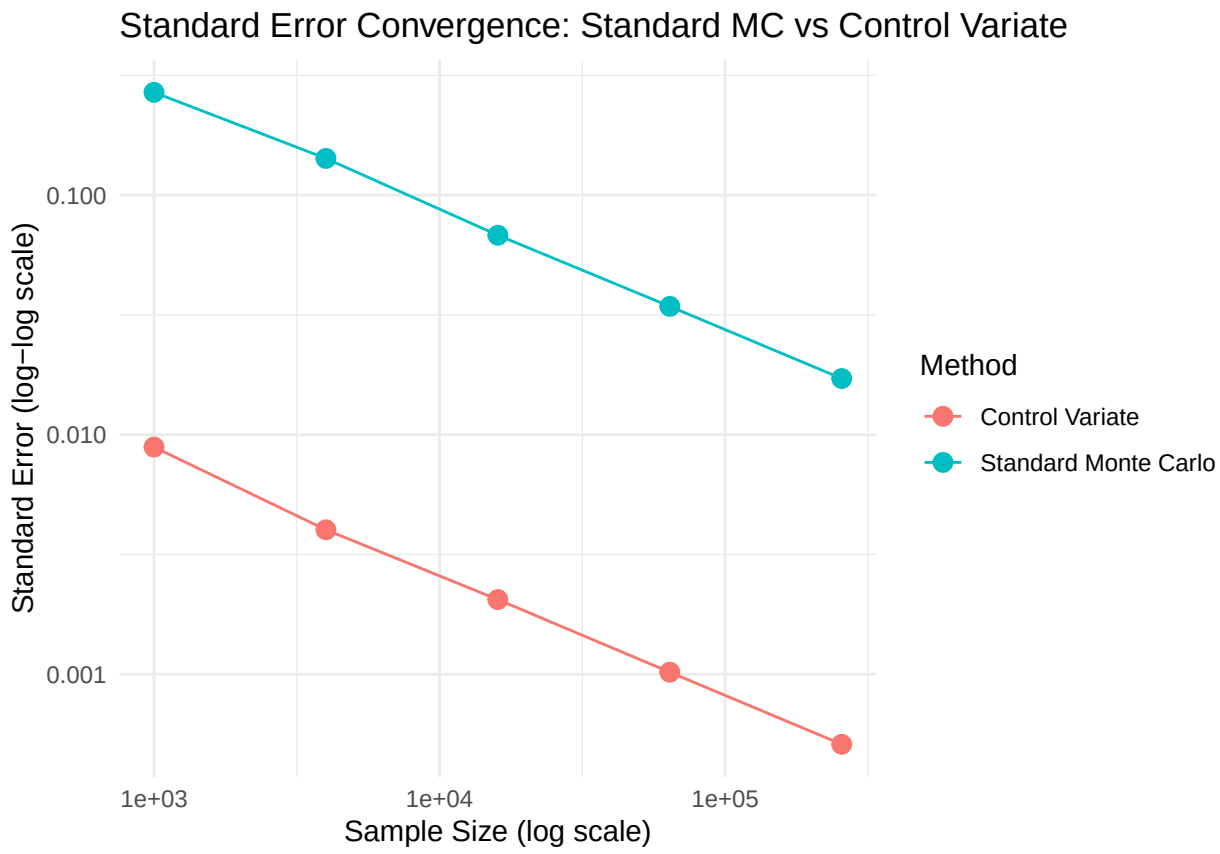
Table 2: Monte Carlo with Control Variate Simulation Results

Sample_Size	Option_Price	Standard_Error	CI_Lower	CI_Upper	Computation_Time_sec
1000	7.17	0.00888	7.15	7.18	0.03
4000	7.16	0.00401	7.15	7.17	0.03
16000	7.16	0.00205	7.16	7.17	0.14
64000	7.16	0.00102	7.16	7.17	0.59
256000	7.17	0.00051	7.16	7.17	2.70

