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Functional Programming	WS 2021	LVA 703025
Exercise Sheet 1, 5 points		Deadline: Wednesday, October 13, 2021, 6am

- Mark your completed exercises in the OLAT course of the PS.
- You can use a template .hs-file that is provided on the proseminar page.
- Upload your modified .hs-file of Exercise 2 in OLAT.
- Your .hs-file should be compilable with ghci.

Exercise 1 *Haskell setup, no points*

Setup a working Haskell environment on your computer and get familiar with `ghci`. To do this follow these steps:

1. Install Haskell, e.g., via `ghcup`.[1]
2. Run `ghci` in a terminal and evaluate the expression `(5 + 2) * 3`.
3. Find and install a suitable text editor for your system to write and edit .hs files.[2] You can try one of the following free editors:
   - Atom[3] (Windows, macOS, Linux)
   - Notepad++[4] (Windows)
   - Gedit[5] (Windows, macOS, Linux)
4. Copy or enter the following code in your text editor and save it to a file called `myProgram.hs`. Be aware to use standard double-quotes (`"`), but neither two single-quotes (`''`) nor fancy-looking double-quotes (“ or ”).
   ```
   hello :: String -> String
   hello xs = "Hello " ++ xs
   ```
5. Load the file in `ghci` with the command `ghci myProgram.hs`
6. Evaluate the expression `hello "World"`
7. Make yourself familiar with `ghci`, in particular try the following commands:
   - `:?` – help
   - `:load name.hs` or `:l name.hs` – load Haskell script `name.hs`
   - `:reload` or `:r` – reload current Haskell script
   - `:edit` or `:e` – edit current Haskell script
   - `:set editor someEditor` – set `someEditor` as preferred editor

   Further investigate what happens if you type `h` and then the tabulator key, or `hel` and then the tabulator key.

You can find links to introductory material about `ghci`, the command line, etc. on the lecture homepage.[6]

[1] https://www.haskell.org/ghcup/
[2] Word processors like Microsoft Word, Apple pages,... are not text editors.
[3] https://atom.io/
[4] https://notepad-plus-plus.org/
[5] https://wiki.gnome.org/Apps/Gedit
[6] http://cl-informatik.uibk.ac.at/teaching/ws21/fp/ghc_setup.php

Solution 1

After the proseminar everyone should have access to a working Haskell-environment and be able to run ghci.

Exercise 2 *Writing simple functions* **5 p.**

1. Define a function `milesToKilometers m = ...` to convert miles into kilometers. (1 point)
2. Define a function `volume r = ...` to compute the volume of a sphere with radius `r`. (1 point)
3. Define a function `average x y = ...` that computes the average of two numbers `x` and `y`. (1 point)
4. Is `average (average x y) z` the average of three numbers `x`, `y` and `z`? (1 point)
5. Define a function `averageVolume r1 r2 = ...` that computes the average volume of two spheres having radius `r1` and `r2`, respectively. (1 point)

Solution 2

```haskell
milesToKilometers m = 1.609344 * m

-- use predefined "pi", or replace "pi" by numerical value 3.141592653589793
volume r = 4 / 3 * pi * r^3

average x y = (x + y) / 2

{-
 average (average x y) z =
 ((x+y) / 2 + z) / 2 =
 x/4 + y/4 + z/2
 which is not the same as the correct average
 x/3 + y/3 + z/3
-}

-- just reuse the previous functions
averageVolume r1 r2 = average (volume r1) (volume r2)
```

Exercise Sheet 2, 10 points Deadline: Wednesday, October 20, 2021, 6am

- Mark your completed exercises in the OLAT course of the PS.
- Upload your .hs-file of Exercise 3 in OLAT.
- Your .hs-file should be compilable with ghci.

Exercise 1 *Typing* **4 p.**

Given the definition

```
plus1 x = x + 1
```

which of the following typing judgments are valid? Justify your answers.

1. `0 :: Bool` (1 point)
2. `head "test" :: Char` (1 point)
3. `'hello' :: String` (1 point)
4. `plus1 :: Integer -> Integer` (1 point)

Solution 1

1. Since `0` is a number but type `Bool` consists of the truth-values `True` and `False`, the typing judgment `0 :: Bool` is not valid.
2. The expression `head "test"` extracts the first character of the string `"test"`, which is `'t'` and has type `Char`. Therefore, the typing judgment `head "test" :: Char` is valid.
3. In Haskell strings are written between double quotes (`"`) and not single quotes (`'`). Therefore, this expression is not even syntactically correct and thus the typing judgment `'hello' :: Char` is invalid.
4. In Haskell, addition of `Integer`s is the function `(+) :: Integer -> Integer -> Integer`. That is, a function taking two `Integer`s as arguments and delivering an `Integer` as result. Inside the definition of `plus1` the second argument of `(+)` is fixed to be `1 :: Integer`. Therefore, the resulting function `plus1` takes a single argument of type `Integer` and delivers an `Integer` as result. Thus, the typing judgment `plus1 :: Integer -> Integer` is valid.

Exercise 2 *Parsing expressions* **3 p.**

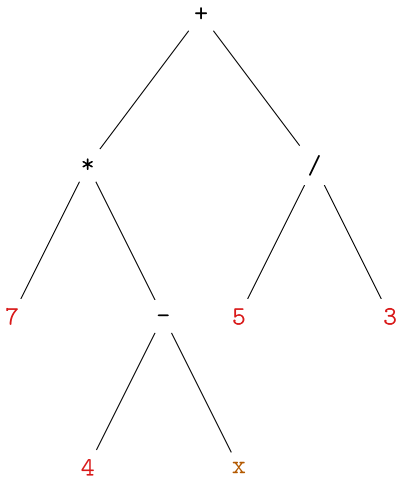
Draw the abstract syntax trees of the following expressions:

1. `7 * (4 - x) + 5 / 3` (1 point)
2. `(x < 10) || (y > 15)` (1 point)
3. `average 5 10 * square 2 + 10` (1 point)

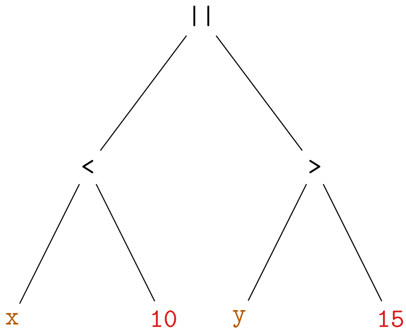
Remark: function applications (e.g., `square 7`) bind stronger than operator applications (e.g., `3 * 4`).

Solution 2

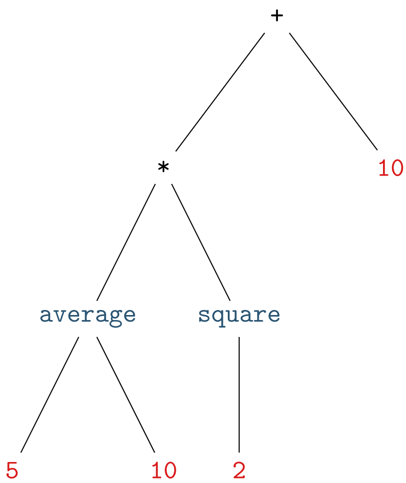
1. `7 * (4 - x) + 5 / 3`



2. `(x < 10) || (y > 15)`



3. `average 5 10 * square 2 + 10`



Exercise 3 *Modelling* **3 p.**

In graphical user interfaces (GUIs) a *menu* typically consists of *items* and submenus. One specific application of such menus is website navigation, where items would consist of a label (the text to click on) and a link (the URL of the website to navigate to when the item is clicked).

1. Give a Haskell datatype definition to model items for website navigation as described above. Moreover, define constants that represent the items OLAT (`https://lms.uibk.ac.at`) and FP (`http://cl-informatik.uibk.ac.at/teaching/ws21/fp`) (1 point)

2. Give a Haskell datatype definition to model menus that may contain up to two items. Moreover, define a constant that represents a menu containing the items for OLAT and FP from above. (1 point)

3. Change your definition from the previous exercise such that a menu may contain an arbitrary number of items. Moreover, define a constant that represents a menu with at least three items and also represent this constant as a tree as shown on the slides of week 2, page 23. (1 point)

Solution 3

1. ```haskell
   data Item = Item String String

   olat = Item "OLAT" "https://lms.uibk.ac.at"
   fp = Item "FP" "http://cl-informatik.uibk.ac.at/teaching/ws21/fp"
   ```

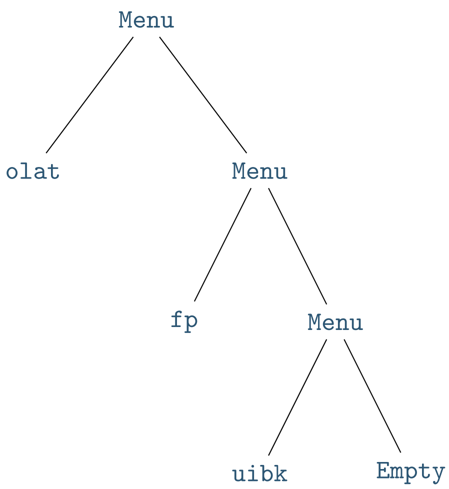
2. ```haskell
   data Menu2 = Menu2Empty          -- the empty menu
              | Menu2One Item       -- a menu with a single item
              | Menu2Two Item Item  -- a menu with two items
     deriving Show

   menu2 = Menu2Two olat fp
   ```

3. ```haskell
     data MenuList = Empty | Menu Item MenuList deriving Show

     uibk = Item "UIBK" "https://www.uibk.ac.at"
     menuList = Menu olat
                  (Menu fp
                    (Menu uibk Empty)
                  )
   ```

   - Tree representation


   ```

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Exercise Sheet 3, 10 points | Deadline: Wednesday, October 27, 2021, 6am

- Mark your completed exercises in the OLAT course of the PS.
- You can use a template .hs-file that is provided on the proseminar page.
- Upload your .hs-file(s) of Exercises 2 and 3 in OLAT.
- Your .hs-file should be compilable with ghci.

