Generalized Control Variate Methods for Pricing Asian Options

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Abstract

The conventional control variate method proposed by Kemna and Vorst (1990) to evaluate Asian options can be interpreted as a particular selection of linear martingale controls. We construct another martingale control variate method based on an option price approximation, which outperforms the conventional control variate method. It is straightforward to generalize such linear control to a nonlinear situation like the American Asian option problem. From the variance analysis of martingales, the performance of control variate methods depends on the distance between the approximate martingale and the optimal martingale. This measure becomes helpful for the design of control variate methods for complex problems such as Asian option under stochastic volatility models. We demonstrate multiple choices of controls and test them under MC/QMC simulations. QMC methods do not work well without a control. After adding a control, the variance reduction ratios raise up to 260 times for randomized QMC and 60 times for MC simulations.

1 Introduction

Control variate methods have been widely used for computational finance as a mean of variance reduction. Perhaps the most successful application is to evaluate continuous-time arithmetic-average Asian options. Kemna and Vorst [13] employed a control being a discounted counterpart geometric-average Asian option payoff less than its price. This method is efficient because (1) the correlation between the arithmetic average and the geometric average is very high and (2) the counterpart geometric average Asian option price has a closed-form solution. Therefore searching for a highly correlated payoff random variable with a possible closed-form expectation becomes a criteria for the construction of control variates. There are sophisticate techniques such as stratified sampling or Brownian bridge developed to combine with the control variate method in order to enhance variance reduction. See [7] and references therein for more discussions. However for complex problems such as American style or the situation where the volatility are random, it becomes less transparent to construct a control variate because the searching criteria may not be easy to satisfy.

In this paper we investigate the conventional control variate method proposed by Kemna and Vorst as a linear control. The martingale representation of the control is used for variance analysis. It turns out that the variance induced from the conventional control variate method is a projection from a stochastic control parameter space. Moreover, we are able to build up many martingale controls based on option price approximations. The martingale control constructed from a more accurate price approximation is more correlated to the original discounted payoff and the martingale control has mean zero. This idea bridges up the studies of control variate methods to option price approximations. There is an enormous literature studying arithmetic-average Asian option prices even under the Black-Scholes model. See [19, 21] for example. Using a recent approximation by Zhang [21], it is easy to construct a new martingale control for the use of variance reduction, which outperforms the conventional control variate in terms of generating smaller standard errors. In particular, these martingale controls are applicable to nonlinear problems such as American Asian options. As a matter of fact, there is only one uncertainty induced by Brownian motion under the Black-Scholes model. Hence martingale control defined on the same uncertainty are expected to reduce the variance. These are confirmed by many numerical results in this paper.

In the context of stochastic volatility models, searching for control variate is less known and studied. Clewlow and Carverhill [1] applied a financial intuition of delta hedging to build up some portfolio possibly with other Greeks hedge as controls in order to evaluate option prices. They found that the variance obtained by this control variate under an one-factor stochastic volatility model [10] can be significantly reduced. By means of the least-squares method [14], Potters, Bouchaud, and Sestovic [15] proposed an optimal hedged Monte Carlo method to price options under the "historical" probability measure. Later, Pochart and Bouchaud [16] extended the same idea to problems when shortfall risk and transaction cost constraints are considered. Heath and Platen [9] and Fouque and Han [3] used different approximate European option prices to approximate *delta* portfolio under stochastic volatility models and showed significant variance reduction performance. Moreover, in [4], Fouque and Han obtained an asymptotic result to characterize the variance of a martingale control variate under stochastic volatility models when the *time scales* of driving volatility processes are well-separated.

Based on the construction of hedging martingales under stochastic volatility models, we further investigate the Asian option pricing problem. We find that using the hedging martingale as a control can only partially diminish uncertainties associate with driving volatility processes. In contrast, the conventional control by the geometric-average counterpart can be shown to reduce each part of random sources, though the counterpart option price has no closed-form solution. Therefore we propose a two-step algorithm to overcome this difficulty.

All Monte Carlo methods mentioned so far are fundamentally related to psudo random sequences. However integration methods using the alternative quasi-random sequences (or called low-discrepancy sequences) have drawn lots of attentions in recent years because its theoretical rate of convergence is $\mathcal{O}(1/n^{1-\varepsilon})$ for all $\varepsilon > 0$ subjected to the dimensionality and the regularity of the integrand. We refer to [7], [11] and [12] for further discussions on Monte Carlo and Quasi-Monte Carlo (MC/QMC for short) methods in applications of computational finance. In our numerical results dealing with stochastic volatility models, all estimators without controls perform poorly using randomized QMC. This is not surprise because a nonsmooth call payoff and a high dimension of 300 are encountered in these experiments. However after adding controls QMC methods work very well in all cases. Even in a high dimension regime, the control starts to play the role of a smoother. It can be seen from a variance analysis in Section 4.3 that on average the fluctuation of the control variate is continuous and of small order. In [8], the authors study the smoothing effect by martingale controls. An example to estimate low-biased solutions of an American option by least squares method [14] is given. It shows that using QMC (Niederreiter sequence) actually gives estimates greater than the true option price rather than biased low. Adding a martingale control for variance reduction, all problematic estimates become biased low and very close to true values.

The organization of this paper is the following. In Section 2, we review the conventional control variate method and give a new interpretation. New martingale controls are constructed and easily applied to American Asian option problems. In Section 3, multi-factor stochastic volatility models are introduced. A Singular and regular perturbation method is applied to Asian option problem in Section 4 for pricing geometric-average Asian options. This method outperforms an importance sampling studied in [3]. A variance analysis for a simplified perturbed volatility model is presented. By a combination of martingale control variate method for the geometric-average Asian option and the conventional control variate method for the arithmetic-average Asian option, we propose a two-step method in Section 5. Numerical experiments implemented by Monte Carlo method and quasi-Monte Carlo methods are presented. We test several combinations of control variate methods with quasi-Monte Carlo methods, including the Sobol' sequence and L'Ecuyer type good lattice points together with the Brownian bridge sampling technique.

2 Revisit Control Variate Methods

We consider the estimation of a mathematical expectation $E\{X\}$ by Monte Carlo simulations, where X is a square-integrable random variable defined on a probability space. A control variate method is a variance reduction technique which aims to improve the precision of the estimate obtained from plain Monte Carlo simulations. The basic idea of control variate is to choose an appropriate counterpart square integrable random variable, say Y being centered at zero, and a control parameter λ so that the control variate $X + \lambda Y$ is unbiased, $E\{X\} = E\{X + \lambda Y\}$, and the variance can be reduced, $Var\{X\} > Var\{X + \lambda Y\}$. This can be done by choosing (1) any mean-zero random variable Y correlated with X and (2) the optimal control parameter, deduced from minimizing the variance of the control variate,

$$\lambda^\star = -\frac{Cov(X,Y)}{Var\{Y\}} \tag{1}$$

so that it is easy to see that the new variance can not be greater than the original variance:

$$Var\{X + \lambda^\star Y\} = \left(1 - \rho^{\star^2}\right) Var\{X\} \leq Var\{X\}, \tag{2}$$

where $\rho^\star$ is the correlation coefficient of random variables X and Y. In the case of random variables X and Y being highly correlated, either positive or negative, $|\rho^\star| \approx 1$ and a considerable reduction of variance is expected.
Because calculating the optimal control parameter $\lambda^\star$ requires the exact value of $E\{X\}$, in practice one can use a suboptimal control parameter λ by approximating the right hand side of (1) empirically through the plain Monte Carlo simulations. The error of variances obtained from the optimal control variate parameter $\lambda^\star$ and its perturbation λ^ε is shown below.

Lemma 1 *Given any constant ε such that the suboptimal control parameter $\lambda^\varepsilon = \lambda^\star + \varepsilon$ being a perturbed parameter, the gain of variance from the suboptimal control variate is of order ε^2:*

$$Var\left\{X + (\lambda^\star + \varepsilon)\, Y\right\} - \left(1 - \rho^{\star^2}\right) Var\{X\} = \varepsilon^2\, Var\{Y\}.$$

Proof: It is easy to calculate the variance of the perturbed control variate

$$\begin{aligned}
& Var\left\{X + (\lambda^\star + \varepsilon)\, Y\right\} \\
&= Var\left\{X + \lambda^\star Y\right\} + \varepsilon^2 Var\{Y\} \\
&= \left(1 - \rho^{\star^2}\right) Var\{X\} + \varepsilon^2 Var\{Y\},
\end{aligned}$$

where we have used the definition in (1) and a result in (2).
If the perturbation parameter ε is small, the error of these variances is negligible provided the variance

of the chosen control Y is for instance of $\mathcal{O}(1)$. The simple variance analysis shown in (1) and (2) does not depend on the variance of control Y. It becomes significant while we study a sensitivity analysis over the control parameter shown in Lemma 1. Though the variance of control is not often emphasized in use of control variate methods, we will see in Section 5 that it becomes important for the selection of controls.

