General Information: This assignment contains written and/or programming tasks. Combine all the answers to the written tasks in a single PDF document, named `{lastname}-written.pdf`. You can also scan or take pictures of (readable) handwritten papers. JPEG/PNG image files are accepted in this case and they should be named `{exercisenumber}-{lastname}-written.{jpeg/png}`. Make sure that we can follow the manual calculations. Do not combine too many small steps into one.
The programming tasks have to be solved in *Julia* and the source code files have to be submitted using the following naming scheme: `{exercisenumber}-{lastname}.jl`.

(1) (1 point) Determine the limit if there is one for each of the following sequences:

a) (0.25 points)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n^2+45n-15}{\sqrt{36n^4}-16n-32}$$

Solution:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{5n^2+45n-15}{\sqrt{36}\sqrt{n^4}-16n-32} \\
&= \lim_{n\to\infty} \frac{5n^2+45n-15}{6n^2-16n-32} \\
&= \lim_{n\to\infty} \frac{5+45\frac{1}{n}-15\frac{1}{n^2}}{6-16\frac{1}{n}-32\frac{1}{n^2}} \\
&= \frac{5+\lim_{n\to\infty} 45\frac{1}{n}-\lim_{n\to\infty} 15\frac{1}{n^2}}{6-\lim_{n\to\infty} 16\frac{1}{n}-\lim_{n\to\infty} 32\frac{1}{n^2}} \\
&= \frac{5+45\cdot 0-15\cdot 0}{6-16\cdot 0-32\cdot 0} \\
&= \frac{5}{6}
\end{aligned}$$

b) (0.25 points)

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n}\sin(n)$$

Solution:
$\lim_{n\to\infty}\sin(n)$ is indeterminate, but we know that it can only take values from $I=[-1,1]$. Since $\lim_{n\to\infty}\frac{x}{n}=\frac{x}{\infty}$ with $x\in I$, we can conclude that $\lim_{n\to\infty} b_n = 0$

c) (0.25 points)

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \ln\left(2\frac{n^2}{4}\cdot\frac{1}{n^3}\cdot\sqrt{\frac{1}{n}}\right)$$

Solution:

$$\begin{aligned}
c_n &= \ln\left(2\cdot\frac{1}{4}\cdot n^2\cdot\frac{1}{n^3}\cdot\sqrt{\frac{1}{n}}\right) \\
&= \ln\left(\frac{1}{2}\cdot\frac{1}{n}\cdot\sqrt{\frac{1}{n}}\right)
\end{aligned}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$
$$\lim_{u\to 0} \sqrt{u} = 0$$
$$\lim_{n\to\infty} \left(\frac{1}{2} \cdot \frac{1}{n} \cdot \sqrt{\frac{1}{n}}\right) = 0$$
$$\lim_{v\to 0} \ln(v) = -\infty$$

d) (0.25 points) **Hint**: L'Hôpital's rule states that for two differentiable functions f and g where $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ results in an indeterminate form, the following expression holds: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$.

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} n^5 e^{-n}$$

Solution:
According to L'Hôpital's rule:

$$\begin{aligned} &\lim_{n\to\infty} \frac{n^5}{e^n} \\ \iff &\lim_{n\to\infty} \frac{5n^4}{e^n} \\ \iff &\ldots \\ \iff &\lim_{n\to\infty} \frac{120}{e^n} \end{aligned}$$

Since $\lim_{n\to\infty} e^n = \infty$, we can conclude that $\lim_{n\to\infty} d_n = 0$.

(2) (2 points) Do the following series converge?

a) (0.5 points) Use the ratio test to determine whether the following series converges:

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}.$$

Hint: The ratio test utilizes the limit:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Following cases are distinguished:

- if $L < 1$, the series converges,
- if $L > 1$, the series diverges,
- if $L = 1$, n/a.

Solution:

We have $a_n = \frac{2^n}{n!}$ and $a_{n+1} = \frac{2^{n+1}}{(n+1)!}$:

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \right| \\ &= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{2^n \cdot 2}{(n+1)n!} \cdot \frac{n!}{2^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{2}{(n+1)\cdot 1} \cdot \frac{1}{1} \right| \\ &= \lim_{n\to\infty} \left| \frac{2}{(n+1)} \right| = 0 \end{aligned}$$

Since $L < 1$, the series converges.

b) (0.5 points) Use the integral test to determine whether the following series converges:

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)^2}.$$

Hint: According to the integral test, a series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the integral $\lim_{a\to\infty} \int_1^a f(n)\mathrm{d}n$ converges.