Exercise 1 *Pattern Matching* **2 p.**

Consider the following datatype definitions:

```
data Subject = CS | Math | Physics | Biology
data Programme =
    Bachelor Subject
  | Master Subject
  | Teaching Subject Subject -- teachers need two subjects
data Student = Student
  String -- name
  Integer -- matriculation number
  Bool -- active inscription
  Programme
```

Determine which of the expressions 1.-3. match the patterns in (i) and (ii). For each match give the corresponding substitution. (2 points)

1. `Student "Jane Doe" 243781 True (Teaching Math Physics)`
2. `Student "Max Meyer" 221341 False (Teaching CS Math)`
3. `Student "Mary Smith" 234145 False (Master CS)`

(i) `Student name n _ (Teaching Math _)`

(ii) `Student name n False p@(Master _)`

Solution 1

(i) Only the expression `Student "Jane Doe" 24378391 True (Teaching Math Physics)` matches with substitution `name`/`"Jane Doe"`, `n`/`24378391`.

(ii) Only the expression `Student "Mary Smith" 23416345 False (Master CS)` matches with substitution `name`/`"Mary Smith"`, `n`/`23416345`, `p / Master CS`

Exercise 2 *Function Definitions* **3 p.**

1. Define a function `disj :: Bool -> Bool -> Bool` for computing the disjunction of two Booleans. (1 point)
2. Define a function `sumList :: List -> Integer` that takes a list of integers (as defined in the lecture) and returns the sum of its elements. The sum over an empty list should be 0. (1 point)
3. Define a function `double2nd :: List -> List` that doubles every second element in a given list of integers, i.e., $[1, 7, 9, 3] \rightsquigarrow [1, 14, 9, 6]$. (1 point)

Solution 2

1. Naive solution:
```
disj True True = True
disj True False = True
disj False True = True
disj False False = False
```
Analogous to `conj` on slides:
```
disj False b = b
disj True _ = True
```
Alternative solution:
```
disj False False = False
disj _ _ = True
```
2.
```
sumList Empty = 0
sumList (Cons x xs) = x + sumList xs
```
3.
```
double2nd (Cons x (Cons y xs)) = Cons x (Cons (2*y) (double2nd xs))
double2nd xs = xs
```

Exercise 3 *A Recursive Function* **5 p.**

In this exercise, we will extend the `Expr` datatype from the lecture with variables:
```
data Expr =
    Number Integer
  | Var String
  | Plus Expr Expr
  | Negate Expr
```
We will also need the following datatype to store variable assignments:
```
data Assignment = EmptyA | Assign String Integer Assignment
```

Here, the `EmptyA` constructor corresponds to an empty assignment in which all variables have value 0. The `Assign` constructor takes an assignment and changes the value of one variable to the given integer (see examples below).

1. Write a function `ite :: Bool -> Integer -> Integer -> Integer` such that `ite b x y` returns `x` if `b` is true and `y` otherwise. Use pattern matching on Booleans only. ("`ite`" stands for "if–then–else")(1 point)
2. Write a function `lookupA :: Assignment -> String -> Integer` that returns the value corresponding to the given variable in the given assignment. (1 point)

   Example: Let `myAssn = Assign "x" 1 (Assign "x" 2 (Assign "y" 3 EmptyA))`. Then:
```
lookupA myAssn "x" == 1
lookupA myAssn "y" == 3
lookupA myAssn "z" == 0
```
3. Write a function `eval :: Assignment -> Expr -> Integer` that evaluates the given arithmetic expression under the given variable assignment. (2 points)

   Example: Let `myAssn` be as before. Then:
```
eval myAssn (Plus (Negate (Var "y")) (Number 45)) == 42  -- corresponds to (-y) + 45
```
4. In order to store auxiliary results and avoid computing the same things twice, extend the `Expr` type with a "**let** $x = e_1$ **in** e_2" construct, i.e. the result of e_1 is assigned to the variable x when evaluating e_2, the result of which is then returned. Extend your "`eval`" function accordingly as well. (1 point)

   Example:
   You should be able to write an expression that encapsulates something like this:

$$\textbf{let } x = 2 + 3 \textbf{ in } x + x$$

   How you represent this in your datatype is up to you.

Solution 3

```
ite :: Bool -> Integer -> Integer -> Integer
ite True  x y = x
ite False x y = y
-- later we will see the pre-defined construct "if b then x else y"

lookupA :: Assignment -> String -> Integer
lookupA EmptyA             s  = 0
lookupA (Assign s' x assn) s  = ite (s' == s) x (lookupA assn s)

eval :: Assignment -> Expr -> Integer
eval assn (Number x)    = x
eval assn (Var s)       = lookupA assn s
eval assn (Plus e1 e2)  = eval assn e1 + eval assn e2
eval assn (Negate e)    = -eval assn e
```

For sub-exercise 4: add a constructor “`Let' String Expr' Expr'`” to the `Expr'` type. The `eval'` function is as before except for this additional equation:

```
eval' assn (Let' s e1 e2) = eval' (Assign s (eval' assn e1) assn) e2
```

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Exercise Sheet 4, 10 points — Deadline: Wednesday, November 3, 2021, 6am

- Mark your completed exercises in the OLAT course of the PS.
- You can use a template .hs-file that is provided on the proseminar page.
- Upload your .hs-file(s) of Exercises 1 and 2 in OLAT.
- Your .hs-file should be compilable with ghci.

Exercise 1 *Nested Lists and Either* **5 p.**

1. Study the slides of week 4, pages 19 & 20 to understand the consequences of the definition of the predefined string type.

```
type String = [Char]
(++) :: [a] -> [a] -> [a]    -- and not: String -> String -> String
head :: [a] -> a             -- and not: String -> Char
```

   Given a function `concat :: [[a]] -> [a]`, briefly explain the type of the following six Haskell expressions or give a reason why these expressions result in a type error.

```
e1 = concat [1 :: Int, 2, 3]
e2 = concat ["one", "two", "three"]
e3 = concat [[1 :: Int, 2], [], [3]]
e4 = concat [["one", "two"], [], ["three"]]
e5 = concat e3
e6 = concat e4
```

   (1 point)

2. Define a function `suffixes` that computes the list of all suffixes of a list. Particularly, the following identities should hold:

```
suffixes [1, 2, 3] = [[1,2,3], [2,3], [3], []]
suffixes "hello"   = ["hello", "ello", "llo", "lo", "o", ""]
```

   Hint: structural recursion suffices.

   (1 point)

3. Define a function `prefixes` that computes the list of all prefixes of a list. Particularly, the following identities should hold:

```
prefixes [1, 2, 3] = [[1,2,3], [1,2], [1], []]
prefixes "hello"   = ["hello", "hell", "hel", "he", "h", ""]
```

   Hint: you might need an auxiliary function; structural recursion is not recommended for `prefixes`.

   (2 points)

4. Utilize the `Either` type to create a menu that generates the list of prefixes, suffixes or a meaningful error message depending on its input. Particularly, the following identities should hold:

```
menu 'p' [1,2,3]  = Right [[1,2,3],[1,2],[1],[]]
menu 'p' "hello"  = Right ["hello","hell","hel","he","h",""]
menu 's' [1,2,3]  = Right [[1,2,3],[2,3],[3],[]]
menu 's' "hello"  = Right ["hello","ello","llo","lo","o",""]
menu 'c' "hello"  = Left  "(c) is not supported, use (p)refix or (s)uffix"
```

   (1 point)

Solution 1

```haskell
{-
e1 is not type correct, since [[a]] is not more general than [Int]
e2 :: String   -- substitute a/Char (recall String = [Char], so [String] = [[Char]])
e3 :: [Int]    -- substitute a/Int
e4 :: [String] -- substitute a/String
e5 is not type correct, since [[a]] is not more general than [Int], see e3 and e1
e6 :: String   -- substitute a/Char, see e2 and e4
-}

suffixes :: [a] -> [[a]]
suffixes [] = [[]]
suffixes xs@(_ : ys) = xs : suffixes ys

{-
init is a predefined function that drops the last element of a non-empty list,
i.e., it computes the initial part of a list;
it could also be user-defined or named differently, e.g., dropLast

init :: [a] -> [a]
init [x] = []
init (x : xs) = x : init xs
-}

prefixes :: [a] -> [[a]]
prefixes [] = [[]]
prefixes xs = xs : prefixes (init xs)

menu :: Char -> [a] -> Either String [[a]]
menu 'p' xs = Right (prefixes xs)
menu 's' xs = Right (suffixes xs)
menu a _ = Left ("(" ++ [a] ++ ") is not supported, use (p)refix or (s)uffix")
```

Exercise 2 *Polymorphic Expressions* **5 p.**

1. Define a polymorphic datatype to represent expressions involving addition, multiplication and numbers. In particular `expr1` and `expr2` should be accepted.

   ```haskell
   expr1 = Times (Plus (Number (5.2 :: Double)) (Number 4)) (Number 2)
   expr2 = Plus (Number (2 :: Int)) (Times (Number 3) (Number 4))
   expr3 = Times (Number "hello") (Number "world")
   ```

   Is `expr3` type correct as well? Provide a brief explanation. (1 point)

2. Write a polymorphic function `numbers` that given an expression constructs a list of numbers that occur in the expression. For example `numbers expr1 = [5.2,4,2]` and `numbers expr2 = [2,3,4]`.