In computational finance a well-known example of using the control variate method is the evaluation of a continuous-time and arithmetic-average Asian options under the Black-Scholes model. Based on the risk-neutral pricing theory [18], the fair price of the Asian option, denoted by P_A, is equal to a conditional expectation under the risk-neutral probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, I\!P^\star\right)$

$$P_A(t, S_t, A_t) = I\!E^\star\left\{e^{-r(T-t)}H(A_T)|\mathcal{F}_t\right\} \tag{3}$$

where the underlying risky asset S_t is governed by a geometric Brownian motion defined as

$$dS_t = rS_t dt + \sigma S_t dW_t^\star, \tag{4}$$

the arithmetic-average price process A_t is defined by $A_t = \frac{1}{T}\int_0^t S_s\, ds$. Other notations are defined as follows: t the current time, $T < +\infty$ the maturity, r the risk-free interest rate, σ the volatility, $W_t^\star$ the standard Brownian motion, $H(x)$ the payoff function satisfying the usual integrability condition. For example if $H(x) = \max\{x-K, 0\} \equiv (x-K)^+$ for the strike price $K > 0$ it is a call payoff; if $H(x) = \max\{K-x, 0\} \equiv (K-x)^+$ it is a put payoff.

The control variate method proposed by Kemna and Vorst [13] to evaluate P_A defined in (3) introduces a counterpart geometric-average price process $G_t = \exp(\frac{1}{T}\int_0^t \ln S_s\, ds)$ and a geometric-average Asian option price P_G defined by

$$P_G(t, S_t, G_t) = I\!E^\star\left\{e^{-r(T-t)}H(G_T)|\mathcal{F}_t\right\} \tag{5}$$

such that a control variate for estimating P_A is

$$e^{-r(T-t)}H(A_T) + \lambda\left(e^{-r(T-t)}H(G_T) - P_G(t, S_t, G_t)\right). \tag{6}$$

The success of this control variate attributes at least two facts:

(1) the control

$$e^{-r(T-t)}H(G_T) - P_G(t, S_t, G_t) \tag{7}$$

is highly correlated to the original discounted payoff variable $e^{-r(T-t)}H(A_T)$. This is confirmed by empirical tests. See for instance [7, 13] that correlations between the control variate (7) and $e^{-r(T-t)}H(A_T)$ are close to 1 in many examples. From Lemma 1 a small error in the control parameter does not affect the variance reduction much. Therefore it is also practical to simply use a constant control parameter.

(2) the counterpart geometric-average Asian option price P_G admits a Black-Scholes type closed-form solution and the random variable G_T is defined on the same probability space as A_T so that Monte Carlo simulations for the control variate become very efficient to implement.

Though such application of control variate methods using the geometric average and the arithmetic average seems problem oriented, in next section we reinterpret that the conventional control variate method is essentially a particular choice of option price approximation by P_G to the true price P_A. As a result one can enlarge the class of control variate methods from choices of option price approximations. For example on the merit of a more accurate approximation proposed by Zhang [21], a new control variate can be built up accordingly. In next two sections, we demonstrate applications of these control variate methods to evaluate Asian options and American Asian options.

2.1 Martingale Representation of the Control

By an application of Ito's lemma, the control in (7) has the following martingale representation

$$e^{-rT}H(G_T) - P_G(0, S_0, G_0) = \mathcal{M}(P_G; T) \tag{8}$$

where the process $\mathcal{M}(P_G; t)$ is a zero-centered martingale

$$\mathcal{M}(P_G; t) = \int_0^t e^{-rs} \frac{\partial P_G}{\partial x}(s, S_s = x, G_s)\, \sigma S_s dW_s^\star. \tag{9}$$

Imagine the control parameter λ being a $\mathcal{F}_t$-adapted process and we introduce a generalized control variate

$$e^{-r\,T}H(A_T) + \mathcal{M}(\lambda\, P_G; T). \tag{10}$$

The variance of the control variate (6) conditional at time zero is

$$\begin{aligned} & Var\left\{e^{-r\,T}H(A_T) - P_A(0, S_0, A_0) + \lambda \mathcal{M}(\,P_G; T)\right\} \\ & = Var\left\{\mathcal{M}(P_A; T) + \lambda \mathcal{M}(P_G; T)\right\} \\ & = \int_0^T e^{-2rs}\sigma^2 I\!E^\star \left\{ \left(\frac{\partial P_A}{\partial x}(s, S_s, A_s) + \lambda \frac{\partial P_G}{\partial x}(s, S_s, G_s) \right)^2 S_s^2 \mid \mathcal{F}_0 \right\} ds, \end{aligned} \tag{11}$$

where we have used the linearity of stochastic integrals and the quadratic variation. Imagine the control parameter λ being a $\mathcal{F}_t$-adapted process, one can fully eliminate the variance by the optimal control *process*

$$\lambda_t^\star = -\frac{\partial P_A}{\partial x}(t, S_t, A_t) \,/ \frac{\partial P_G}{\partial x}(t, S_t, G_t)\,. \tag{12}$$

Analog to (1), the optimal control parameter or process requires the exact value of P_A or its delta respectively in order to diminish the variance. We show next that the optimal control parameter $\lambda^\star$ is simply a projection of the optimal control process over the real line $\Re$.

Lemma 2 *For pricing Asian options by conventional control variate method, the optimally reduced variance in (2) can be obtained from solving the following minimization problem*

$$\left(1 - \rho^{\star^2}\right) Var\left\{e^{-r\,T}H(A_T)\right\} = \min_{\lambda \in \Re} Var\left\{ \int_0^T e^{-rs} \left(\frac{\partial P_A}{\partial x}(s, S_s, A_s) + \lambda \frac{\partial P_G}{\partial x}(s, S_s, G_s) \right) \sigma S_s dW_s^\star \right\}.$$

Proof: From (11) the constant minimizer solving the variance above is

$$\lambda^\star \quad = \quad -\frac{Cov\left(\mathcal{M}(P_A; T), \mathcal{M}(P_G; T)\right)}{Var\left(\mathcal{M}(P_G; T)\right)}$$

such that the optimally reduced variance is

$$I\!E^\star \left\{ \int_0^T e^{-2rs} \left(\frac{\partial P_A}{\partial x}(s, S_s, A_s) + \lambda^\star \frac{\partial P_G}{\partial x}(s, S_s, G_s) \right)^2 \sigma^2 S_s^2 ds \right\} = \left(1 - \rho^{\star^2}\right) Var\left\{e^{-r\,T}H(A_T)\right\},$$

where $\rho^\star$ denotes the correlation of $e^{-r\,T}H(A_T)$ and the control $\mathcal{M}(P_G; T)$.
Because the geometric-average control $\mathcal{M}(P_G; T)$ is not perfectly correlated to the counterpart arithmetic-average payoff $e^{-r\,T}H(A_T)$, the variance of the control variate (6) can only be optimally reduced by the factor $1 - \rho^{\star^2}$ by the choice of a constant control parameter $\lambda^\star$.

2.2 Construction of More Linear Controls

Analog to the martingale representation in (8) and (9), the arithmetic-average control $\mathcal{M}(P_A;T)$ is optimal since it is perfectly correlated to the discounted payoff $e^{-rT}H(A_T)$. The conventional control variate method using the control $\mathcal{M}(P_G;T)$ can also be reinterpreted as a method of using the option price P_G to approximate the true price P_A. This can be seen from the variance analysis (11) by choosing $\lambda = -1$

$$Var\left(e^{-rT}H(A_T) - \left(e^{-rT}H(G_T) - P_G(0,S_0,G_0)\right)\right) = Var\left(\mathcal{M}\left(P_A - P_G;T\right)\right).$$

Variance reduction depends on the difference of option prices $P_A - P_G$. This fact opens up an opportunity to build up new controls. Notice that all controls is (linearly) additive to the discounted payoff $e^{-rT}H(A_T)$.