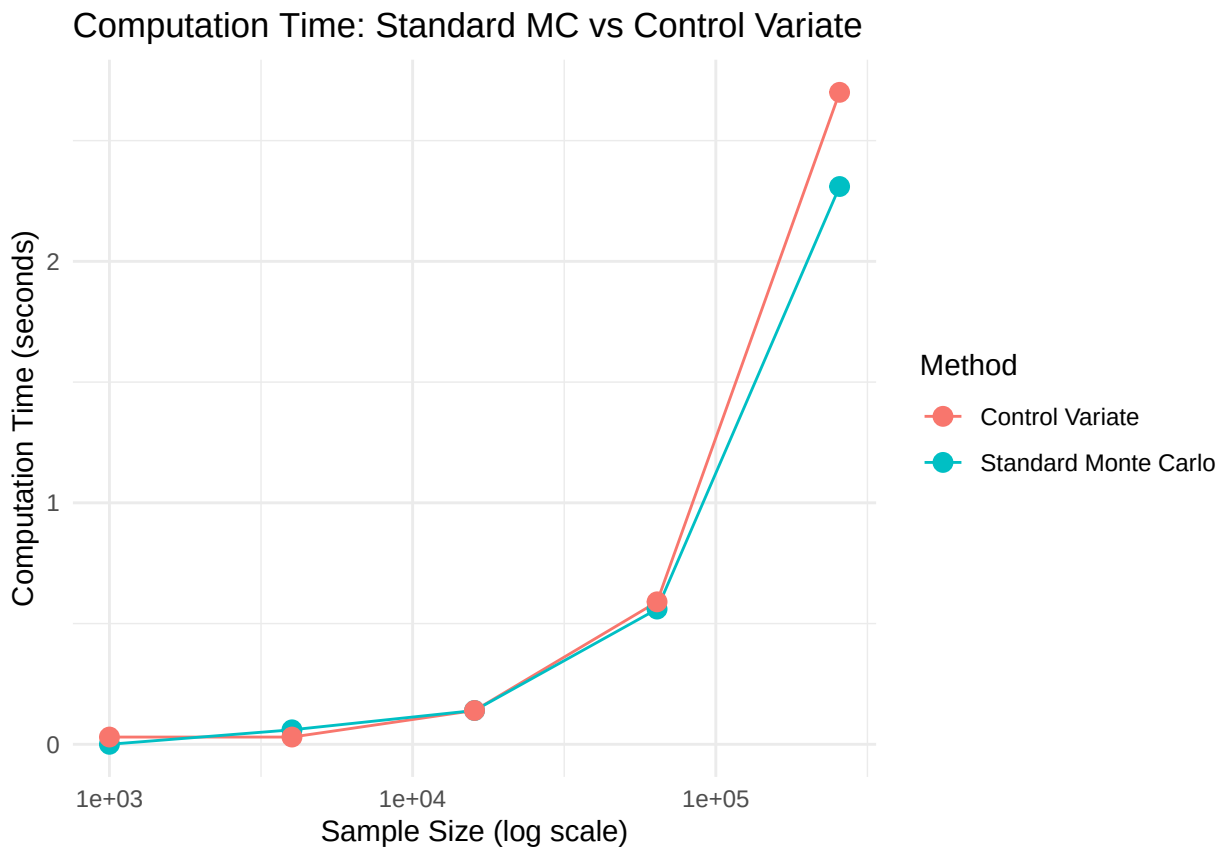
```
# Combine data if not already done
standard_mc_results$Method <- "Standard Monte Carlo"
results_cv$Method <- "Control Variate"
comparison_results <- rbind(standard_mc_results, results_cv)

# Plot Option Price Convergence with 95% CI
ggplot(comparison_results, aes(x = Sample_Size, y = Option_Price, color = Method, group = Method)) +
  geom_line() +
  geom_point(size = 3) +
  geom_ribbon(
    aes(ymin = CI_Lower, ymax = CI_Upper, fill = Method),
    alpha = 0.2,
    color = NA
  ) +
  scale_x_log10() +
  labs(
    title = "Option Price Convergence with 95% CI: Standard MC vs Control Variate",
    x = "Sample Size (log scale)",
    y = "Option Price",
    color = "Method",
    fill = "Method"
  ) +
  theme_minimal()
```




```r
# Plot Standard Error Convergence
ggplot(comparison_results, aes(x = Sample_Size, y = Standard_Error, color = Method, group = Method)) +
  geom_line() +
  geom_point(size = 3) +
  scale_x_log10() +
  scale_y_log10() +
  labs(
    title = "Standard Error Convergence: Standard MC vs Control Variate",
    x = "Sample Size (log scale)",
    y = "Standard Error (log-log scale)",
    color = "Method"
  ) +
  theme_minimal()
```




```r
# Plot Computation Time Comparison
ggplot(comparison_results, aes(x = Sample_Size, y = Computation_Time_sec, color = Method, group = Metho
  geom_line() +
  geom_point(size = 3) +
  scale_x_log10() +
  labs(
    title = "Computation Time: Standard MC vs Control Variate",
    x = "Sample Size (log scale)",
    y = "Computation Time (seconds)",
    color = "Method"
  ) +
  theme_minimal()
```



2.4 Analysis of Results: Comparing Standard Monte Carlo (MC) and Control Variate (CV)

The comparison of the Standard Monte Carlo method and the Control Variate method reveals clear advantages of the latter in terms of convergence rate, variance reduction, and overall efficiency.