   Also provide a type for your function that is as general as possible. (1 point)

3. Write a polymorphic function `eval` to evaluate an expression. For example `eval expr1 = 18.4` and `eval expr2 = 14`.

   Also provide a type for your function that is as general as possible. (1 point)

4. Write a polymorphic function `exprToString` that converts an expression into a string that represents the expression. The string should insert parentheses only if they are required. For example:

   - `exprToString expr1 = "(5.2 + 4.0) * 2.0"`
   - `exprToString expr2 = "2 + 3 * 4"`

   Also provide a type for your function that is as general as possible. (2 points)

Solution 2

```haskell
data Expr a =
    Number a
  | Times (Expr a) (Expr a)
  | Plus (Expr a) (Expr a)
  deriving Show

numbers :: Expr a -> [a]
numbers (Number x) = [x]
numbers (Times e1 e2) = numbers e1 ++ numbers e2
numbers (Plus e1 e2)  = numbers (Times e1 e2)

eval :: Num a => Expr a -> a
eval (Number x) = x
eval (Times e1 e2) = eval e1 * eval e2
eval (Plus e1 e2) = eval e1 + eval e2

paren :: Expr a -> String -> String
paren (Plus _ _) s = "(" ++ s ++ ")"
paren _ s = s

exprToString :: Show a => Expr a -> String
exprToString (Number x) = show x
exprToString (Plus e1 e2) =
  exprToString e1 ++ " + " ++ exprToString e2
exprToString (Times e1 e2) =
  paren e1 (exprToString e1) ++ " * " ++ paren e2 (exprToString e2)
```

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Functional Programming	WS 2021	LVA 703025
Exercise Sheet 5, 10 points		Deadline: Wednesday, November 10, 2021, 6am

- Mark your completed exercises in the OLAT course of the PS.
- You can start from `template_05.hs` provided on the proseminar page.
- Upload your .hs-file(s) of Exercises 1 and 2 in OLAT.
- Your .hs-file(s) should be compilable with ghci.

Exercise 1 *Lists and Tuples* **5 p.**

1. Implement a Haskell function `mergeLists` that takes two lists and combines them into a list of pairs. If the lists do not have the same length, the rest of the longer list is ignored. Make sure that the type of your function is as general as possible. (1 point)

   Example: `mergeLists [1,2,3,4] ['a','b','c'] == [(1,'a'),(2,'b'),(3,'c')]`

2. Implement a Haskell function `calculateAge :: (Int, Int, Int) -> Int` that, given a birthday as a triple of day, month and year, computes the current age in years (as of November 10, 2021). (1 point)

   Example: `calculateAge (10, 11, 2021) == 0`

3. Implement a Haskell function

   ```
   convertDatesToAges :: [(String, (Int, Int, Int))] ->  [(String, Int)]
   ```

   that converts a list of names and birthdays into a list of names and ages. That is, it takes a list of pairs with names and dates and computes a list of pairs with names and ages. (1 point)

4. Implement a Haskell function `getOtherPairValue` that takes a pair of name and age as well as a second argument that can be either a name *or* an age and returns the part of the input that wasn't the second argument. If the second argument does not match either element of the input-pair the function should return nothing. (2 points)

   Example: When invoking `getOtherPairValue` on the pair `("Hans", 45)` together with the second argument `45` (wrapped in an appropriate type), the result should be `"Hans"` (wrapped in an appropriate type).

Hint: You might find the `ite` (if-then-else) function of sheet 3 useful for some of the above exercises.

```
ite :: Bool -> a -> a -> a
ite True x y = x
ite False x y = y
```

Solution 1

1. ```
   mergeLists :: [a] -> [b] -> [(a, b)]
   mergeLists (x : xs) (y : ys) = (x, y) : mergeLists xs ys
   mergeLists _ _ = []
   ```

2. ```
   calculateAge :: (Int, Int, Int) -> Int
   calculateAge (d, m, y) = ite (m < 11 || m <= 11 && d <= 10) (2021 - y) (2020 - y)
   ```

3. `convertDatesToAges :: [(String, (Int, Int, Int))] ->  [(String, Int)]`
   `convertDatesToAges ((x, y) : xs) = (x, calculateAge y) : convertDatesToAges xs`
   `convertDatesToAges [] = []`

4. `getOtherPairValue :: (String, Int) -> Either String Int -> Maybe (Either String Int)`
   `getOtherPairValue (x, y) (Right r) = ite (y == r) (Just (Left x)) Nothing`
   `getOtherPairValue (x, y) (Left l) = ite (x == l) (Just (Right y)) Nothing`

Exercise 2 *Equational Reasoning, Lists, and Tuples* **5 p.**

Consider the following Haskell functions

```
addPair (x, y)    =  x + y
addList []        =  []
addList (x : xs) =  addPair x : addList xs
```

1. Write down the types of the functions above and explain how you were able to derive them. (1 point)

2. Evaluate the result of the following expression step-wise (that is, only using one of the above equations at a time) by hand. (2 points)

   `addList ((1,2):(2,1):[])`

   Remark: For clarification here is an example of another step-wise evaluation (this process is called *equational reasoning*; the definition of `lastElem` can be found on page 3 of the slides of week 4):

   ```
   lastElem (Cons 1 (Cons 2 Empty)) = lastElem (Cons 2 Empty)
                                    = 2
   ```

3. Implement a Haskell function `fstList :: [(a, b)] -> [a]` that collects all the first elements of a list of pairs. (1 point)

   Examples:
   ```
   fstList [(1,'a'),(2,'b'),(3,'c')] == [1,2,3]
   fstList [("hello","world")] == ["hello"]
   fstList [] == []
   ```

4. Implement a Haskell function `lengthSumMax :: (Num a, Ord a) => [a] -> (Int, a, a)` that, given a list of non-negative numbers, computes its length, the sum of all its elements and the maximum of all its elements and returns those three values as a triple. (1 point)

   Examples:
   ```
   lengthSumMax [] == (0,0,0)
   lengthSumMax [0,1,0,2,0] == (5,3,2)
   ```

Solution 2

1.
   - The type of `addPair` is `Num a => (a, a) -> a`. Since addition `(+)` is used on `x` and `y` both have to be of the same type `a` that is an instance of `Num`. Moreover, the input of `addPair` is a pair.
   - The type of `addList` is `Num a => [(a, a)] -> [a]`. Since `addList` is defined by pattern-matching on lists on its input, its input has to be a list. Moreover, since to each element of the input list the function `addPair` is applied, the list-elements have to be pairs of type `(a, a)` where `Num a`.

2.
   ```
   addList ((1,2):(2,1):[]) = addPair (1,2) : addList ((2,1):[])
                            = (1 + 2) : addList ((2,1):[])
                            = 3 : addList ((2,1):[])
                            = 3 : addPair (2,1) : addList []
                            = 3 : (2 + 1) : addList []
                            = 3 : 3 : addList []
                            = 3 : 3 : []
   ```

3. `fstList :: [(a, b)] -> [a]`
   ```
   fstList ((x, y) : xs) = x : fstList xs
   fstList [] = []
   ```

4. `lengthSumMax :: (Num a, Ord a) => [a] -> (Int, a, a)`
   ```
   lengthSumMax [] = (0, 0, 0)
   lengthSumMax (x : xs) = lsmHelper x (lengthSumMax xs)

   lsmHelper x (l, s, m) = (l + 1, s + x, max m x)
   ```

- Mark your completed exercises in the OLAT course of the PS.
- You can start from `template_06.hs` provided on the proseminar page.
- Your .hs-file(s) should be compilable with ghci and be uploaded in OLAT.

Exercise 1 *Functions on Numbers* **4 p.**

1. Implement a Haskell function `dividesRange :: Integer -> Integer -> Integer -> Bool` that checks whether there is any divisor of a number within a given range: `dividesRange n l u` should be true iff there is some `x` such that `l` ≤ `x` ≤ `u` and `x` divides `n`. (1 point)

   Example: `dividesRange 629 15 25 == True` since $15 \leq 17 \leq 25$ and 17 divides 629.

   Hint: You can use the built-in functions `div` or `mod` for checking divisibility of two numbers: `div x y` and `mod x y` compute the quotient and the remainder of the integral division of `x` by `y`, respectively. E.g., `div 25 4 = 6` and `mod 25 4 = 1`, since $25 = 6 \cdot 4 + 1$.

2. Implement a Haskell function `prime :: Integer -> Bool` to determine whether a number is prime. Recall: n is a prime number if $n \geq 2$ and n has exactly two divisors. (1 point)

   Example: `prime 7793 == True` and `prime 7797 == False`.

3. Implement a Haskell function `generatePrime :: Integer -> Integer` which takes a number `d` as input and computes a prime number with at least `d` digits. (1 point)

   Valid examples: `generatePrime 4 == 1009` and `generatePrime 8 == 10000019`.

4. How far can you increase `d` such that `generatePrime d` is computed within 1 minute? If this value is below 10, then improve your algorithm `prime`. (1 point)

   Hint: Implement a square root function directly on integers, i.e., without using `sqrt :: Double -> Double`. It does not matter if you implement $\lfloor\sqrt{n}\rfloor$ or $\lceil\sqrt{n}\rceil$. For instance, the integer square root of 27 can either be 5 or 6.

Solution 1

```haskell
divides :: Integer -> Integer -> Bool
divides x y = mod y x == 0

dividesRange :: Integer -> Integer -> Integer -> Bool
dividesRange n l u
  | l > u = False
  | otherwise  = divides l n || dividesRange n (l + 1) u

prime :: Integer -> Bool
prime n = n >= 2 && not (dividesRange n 2 (sqrtIntBisect n))

generatePrime :: Integer -> Integer
generatePrime d = gen (10^(d-1) + 1) where
  gen n
     | prime n = n
     | otherwise = gen (n+2)

-- sqrtIntVariant x computes the floor of sqrt(x)

-- simple version just counts x upwards until x^2 > n
sqrtIntSlow :: Integer -> Integer
sqrtIntSlow n = main 0 where
  main x
    | x * x > n = x - 1
    | otherwise = main (x + 1)

-- bisection version finds correct value by
-- iteratively dividing a search interval [l,r] in two halves
sqrtIntBisect :: Integer -> Integer
sqrtIntBisect x = s 0 x where
  s l u -- we have lower and bounds l and u satisfying l^2 <= x <= u^2
    | u - l > 1 =
        let m = div (u + l) 2;
            m2 = m * m
         in if m2 > x then s l m
            else if m2 < x then s m u
            else m
    | u == l = l
    | otherwise = if u^2 <= x then u else l
```

If `prime` is defined as `prime n = n >= 2 && not (dividesRange n 2 (n - 1))`, then `generatePrime` cannot compute `generatePrime 10` within one minute (using a computer with a 3.1 GHz Intel Core i7). With the current solution `generatePrime 14` is computed in around 1 minute.