Approximating Asian option prices has been studied for nearly twenty years. One is possible to build up a martingale control by selecting a better price approximation. For example we introduce a control $\mathcal{M}(P_Z;T)$, where P_Z is an approximate option price proposed by Zhang [21] based on a perturbation expansion on the singularity at the diffusion coefficient. For an Asian call option with the strike price K, we consider only the leading order expansion in [21] as

$$P_Z(t,S_t,A_t) = \frac{S_t}{T}\,f(\xi_t, T-t), \tag{13}$$

where the variable $\xi_t = \frac{T\,(K-A_t)}{S_t}e^{-r(T-t)} - \frac{1-e^{-r(T-t)}}{r}$ and the function $f(\xi,\tau) = -\xi\mathcal{N}(-\xi/\sqrt{2\tau}) + \sqrt{\frac{\tau}{\pi}}e^{-\xi^2/(4\tau)}$. The cumulative normal integral function is denoted by $\mathcal{N}(\cdot)$. A new control is defined by

$$\mathcal{M}(P_Z;t) = \int_0^t e^{-rs}\frac{\partial P_Z}{\partial x}\left(s, S_s = x, A_s\right)\sigma S_s dW_s^\star. \tag{14}$$

In Table 1 we illustrate effects of three controls, including the conventional geometric-average Asian option as in (7), its martingale representation $\mathcal{M}(P_G;T)$ as in (8), and the martingale control by Zhang's approximation $\mathcal{M}(P_Z;T)$ as in (14), under the in-the-money situation with various volatility. Because these corresponding estimators are unbiased, we shall focus on standard errors. Since the first two controls are equivalent, it is observed that standard errors are roughly of the same order. The last control by $\mathcal{M}(P_Z;T)$ produces the smallest standard errors. This phenomenon also indicates that on average P_Z provides a better approximation to the true option price P_A than P_G.

We have proposed a constructive way to build up martingale control $\mathcal{M}(P;T)$ where P is any price approximation to P_A. For instance if $P = P_G$, we recover the conventional control variate from (8) because the discounted process $e^{-rt}P_G(t,S_t,G_t)$ is a martingale. When we choose the approximation $P = P_Z$, the representation of the equivalent process-type control is no longer suitable for the use as a control. This is because $e^{-rt}P_Z(t,S_t,A_t)$ is not a martingale which can be seen from the definition (13) and the probability representation of the transformed function function

$$f(\xi, T-t) = E\left\{max\left(-\varphi_T, 0\right) \mid \varphi_t = \xi\right\}, \tag{15}$$

where the process φ_t is governed by the stochastic differential equation

$$d\varphi_t \;=\; \frac{\sigma}{r}(1-e^{-r(T-t)})dW_t,$$

where W_t is a Brownian motion. Notice that this W_t can be defined on a different probability space than $W_t^\star$ in (4). This fact implies that martingale controls are more general than conventional controls

Table 1: Comparisons of three control variates to estimate the arithmetic-average Asian call option prices with a fixed strike price while constant volatility varies from 10% to 70 %. Other model parameters are chosen by $S_0 = 65, K = 55, r = 0.06, T = 1$ year. In column 2, CV denotes the conventional control as in (7); in column 3, $\mathcal{M}(P_G;T)$ denotes the equivalent martingale representation as in (8); in column 4, $\mathcal{M}(P_Z;T)$ denotes the new control as in (14). Sample means and standard errors in parenthesis are shown in pair. Monte Carlo simulations are implemented under the sample size $N = 10000$ and the discretized time step 200.

σ	CV	$\mathcal{M}(P_G;T)$	$\mathcal{M}(P_Z;T)$
10	11.2920 (0.00052)	11.2910 (0.00051)	11.2920 (0.00019)
20	11.4010 (0.0018)	11.4020 (0.0021)	11.4010 (0.0013)
30	11.8950 (0.0038)	11.8930 (0.0046)	11.8960 (0.0031)
40	12.6820 (0.0066)	12.6840 (0.0081)	12.6790 (0.0053)
50	13.6230 (0.012)	13.6070 (0.012)	13.6280 (0.0079)
60	14.6670 (0.017)	14.6270 (0.017)	14.6870 (0.011)
70	15.7910 (0.024)	15.7310 (0.026)	15.7600 (0.014)

where concrete correlated processes must be postulated.
So far we have discussed only the European Asian options. Conventional control variate method [13] is reviewed and expanded to martingale control variates. It is readily observed that these controls are in linear fashion. Next, we apply these martingale controls in a nonlinear fashion to the situation where early exercise is permitted.

2.3 Pricing American Asian Options: Nonlinear Control

This section concerns the American Asian option pricing problem under the Black-Scholes model. We study the price of an American Asian call option with a fixed strike price K as an example but other payoffs can be treated similarly. The Asian call option price solves an optimal stopping time problem

$$P_{A,A}(t, S_t, A_t) = ess \sup_{t \le \tau \le T} I\!E^\star \left\{ e^{-r\,(\tau - t)} (A_\tau - K)^+ | \mathcal{F}_t \right\}, \tag{16}$$

with τ being a bounded stopping time between the current time t and the maturity T. Solving an optimal stopping problem by simulation, known as a primal approach, is challenging though there exist some methods like least squares [14], etc. Because the primal approach is less related to control variate. We will focus directly to the dual approach to solving the problem (16). On the merit of the dual formulation [5, 17], we have an equivalent expression at time zero:

$$P_{A,A}(0, S_0, A_0) = \inf_{\mathcal{M} \in H_0^1} I\!E^\star \left\{ \sup_{0 \le t \le T} \left((A_t - K)^+ - \mathcal{M}_t \right) | \mathcal{F}_0 \right\}, \tag{17}$$

where martingales in the space H_0^1 are uniformly integrable and centered at zero at time zero. Hence given a suitable martingale, a high-biased estimate of the American Asian price $P_{A,A}$ can be obtained. It turns out the optimal martingale which gives a pathwise equality [5]

$$P_{A,A}(0, S_0, A_0) = \sup_{0 \le t \le T} \left((A_t - K)^+ - \mathcal{M}_t^\star \right)$$

is $\mathcal{M}_t^\star = \mathcal{M}(P_{A,A}; t)$. The price and variance error analysis are also given in [5] . Heuristically we like to propose some suboptimal martingales using $P_{A,A}$ price approximation such as $\mathcal{M}(P_G; t)$ or $\mathcal{M}(P_Z; t)$.

Table 2: Comparisons of two martingale control variates to estimate American arithmetic-average Asian call option while the fixed strike price K takes value of 45, 50 and 55. Other model parameters are chosen by $S_0 = 55, r = 0.1, \sigma = 40\%, T = 1$ year. In column 2, $\mathcal{M}(P_G; t)$ denotes the the martingale control as in (8); and those MADs in column 3 are computed based on results in column 2. In column 4, $\mathcal{M}(P_Z; t)$ denotes the martingale control as in (14) and column 5 show associated MADs. Sample means and standard errors in parenthesis are shown in columns 2 and 4. Monte Carlo simulations are implemented under the sample size $N = 5000$ and the discretized time step 500.

K	$\mathcal{M}(P_G; t)$	MAD	$\mathcal{M}(P_Z; t)$	MAD
45	8.3598 (0.0113)	0.2451	8.3671 (0.0051)	0.1455
50	5.5622 (0.0111)	0.2536	5.5675 (0.0063)	0.1880
55	3.5361 (0.0113)	0.7600	3.5330 (0.0065)	0.7355

In Table 2 we illustrate effect of two martingale controls, including the martingale control by the European geometric-average Asian option counterpart $\mathcal{M}(P_G; t)$ and the martingale control by Zhang's approximation $\mathcal{M}(P_Z; t)$, with various strike prices. Notice that the standard errors, shown in parenthesis and MAD (the mean absolute deviation from the mean) are important factors to indicate how large the high biased estimate deviated from the true price. See [5] for the relation of price bound and variance error and [17] for the definition of MAD. We observe that all standard errors and MADs from Zhang's method are all smaller than the typical approximation by the geometric-average Asian option. These numerical results also support that $\mathcal{M}(P_Z; t)$ is a better control than $\mathcal{M}(P_G; t)$. Note that $\mathcal{M}(P_G; t) = e^{-rt}H(G_t) - P_G(0, S_0, G_0; t)$, where $P_G(0, S_0, G_0; t)$ denotes the price of a geometric-average Asian option price with the maturity t rather than T. One can in principle replace this martingale control by its process representation, but this does not reduce any computational burden. In the situation of pricing European Asian options, the martingale at maturity T, $\mathcal{M}(P_G; T)$, is only needed for the linear control. The computational complexity for $e^{-rT}H(G_T) - P_G(0, S_0, G_0; T)$ is indeed much less than $\mathcal{M}(P_G; T)$, where the delta $\frac{\partial P_G}{\partial S_t}$ must be computed along each simulated path. In contrast the martingale control process $\{\mathcal{M}(P_G; t)\}_{0 \leq t \leq T}$ is required to estimate the high-biased solution. Therefore the computational complexity for $\mathcal{M}(P_G; t)$ and $e^{-rt}H(G_t) - P_G(0, S_0, G_0; t)$ are roughly the same.

