(1) a)

$$\int \frac{1}{\sqrt[3]{x}} + \cos(7-4x)\,\mathrm{d}x = \int \frac{\mathrm{d}x}{\sqrt[3]{x}} + \int \cos(7-4x)\,\mathrm{d}x$$

$$\text{Subst. for second part: } u = 7-4x, \mathrm{d}x = \frac{\mathrm{d}u}{-4}$$

$$= \int x^{-\frac{1}{3}}\,\mathrm{d}x + \int \frac{\cos(u)}{-4}\,\mathrm{d}u$$

$$= \int x^{-\frac{1}{3}}\,\mathrm{d}x - \frac{1}{4}\int \cos(u)\,\mathrm{d}u$$

$$= \frac{3}{2}x^{\frac{2}{3}} + C_1 - \frac{1}{4}\sin(u) + C_2$$

Resubstitution:

$$= \frac{3}{2}x^{\frac{2}{3}} + C_1 - \frac{1}{4}\sin(7-4x) + C_2$$

$$= \frac{3}{2}\sqrt[3]{x^2} - \frac{1}{4}\sin(7-4x) + C$$

b)

$$\int \frac{x^2+1}{x^3+3x}\,\mathrm{d}x = \text{Substitution: } u = x^3+3x, \mathrm{d}x = \frac{\mathrm{d}u}{3x^2+3}$$

$$= \int \frac{x^2+1}{u} \cdot \frac{1}{3x^2+3}\mathrm{d}u$$

$$= \frac{1}{3}\int \frac{x^2+1}{u} \cdot \frac{1}{x^2+1}\mathrm{d}u$$

$$= \frac{1}{3}\int \frac{1}{u}\mathrm{d}u$$

$$= \frac{1}{3}\ln|u| + C$$

Resubstitution:

$$= \frac{1}{3}\ln\left|x^3+3x\right| + C$$

c)

$$\int \frac{x-1}{x^2-1}\,\mathrm{d}x = \int \frac{x-1}{(x+1)(x-1)}\,\mathrm{d}x$$

$$= \int \frac{1}{x+1}\,\mathrm{d}x$$

$$= \ln(x+1) + C$$

d)

$$\int \sin(x)\cdot x\,\mathrm{d}x = \text{partial integration with: } f(x) = x, g'(x) = \sin(x)$$

$$\to f(x) = x, f'(x) = 1, g'(x) = \sin x, g(x) = -\cos x$$

$$= -x\cdot\cos x - \int 1\cdot(-\cos x)\,\mathrm{d}x$$

$$= -x\cdot\cos x + \sin x + C$$

(2)

$$
\begin{aligned}
\int_0^8 \sqrt{1+x^2}\,\mathrm{d}x &= [x = \sinh u, \mathrm{d}x = \cosh u \,\mathrm{d}x] \\
&= \int_0^{\sinh^{-1}(8)} \cosh u^2 \,\mathrm{d}u \\
&= \frac{1}{2}\int_0^{\sinh^{-1}(8)} \cosh(2u) + 1 \,\mathrm{d}u \\
&= \frac{u}{2} + \frac{\sinh 2u}{4}\Big|_0^{\sinh^{-1}(8)} \\
&= \frac{u}{2} + \frac{\sinh 2u}{4}\Big|_0^{\sinh^{-1}(8)} \\
&= \frac{1}{2}\left(\sinh^{-1}(8) + \frac{1}{2}\sinh\left(2\sinh^{-1}(8)\right)\right) \\
&= \frac{1}{2}\left(\sinh^{-1}(8) + 8\cosh\left(\sinh^{-1}(8)\right)\right) \\
&= \frac{1}{2}\left(\sinh^{-1}(8) + 8\sqrt{65}\right) \\
&\approx 33.637
\end{aligned}
$$

```julia
# Approximate (*) with adaptive Gauss-Kronrod quadrature
using QuadGK
quadgk(x -> sqrt(1 + x^2), 0.0, 8.0)
```

(3) A sketch is given in Figure 1.

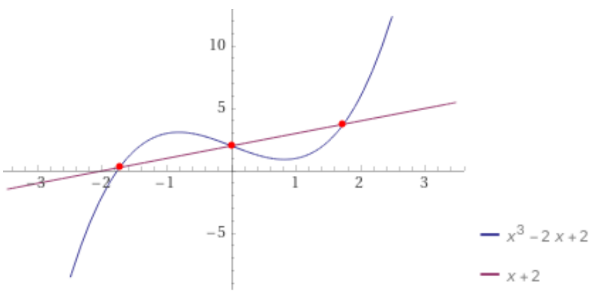

Figure 1: The enclosed area between the polynomial function and the line should be computed