Exercise 2 *Heron's method* **2 p.**

Heron's method is an ancient (but very efficient) method to approximate the square root of a given non-negative real number x. It works like this: We recursively define the following sequence of numbers:

$$y_0 = x \qquad\qquad y_{n+1} = \begin{cases} 0 & \text{if } y_n = 0 \\ \frac{1}{2}(y_n + \frac{x}{y_n}) & \text{otherwise} \end{cases}$$

Mathematically, this sequence converges monotonically to $\sqrt{x}$ but never actually reaches it (unless $x = 0$ or $x = 1$), giving successively better and better approximations to $\sqrt{x}$.

However, due to the finite precision of the `Double` type, doing this computation in Haskell, you will always find that at some point $y_{n+1} == y_n$.

Your task is to write a function `heron :: Double -> [Double]` that outputs the sequence of numbers $[y_0, \ldots, y_n]$, where n is the smallest number such that $y_{n+1} == y_n$. (2 points)

Examples:

```
heron 0 == [0.0]
heron 1 == [1.0]
heron 2 == [2.0, 1.5, 1.4166666666666665, 1.4142156862745097,
            1.4142135623746899, 1.414213562373095]
```

Solution 2

```
heron :: Double -> [Double]
heron x = aux x
  where aux y = let y' = if y == 0 then 0 else 0.5 * (y + x / y)
                in  if y' == y then [y] else y : aux y'
```

Exercise 3 *Fibonacci numbers* **4 p.**

The Fibonacci numbers $(a_n)_{n\in\mathbb{N}} = 0, 1, 1, 2, 3, 5, 8, \ldots$ are defined by the recurrence

$$a_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ a_{n-1} + a_{n-2} & \text{otherwise} \end{cases}$$

They can also be computed more efficiently by the following recurrence:

$$a_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \text{ or } n = 2 \\ a^2_{\lfloor \frac{n}{2} \rfloor} + a^2_{\lfloor \frac{n}{2} \rfloor + 1} & \text{if } n \text{ is odd} \\ (2a_{\lfloor \frac{n}{2} \rfloor + 1} - a_{\lfloor \frac{n}{2} \rfloor})a_{\lfloor \frac{n}{2} \rfloor} & \text{if } n \text{ is even} \end{cases}$$

You will implement the computation of the a_n given a non-negative integer n as an input in three different ways:

1. Write a function `fib :: Integer -> Integer` that computes a_n using the naïve recurrence $a_{n+2} = a_{n+1} + a_n$ and a function `fib' :: Integer -> Integer` that does the same using the more complicated recurrence given above.

   Test your functions on increasingly large values and see how much time they take.
   Explain the results. (2 points)

   Hint:

   - If you run the command `:set +s` in GHCi, it will print how long each evaluation took.[1]
   - If an evaluation takes too long, you can abort it using the key combination `Ctrl + C`.
   - Recall that in Haskell, $\lfloor \frac{n}{2} \rfloor$ is written as `div n 2`. Also note that the following pre-defined functions exist: `even :: Integer -> Bool` and `odd :: Integer -> Bool`

2. Write a function `fibFast :: Integer -> Integer` that does the same as `fib'` but internally remembers all values that have already been computed in a lookup table. Again check how long it takes on increasingly large inputs. (2 points)

   Hint:

   - You will need an auxiliary function

     `fibFastAux :: Integer -> [(Integer, Integer)] -> (Integer, [(Integer, Integer)])`

[1] Note that benchmarking functions in GHCi like this is not particularly accurate for a number of reasons: the code is not compiled but only interpreted, and many optimisations that GHC normally does are not performed. But for the scope of this exercise, this is fine.

that takes a number n and a lookup table (consisting of pairs (i, a_i)) and returns both the result a_n and a (possibly bigger) lookup table. If the pair for n is already in the table, the stored value of a_n should be returned – otherwise the recurrence should be used to recursively compute the value of a_n, and the new pair (n, a_n) is then stored in the table.

- The pre-defined function `lookup :: Eq a => a -> [(a, b)] -> Maybe b` will be useful to lookup values in the table.
- The numbers involved grow *very* fast, so even the printing takes a lot of time. For more consistent results, try showing the number of digits of the result instead of the actual result, e.g. `length (show (fib' 10000))` or `length (show (fibFast 10000))`.

Solution 3

1. For `fib`, every function call spawns two new recursive calls, each decreasing the argument by a constant amount. Thus, we get exponentially many calls (roughly 2^{n}).

   Consequently, `fib'` is very inefficient and takes fairly long even on small examples like `fib 35`.

   In `fib'`, every function call again spawns two recursive calls, but we (roughly) halve the argument with every call. Thus, we get roughly $2^{log_2\ n} = n$ calls, i.e. linearly many.

   This function also starts to hit performance problems around $n = 2 \cdot 10^6$. But the important observation is that it is much faster than `fib`.

```haskell
fib :: Integer -> Integer
fib 0 = 0
fib 1 = 1
fib n = fib (n - 1) + fib (n - 2)


fib' :: Integer -> Integer
fib' n
  | n == 0            = 0
  | n == 1 || n == 2  = 1
  | odd n             = a ^ 2 + b ^ 2
  | even n            = (2 * b - a) * a
  where a = fib' (div n 2)
        b = fib' (div n 2 + 1)
```

2. The number of actual recursive calls is hard to estimate here, but some experimentation shows that it is roughly logarithmic in `n`. This is much better than what we had before, and consequently, the function is much faster.

   Note however that this does not mean that the overall running time of `fibFast` is logarithmic - the numbers grow exponentially, so in the end, the running time is still super-linear.

```haskell
fibFast :: Integer -> Integer
fibFast n = fst (fibFastAux n [])
  where
   fibFastAux :: Integer -> [(Integer,Integer)] -> (Integer, [(Integer,Integer)])
   fibFastAux n table | n <= 2 = (if n == 0 then 0 else 1, table)
   fibFastAux n table =
        case lookup n table of
            Just res -> (res, table)
            Nothing ->
              let (a, table1) = fibFastAux (div n 2) table
                  (b, table2) = fibFastAux (div n 2 + 1) table1
                  res = if odd n then a ^ 2 + b ^ 2 else (2 * b - a) * a
              in  (res, (n, res) : table2)

{-
  As a bonus, the following function computes the number of recursive calls
  of "fibFast".
-}
fibFastCalls :: Integer -> Integer
fibFastCalls n = fst (fibFastAux n [])
  where fibFastAux n table | n <= 2 = (1, table)
        fibFastAux n table =
          if elem n table then
            (1, table)
          else
            let (m1, table1) = fibFastAux (div n 2) table
                (m2, table2) = fibFastAux (div n 2 + 1) table1
            in  (m1 + m2 + 1, n : table2)
```

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Functional Programming	WS 2021	LVA 703025
Exercise Sheet 7, 10 points		Deadline: Wednesday, November 24, 2021, 6am

- Mark your completed exercises in the OLAT course of the PS.
- You can start from `template_07.hs` provided on the proseminar page.
- Your .hs-file(s) should be compilable with ghci and be uploaded in OLAT.

Exercise 1 *Rational Numbers* **5 p.**

Implement rational numbers in Haskell. Here, rational numbers are represented by two integers, the numerator and the denominator. For instance the rational number $\frac{-3}{5}$ is represented as `Rat (-3) 5` when using the following data type definition.

```
data Rat = Rat Integer Integer
```

1. Implement a normalisation function for rational numbers, so that all of `Rat 2 4`, `Rat (-1) (-2)` and `Rat 1 2` get transformed into the same internal representation. (1 point)

   Hint: the Prelude already contains a function `gcd`.

2. Make `Rat` an instance of `Eq` and `Ord`. Of course, `Rat 2 4 == Rat 1 2` should be valid. (1 point)

3. Make `Rat` an instance of `Show`. Make sure that `show r1 == show r2` whenever `r1 == r2` for two rational numbers `r1` and `r2`. In particular, `show (Rat 1 2) == show (Rat 2 4)` should evaluate to true. Moreover, integers should be represented without division operator. (1 point)

   Examples: `show (Rat (-4) (-1)) == "4"` and both `show (Rat (-3) 2)` and `show (Rat 3 (-2))` result in `"-3/2"`.

4. Make `Rat` an instance of `Num`. See `https://hackage.haskell.org/package/base-4.16.0.0/docs/Prelude.html#t:Num` for a detailed description of this type class. (2 points)

Solution 1

```haskell
data Rat = Rat Integer Integer

normaliseRat :: Rat -> Rat
normaliseRat (Rat n d) = normaliseSign (n `div` g) (d `div` g) where
  g = gcd n d
  normaliseSign n d
    | d < 0 = Rat (negate n) (negate d)
    | otherwise = Rat n d

instance Eq Rat where
  Rat n1 d1 == Rat n2 d2 = n1 * d2 == n2 * d1

instance Ord Rat where
  r1 <= r2 = let
    Rat n1 d1 = normaliseRat r1
    Rat n2 d2 = normaliseRat r2
    in n1 * d2 <= n2 * d1

instance Show Rat where
  show r = pretty (normaliseRat r) where
    pretty (Rat n d)
      | d == 1 = show n
      | otherwise = show n ++ "/" ++ show d

createRat n d = normaliseRat (Rat n d)

instance Num Rat where
  Rat n1 d1 + Rat n2 d2 = createRat (n1 * d2 + n2 * d1) (d1 * d2)
  Rat n1 d1 * Rat n2 d2 = createRat (n1 * n2) (d1 * d2)
  negate (Rat n d) = Rat (negate n) d
  fromInteger i = Rat i 1
  abs (Rat n d) = Rat (abs n) (abs d)
  signum r
    | r < 0 = -1
    | r > 0 = 1
    | otherwise = 0
```

Exercise 2 *Class and Data* **5 p.**

1. Create a new data type `Ingredient`, which should hold the name, the amount and the unit of the ingredient. For the unit there exists ML (milliliters), G (gram) and PC (pieces). Also make `Ingredient` an instance of `Show` to show an ingredient like this `"200 ML of Milk"`. (1 point)

2. Create a new `class Price` with a function `getPrice`, which should return a price in euro as a `Float`. Create now an instance for your `Ingredient` data type. The price of an ingredient is defined independently of the name of the ingredient as follows: 1 ML costs 0.12 cent, 1 PC costs 75 cent and 1 G costs 0.095 cent.