Currently only one random source namely the Brownian motion $W_t^\star$ is encountered, general martingale controls are in the form of stochastic integrals with respect to that Brownian motion. Next we introduce multi-factor stochastic volatility models, where multiple Brownian motions are involved. The choice of martingale controls becomes difficult to make. We shall introduce multi-factor stochastic volatility model and the perturbation technique in Section 3. In Section 4 the singular and regular perturbation technique is applied to simplify the choice of martingales for the geometric-average Asian options. In Section 5 control variates to estimate arithmetic-average Asian options are discussed. A combination of the conventional control variate method and the martingale control turns out to be a much better choice than applying martingale controls alone.

3 Monte Carlo Pricing under Multi-factor Stochastic Volatility Models

Under a risk-neutral probability measure $I\!P^\star$ parametrized by the combined market price volatility premium (Λ_1, Λ_2), we consider the following multi-factor stochastic volatility models defined by

$$\begin{aligned}
dS_t &= rS_t dt + \sigma_t S_t dW_t^{(0)*}, \\
\sigma_t &= f(Y_t, Z_t), \\
dY_t &= \left[\frac{1}{\varepsilon} c_1(Y_t) + \frac{g_1(Y_t)}{\sqrt{\varepsilon}} \Lambda_1(Y_t, Z_t)\right] dt + \frac{g_1(Y_t)}{\sqrt{\varepsilon}} \left(\rho_1 dW_t^{(0)*} + \sqrt{1-\rho_1^2} dW_t^{(1)*}\right), \\
dZ_t &= \left[\delta c_2(Z_t) + \sqrt{\delta} g_2(Z_t) \Lambda_2(Y_t, Z_t)\right] dt \\
&\quad + \sqrt{\delta} g_2(Z_t) \left(\rho_2 dW_t^{(0)*} + \rho_{12} dW_t^{(1)*} + \sqrt{1-\rho_2^2-\rho_{12}^2} dW_t^{(2)*}\right),
\end{aligned} \tag{18}$$

where S_t is the underlying asset price process with a constant risk-free interest rate r. Its random volatility σ_t is driven by two stochastic processes Y_t and Z_t varying on the *time scales* ε and $1/\delta$, respectively. The vector $\left(W_t^{(0)*}, W_t^{(1)*}, W_t^{(2)*}\right)$ consists of three independent standard Brownian motions. Instant correlation coefficients ρ_1, ρ_2, and ρ_{12} satisfy $|\rho_1| < 1$ and $|\rho_2^2 + \rho_{12}^2| < 1$. The volatility function f is assumed to be smoothly bounded. Coefficient functions of processes Y_t and Z_t, namely (c_1, g_1, Λ_1) and (c_2, g_2, Λ_2) are assumed to be smooth such that they satisfy the existence and uniqueness conditions for the strong solutions of stochastic differential equations. Mean-reverting processes such as Ornstein-Uhlenbeck (OU) processes or square-root processes are typical examples to model driving volatility processes [6, 10]. Under this setup, the joint process (S_t, Y_t, Z_t) is Markovian.
Given the multi-factor stochastic volatility model (18) under $I\!P^\star$, the price of a plain European option with the integrable payoff function H and expiry T is defined by

$$P^{\varepsilon,\delta}(t,x,y,z) = I\!E^\star{}_{t,x,y,z}\left\{e^{-r(T-t)} H(S_T)\right\},$$

where $I\!E^\star{}_{t,x,y,z}$ is a short notation for the expectation with respect to $I\!P^\star$ conditioning on the current states $S_t = x, Y_t = y, Z_t = z$. A basic Monte Carlo simulation approximates the option price $P^{\varepsilon,\delta}(0, S_0, Y_0, Z_0)$ at time 0 by the sample mean

$$\frac{1}{N} \sum_{i=1}^{N} e^{-rT} H(S_T^{(i)}) \tag{19}$$

where N is the total number of sample paths and $S_T^{(i)}$ denotes the i-th simulated stock price at time T. Variance reduction techniques are particularly important to accelerate the computing efficiency of the basic Monte Carlo pricing estimator (19). Next we briefly review the construction of a generic algorithm, i.e. martingale control variate method, recently proposed and analyzed by Fouque and Han [3, 4].

3.1 Construction of Martingale Control Variates

Assuming that the European option price $P^{\varepsilon,\delta}(t, S_t, Y_t, Z_t)$ is twice differentiable in state space and once differentiable in time, we apply Ito's lemma to its discounted price $e^{-rt}P^{\varepsilon,\delta}$, then integrate from time 0 to the maturity T. After canceling out the pricing partial differential equation in some non-martingale terms and using the fact that $P^{\varepsilon,\delta}(T, S_T, Y_T, Z_T) = H(S_T)$, the following martingale representation can be obtained

$$P^{\varepsilon,\delta}(0, S_0, Y_0, Z_0) = e^{-rT} H(S_T) - \mathcal{M}_0(P^{\varepsilon,\delta}) - \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_1(P^{\varepsilon,\delta}) - \sqrt{\delta}\mathcal{M}_2(P^{\varepsilon,\delta}), \tag{20}$$

where centered martingales are given by

$$\mathcal{M}_0(P^{\varepsilon,\delta}) = \int_0^T e^{-rs}\frac{\partial P^{\varepsilon,\delta}}{\partial x}(s, S_s, Y_s, Z_s) f(Y_s, Z_s) S_s dW_s^{(0)*}, \tag{21}$$

$$\mathcal{M}_1(P^{\varepsilon,\delta}) = \int_0^T e^{-rs}\frac{\partial P^{\varepsilon,\delta}}{\partial y}(s, S_s, Y_s, Z_s) g_1(Y_s) d\tilde{W}_s^{(1)*}, \tag{22}$$

$$\mathcal{M}_2(P^{\varepsilon,\delta}) = \int_0^T e^{-rs}\frac{\partial P^{\varepsilon,\delta}}{\partial z}(s, S_s, Y_s, Z_s) g_2(Z_s) d\tilde{W}_s^{(2)*}, \tag{23}$$

where the Brownian motions are

$$\tilde{W}_s^{(1)*} = \rho_1 W_s^{(0)*} + \sqrt{1-\rho_1^2} W_s^{(1)*},$$
$$\tilde{W}_s^{(2)*} = \rho_2 W_s^{(0)*} + \rho_{12} W_s^{(1)*} + \sqrt{1-\rho_1^2-\rho_{12}^2} W_s^{(2)*}.$$

These martingales $\mathcal{M}_0, calM_1$, and $\mathcal{M}_3$ play the role of "perfect" controls of Monte Carlo simulations. Namely, if the martingales (21), (22), and (23) can be exactly computed, then one can just generate one sample path to evaluate the option price through Equation (20). Unfortunately the gradient components $\left(\frac{\partial P^{\varepsilon,\delta}}{\partial x}, \frac{\partial P^{\varepsilon,\delta}}{\partial y}, \frac{\partial P^{\varepsilon,\delta}}{\partial z}\right)$ of the option price appearing in the martingales is not possibly known in advance while the option price $P^{\varepsilon,\delta}$ itself is exactly what we want to estimate. However, one can choose an approximate option price to substitute $P^{\varepsilon,\delta}$ used in the martingales (21), (22) and (23), and still retain their martingale properties.

When time scales ε and $1/\delta$ are well separated; namely, $0 < \varepsilon, \delta \ll 1$, the *zeroth* order approximation of the Black-Scholes type can be found in [6]

$$P^{\varepsilon,\delta}(t, x, y, z) \approx P_{BS}(t, x; \bar{\sigma}(z)) \tag{24}$$

with its pointwise accuracy of order $\mathcal{O}\left(\sqrt{\varepsilon}, \sqrt{\delta}\right)$ for continuous piecewise payoffs and $P_{BS}(t, x; \bar{\sigma}(z))$ solves the Black-Scholes partial differential equation with the constant volatility $\bar{\sigma}(z)$ and the terminal condition $P_{BS}(T, x) = H(x)$. Note that this approximation $P_{BS}(t, x; \bar{\sigma}(z))$ is y-variable independent, where y is the initial value of the fast varying process Y_t. The z-dependent *effective volatility* $\bar{\sigma}(z)$ is defined as the square root of an average of the variance function f^2 with respect to a limiting distribution of Y_t:

$$\bar{\sigma}^2(z) = \int f^2(y, z) d\Phi(y) = \left\langle f^2(y, z)\right\rangle, \tag{25}$$

where $\Phi(y)$ denotes the invariant distribution of the fast varying process Y_t while setting the volatility premium Λ_1 as zero. This is because, in the drift term of the dY_t equation in (18), $\frac{1}{\varepsilon}$ term dominates $\frac{1}{\sqrt{\varepsilon}}$ term when $\varepsilon \ll 1$. We use the bracket $< \cdot >$ to represent such average. In the OU case, we choose that $c_1(y) = m_1 - y$ and $g_1(y) = \nu_1\sqrt{2}$ with $\Lambda_1 = 0$ such that $1/\varepsilon$ is the rate of mean reversion, m_1 is the long run mean, and ν_1 is the long run standard deviation. Its invariant distribution Φ is normal with mean m_1 and variance ν_1^2. We refer to [6] for detailed discussions.

