General Information: This assignment contains written and/or programming tasks. Combine all the answers to the written tasks in a single PDF document, named `{lastname}-written.pdf`. You can also scan or take pictures of (readable) handwritten papers. JPEG/PNG image files are accepted in this case and they should be named `{exercisenumber}-{lastname}-written.{jpeg/png}`. Make sure that we can follow the manual calculations. Do not combine too many small steps into one. The programming tasks have to be solved in *Julia* and the source code files have to be submitted using the following naming scheme: `{exercisenumber}-{lastname}.jl`.

(1) a) (0.5 points) Given the function

$$y(x) = e^{-3x}\,,$$

determine if the given function is a solution to the following differential equation (detail your answer)

$$y'' + 2y' = 3y$$

Solution:

$$\begin{aligned} y' &= -3e^{-3x} \\ y'' &= 9e^{-3x} \end{aligned}$$

$$\begin{aligned} 9e^{-3x} + 2(-3)e^{-3x} &= 3e^{-3x} \\ 9e^{-3x} - 6e^{-3x} &= 3e^{-3x} \\ 3e^{-3x} &= 3e^{-3x} \qquad \square \end{aligned}$$

b) (0.5 points) Show that the function

$$y = x\cos(\ln|x|)$$

is a solution to the differential equation

$$x^2y'' - xy' + 2y = 0$$

Solution:

$$\begin{aligned} y' &= \cos(\ln|x|) - x\sin(\ln|x|)\frac{1}{x} = \cos(\ln|x|) - \sin(\ln|x|) \\ y'' &= -\sin(\ln|x|)\frac{1}{x} - \cos(\ln|x|)\frac{1}{x} \end{aligned}$$

$$\left.\begin{aligned} x^2y'' &= -x\cos(\ln|x|) - x\sin(\ln|x|) \\ -xy' &= -x\cos(\ln|x|) + x\sin(\ln|x|) \\ 2y &= 2x\cos(\ln|x|) \end{aligned}\right\} +$$

$$x^2y'' - xy' + 2y = -2x\cos(\ln|x|) + 2x\cos(\ln|x|) = 0 \qquad \square$$

c) (1 point) Show that the function

$$y(x) = x^3(C + \ln|x|)$$

is a solution to the differential equation

$$xy' - 3y = x^3$$

also, find the particular solution for the given initial condition

$$y(1) = 17$$

.

Solution:

$$\begin{aligned}
y' &= 3x^2(C + \ln|x|) + x^3(\frac{1}{x}) = 3x^2(C + \ln|x|) + x^2 \\
xy' - 3y &= 3x^3(C + \ln|x|) + x^3 - 3x^3(C + \ln|x|) = x^3 \qquad \square \\
y(1) &= 1^3(C + \ln|1|) = C \\
y(1) &= 17 \rightarrow C = 17
\end{aligned}$$

(2) (2 points) The following differtial equation is given:

$$y' = \frac{x + e^{2x}}{y}.$$

Compute the general solution $y(x)$ by separation of variables.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x + e^{2x}}{y} \\
y\ \frac{dy}{dx} &= (x + e^{2x}) \\
\int y\ \frac{dy}{dx} dx &= \int (x + e^{2x})\ dx \\
\int y\ dy &= \int (x + e^{2x})\ dx \\
\frac{y^2}{2} &= \frac{x^2}{2} + \frac{e^{2x}}{2} + C_1 \\
y^2 &= x^2 + e^{2x} + C_2 \\
y(x)_{1,2} &= \pm\sqrt{x^2 + e^{2x} + C_2}
\end{aligned}$$

(3) (1, 1, 0.5, 0.5 points) Explicit Euler Method to solve the ordinary differential equations:

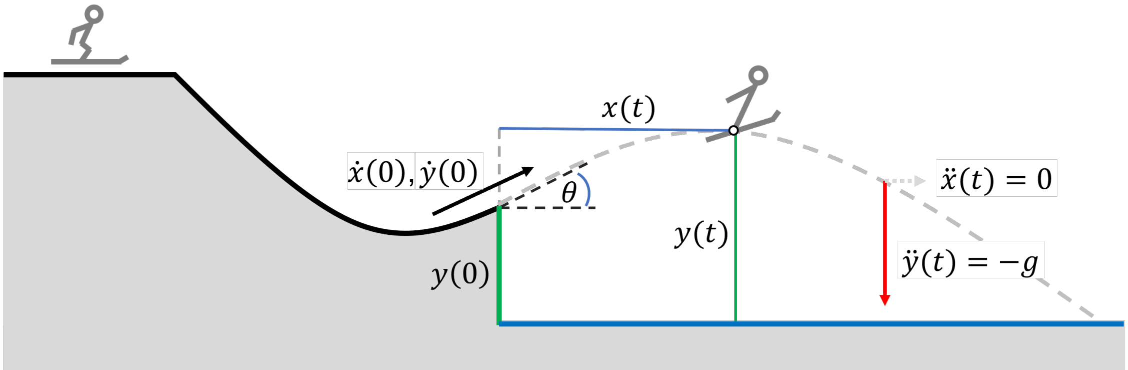
$$\begin{aligned}
\ddot{y}(t) &= -g \\
\ddot{x}(t) &= 0.
\end{aligned}$$