   Modify the `Show` instance of `Ingredient` of the previous part in a way that additionally the price of the ingredient is displayed in Euro, e.g., `"200 ML of Milk, cost: 0.24 EUR"` (2 points)

3. Create a data type `Recipe`, where you can store an arbitrary amount of ingredients. Make `Recipe` an instance of `Price` which calculates the total costs of the ingredients. Moreover, make `Recipe` an instance of `Show` such that a recipe is displayed in a form like this: ”200.0 ML of Milk, cost: 0.24 EUR - 200.0 G of Sugar, cost: 0.19 EUR - 3.0 PC of Egg, cost: 2.25 EUR - Price of the Recipe: 2.68 EUR”. (2 points)

Solution 2

```haskell
data Unit = ML | G | PC deriving Show

data Ingredient = Ing String Float Unit

class Price a where
  getPrice :: a -> Float

instance Price Ingredient where
  getPrice (Ing s x ML) = x * 0.12 / 100
  getPrice (Ing s x G) = x * 0.095 / 100
  getPrice (Ing s x PC) = x * 75 / 100

instance Show Ingredient where
  show (Ing s x u) = show x ++ " " ++ show u ++ " of " ++ s ++ ", cost: " ++
show (getPrice (Ing s x u)) ++ " EUR"

data Recipe = Recipe [Ingredient]

instance Price Recipe where
  getPrice (Recipe ings) = cost ings where
    cost [] = 0
    cost (i : is) = getPrice i + cost is

instance Show Recipe where
  show (r@(Recipe ings)) = showIngs ings ++ "Price of the Recipe: " ++ show (getPrice r) ++
" EUR"
    where showIngs [] = ""
          showIngs (i : is) = show i ++ " - " ++ showIngs is
```

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- You can start from `template_08.hs` provided on the proseminar page.
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Exercise 1 *Partial Application and Sections* **3 p.**

Consider the following functions:

```
div1 = (/)
div2 = (2 /)
div3 = (/ 2)
eqTuple f = (\(x, y) -> f x == f y)
eqTuple' f (x, y) = f x == f y
```

1. Explain what these functions do and give the most general type signature for each function (do not use GHCi to find the type signatures). Give an example that shows the difference between `div2` and `div3` and explain why they are different. (1 point)
2. We say that a Haskell function `f` is equal to a Haskell function `g`, whenever `f x1 .. xN = g x1 .. xN` for all inputs `x1`, . . . , `xN`. Based on this definition, are the functions `eqTuple` and `eqTuple'` equal? Justify your answer. (1 point)
3. Which of the functions `foo1 x y = y / x` and `foo2 x y = (\u v -> v / u) y x` are equal to `div1` from above? Justify your answer. (1 point)

Solution 1

1.
```
-- takes two fractional numbers x and y and computes x / y
div1 :: Fractional a => a -> a -> a
div1 = (/)

-- takes a fractional number x and computes 2 / x
div2 :: Fractional a => a -> a
div2 = (2 /)

-- takes a fractional number x and computes x / 2
div3 :: Fractional a => a -> a
div3 = (/ 2)

-- takes a function f and yields a function that
-- takes a pair (x, y) and computes f x == f y
eqTuple :: Eq b => (a -> b) -> (a, a) -> Bool
eqTuple f = (\(x, y) -> f x == f y)

-- takes a function and a pair (x, y) and computes f x == f y
eqTuple' :: Eq b => (a -> b) -> (a, a) -> Bool
eqTuple' f (x, y) = f x == f y
```

The following example shows that `div2` and `div3` are not equal: `div2 5` $\neq$ `div3 5`. We use partial application to provide a single argument of `(/)`: In `div2` we provide `2` as the first argument (the numerator) and in `div3` we provide `2` as the second argument (the denominator).

2. The functions `eqTuple` and `eqTuple'` are indeed equal: on every input function `f` and pair `(x, y)` they return the same value `f x == f y`.

3. The function `foo1` is not equal to `div1`, since `foo1 1 2 = 2 / 1` $\neq$ `1 / 2 = div1 1 2`.

   The function `foo2` is equal to function `div1`, since `foo2 x y` = `(\u v -> v / u) y x` = `(\v -> v / y) x` = `x / y` = `div1 x y` for all `x` and `y`.

Exercise 2 *Higher-Order Functions and Lambdas* **5 p.**

1. Implement a recursive higher-order function `fan :: (a -> Bool) -> [a] -> [[a]]` that takes a predicate[1] and a list, and "fans out" the list into a list of sublists such that for each sublist either *each* element satisfies the predicate or *none* does. Moreover, your implementation should satisfy the equation

   `concat (fan p xs) == xs` (3 points)

   that is, concatenating the result of a call to `fan` results in the original list.

   Examples:
   ```
   fan undefined [] == []
   fan even [1..5] == [[1],[2],[3],[4],[5]]
   fan (== 'T') "This is a Test" == ["T","his is a ","T","est"]
   ```

2. Use `fan` from exercise 1 together with some lambda expression to implement a function

   ```
   splitOnNumbers = fan (\x -> ... x ...)
   ```

   that splits a given text into numbers and non-numbers. (1 point)

   Example: `splitOnNumbers "8 out of 10 cats" == ["8"," out of ","10"," cats"]`

   Hint: Recall that `Char` is an instance of `Ord`.

3. Use `fan` from exercise 1 to implement a function `splitBy :: (a -> Bool) -> [a] -> [[a]]` that splits a given list into sublists such that only parts remain that do not satisfy the given predicate. (1 point)

   Example: `splitBy (== '\n') "Just\nsome\nlines\n" == ["Just","some","lines"]`

Hint: If you did not manage to implement `fan` of exercise 1, you may use the following implementation in exercises 2 and 3:

```
import Data.List
fan p = groupBy (\x y -> p x == p y)
```

Solution 2

1. ```
   fan :: (a -> Bool) -> [a] -> [[a]]
   fan p [] = []
   fan p [x] = [[x]]
   fan p (x:xs)
     | p x == p y = (x:y:ys) : yss
     | otherwise  = [x] : (y:ys) : yss
     where
       ((y:ys):yss) = fan p xs
   ```

2. `splitOnNumbers = fan (\x -> '0' <= x && x <= '9')`

3. `splitBy p = filter (not . p . head) . fan p`

[1] Functions that return `Bool`s are sometimes called *predicates*, since they "decide" whether their input satisfies some property. For example `even` from the `Prelude` is a predicate on integers.
   ```

Exercise 3 *The `foldr` Function* **2 p.**

Implement the following functions using `foldr` instead of recursion. In the process, you may find lambda expressions useful.

1. Consider a function that converts a list of digits (represented as `Integer`s) into an `Integer`:

   ```
   dig2int :: [Integer] -> Integer
   dig2int [] = 0
   dig2int (x:xs) = x + 10 * dig2int xs
   ```

   Examples: `dig2int [2,1,5] == 512`

   Implement a variant `dig2intFold` of `dig2int` using `foldr`. (1 point)

2. Consider a function `suffs` that computes all suffixes of a list, from longest to shortest:

   ```
   suffs :: [a] -> [[a]]
   suffs [] = [[]]
   suffs (y @ (_ : xs)) = y : suffs xs
   ```

   Examples:

   ```
   suffs [1,2] = [[1,2], [2], []]
   suffs "hello" = ["hello", "ello", "llo", "lo", "o", ""]
   ```

   Implement a variant `suffsFold` of `suffs` using `foldr`. (1 point)

Solution 3

1. ```
   dig2intFold :: [Integer] -> Integer
   dig2intFold = foldr (\x acc -> x + 10 * acc) 0
   ```

2. ```
   suffsFold :: [a] -> [[a]]
   suffsFold = foldr (\x acc -> (x : head acc) : acc) [[]]
   ```

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- You can start from `template_09.hs` provided on the proseminar page.
- Your .hs-file(s) should be compilable with ghci and be uploaded in OLAT.

Exercise 1 *The Caesar cipher* **6 p.**

The well known Caesar cipher[1] encodes a string by shifting each character n times (for some n).

1. Write a function `shift :: Int -> Char -> Char` that applies a shift factor to a lower-case letter (letters `'a'` – `'z'`). Other characters should be ignored by your function.

   Hint: `fromEnum` and `toEnum` can be used to convert between characters and `Int` values (corresponding to Unicode).

   Examples: `shift 5 'a' = 'f'`, `shift 21 'f' = 'a'`, `shift 5 '.' = '.'`

   Using `shift` and list-comprehensions define a function `encode :: Int -> String -> String` shifting each lower-case letter in a given string.