Because the approximate option price $P_{BS}(t,x;\bar\sigma(z))$ is independent of y, the term $\mathcal{M}_1(P_{BS})$ diminishes. Since the approximate martingale $\mathcal{M}_2^{(i)}(P_{BS})$ for (23) is small of order $\sqrt{\delta}$. Intuitively, we can neglect this term as well. We then select the stochastic integral $\mathcal{M}_0(P_{BS})$ as the major control for the estimator (19) and formulate the following martingale control variate estimator:

$$\frac{1}{N}\sum_{i=1}^{N}\left[e^{-rT}H(S_T^{(i)}) - \mathcal{M}_0^{(i)}(P_{BS})\right]. \tag{26}$$

This is the approach taken by Fouque and Han [3], in which the proposed martingale control variate method is numerically superior to an importance sampling method in [2] for pricing European options. An asymptotic analysis of the martingale control variate, shown in Theorem 1 in [4], guarantees that: Under OU-type processes to model (Y_t, Z_t) in (18) with $0 < \varepsilon, \delta \ll 1$, the variance of the martingale control variate for European options is small of order ε and δ; namely

$$\text{Var}\ \left(e^{-rT}H(S_T) - \mathcal{M}_0(P_{BS})\right) = \mathcal{O}(\max\{\varepsilon,\delta\}). \tag{27}$$

Moreover the financial interpretation of the martingale control term

$$\mathcal{M}_0(P_{BS}) = \int_0^T e^{-rs}\frac{\partial P_{BS}}{\partial x}(s,S_s;\bar\sigma(Z_s))f(Y_s,Z_s)S_s dW_s^{(0)*}$$

corresponds to the cumulative cost of a *delta* hedging strategy. This martingale control variate method can be easily extended to hitting time problems like barrier options and optimal stopping time problems like American options. Numerical results and some variance analysis are discussed in [4].

4 Variance Reduction for Asian Options: Geometric-Average Case

A general form of payoffs for geometric-average Asian options (GAO) consists of a fixed strike K_2, a floating strike S_T, a coefficient K_1, and a geometric-average of stock prices. For example, the price at time t of a GAO call option is defined by

$$I\!E^\star\left\{e^{-r(T-t)}(G_T - K_1 S_T - K_2)^+ \mid \mathcal{F}_t\right\},$$

where $\mathcal{F}_t$ denotes the filtration generated by the process $(S_s, Y_s, Z_s)_{0\le s\le t}$. The random variable G_T denotes the geometric average of stock prices up to time T

$$G_T = \exp\left(\frac{1}{T}\int_0^T \ln S_t dt\right).$$

We introduce the running sum process $L_t = \int_0^t \ln S_u du$ or, in its differential form,

$$dL_t = \ln S_t dt, \tag{28}$$

such that the joint dynamics (S_t, Y_t, Z_t, L_t) is Markovian. Hence we denote the call price of GAO by

$$P_G^{\varepsilon,\delta}(t,x,y,z,L) = I\!E^\star{}_{t,x,y,z,L}\left\{e^{-r(T-t)}\left(\exp\left(\frac{L_T}{T}\right) - K_1 S_T - K_2\right)^+\right\}, \tag{29}$$

by assuming $(S_t = x, Y_t = y, Z_t = z, L_t = L)$. A basic Monte Carlo simulation consists in generating N independent trajectories governed by equations (18) and (28), and averaging out the discounted sample payoffs in order to obtain an unbiased GAO price estimator:

$$P_G^{\varepsilon,\delta} \approx P_G^{MC} = \frac{e^{-r(T-t)}}{N} \sum_{k=1}^{N} \left(\exp\left(\frac{L_T^{(k)}}{T} \right) - K_1 \, S_T^{(k)} - K_2 \right)^+ . \tag{30}$$

4.1 Martingale Control Variate for Geometric Average Asian Options

The construction of martingale control variates for GAO price is similar to the procedure presented in Section 3. We first apply Ito's lemma to the discounted GAO price then integrate out the time variable. Therefore, the following martingale representation is obtained

$$\begin{aligned} P_G^{\varepsilon,\delta}(0, S_0, Y_0, Z_0, L_0) = {} & e^{-rT} \left(\exp\left(\frac{L_T}{T} \right) - K_1 \, S_T - K_2 \right)^+ - \mathcal{M}_0(P_G^{\varepsilon,\delta}) - \frac{1}{\sqrt{\varepsilon}} \mathcal{M}_1(P_G^{\varepsilon,\delta}) \\ & - \sqrt{\delta} \mathcal{M}_2(P_G^{\varepsilon,\delta}), \end{aligned} \tag{31}$$

where stochastic integrals $\mathcal{M}_0, \mathcal{M}_1$ and $\mathcal{M}_2$ are defined the same as in (21) - (23) but with $P_G^{\varepsilon,\delta}$ in these martingales instead. The martingale control variate estimator for the GAO price is then formulated by

$$P_G^{\varepsilon,\delta}(0, S_0, Y_0, Z_0, L_0) \approx \frac{1}{N} \sum_{k=1}^{N} \left[e^{-rT} \left(\exp\left(\frac{L_T^{(k)}}{T} \right) - K_1 \, S_T^{(k)} - K_2 \right)^+ - \mathcal{M}_0^{(k)}(P_{BS}^G) \right], \tag{32}$$

provided that the "homogenized" GAO price P_{BS}^G, as an approximation to the true GAO price, can be easily calculated. For the case of fixed-strike GAOs. i.e. $K_1 = 0$, it is shown in [2] that $P_{BS}^G(t, x, L; \bar{\sigma}(z))$ admits a closed-form solution

$$\begin{aligned} & P_{BS}^G(t, x, L; \bar{\sigma}(z)) \\ & = \exp\left(\frac{L - t \ln x}{T} + \ln x + R(t, T, z) \right) \mathcal{N}(d_1(x, z, L)) - K e^{-r(T-t)} \mathcal{N}(d_2(x, z, L)), \end{aligned} \tag{33}$$

where

$$\begin{aligned} R(t, T, z) &= \left(r - \frac{\bar{\sigma}^2(z)}{2} \right) \frac{(T-t)^2}{2T} + \bar{\sigma}^2(z) \, \frac{(T-t)^3}{6T^2} - r(T-t), \\ d_1(x, z, L) &= \frac{T \ln(x/K) + L - t \ln x + (r - \bar{\sigma}^2(z)/2)(T-t)^2/2 + \bar{\sigma}^2(z) \frac{(T-t)^3}{3T}}{\bar{\sigma}(z) \sqrt{\frac{(T-t)^3}{3}}} \\ d_2(x, z, L) &= d_1(x, z, L) - \bar{\sigma}(z) \sqrt{\frac{(T-t)^3}{3T^2}}. \end{aligned} \tag{34}$$

The probabilistic representation of the homogenized GAO price $P_{BS}^G(t, x, L; \bar{\sigma}(z))$ is

$$P_{BS}^G(t, \bar{S}_t = x, \bar{L}_t = L; \bar{\sigma}(z)) = \tilde{I\!E} \left\{ e^{-r(T-t)} \left(e^{\bar{L}_T / T} - K \right)^+ \mid \bar{S}_t = x, \bar{L}_t = L \right\}, \tag{35}$$

Table 3: Parameters used in the two-factor stochastic volatility model (18).

r	m_f	m_s	ν_f	ν_s	ρ_1	ρ_2	ρ_{12}	Λ_f	Λ_s	$f(y,z)$
10%	-0.8	-0.6	0.7	1	-0.2	-0.2	0	0	0	$\exp(y+z)$

Table 4: Initial conditions and Asian call option parameters.