1. Option Price Convergence

The Option Price Convergence plot highlights the convergence of option price estimates as the sample size increases. The control variate method demonstrates significantly tighter confidence intervals compared to the standard Monte Carlo method.

For the CV method, the option price estimate stabilizes more rapidly and converges to a consistent value, even at smaller sample sizes. In contrast, the standard MC method requires larger sample sizes to achieve comparable stability. Both methods ultimately converge to approximately the same option price as the sample size increases ($N = 256,000$), confirming the consistency of their final estimates.

The tighter confidence intervals produced by the CV method highlight its ability to improve precision, particularly when computational resources are limited or fewer samples are used.

2. Standard Error Convergence

The Standard Error Convergence plot examines the convergence of standard error on a log-log scale. Both methods exhibit the expected Monte Carlo convergence rate of $O(1/\sqrt{N})$, as evidenced by the linear trends in the plot. However, the CV method consistently achieves lower standard errors compared to the standard MC method at all sample sizes.

For instance: at $N = 1000$, the CV method achieves a standard error more than 10 times smaller than the standard MC method. Even as N increases to $256,000$, the CV method maintains a significant advantage in standard error reduction.

This reduction in standard error reflects the CV method's effectiveness in reducing variance, which is critical in applications requiring high precision. The CV technique allows for comparable accuracy with far fewer samples, making it computationally more efficient than standard MC.

3. Computation Time

The third plot compares the computation times for the two methods. As expected, the computation time increases linearly with sample size for both approaches. The CV method incurs slightly higher computational costs compared to standard MC, reflecting the additional calculations required for the control variate adjustment. For example, at $N = 256,000$, the computation time for CV is marginally higher than that of standard MC.

Despite this minor overhead, the significant reduction in variance achieved by the CV method more than compensates for the slightly higher computation cost. This trade-off makes the CV technique highly effective in scenarios where variance reduction is a priority.

2.5 Further Discussion on Variance Reduction Techniques

In addition to the control variate method, variance reduction can be further enhanced by combining it with other techniques such as stratified sampling and the Brownian bridge construction. These methods introduce additional refinements to the Monte Carlo simulation process, addressing specific sources of randomness and variability in the simulation.

Stratified sampling divides the sample space into non-overlapping strata and ensures that an equal number of samples are drawn from each stratum. This method reduces variance by minimizing randomness in the sampling process.

Mathematically, the variance of the stratified sampling estimator can be expressed as:

$$\mathrm{Var}(\hat{\theta}_{\text{stratified}}) = \sum_{k=1}^{L} \frac{w_k^2}{n_k} \mathrm{Var}(\theta_k),$$

where: - L is the number of strata, - w_k is the weight of the k-th stratum, - n_k is the number of samples in the k-th stratum, - $\mathrm{Var}(\theta_k)$ is the variance within the k-th stratum.

By ensuring that n_k is proportional to w_k, stratified sampling can achieve a lower variance compared to direct sampling. A common application of stratified sampling in Monte Carlo simulations is in the context of Brownian motion, where stratification is often applied to the first step of the simulation, Z_1, which introduces the most variability in the path. This step simplifies the implementation and ensures that the variability in the simulation is minimized from the outset. However, stratified sampling has its limitations. It is computationally more expensive than standard Monte Carlo sampling because it requires the sample space to be divided and the simulation to be carefully managed within each stratum.

The Brownian bridge construction is an effective technique for variance reduction that refines the simulation of Brownian paths by interpolating intermediate points based on the known values at the start and end of the process. By introducing conditional dependencies among simulated points, this method ensures consistency with the overall process, making it particularly useful when the terminal value of the asset significantly influences the option payoff. Moreover, the Brownian bridge is computationally efficient for high-dimensional problems, as it reduces the number of independently generated random variables and complements the control variate method by improving the accuracy of simulated paths.

Despite its advantages, the implementation of the Brownian bridge introduces added complexity. Interpolation of intermediate points and adjustments to the simulation framework require careful execution.

Additionally, the variance reduction achieved depends on the characteristics of the option and the underlying asset paths, and in some cases, the benefits may be marginal relative to the additional implementation effort.