   Example: `encode 5 "here is an example." = "mjwj nx fs jcfruqj."` (1 point)

2. The key to having a program crack the Caesar cipher is the observation that some letters appear more frequently than others in English text. Below is a list of approximate frequencies (in percent) of the 26 letters of our alphabet (source: https://en.wikipedia.org/wiki/Letter_frequency):

   ```
   freqList = [8.2, 1.5, 2.8, 4.3, 13, 2.2, 2, 6.1, 7, 0.15, 0.77, 4, 2.4, 6.7,
               7.5, 1.9, 0.095, 6, 6.3, 9.1, 2.8, 0.98, 2.4, 0.15, 2, 0.074]
   ```

   If we measure how well a given frequency distribution matches up with the expected distribution, e.g. by using the chi-squared statistic, we can choose the shift factor that produces the best match for our decoding. Implement the following functions to help you achieve this task in the next item.

   (a) `count :: Char -> String -> Int` which returns the number of occurrences of a particular character in a string and `percent :: Int -> Int -> Float` that calculates the percentage of one integer with respect to another. (1 point)

   Examples: `count 'e' "example" = 2`, `percent 1 3 = 33.333336`

   (b) `freqs :: String -> [Float]` which computes the list of frequencies for a given string.

   Hint: `['a'..'z']` produces a list of the 26 lower-case letters. (1 point)

   Example: `freqs "abbcccdddd" = [10.0,20.0,30.000002,40.0,0.0,...,0.0]`

   (c) `chisqr :: [Float] -> [Float] -> Float` which, given a list of observed frequencies os and expected frequencies es, computes the *chi-square statistic*: (1 point)

   $$\sum_{i=0}^{n-1} \frac{(os_i - es_i)^2}{es_i}$$

   Note that smaller results of the chi-square statistic indicate better matches between observed and expected frequencies.

[1] https://en.wikipedia.org/wiki/Caesar_cipher

(d) `rotate :: Int -> [a] -> [a]` which rotates the elements of a list n places to the left, wrapping around the end of the list, and assuming that $0 \leqslant n \leqslant$ length of the list and the function `positions :: Eq a => a -> [a] -> [Int]` which returns the list of all positions at which a value occurs in a list. (1 point)

Hint: For `positions` first pair all elements in the list with their position using `zip`.

Examples: `rotate 3 [1,2,3,4,5] = [4,5,1,2,3]`, `positions 3 [3,1,3,3] = [0,2,3]`

3. Write a function `crack :: String -> String` which attempts to decode a Caesar cipher-encoded string by first computing the frequency list of the string, then calculating the chi-square statistic of each possible rotation of the frequency list with respect to the frequencies given in `freqList`, and finally taking the position of the minimum chi-square value as the shift factor for decoding. (In the unlikely case that there are multiple minimum positions, simply pick one.) (1 point)

   Use your cracking function to decode the following text:

   rkcuovv sc pex

Solution 1

```
shift :: Int -> Char -> Char
shift n c
  | c >= 'a' && c <= 'z' = toEnum $ (fromEnum c - fromEnum 'a' + n) `mod` 26 +
(fromEnum 'a')
  | otherwise = c

encode :: Int -> String -> String
encode n xs = [shift n x | x <- xs]

freqList = [8.2, 1.5, 2.8, 4.3, 13, 2.2, 2, 6.1, 7, 0.15, 0.77, 4, 2.4, 6.7,
            7.5, 1.9, 0.095, 6, 6.3, 9.1, 2.8, 0.98, 2.4, 0.15, 2, 0.074]

count :: Char -> String -> Int
count c xs = length $ filter ((==) c) xs

percent :: Int -> Int -> Float
percent n m = fromIntegral n / fromIntegral m * 100

freqs :: String -> [Float]
freqs xs = [percent (count c xs) n | c <- ['a'..'z']]
  where n = length (filter (\c -> c >= 'a' && c <= 'z') xs)

chisqr :: [Float] -> [Float] -> Float
chisqr os es = sum [(o - e)^2 / e | (o, e) <- zip os es]

rotate :: Int -> [a] -> [a]
rotate n xs = drop n xs ++ take n xs

positions :: Eq a => a -> [a] -> [Int]
positions x xs = [i | (x', i) <- zip xs [0..], x' == x]

crack :: String -> String
-- factor is assumed to be the shift factor used in the encoding,
-- so for decoding we use it as negative shift
crack xs = encode (-factor) xs
  where
    factor = head $ positions (minimum chis) chis
    chis = [chisqr (rotate n table) freqList | n <- [0..25]]
    table = freqs xs

encString = "rkcuovv sc pex"
-- crack encString == "haskell is fun"
```

Exercise 2 *Bernoulli numbers* **4 p.**

The Bernoulli numbers are a sequence of rational numbers defined like this:

$$B_0 = 1 \qquad\qquad B_n = \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{k-n-1} \text{ if } n > 0$$

Here, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ denotes the binomial coefficient, where $n! = 1 \cdot \ldots \cdot n = \prod_{i=1}^{n} i$ denotes the factorial. The following table lists the first few values of B_n:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	...
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$	...

Note that `Rational` is the type of rational numbers from the Haskell standard library. It has all the type class instances you would expect from it, including most notably a division operator `(/)` and a conversion from integers to rationals `fromInteger :: Integer -> Rational`. There is also the operator

```
(%) :: Integer -> Integer -> Rational
```

that turns a numerator a and denominator b into the rational number $\frac{a}{b}$, i.e. `1 % 2` corresponds to $\frac{1}{2}$.

1. Write functions `fact :: Integer -> Integer` and `binom :: Integer -> Integer -> Integer` that compute the factorial (resp. binomial coefficients) for non-negative inputs. (1 point)

2. Write a function `bernoulli :: Integer -> Rational` such that `bernoulli n` $= B_n$. What is the largest n for which your function still finishes in a reasonable amount of time? (1 point)

   Example: `map bernoulli [0..6] == [1%1, (-1)%2, 1%6, 0%1, (-1)%30, 0%1, 1%42]`

3. Write a function `bernoullis :: Integer -> [Rational]` that, given an integer $n \geq 0$, computes the list of the Bernoulli numbers B_0 to B_n, i.e. `bernoullis n == map bernoulli [0..n]`. Implement it in a more efficient way than just calling `bernoulli` n times! Avoid recomputing results that you have already computed! (1 point)

4. Looking at the sequence of Bernoulli numbers, it seems that starting with B_3, every B_i with i odd is 0. It also seems that in the sequence of the remaining ones, the sign keeps alternating in every step (i.e. B_2, B_6, B_{10}, etc. are positive, B_4, B_8, B_{12}, etc. are negative).

   Write two Haskell functions `check1 :: Integer -> Bool` and `check2 :: Integer -> Bool` that check whether these conjectures are true for all B_n up to a given number n! (1 point)

Trivia: Ada Lovelace is widely credited with having written the first non-trivial computer program in 1842 – and it was an algorithm for computing Bernoulli numbers!

Solution 2

```haskell
fact :: Integer -> Integer
fact n = product [1..n]

binom :: Integer -> Integer -> Integer
binom n k = fact n `div` (fact k * fact (n - k))
-- alternatively: binom n k = product [n-k+1..n] `div` fact k

bernoulli :: Integer -> Rational
bernoulli 0 = 1
bernoulli n = sum [bernoulli k * (binom n k % (k - n - 1)) | k <- [0..n-1]]
-- "bernoulli 22" already takes over 10 seconds to compute, and the time seems to roughly double with e

bernoullis :: Integer -> [Rational]
bernoullis 0 = [1]
bernoullis n = bs ++ [b]
  where bs = bernoullis (n - 1)
        b  = sum [bk * (binom n k % (k - n - 1)) | (k, bk) <- zip [0..n] bs]
-- with this, we can compute e.g. "last (bernoulli 500)" within about 5 seconds

-- One can also define the infinite list of all Bernoulli numbers fairly elegantly:
bernoullis' :: [Rational]
bernoullis' = 1 : map f [1..]
  where f n = sum [bk * (binom n k % (k - n - 1)) | (k, bk) <- zip [0..n-1] bernoullis']

-- Transforms a list [x0, x1, x2, ...] into the list [x0, x2, x4, ...]
dropEveryOther :: [a] -> [a]
dropEveryOther (x : y : xs) = x : dropEveryOther xs
dropEveryOther xs = xs

check1 :: Integer -> Bool
check1 n = all (== 0) bs
  where bs = tail (dropEveryOther (tail (bernoullis n)))

check2 :: Integer -> Bool
check2 n = all (\(x, y) -> signum x == -signum y) (zip bs (tail bs))
  where bs = tail (dropEveryOther (bernoullis n))
```

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Functional Programming	WS 2021	LVA 703025
Exercise Sheet 10, 10 points		Deadline: Wednesday, January 12, 2022, 6am

- Mark your completed exercises in the OLAT course of the PS.
- You can start from `template_10.tgz` provided on the proseminar page.
- Your .hs-files should be compilable with ghci and be uploaded in OLAT.

Exercise 1 *Connect Four* **10 p.**

In this exercise we want to extend the implementation of Connect Four from the lecture in various ways. Note that all sub-tasks can be solved independently.