$\$S_0$	Y_0	Z_0	L_0	$\$K_2 = K$	T years
100	-1	-0.5	0	110	1

where $\bar{S}_t$ and $\bar{L}_t$ are governed by

$$\begin{aligned} d\bar{S}_t &= r\bar{S}_t dt + \bar{\sigma}(z)\bar{S}_t d\tilde{W}_t, \\ d\bar{L}_t &= \ln \bar{S}_t dt, \end{aligned} \tag{36}$$

respectively and $Z_t = z$. Let $\tilde{W}_t$ denote a Brownian motion under a probability measure $\tilde{P}$, under which the conditional expectation is defined. These derivation can also be found in [2].
For other GAO payoffs such as call or put of the floating strike, i.e. $K_2 = 0$, one can derive similar formula. We omit these cases to limit the length of this paper.

4.2 Numerical Results for Pricing GAO

We present numerical results from Monte Carlo simulations to evaluate fixed-strike GAO prices in this section. Parameters in our model are shown in Table 3. Other values (initials conditions and option parameters) are given in Table 4. Parameters chosen in these tables are exactly the same used in an numerical example in [2] in order to compare the efficiency of Monte Carlo methods. Sample paths in (32) are simulated based on the Euler scheme to discretize equations (18) and (28) with time step $\Delta t = 0.005$, the stochastic integral $\mathcal{M}_0$ is approximated by a Riemann sum, and the number of total paths are 5000. As demonstrated in [2], results of variance reduction ratios obtained from an importance sampling technique versus the basic Monte Carlo method are now listed on the third column of Table 5. The variance reduction ratios obtained from the martingale control variate method are listed on the last column of Table 5. We find that the martingale control variate method outperforms the importance sampling method for all cases.

4.3 Variance Analysis of a Perturbed Volatility

Based on the fact that random volatility is fluctuating around its long run mean, we study a variance analysis for a simplified model which is helpful to explain the effect of martingale control. Let's assume that under the risk-neutral probability measure $S_t^{\varepsilon,\delta}$ is a perturbed risky asset defined by

$$dS_t^{\varepsilon,\delta} = rS_t^{\varepsilon,\delta}dt + \sigma_t^{\varepsilon,\delta}S_t^{\varepsilon,\delta}dW_t^*, \tag{37}$$

where the perturbed volatility is $\sigma_t^{\varepsilon,\delta} = \bar{\sigma} + \sqrt{\varepsilon}g_t + \sqrt{\delta}h_t$, $\bar{\sigma} > 0$ denotes the long run mean volatility, ε and δ are small parameters, and perturbed functions $\{g_t, h_t\}_{0\leq t\leq T}$ are assumed to be deterministic and bounded such that $\sigma_t^{\varepsilon,\delta} > 0, 0 \leq t \leq T$. A geometric-average Asian option is defined by

$$P_G^{\varepsilon,\delta}\left(t, S_t^{\varepsilon,\delta}\right) = I\!E^{\star}{}_{t,S_t^{\varepsilon,\delta}}\left\{e^{-r(T-t)}H(L_T^{\varepsilon,\delta})\right\}, \tag{38}$$

Table 5: Comparison of variance reduction ratios for various time scales ε and δ. V^{MC} denotes the sample variance obtained from the basic Monte Carlo method. $V^{IS}(\tilde{P}_G)$ denotes the sample variance computed by an importance sampling with the first-order price approximation $\tilde{P}_G$. This technique and its several numerical variance reduction ratios $V^{MC}/V^{IS}(\tilde{P}_G)$ can be found in [2]. $V^{MC+CV}(P^G_{BS})$ denotes the sample variance computed by the martingale control variate method with the zeroth-order GAO price approximation P^G_{BS} in (33).

ε	δ	$V^{MC}/V^{IS}(\tilde{P}_G)$	$V^{MC}/V^{MC+CV}(P^G_{BS})$
1/100	0.05	7.6320	26.0610
1/75	0.1	5.6264	24.2428
1/50	0.5	5.3007	12.2437
1/25	1	3.9444	10.4226

where we denote the running sum process $L^{\varepsilon,\delta}_T = \frac{1}{T}\int_0^T \ln S^{\varepsilon,\delta}_t dt$ and the function H is assumed to be smooth and bounded.

Theorem 3 *(***Variance Analysis***) Given conditions described above, for any fixed initial state $(0, S^{\varepsilon,\delta}_0)$, there exist $\varepsilon > 0$ and $\delta > 0$ small enough and a positive constant C such that*

$$Var\left(e^{-rT} H\left(L^{\varepsilon,\delta}_T\right) - \mathcal{M}_0(P^G_{BS})\right) \leq C \max\{\varepsilon, \delta\},$$

where P^G_{BS} is defined as in (35) and (36) except that the homogenized volatility $\bar{\sigma}$ is chosen as constant.
Proof: (We suppress the sub-scripts under the expectation $I\!E^\star$ hereafter in this lemma.) Taking the pathwise derivative [7] for option price $P^{\varepsilon,\delta}_G$ with respect to the stock price, the chain rule can be applied and we obtain

$$\begin{aligned} \frac{\partial P^{\varepsilon,\delta}_G}{\partial S^{\varepsilon,\delta}_t}(t, S^{\varepsilon,\delta}_t, L^{\varepsilon,\delta}_t) &= I\!E^\star\left\{e^{-r(T-t)} H'\left(L^{\varepsilon,\delta}_T\right) \int_t^T \frac{\partial \ln S^{\varepsilon,\delta}_s}{\partial S^{\varepsilon,\delta}_t} ds \mid \mathcal{F}_t\right\} \\ &= \frac{T-t}{S^{\varepsilon,\delta}_t T} I\!E^\star\left\{e^{-r(T-t)} H'\left(L^{\varepsilon,\delta}_T\right) \mid \mathcal{F}_t\right\}. \end{aligned} \tag{39}$$

Similarly, given the state variable $S^{\varepsilon,\delta}_t$ and $L^{\varepsilon,\delta}_t$ the *delta* of the geometric average Asian option with the constant volatility $\bar{\sigma}$ has the following decomposition

$$\frac{\partial P^G_{BS}}{\partial \bar{S}_t}(t, \bar{S}_t = S^{\varepsilon,\delta}_t, \bar{L}_t = L^{\varepsilon,\delta}_t) = \frac{T-t}{\bar{S}_t T} I\!E^\star\left\{e^{-r(T-t)} H'\left(\bar{L}_T\right) \mid \mathcal{F}_t\right\}, \tag{40}$$

where the constant-volatility stock price $\bar{S}_t$ satisfies

$$d\bar{S}_t = r\bar{S}_t dt + \bar{\sigma}\bar{S}_t dW^*_t$$

with the running sum $\bar{L}_t = \int_0^t \ln \bar{S}_s ds$. Conditional on the driving volatility processes, the absolute difference between $\frac{\partial P^{\varepsilon,\delta}_G}{\partial S^{\varepsilon,\delta}_t}$ and $\frac{\partial P^G_{BS}}{\partial \bar{S}_t}$ is equal to

$$\begin{aligned} &\frac{\partial P^{\varepsilon,\delta}_G}{\partial S^{\varepsilon,\delta}_t}(t, S^{\varepsilon,\delta}_t, L^{\varepsilon,\delta}_t) - \frac{\partial P^G_{BS}}{\partial \bar{S}_t}(t, \bar{S}_t = S^{\varepsilon,\delta}_t, \bar{L}_t = L^{\varepsilon,\delta}_t) \\ &= \frac{T-t}{S^{\varepsilon,\delta}_t T} I\!E^\star\left\{e^{-r(T-t)} H''\left(\hat{L}^{\varepsilon,\delta}_T\right)\left(L^{\varepsilon,\delta}_T - \bar{L}_T\right) \mid \mathcal{F}_t\right\}, \end{aligned} \tag{41}$$

which is obtained by applying the Mean-Value Theorem. The inner difference conditional on $\mathcal{F}_t$ is

$$L_T^{\varepsilon,\delta} - \hat{L}_T = \frac{1}{T}\int_t^T \left(\ln S_t^{\varepsilon,\delta} - \ln \bar{S}_t\right) dt \tag{42}$$
$$= \frac{1}{T}\left[\int_t^T \int_t^u \frac{\bar{\sigma}^2 - \sigma_s^{\varepsilon,\delta 2}}{2} ds\, du - \int_t^T \int_t^u \varepsilon\, g_s + \delta\, h_s\, dW_s^* \, du\right].$$

