Exercise group 47 Pen and Paper

1. $N = 100$ test samples trained
$K = 5$ " " misclassified
90% of confidence interval $\rightarrow \alpha = 0.1$ ~~confidence~~ $\rightarrow$ error
Classification accuracy $\rightarrow$ Binomial distribution $\rightarrow$ HINT

Wald method (commonly recommended on textbooks but the most biased)

$\hookrightarrow$ C.I. (confidence intervals) $= \hat{p} \pm z \sqrt{\dfrac{\hat{p}(1-\hat{p})}{N}} = 0.95 \pm 0.0207 = \begin{cases} 0.9707 \\ 0.9293 \end{cases}$

$\hookrightarrow$ $CI \in [0.9293, 0.9707]$

$$\hat{p} = \frac{N-K}{N} = 1 - \frac{K}{N} = 1 - \frac{5}{100} = 0.95$$

$$z = 1 - \frac{1}{2}\alpha = 1 - \frac{1}{2} 0.1 = 0.95$$

2. $K = 3$ test samples misclassified

$CI = 0.97 \pm 0.0162 = \begin{cases} 0.9862 \\ 0.9538 \end{cases} \rightarrow CI \in [0.9538, 0.9862]$

$\hookrightarrow$ better than the previous

$$\hat{p} = 1 - \frac{3}{100} = 0.97$$

2.

$$\mu_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \Sigma_0 : \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Sigma_1 : \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}$$

$$w = S_w^{-1} (\mu_1 - \mu_0) = (\Sigma_0 + \Sigma_1)^{-1} (\mu_1 - \mu_0)$$

regularized solution

$$w = (\Sigma_0 + \Sigma_1 + \lambda I)^{-1} (\mu_1 - \mu_0) \quad \text{with } \lambda > 0$$

alternatively as scale w is not relevant but the direction

$$w = \lambda (\Sigma_0 + \Sigma_1 + \lambda I)^{-1} (\mu_1 - \mu_0)$$

λ avoids the convergence of w to zero when $\lambda \to \infty$

a) When $\lambda = 100$

$$w = 100 \left[2\begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix} \right]^{-1} \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] =$$

$$= 100 \begin{pmatrix} 106 & -4 \\ -4 & 104 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = 100 \begin{pmatrix} 0.0094 & 0.004 \\ 0.004 & 0.0096 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} =$$

$$= \begin{pmatrix} -0.9811 \\ -0.9993 \end{pmatrix}$$

b) $$w = \lambda \left[2\begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right]^{-1} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 6+\lambda & -4 \\ -2 & 4+\lambda \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -1 \end{pmatrix} =$$

$$\begin{pmatrix} 6+\lambda & -4 \\ -2 & 4+\lambda \end{pmatrix}^{-1} = \frac{1}{\underbrace{(6+\lambda)(4+\lambda) - (-4)(-2)}} \begin{pmatrix} 4+\lambda & 2 \\ 4 & 6+\lambda \end{pmatrix} = \frac{1}{(\lambda+2)(\lambda+8)} \begin{pmatrix} 4+\lambda & 2 \\ 4 & 6+\lambda \end{pmatrix}$$

$$(6+\lambda)(4+\lambda) - 8 = 24 + 6\lambda + 4\lambda + \lambda^2 - 8 = \lambda^2 + 10\lambda + 16 = (\lambda+2)(\lambda+8)$$

$$w = \frac{\lambda}{(\lambda+2)(\lambda+8)} \begin{pmatrix} 4+\lambda & 2 \\ 4 & 6+\lambda \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} =$$

$$= \begin{bmatrix} \frac{\lambda(4+\lambda)}{(\lambda+2)(\lambda+8)} & \frac{2\lambda}{(\lambda+2)(\lambda+8)} \\ \frac{4\lambda}{(\lambda+2)(\lambda+8)} & \frac{\lambda(6+\lambda)}{(\lambda+2)(\lambda+8)} \end{bmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\lim_{\lambda\to\infty} w = \{\text{L'Hopital rule}\} = \begin{bmatrix} \frac{(4+\lambda)+\lambda+4}{2(\lambda+8)+(\lambda+2)8} & \frac{2}{2(\lambda+8)+(\lambda+2)8} \\ \frac{4}{2(\lambda+8)+(\lambda+2)8} & \frac{(6+\lambda)+\lambda+6}{2(\lambda+8)+(\lambda+2)8} \end{bmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$= \begin{bmatrix} \frac{2(\lambda+4)}{2(\lambda+8)+(\lambda+2)8} & \frac{2}{2(\lambda+8)+(\lambda+2)8} \\ \frac{2}{2(\lambda+8)+(\lambda+2)8} & \frac{2(\lambda+6)}{2(\lambda+8)+(\lambda+2)8} \end{bmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{bmatrix} \frac{\lambda+4}{(\lambda+8)+4(\lambda+2)} & \frac{1}{(\lambda+8)+4(\lambda+2)} \\ \frac{1}{(\lambda+8)+4(\lambda+2)} & \frac{\lambda+6}{(\lambda+8)+(\lambda+2)4} \end{bmatrix}$$

$$\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \{\text{L'Hopital}\} = \begin{bmatrix} \frac{4}{8+(4\cdot 2)} & \cancel{\frac{0}{8+4+2}} \\ \cancel{0} & \frac{6}{8+2\cdot 4} \end{bmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} =$$

$$= \begin{bmatrix} 1/4 & 0 \\ 0 & 3/8 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \cancel{\begin{bmatrix} -1/4 \\ -3/8 \end{bmatrix}}$$

correct result $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$