1. The user-interface does not check whether input moves are valid: it is not checked whether the input from the user really is a number, and whether this number is a valid move. Both cases may lead to unintended behavior or crashes of the programs. Therefore, you should modify the user-interface in a way that it repeatedly asks for input until a valid move has been entered, e.g. as follows:

```
Choose one of [0,1,2,3,4,5,6]: five
five is not a valid move, try again: 8
8 is not a valid move, try again: 3
... accept and continue ...
```

(2 points)

2. Modify the user interface so that after a match has been completed, it asks whether another round should be played. If so, the starting player should be switched. Clearly, this also requires a change in the type of `initialState`. (2 points)

3. Extend the implementation so that it can save and load games, e.g., via file `connect4.txt`. The user interface might look like this:

```
Welcome to Connect Four
(n)ew game or (l)oad game: l
... game starts by loading state from connect4.txt ...
Choose one of [0,2,3,5,6] or (s)ave: s
... game is saved in file connect4.txt and program quits ...
```

For the implementation, note that `read . show = id` and that one can automatically derive `Read`-instances in datatype definitions. (2 points)

4. Modify the function `winningPlayer` in the game logic, so that also diagonals are taken into account.

(2 points)

5. Extend the implementation so that it can give hints. To be more precise, the user interface should inform the current player, whenever she can win within 1 or 2 moves by providing a hint. Winning in 2 moves means that after following the move from the hint, you will win the game no matter how the opponent moves in between your moves. In that case the player can type "`h`" to see a first move that leads to success.

```
Choose one of [0,2,3,5,6] or see (h)int to win within 2 moves: h
Hint: Drop a piece in column 2
Choose one of [0,2,3,5,6]: 3
... the game continues since the user is not forced to follow hints
```

(2 points)

Solution 1

```haskell
{- user interface with I/O -}
module Main(main) where

import Text.Read
import System.IO
import Logic

file :: FilePath
file = "connect4.txt"

startPlayer :: Player
startPlayer = 1

main :: IO()
main = do
  hSetBuffering stdout NoBuffering
  putStrLn "Welcome to Connect Four"
  newOrLoad "(n)ew game or (l)oad game: "

newOrLoad :: [Char] -> IO ()
newOrLoad str = do
  putStr str
  answer <- getLine
  if elem answer ["n","new"] then newGame startPlayer
  else if elem answer ["l","load"] then loadGame
  else newOrLoad "unknown answer, please type \"n\" or \"l\": "

loadGame :: IO()
loadGame = do
  stateStr <- readFile file
  let (startPlayer,state) = read stateStr
  game startPlayer state

newGame :: Player -> IO()
newGame startPlayer = game startPlayer (initState startPlayer)

game :: Player -> State -> IO ()
game startPlayer state = do
  putStrLn $ showState state
  case winningPlayer state of
    Just player ->
      do putStrLn $ showPlayer player ++ " wins!"
         anotherRound startPlayer
    Nothing -> let moves = validMoves state in
      if null moves then do
         putStrLn "Game ends in draw."
         anotherRound startPlayer
      else do
        codeMove <- getMove state moves
        case codeMove of
          Right move -> game startPlayer (dropTile state move)
          Left code ->
            if code == 's'
            then saveGame startPlayer state
            else error $ "unknown internal code: " ++ [code]
```

```haskell
saveGame :: Player -> State -> IO ()
saveGame startPlayer state = do
  writeFile file $ show (startPlayer, state)
  putStrLn "Game saved. Good Bye!"

getMove :: State -> [Move] -> IO (Either Char Move)
getMove state moves = let
  hint = winMoves state
  (hintStr, hintStr2) = case hint of
    Nothing -> ("", "I don't have any hints for you")
    Just col -> (", (h)int", "Hint: Drop a piece in column " ++ show col)
 in do
  putStr $ "Choose one of " ++ show moves ++ hintStr ++ " or (s)ave game: "
  moveStr <- getLine
  if moveStr `elem` ["s","save"] then return $ Left 's'
  else if moveStr `elem` ["h", "hint"] then do
    putStrLn hintStr2
    getMove state moves
  else case extractMove moveStr of
    Nothing -> do
      putStrLn $ moveStr ++ " is not a valid move"
      getMove state moves
    Just move -> return (Right move)
  where
    extractMove moveStr = do
      move <- readMaybe moveStr
      if move `elem` moves then return move else Nothing

anotherRound :: Player -> IO ()
anotherRound startPlayer = anotherRoundMain
  "Another round, (y)es or (n)o: "
 where
  anotherRoundMain str = do
    putStr str
    answer <- getLine
    if elem answer ["y", "yes"]
      then newGame (otherPlayer startPlayer)
      else if elem answer ["n", "no"]
        then return ()
        else
          anotherRoundMain
          $  "I don't understand \""
          ++ answer
          ++ "\", please type \"y\" or \"n\": "

{- logic of Connect Four -}

module Logic(State, Move, Player,
  initState, showPlayer, showState, otherPlayer,
  winningPlayer, validMoves, dropTile, winMoves) where

import Data.List
import Data.Maybe


type Tile   = Int   -- 0, 1, or 2
type Player = Int   -- 1 and 2
type Move   = Int   -- column number
```

```haskell
data State = State Player [[Tile]]  deriving (Show,Read)-- list of rows

empty :: Tile
empty = 0

numRows, numCols :: Int
numRows = 6
numCols = 7

initState :: Player -> State
initState startPlayer = State startPlayer
  (replicate numRows (replicate numCols empty))

otherPlayer :: Player -> Player
otherPlayer = (3 -)

dropTile :: State -> Move -> State
dropTile (State player rows) col = State
  (otherPlayer player)
  (reverse $ dropAux $ reverse rows)
    where
      dropAux (row : rows) =
        case splitAt col row of
         (first, i : last) ->
           if i == empty
             then (first ++ player : last) : rows
             else row : dropAux rows

validMoves :: State -> [Move]
validMoves (State _ rows) =
  map fst . filter ((== empty) . snd) . zip [0..] $ head rows

showPlayer :: Player -> String
showPlayer 1 = "X"
showPlayer 2 = "O"

showTile :: Tile -> Char
showTile t = if t == empty then '.' else head $ showPlayer t

showState :: State -> String
showState (State player rows) =
  unlines $ map (head . show) [0 .. numCols - 1] :
    map (map showTile) rows
     ++ ["\nPlayer " ++ showPlayer player ++ " to go"]

transposeRows ([] : _) = []
transposeRows xs = map head xs : transposeRows (map tail xs)

diagonals :: [[Tile]] -> [[Tile]]
diagonals []    = []
diagonals rows@(row : remRows) =
  if length row < winNum || length rows < winNum then
    []
  else
    checkableCs ++ diagonals remRows
  where
    winNum          = 4
    numCheckableCs = numCols - winNum
```

```haskell
    extract          = zipWith (!!) rows
    checkableCs     =
      concatMap (\c -> let ps = map (+ c) [0 .. winNum-1]
                       in [extract ps, extract $ reverse ps])
        [0 .. numCheckableCs]

winningLine :: Player -> [Tile] -> Bool
winningLine player [] = False
winningLine player row = take 4 row == replicate 4 player
  || winningLine player (tail row)

winningPlayer :: State -> Maybe Player
winningPlayer (State player rows) =
  let oplayer = otherPlayer player
      longRows = rows ++ transposeRows rows ++ diagonals rows
    in if any (winningLine oplayer) longRows
      then Just oplayer
      else Nothing

winMove :: State -> Move -> Bool
winMove state = isJust . winningPlayer . dropTile state

winMoves :: State -> Maybe Move
winMoves state = find (checkOnlyWins . dropTile state) moves
 where
  moves = validMoves state
  checkOnlyWins state =
    let opMoves = validMoves state
    in  (isJust $ winningPlayer state)
          || all
               (\m ->
                 let afterOp         = dropTile state m
                     pMovesAfterOp = validMoves afterOp
                 in  (isNothing $ winningPlayer afterOp)
                       && (any (winMove afterOp) pMovesAfterOp)
               )
               opMoves && not (null opMoves)
```

Functional Programming	WS 2021	LVA 703025
Exercise Sheet 11, 10 points		Deadline: Wednesday, January 19, 2022, 6am

- Mark your completed exercises in the OLAT course of the PS.
- You can start from `template_11.hs` provided on the proseminar page.
- Your .hs-file(s) should be compilable with ghci and be uploaded in OLAT.

Exercise 1 *Evaluation Strategies and Kinds of Recursion* **5 p.**

1. Given the four functions:

```
double x = x * 2
square x = x * x
add2times x y = x + double y
func x y = square x + add2times y x
```

   Evaluate each of the following expressions step-by-step under the three evaluation strategies call-by-value, call-by-name, and call-by-need. (3 points)

   (a) `add2times (5+2) 8`

   (b) `double (square 5)`

   (c) `func (2+2) 4`

2. For each of the following functions, specify which kind of recursion they use: (1 point)

   (a)
```
squareList [] = []
squareList (x:xs) = x*x : squareList xs
```

   (b)
```
doubleTimes x 0 = x
doubleTimes x y = doubleTimes (x+x) (y-1)
```

   (c)
```
add2List [] = 0
add2List (x:xs) = x + add2List xs
```

   (d)
```
average :: [Double] -> Double
average xs = aux xs 0 (fromIntegral (length xs))
  where aux [] s c = s / c
        aux (x:xs) s c = aux xs (s+x) c
```

3. Implement two variants of a function that takes a string and produces an upper case version of it: `stringToUpperTail` using tail recursion and `stringToUpperGuarded` using guarded recursion. For example `stringToUpperTail "Hello" = stringToUpperGuarded "Hello" = "HELLO"`. (1 point)

Solution 1

First notice that only for pattern matching the evaluation proceeds from left to right, i.e., evaluating the Haskell expression `let f 0 0 = 0 in f (error "left") (error "right")` leads to the error `"left"` and cannot result in the `"right"` error. By contrast, for the builtin arithmetic functions the argument evaluation is not strictly restricted from left to right, e.g., invoking `error "left" + error "right"` and `error "left" * error "right"` will result in different error messages. Therefore, in the following solution we freely chose to evaluate the arguments of builtin arithmetic functions from right to left. Having chosen left to right is also perfectly fine.

1. (a) `add2times (5+2) 8`

   - call-by-value

```
add2times (5+2) 8 = add2times 7 8
                  = 7 + double 8
                  = 7 + (8 * 2)
                  = 7 + 16
                  = 23
```

   - call-by-name

```
add2times (5+2) 8 = (5 + 2) + double 8
                  = (5 + 2) + (8 * 2)
                  = (5 + 2) + 16
                  = 7 + 16
                  = 23
```

   - call-by-need

     Same solution as call-by-name.