The variance of the control variate is

$$Var\left(e^{-rT} H\left(L_T^{\varepsilon,\delta}\right) - \mathcal{M}_0(P_{BS}^G)\right) \tag{43}$$
$$= Var\left(\int_0^T e^{-rs}\left(\frac{\partial P_G^{\varepsilon,\delta}}{\partial x} - \frac{\partial P_{BS}^G}{\partial x}\right)(s, S_s^{\varepsilon,\delta}, L_s^{\varepsilon,\delta})\sigma^{\varepsilon,\delta} S_s^{\varepsilon,\delta} dW_s^*\right)$$
$$= I\!E^\star\left\{\int_0^T e^{-2rs}\left(\frac{\partial P_G^{\varepsilon,\delta}}{\partial x} - \frac{\partial P_{BS}^G}{\partial x}\right)^2 (s, S_s^{\varepsilon,\delta}, L_s^{\varepsilon,\delta})\sigma^{\varepsilon,\delta 2} S_s^{\varepsilon,\delta 2} ds\right\}$$
$$\leq C\int_0^T I\!E^\star\left\{\left(I\!E^\star\left\{e^{-r(T-t)} H''\left(\hat{L}_T^{\varepsilon,\delta}\right)\left(L_T^{\varepsilon,\delta} - \bar{L}_T\right) \mid \mathcal{F}_s\right\}\right)^2 \mid \mathcal{F}_0\right\} ds, \tag{44}$$

where we substitute (41) and (42) into (43) and C is some constant because $\sigma^{\varepsilon,\delta}$ is assumed to be bounded. The nested expectation defined in (44) can be bounded above by

$$C I\!E^\star\left\{\left(L_T^{\varepsilon,\delta} - \bar{L}_T\right)^2 \mid \mathcal{F}_0\right\}$$
$$\leq C\left(\int_0^T \int_0^t I\!E^\star\left\{\left(\frac{\bar{\sigma}^2 - \sigma_s^{\varepsilon,\delta 2}}{2}\right)^2 \mid \mathcal{F}_0\right\} ds\, dt + \int_0^T I\!E^\star\left\{\left(\int_0^t \varepsilon\, g_s + \delta\, h_s\, dW_s^*\right)^2 \mid \mathcal{F}_0\right\} dt\right)$$
$$\leq C \max\{\varepsilon, \sqrt{\varepsilon\,\delta}, \delta\},$$

where the integrability of g and h and Ito isometry property are used to obtain the estimate. The notation C has been abused to denote some constant independent of parameters ε and δ. We therefore conclude that the variance of martingale control variate is of $\mathcal{O}(\varepsilon, \delta)$.
An argument to treat the general multi-factor model (18) will be studied in a separate paper.

5 Variance Reduction for Asian Options: Arithmetic-Average Case

Similarly to the GAO case, it is convenient to introduce a running sum process $I_t = \int_0^t S_u du$, or its differential form,

$$dI_t = S_t dt,$$

such that the joint dynamics (S_t, Y_t, Z_t, I_t) defined in (18) is Markovian. Under the risk-neutral probability measure $I\!P^\star$ the price of an arithmetic average Asian call option is given by

$$P_A^{\varepsilon,\delta}(t, x, y, z, I) = I\!E_{t,x,y,z,I}^*\left\{e^{-r(T-t)}\left(\frac{I_T}{T} - K_1\, S_T - K_2\right)^+\right\},$$

conditioning on $S_t = x, Y_t = y, Z_t = z, I_t = I$. We use this type of options as typical examples when we discuss variance reduction of Monte Carlo simulations. A basic Monte Carlo simulation for pricing arithmetic-average Asian options (AAO in short) with N replications is given by

$$P_A^{\varepsilon,\delta} \approx P_A^{MC} = \frac{e^{-r(T-t)}}{N} \sum_{k=1}^{N} \left(\frac{I_T^{(k)}}{T} - K_1\, S_T^{(k)} - K_2 \right)^+ . \tag{45}$$

5.1 Two-Step Control Variate Method

It becomes straightforward to derive the generalized control variates method by directly applying the martingale representation theorem to AAOs such that

$$\begin{aligned} P_A^{\varepsilon,\delta}(0, S_0, Y_0, Z_0, I_0) = e^{-rT} \left(\exp\left(\frac{I_T}{T}\right) - K_1\, S_T - K_2 \right)^+ - \mathcal{M}_0(P_A^{\varepsilon,\delta}; T) - \frac{1}{\sqrt{\varepsilon}} \mathcal{M}_1(P_A^{\varepsilon,\delta}; T) \\ - \sqrt{\delta} \mathcal{M}_2(P_A^{\varepsilon,\delta}; T), \end{aligned} \tag{46}$$

where stochastic integrals $\mathcal{M}_0, \mathcal{M}_1$ and $\mathcal{M}_2$ are defined the same as before. Similar to GAO cases, one can formulate a martingale control variate estimator

$$P_A^{\varepsilon,\delta}(0, S_0, Y_0, Z_0, I_0) \approx \frac{1}{N} \sum_{k=1}^{N} \left[e^{-rT} \left(\frac{I_T^{(k)}}{T} - K_1\, S_T^{(k)} - K_2 \right)^+ - \mathcal{M}_0^{(k)}(P_{BS}^A) \right], \tag{47}$$

with P_{BS}^A the homogenized AAO price under some constant volatility, which does not admit a closed-form solution. Heuristically one might choose an price approximation such as P_G in (5) or P_Z in (13) to substitute the true homogenized AAO price P_{BS}^A. Notice that these homogenized price approximations are y-variable independent as the fast varying process Y_t in (18) is averaged out. Assume that there is no correlations between Brownian motions; namely all ρ's are zero. Let $\tilde{P}$ be a favorable homogenized price approximation such as P_G or P_Z so that the martingale control $\mathcal{M}_0(\tilde{P}; T)$ is used. The variance of such control variate is

$$\begin{aligned} & Var\left(e^{-rT} \left(\frac{I_T}{T} - K_1\, S_T - K_2 \right)^+ - \mathcal{M}_0(\tilde{P}; T) \right) \\ & = \left\langle \mathcal{M}_0(P_A^{\varepsilon,\delta} - \tilde{P}; T) \right\rangle_Q + \frac{1}{\varepsilon} \left\langle \mathcal{M}_1(P_A^{\varepsilon,\delta}; T) \right\rangle_Q + \delta \left\langle \mathcal{M}_2(P_A^{\varepsilon,\delta} - \tilde{P}; T) \right\rangle_Q , \end{aligned} \tag{48}$$

where $\langle \cdot \rangle_Q$ denotes the expectations of a quadratic variation. Notice that the second term above is with a large coefficient $\frac{1}{\varepsilon}$ as ε is a small parameter. This term is completely not affected by the martingale control. To reduce the variance (48), one prefers to find a x, y, z-dependent price approximation so that each quadratic variation can be possibly reduced.
The conventional control

$$e^{-r(T-t)} \left(\frac{L_T}{T} - K_1\, S_T - K_2 \right)^+ - P_G^{\varepsilon,\delta}(0, S_0, Y_0, Z_0) \tag{49}$$

has the advantage to affect each martingale terms. This can be seen from the variance of the control variate

$$\begin{aligned} & Var\left(e^{-rT} \left(\frac{I_T}{T} - K_1\, S_T - K_2 \right)^+ - \lambda \left(e^{-rT} \left(\frac{L_T}{T} - K_1\, S_T - K_2 \right)^+ - P_G^{\varepsilon,\delta}(0, S_0, Y_0, Z_0) \right) \right) \\ & = \left\langle \mathcal{M}_0(P_A^{\varepsilon,\delta} - \lambda P_G^{\varepsilon,\delta}; T) \right\rangle_Q + \frac{1}{\varepsilon} \left\langle \mathcal{M}_1(P_A^{\varepsilon,\delta} - \lambda P_G^{\varepsilon,\delta}; T) \right\rangle_Q + \delta \left\langle \mathcal{M}_2(P_A^{\varepsilon,\delta} - \lambda P_G^{\varepsilon,\delta}; T) \right\rangle_Q . \end{aligned} \tag{50}$$

Comparing with the previous variance (48) reduced by a martingale control, the conventional control variate apparently has the potential to diminish every quadratic variation, in particular the large order term $\frac{1}{\varepsilon}\left\langle \mathcal{M}_1(P_A^{\varepsilon,\delta};T)\right\rangle_Q$. The drawback of using the conventional control (49) is that the GAO price does not admit a closed-form solution. Therefore it is reasonable to use a two-step algorithm for variance reduction:
Step 1: Estimate $P_G^{\varepsilon,\delta}(0,S_0,Y_0,Z_0)$ by a martingale control $\mathcal{M}_0(P_{BS}^G;T)$ such that

$$P_G^{\varepsilon,\delta}(0,S_0,Y_0,Z_0) = \mathbb{E}^\star\left\{e^{-rT}\left(\frac{L_T^{(k)}}{T} - K_1\, S_T^{(k)} - K_2\right)^+ - \mathcal{M}_0(P_{BS}^G;T)\right\}.$$

Step 2: Estimate $P_G^{\varepsilon,\delta}(0,S_0,Y_0,Z_0)$ by the conventional control (49) such that

$$\begin{aligned} P_A^{\varepsilon,\delta}(0,S_0,Y_0,Z_0) = \quad & \mathbb{E}^\star\left\{e^{-rT}\left(\frac{I_T}{T} - K_1\, S_T - K_2\right)^+\right. \\ & \left. -\lambda\left(e^{-rT}\left(\frac{L_T}{T} - K_1\, S_T - K_2\right)^+ - P_G^{\varepsilon,\delta}(0,S_0,Y_0,Z_0)\right)\right\}. \end{aligned}$$

The variance analysis for this two-step algorithm is the following. In Step 1, it is implied from Theorem 3 that the variance is small of $\mathcal{O}(\varepsilon,\delta)$. In Step 2, it is seen from (50) and (48) that the variance of conventional control variate is smaller than a martingale control variate.