The initial values (i.e. at $t = 0$), are the vertical and horizontal velocity

$$\begin{aligned}
\dot{y}(0) &= v \sin\theta \\
\dot{x}(0) &= v \cos\theta
\end{aligned}$$

and the height (and distance) of the skier at takeoff

$$\begin{aligned}
y(0) &= h_0 \\
x(0) &= 0.
\end{aligned}$$



a) We want to solve the ordinary differential equations for the timesteps $t_1 = h, t_2 = 2h$, with $h = \frac{1}{4}$ by using the explicit euler method.
First, we compute the initial values $\dot{x}_0, \dot{y}_0, x_0, y_0$ from the following parameters

$$h_0 = 1,\ \theta = \frac{\pi}{4},\ v = 2\sqrt{2}.$$

$$\begin{aligned}
\dot{x}(0) &= 2\sqrt{2}\cos(\pi/4) = 2 = \dot{x}_0 \\
\dot{y}(0) &= 2\sqrt{2}\sin(\pi/4) = 2 = \dot{y}_0 \\
x(0) &= 0 = x_0 \\
y(0) &= h_0 = 1 = y_0
\end{aligned}$$

For simplicity we assume $g = 10m/s^2$.

1. iteration t_1

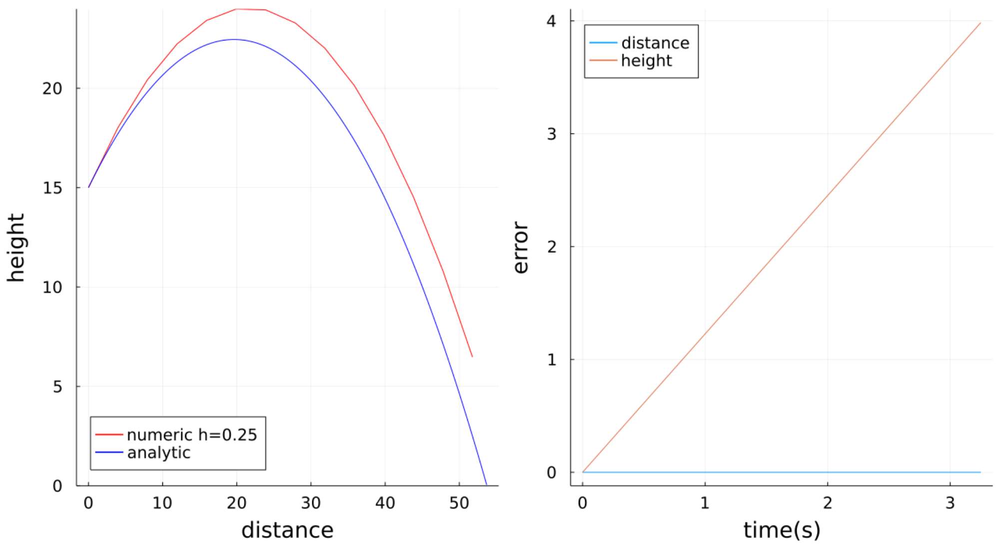
$$\begin{aligned}
\dot{x}_1 &= \dot{x}_0 + h\ddot{x}(t_0) = 2 + \frac{1}{4} \cdot 0 = 2 \\
\dot{y}_1 &= \dot{y}_0 + h\ddot{y}(t_0) = 2 + \frac{1}{4} \cdot (-10) = -\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
x_1 &= x_0 + h\dot{x}_0 = 0 + \frac{1}{4} \cdot 2 = \frac{1}{2} \\
y_1 &= y_0 + h\dot{y}_0 = 1 + \frac{1}{4} \cdot 2 = \frac{3}{2}
\end{aligned}$$

2. iteration t_2

$$\begin{aligned}
\dot{x}_2 &= \dot{x}_1 + h\ddot{x}(t_1) = 2 + \frac{1}{4} \cdot 0 = 2 \\
\dot{y}_2 &= \dot{y}_1 + h\ddot{y}(t_1) = -\frac{1}{2} + \frac{1}{4} \cdot (-10) = -3
\end{aligned}$$

$$\begin{aligned}
x_2 &= x_1 + h\dot{x}_1 = \frac{1}{2} + \frac{1}{4} \cdot 2 = 1 \\
y_2 &= y_1 + h\dot{y}_1 = \frac{3}{2} + \frac{1}{4} \cdot -\frac{1}{2} = \frac{11}{8}
\end{aligned}$$



b) see implementation `ski_jump_euler.jl`

c) Using smaller timesteps increases the accuracy of the approximation. The explict euler method extrapolates along the tangent of the function (its slope is given by the first derivative).
Because the horizontal velocity does not change over time the method produces an exact approximation (ignoring numerical errors). That is, the solution x is the area of the rectangle $\dot{x}(0) \cdot t$ (the method computes it as the sum of smaller rectangles $\dot{x}(0) \cdot h$). In constrast the vertical velocity linearly changes over time, which means that the height has a quadratic time dependency. As a result, we introduce a local discretization error which accumulates linearly over time.