Solution:
We have $f(n) = \frac{1}{(n+2)^2}$:

$$\begin{aligned} &\lim_{a\to\infty} \int_1^a \frac{1}{(n+2)^2}\mathrm{d}n \\ =&\lim_{a\to\infty} \int_3^{a+2} \frac{1}{u^2}\mathrm{d}u \quad \text{substitution: } u = n+2 \text{ and } \mathrm{d}u = \mathrm{d}n \\ =&\lim_{a\to\infty} \int_3^{a+2} u^{-2}\mathrm{d}u \quad \text{substitution: } u = n+2 \text{ and } \mathrm{d}u = \mathrm{d}n \\ =&\lim_{a\to\infty} -\frac{1}{u}\Big|_3^{a+2} \\ =&\lim_{a\to\infty} -\frac{1}{a+2} - (-\frac{1}{3}) \\ =&-\frac{1}{\infty+2} + \frac{1}{3} \\ =&0 + \frac{1}{3} \end{aligned}$$

The series converges.

c) (0.5 points) Use the direct comparison test to determine whether the following series converges:

$$\sum_{n=1}^{\infty} \frac{n^2}{n^4+n+4}.$$

Hint: The direct comparison test states that if the infinite series $\sum_{n=0}^{\infty} b_n$ converges and $0 \le a_n \le b_n$, then the infinite series $\sum_{n=0}^{\infty} a_n$ converges as well.

Solution:
We have $a_n = \sum_{n=1}^{\infty} \frac{n^2}{n^4+n+4}$ and let $b_n = \sum_{n=1}^{\infty} \frac{n^2}{n^4}$. The condition $0 \leq a_n \leq b_n$ is satisfied. After simplification, we can see that $b_n = \frac{1}{n^2}$ is a p-series with $p > 1$ for which we know that it converges. According to the direct comparison test a_n converges as well.

d) (0.5 points) Use the alternating series test to determine whether the following series converges:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+3}.$$

Hint: The alternating series test states that a series which can be rewritten as $\sum_{n=1}^{\infty}(-1)^n a_n$ converges if the following conditions are met:

i. $|a_n|$ decreases monotonically (check if $|a_{n+1}| \leq |a_n|$)
ii. $\lim_{n\to\infty} a_n = 0$

Solution:
Rewriting the series:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{2n+3} &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n+3} \\ &= \sum_{n=1}^{\infty}(-1)^{n}(-1)\frac{1}{2n+3} \\ &= \sum_{n=1}^{\infty}(-1)^{n} \cdot \frac{-1}{2n+3}.\end{aligned}$$

We have $a_n = \frac{-1}{2n+3}$ and check whether the conditions hold:

i. The sequence $|a_n|$ decreases since $\frac{1}{2n+5} \leq \frac{1}{2n+3}$.
ii. $\lim_{n\to\infty} \frac{-1}{2n+3} = 0$ also holds.

Since both conditions are met, the series converges.

(3) (2 points) A differentiable function $f(x)$ can be approximated at a point a using the *Taylor series*:

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots, = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

with $f^{(k)}(a)$ the k-th derivative of f at the point a.

a) (1.0 point) Write the sum of the first 8 terms of the Taylor series of the function $f(x) = \sin(x)$ at the point $a = 0$. Show that this series can be written with the following formula:

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}. \tag{1}$$

Solution:

The sum of the 8 first terms is:

$$\sin(0) + \frac{\sin'(0)}{1!}(x-0) + \frac{\sin''(0)}{2!}(x-0)^2 + \frac{\sin'''(0)}{3!}(x-0)^3$$
$$+ \frac{\sin^{(4)}(0)}{4!}(x-0)^4 + \cdots + \frac{\sin^{(7)}(0)}{7!}(x-0)^7 \,.$$

The nth derivatives of $\sin(0)$ for $n = 0, \ldots, 7$ are:

$$\begin{aligned}
&\sin(0) = 0\,, \\
&\sin'(0) = 1\,, \\
&\sin''(0) = 0\,, \\
&\sin'''(0) = -1\,, \\
&\sin^{(4)}(0) = 0\,, \\
&\sin^{(5)}(0) = 1\,, \\
&\sin^{(6)}(0) = 0\,, \\
&\sin^{(7)}(0) = -1\,,
\end{aligned}$$

Therefore we can simplify:

$$\begin{aligned}
&0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 - \frac{1}{7!}x^7 \\
&= \frac{x}{1!} - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \\
&= \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\
&= \sum_{k=1}^{7} \frac{x^{k+k-1} \cdot (-1)^{k-1}}{(k+k-1)!} \\
&= \sum_{k=0}^{6} \frac{x^{(k+1)+(k+1)-1} \cdot (-1)^{(k+1)-1}}{((k+1)+(k+1)-1)!} \\
&= \sum_{k=0}^{6} \frac{x^{2k+1} \cdot (-1)^k}{(2k+1)!}
\end{aligned}$$