5.2 Numerical Results for Pricing AAO

In this section, we will compare efficiencies for pricing Asian call option using different control variates developed above, combined with Monte Carlo and quasi-Monte Carlo methods. The C++ on Unix is our programming language in the following examples. The pseudo random number generator used is $ran2()$ in [20]. In our comparisons, the sample sizes for MC method are 10240, 20480, 40960, 81920, 163840, and 327680, respectively; and those for Sobol' sequence related methods are 1024, 2048, 4096, 8192, 16384, and 32768, respectively, each with 10 random shifts; and the sample sizes for LTLRP related methods are 1021, 2039, 4093, 8191, 16381, and 32749, respectively, and again, each with 10 random shifts. We divide the time interval $[0,T]$ into $m = 128$ subintervals. In the following Tables, column 1 contains the numbers of points, numbers without parentheses in the MC column are option values, and numbers within parentheses in the same column are the corresponding standard errors, numbers in the QMC columns are variance reduction ratios.
An arithmetic-average Asian call option with a fixed strike is considered. The payoff variable is $\max(\frac{1}{T}\int_0^T S_t dt - K, 0) = \left(\frac{1}{T}\int_0^T S_t dt - K\right)^+$. We take input parameters defined in Table 3 and 4 as follows: $S_0 = \$100$, $K = \$110, r = 0.1 = 10\%$, $T = 1$ year, $m_1 = -0.8$, $m_2 = -0.6$, $\nu_1 = 0.7$, $\nu_2 = 1.0$, $\rho_1 = \rho_2 = -0.2$, $\rho_{12} = 0.0$, $y_0 = -1.0$, $z_0 = -0.5$, $1/\varepsilon = 75.0$, $\delta = 0.1$. In the following tables we demonstrate numerics of various variance reductions based on two different control variate methods.

In Tables 6 and 7, martingale control $\mathcal{M}_0(P_G;T)$ is used to construct the control variate estimator (47), where we use P_G rather than P_{BS}^A. We observe that the variance reduction ratio is about 23 for pseudo-random sequences by using this control variate technique. Without a control, the variance reduction ratios for Sobol' sequence vary from 1.3 to 5.8 and those for LTLRP are from 1.2 to 5.5; the Brownian bridge (BB in short) sampling actually worsen the efficiency for the Sobol sequence, and it

Table 6: Comparison of simulated Asian option values and variance reduction ratios for $1/\varepsilon = 75.0$, $\delta = 0.1$.

N	MC	MC+CV	Sobol'	Sobol+CV	Sobol+BB	Sobol+CV+BB
1024	7.770(0.1508)	24.2	2.1	28.2	1.9	6.5
2048	7.659(0.1063)	22.9	2.7	62.5	2.8	4.5
4096	7.784(0.0759)	22.9	4.7	76.5	1.2	6.3
8192	7.687(0.0528)	22.7	3.4	53.9	1.3	8.6
16384	7.702(0.0374)	22.9	1.3	55.2	0.7	5.9
32768	7.700(0.0265)	22.9	5.8	74.0	1.1	10.9

Table 7: Comparison of simulated Asian option values and variance reduction ratios for $1/\varepsilon = 75.0$, $\delta = 0.1$ (continued).

N	LTLRP	LTLRP+CV	LTLRP+BB	LTLRP+CV+BB
1021	2.8	98.7	3.9	80.3
2039	1.2	55.6	2.3	99.3
4093	1.9	42.6	2.0	37.6
8191	5.5	137.2	3.6	102.8
16381	3.1	44.7	6.9	60.2
32749	2.8	69.7	2.1	29.5

is a little bit better for the LTLRP case. [7] gives details about BB sampling. When combined with control variate, the variance reduction ratios for both QMC sequences are increased: for the Sobol' sequence vary from 28.2 to 76.5, and those for L'Ecuyer type lattice rule points are from 55.6 to 137.2.

In Tables 8 and 9, the two-step method is used to construct the control variate estimator. We observe that the variance reduction ratio is about 60 for pseudo-random sequences by using the control variate technique. Without a control, variance reduction ratios for Sobol' sequence vary from 1.3 to 5.8 and those for LTLRP are from 1.2 to 5.5; the BB sampling actually worsen the efficiency for the Sobol sequence, and it is a little bit better for the LTLRP case. When combined with control variate, the variance reduction ratios for the Sobol' sequence vary from 37.7 to 201.5, and those for L'Ecuyer type lattice rule points are from 78.9 to 265.2.

From Table 6 to Table 9 we see that indeed the two-step method performs better than the martingale control variate method. This is because the two-step method is able to eliminate more quadratic variations as explained in last section.

Notice that this pricing problem under a stochastic volatility model is a high dimensional problem for QMC methods. It has dimension 3×128=384, the multiplication of 100 time discretization and 3 state variables (S_t, Y_t, Z_t). We see in these tables that QMC sequences like Sobol or LTLPR does not work well under the plain estimator without any control. After adding controls, these randomized QMC methods generate significant reduction on variances, better than Monte Carlo methods. We think that this is a strong evidence that the controls act as smoothers for the QMC methods. At least for GAO cases, Theorem 3 can give a good implication because it says on average a (continuous) control variate is small order of ε or δ. A detail account for analyzing this smoothing effect is left as a future research.

Table 8: Comparison of simulated Asian option values and variance reduction ratios by two-step method for $1/\varepsilon = 75.0$, $\delta = 0.1$.

N	MC	MC+CV	Sobol'	Sobol+CV	Sobol+BB	Sobol+CV+BB
1024	7.770(0.1508)	63.4	2.1	135.8	1.9	109.8
2048	7.659(0.1063)	59.5	2.7	86.5	2.8	98.6
4096	7.784(0.0759)	60.2	4.7	201.5	1.2	94.4
8192	7.687(0.0528)	61.8	3.4	83.6	1.3	89.1
16384	7.702(0.0374)	61.0	1.3	37.7	0.7	47.7
32768	7.700(0.0265)	60.8	5.8	100.7	1.1	37.7

Table 9: Comparison of simulated Asian option values and variance reduction ratios two-step method for $1/\varepsilon = 75.0$, $\delta = 0.1$ (continued).

N	LTLRP	LTLRP+CV	LTLRP+BB	LTLRP+CV+BB
1021	2.8	135.1	3.9	93.4
2039	1.2	78.9	2.3	149.7
4093	1.9	91.3	2.0	139.3
8191	5.5	265.2	3.6	209.6
16381	3.1	139.1	6.9	323.1
32749	2.8	181.7	2.1	115.6

The study of computing Greeks such as delta is not presented here. We refer to our separate paper [8] where computing delta by control variate methods and pathwise differentiation is discussed for European options.

6 Conclusion

In this paper, we study the option pricing problems for Asian options by simulation methods. We revisit the conventional control variate method and give a new interpretation of this method. A new class of martingale controls is proposed and is easy to be generalized to nonlinear situation such as to calculate high-biased solutions of American Asian options. It is natural to evaluate the effectiveness of controls by measuring expectations of their quadratic variations. This becomes helpful for designing control variates of general problems. We propose a two-step method for pricing Asian options under a class of multi-factor stochastic volatility models. Numerically, we implement control variate methods with Monte Carlo and quasi-Monte Carlo methods. For stochastic volatility models, our tests show that quasi-Monte Carlo methods with control variates are much more efficient than Monte Carlo methods even in high dimensional regimes.

Acknowledgement: We are grateful to a financial support by the Institute for Mathematics and its Applications (IMA) at the University of Minnesota, USA. The work of C-H Han is partially supported by NSC grant 94-2119-M-007-008, Taiwan; and the work of Y. Lai was partially supported by an Natural Sciences and Engineering Research Council (NSERC) of Canada grant. C.H. Han is grateful for a discussion with Professor Ken Seng Tan at University of Waterloo.

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