Both functions are given as

$$f(x) = x^3 - 2x + 2, g(x) = x + 2$$

$$
\begin{aligned}
A = A_{\mathrm{I}} + A_{\mathrm{II}} &= \int f(x) - g(x)\,\mathrm{d}x \\
&= \int_{-\sqrt{3}}^{0} f(x) - g(x)\,\mathrm{d}x + \int_{0}^{\sqrt{3}} g(x) - f(x)\,\mathrm{d}x \\
&= \int_{-\sqrt{3}}^{0} (x^3 - 2x + 2) - (x+2)\,\mathrm{d}x + \int_{0}^{\sqrt{3}} (x+2) - (x^3 - 2x + 2)\,\mathrm{d}x \\
&= \int_{-\sqrt{3}}^{0} x^3 - 3x\,\mathrm{d}x + \int_{0}^{\sqrt{3}} -x^3 + 3x)\,\mathrm{d}x \\
&= \left[\frac{x^4}{4} - \frac{3x^2}{2}\right]_{-\sqrt{3}}^{0} + \left[-\frac{x^4}{4} + \frac{3x^2}{2}\right]_{0}^{\sqrt{3}} \\
&= \left[\frac{x^2(x^2-6)}{4}\right]_{-\sqrt{3}}^{0} + \left[\frac{x^2(6-x^2)}{4}\right]_{0}^{\sqrt{3}} \\
&= (0 - \frac{3(3-6)}{4}) + (\frac{3(6-3)}{4} - 0) \\
&= 4.5
\end{aligned}
$$

(4) a) The velocity is given by

$$
\begin{aligned}
v(t) = v(t) - v(0) &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} v(\tau)\mathrm{d}\tau = \int_0^t v'(\tau)\mathrm{d}\tau \\
= \int_0^t a(\tau)\mathrm{d}\tau &= \begin{cases} 1.5t & 0 \le t \le 60 \\ 90 & 60 < t \le 150 \\ 90 - \frac{t^2}{300} + t - 75 & 150 < t \le 300 \end{cases}
\end{aligned}
$$

and plotted in Figure 2.

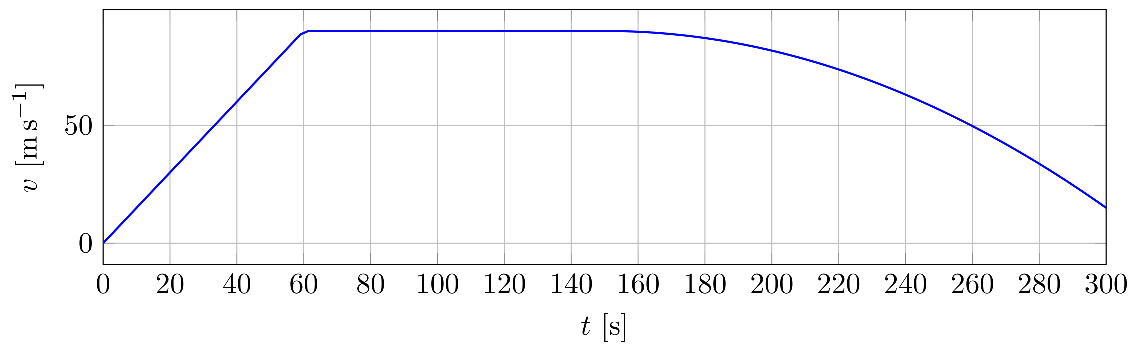

Figure 2: Velocity $v(t)$.

The position is given by

$$
\begin{aligned}
x(t) = x(t) - x(0) &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau)\mathrm{d}\tau = \int_0^t x'(\tau)\mathrm{d}\tau \\
= \int_0^t v(\tau)\mathrm{d}\tau &= \begin{cases} 0.75t^2 & 0 \le t \le 60 \\ 2700 + 90t - 5400 & 60 < t \le 150 \\ 10800 - \frac{t^3}{900} + \frac{t^2}{2} + 15t - 9750 & 150 < t \le 300 \end{cases}
\end{aligned}
$$

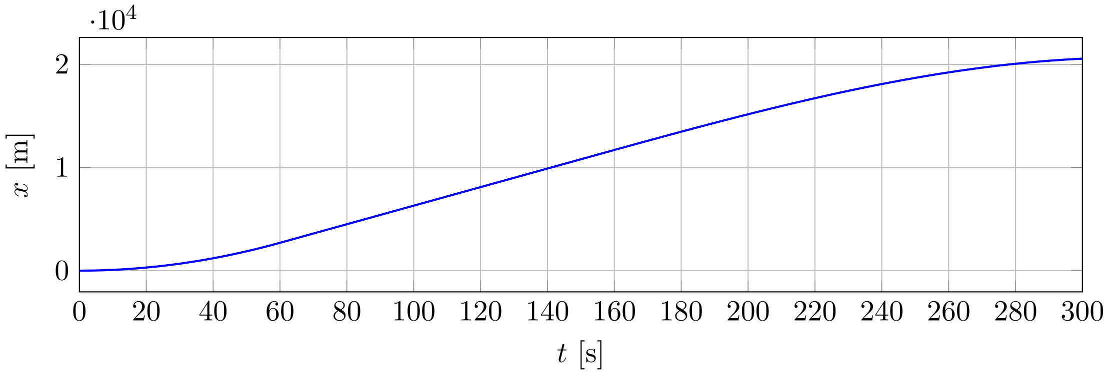

Figure 3: Position $x(t)$.