   (b) `double (square 5)`

   - call-by-value

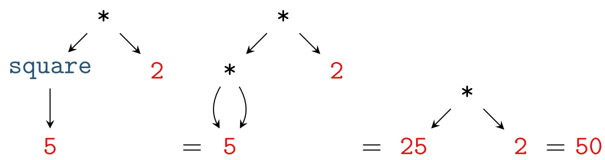
```
double (square 5) = double (5*5)
                  = double 25
                  = 25 * 2
                  = 50
```

   - call-by-name

```
double (square 5) = (square 5) * 2
                  = (5 * 5) * 2
                  = 25 * 2
                  = 50
```

   - call-by-need

`double (square 5)` =



   (c) `func (2+2) 4`

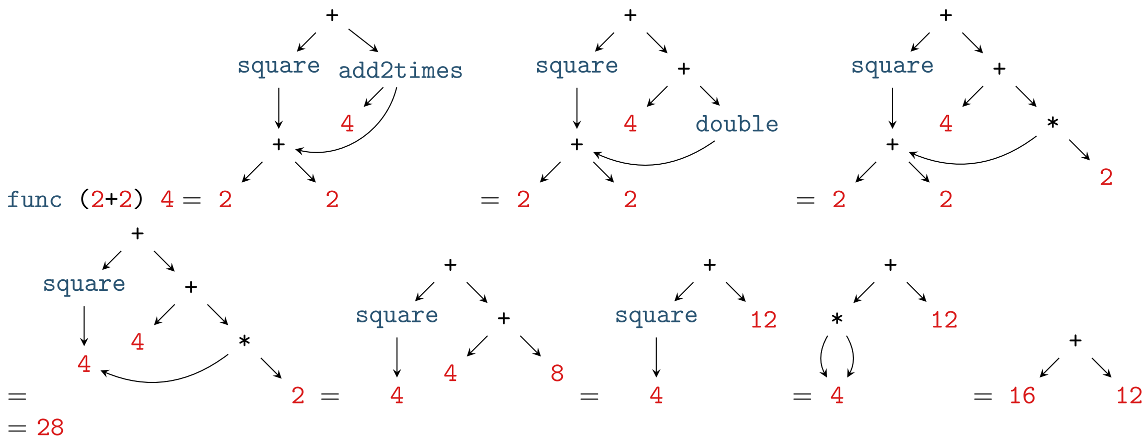
   - call-by-value

```
func (2 + 2) 4 = func 4 4
               = square 4 + add2times 4 4
               = square 4 + (4 + double 4)
               = square 4 + (4 + (4 * 2))
               = square 4 + (4 + 8)
               = square 4 + 12
               = (4 * 4) + 12
               = 16 + 12
               = 28
```

   - call-by-name

```
func (2 + 2) 4 = square (2 + 2) + add2times 4 (2 + 2)
               = square (2 + 2) + (4 + double (2 + 2))
               = square (2 + 2) + (4 + ((2 + 2) * 2))
               = square (2 + 2) + (4 + 4 * 2)
               = square (2 + 2) + (4 + 8)
               = square (2 + 2) + 12
               = ((2 + 2) * (2 + 2)) + 12
               = ((2 + 2) * 4) + 12
               = (4 * 4) + 12
               = 16 + 12
               = 28
```

- call-by-need



2. (a)
```
squareList [] = []
squareList (x:xs) = x*x : squareList xs
```
Guarded Recursion

(b)
```
doubleTimes x 0 = x
doubleTimes x y = doubleTimes (x+x) (y-1)
```
Tail Recursion

(c)
```
add2List [] = 0
add2List (x:xs) = x + add2List xs
```
Linear Recursion (no Tail Recursion)

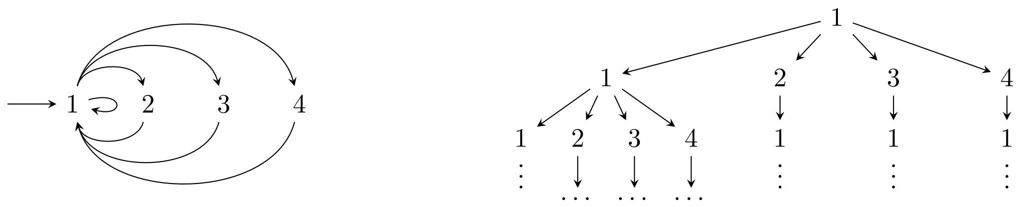
(d)
```
average :: [Double] -> Double
average xs = aux xs 0 (fromIntegral (length xs))
  where aux [] s c = s / c
        aux (x:xs) s c = aux xs (s+x) c
```
Tail Recursion

3.
```
stringToUpperTail :: String -> String
stringToUpperTail = reverse . aux []
  where
    aux acc [] = acc
    aux acc (x:xs) = aux (toUpper x : acc) xs

stringToUpperGuarded :: String -> String
stringToUpperGuarded [] = []
stringToUpperGuarded (x:xs) = toUpper x : stringToUpperGuarded xs
```

Exercise 2 *Lazyness and Infinite Data Structures* **5 p.**

A rooted graph consists of a set of edges between nodes – of the form (source, target) – and additionally has a distinguished node called root. For instance, Figure 1a contains a rooted graph with distinguished node 1 and edges $\{(1,1),(1,2),(1,3),(1,4),(2,1),(3,1),(4,1)\}$.
One way of representing (possibly infinite) rooted graphs is to use (possibly infinite) trees, the so-called *unwinding* of a graph. For example the rooted graph of Figure 1a can be represented by the unwinding shown in Figure 1b.



(a) (b)

Figure 1: A graph and its unwinding

In this exercise graphs and (infinite) trees are represented by the following Haskell type definitions:

```
type Graph a = [(a, a)]
type RootedGraph a = (a, Graph a)
data Tree a = Node a [Tree a] deriving (Eq, Show)
```

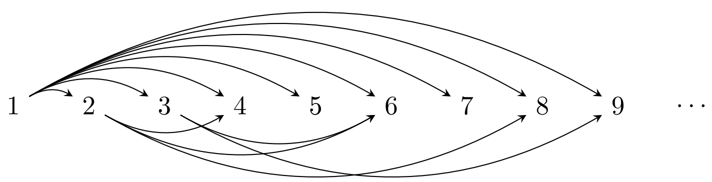
1. Implement a function `unwind :: Eq a => RootedGraph a -> Tree a` that converts a rooted graph into its tree representation. (1 point)

2. Implement a function `prune :: Int -> Tree a -> Tree a` such that `prune n t` results in a pruned tree where only the first `n` layers of the input tree are present. For example invoking `prune 2` on the infinite tree in Figure 1b drops all parts that are depicted by . . . and $\vdots$, and `prune 0` would return a tree that just contains the root node 1.

   Consider the tree that results from unwinding the rooted graph `(z, [(x,z), (z,x), (x,y), (y,x)])`, a figure of eight: $\longrightarrow z \rightleftarrows x \rightleftarrows y$. What is the result of `prune 4` on this tree? (1 point)

3. Implement a function `narrow :: Int -> Tree a -> Tree a` that restricts the number of successors for each node of a tree to a given maximum (by dropping any surplus successors). For example, when calling the function `narrow 1` on the tree $1 \to 2, 3$, the result would be the tree $1 \to 2$. (1 point)

4. Define an infinite tree `mults :: Tree Integer` that represents the graph where every natural number, starting from 1 points to all its multiples: (1 point)



5. Describe the results of evaluating each of the following three expressions: `narrow 4 $ prune 2 mults`, `narrow 1 mults`, and `prune 1 mults`. (1 point)

Solution 2

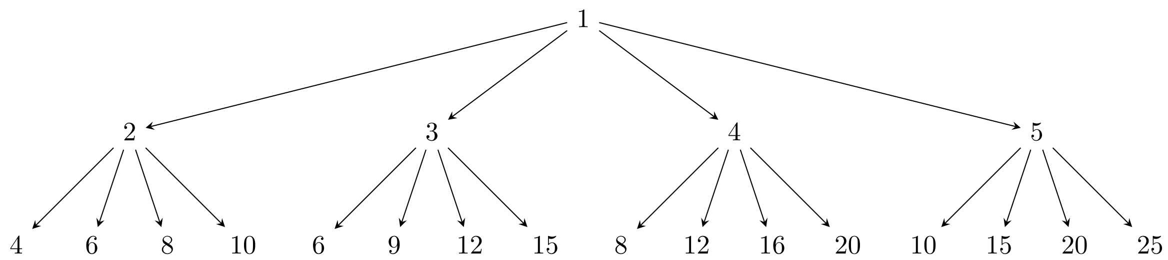
1.
```
unwind :: Eq a => RootedGraph a -> Tree a
unwind (n, g) = Node n (map (\ s -> unwind (s, g)) successors)
  where successors = map snd (filter ( (== n) . fst) g)
```

2. ```haskell
   prune :: Int -> Tree a -> Tree a
   prune 0 (Node x ts) = Node x []
   prune n (Node x ts) = Node x (map (prune (n - 1)) ts)
   ```

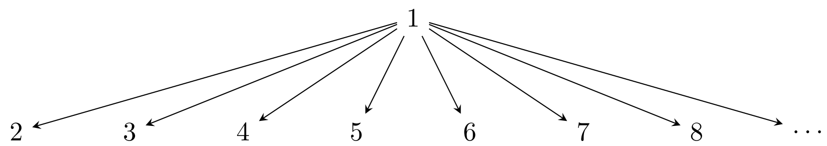
3. ```haskell
   narrow :: Int -> Tree a -> Tree a
   narrow n (Node x ts) = Node x $ map (narrow n) $ take n ts
   ```

4. ```haskell
   mults :: Tree Integer
   mults = go 1
     where go i = Node i $ map go $ map (i*) [2..]
   ```

5.
   - The expression `narrow 4 $ prune 2 mults`

   - The expression `narrow 1 mults` yields an infinite tree that structurally resembles the infinite list of powers of 2: $1 \longrightarrow 2 \longrightarrow 4 \longrightarrow 8 \longrightarrow 16 \rightarrow 32 \rightarrow \cdots$
   - The expression `prune 1 mults` yields the following infinite tree:

     
   ```