and plotted in Figure 3. After $t = 300$ seconds the position is

$$x(300) = 20550\,.$$

b) See *Julia* file `acceleration.jl`.

c) We approximate the area under the curve with rectangles. If the step size is 1 there is nothing to do. If the step size increases to 5 we have to multiply the function values a_j and v_j with a factor of 5 before the summation, cf. Figure 4.

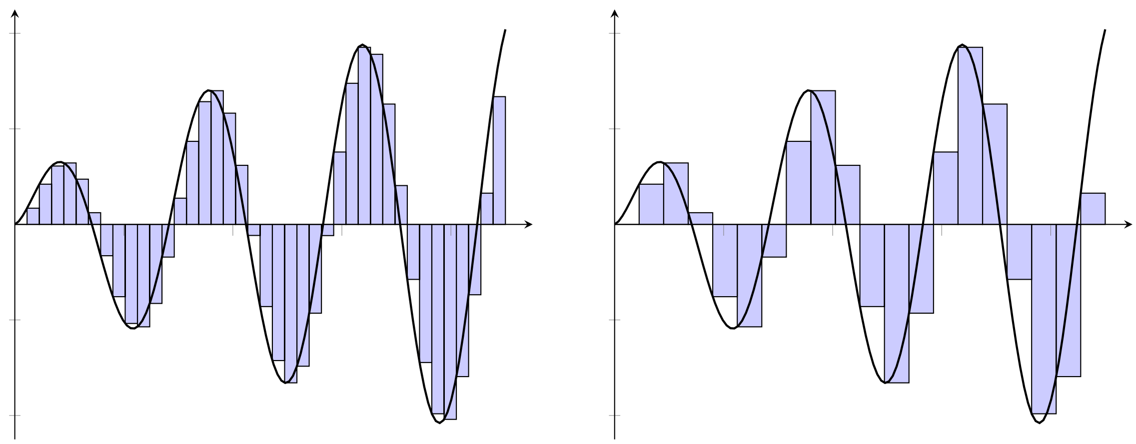

Figure 4: The region under the function curve is partitioned into rectangles. The area of each of these shapes is calculated depending on a smaller (left) or bigger (right) step size. Adding all of these small areas together yields a numerical approximation for a definite integral.

(5) a)

$$\begin{aligned}
V &= \int_0^8 \int_{-4}^4 -\sqrt{\frac{5}{4}}x^2 + 2y + 80 \,\mathrm{d}y \,\mathrm{d}x \\
&= -\sqrt{\frac{5}{4}} \cdot 8 \int_0^8 x^2 \mathrm{d}x + 2 \cdot 8 \int_{-4}^4 y \mathrm{d}y + 80 \cdot 8 \cdot 8 \\
&= -4\sqrt{5}\frac{x^3}{3}\bigg|_{x=0}^{8} + 16\frac{y^2}{2}\bigg|_{y=-4}^{4} + 5120 \\
&= 5120 - \frac{2048\sqrt{5}}{3}
\end{aligned}$$

b)

$$\partial_x f = -\sqrt{5}x$$
$$\partial_y f = 2$$
$$(1, 0, \partial_x f)^\intercal \times (0, 1, \partial_y f)^\intercal = (1, 0, -\sqrt{5}x)^\intercal \times (0, 1, 2)^\intercal = (\sqrt{5}x, -2, 1)^\intercal$$
$$\left|(\sqrt{5}x, -2, 1)^\intercal\right| = \sqrt{5x^2 + (-2)^2 + 1} = \sqrt{5}\sqrt{1 + x^2}$$

$$\begin{aligned}
S &= \int_0^8 \int_{-4}^4 |(1, 0, \partial_x f)^T \times (0, 1, \partial_y f)^T| \,\mathrm{d}x \,\mathrm{d}y \\
&= \sqrt{5} \int_0^8 \int_{-4}^4 \sqrt{1 + x^2} \mathrm{d}y \mathrm{d}x \\
&= 8\sqrt{5} \int_0^8 \sqrt{1 + x^2} \mathrm{d}x \\
&\overset{(\star)}{=} 4\sqrt{5}\left(\sinh^{-1}(8) + 8\sqrt{65}\right)
\end{aligned}$$