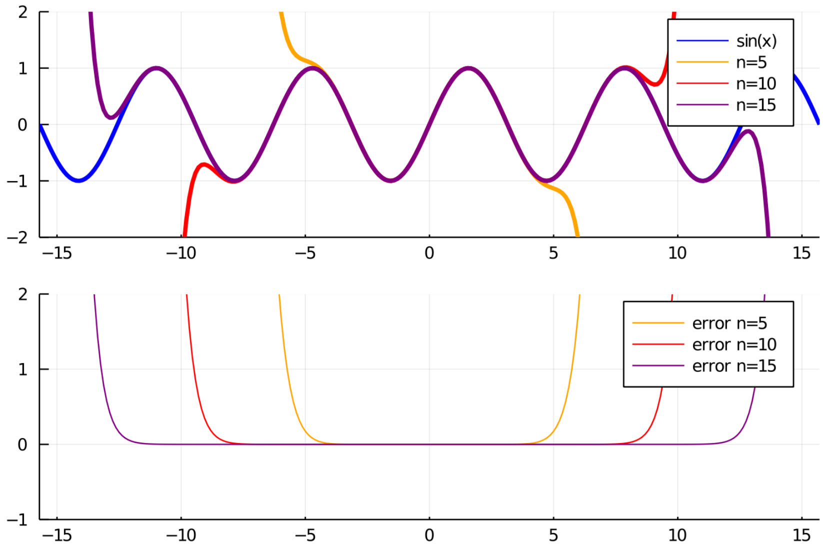

Figure 1: Plot of function $\sin(x)$ (blue) and its Taylor series approximation for different values of n (top). Absolute error between $\sin(x)$ and the Taylor series (bottom).

b) (0.5 points) Using Julia, implement the formula and plot the Taylor series for $n = 5$, $n = 10$, and $n = 15$ (see Figure below). Use the template `sine.jl`.

Solution:
see `sine_solution.jl`

c) (0.5 points) For each value of n, plot the absolute error between the Taylor series approximation and the real function $\sin(x)$. Use the template `sine.jl`.

Solution:
see `sine_solution.jl`

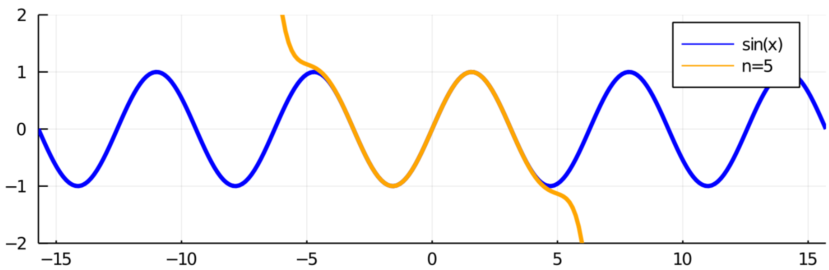

Figure 2: Plot of function $\sin(x)$ (blue) and its Taylor series approximation for $n = 5$.

(4) (2 points) A periodic piecewise continuous function f on the interval $[-\pi, \pi]$ has a *Fourier Series Representation*:

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k sin(kx),$$

with the following coefficients:

$$
\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\mathrm{d}x, \quad k \geq 0 \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\mathrm{d}x, \quad k \geq 1.
\end{aligned}
$$

Compute the terms for $k \in \{1, 3, 5\}$ of the Fourier Series representation for $f(x)$ with the period 2π:

$$
f(x) = \begin{cases} 1 & -\pi < x < 0 \\ -1 & 0 < x < \pi \end{cases} .
$$

Copy your resulting a_0, a_k, b_k into the respective place in the `fourier.jl` file and compare your result to Figure 3.

Solution:
Compute a_k:

$$
\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\mathrm{d}x, \quad k \geq 0 \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos(kx)\mathrm{d}x + \int_{0}^{\pi} f(x)\cos(kx)\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} 1\cos(kx)\mathrm{d}x + \int_{0}^{\pi} (-1)\cos(kx)\mathrm{d}x\right] \\
&= -\frac{2}{\pi}\left[\int_{0}^{\pi} \cos(kx)\mathrm{d}x\right] \\
&= -\frac{2}{\pi}\left[-\frac{1}{k}\sin(kx)\right]_0^{\pi} \\
&= -\frac{2}{\pi}\left[-\frac{1}{k}\sin(k\pi) - (-\frac{1}{k}\sin(k\cdot 0))\right] \\
&= -\frac{2}{\pi}\left[-\frac{1}{k}\sin(k\pi) + \frac{1}{k}\sin(0)\right] \\
&= -\frac{2}{\pi}\left[-\frac{1}{k}0 + \frac{1}{k}0\right] = 0
\end{aligned}
$$

Factor a_k evaluates to 0 for every possible k.

Compute b_k:

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\mathrm{d}x, \quad k \geq 1 \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin(kx)\mathrm{d}x + \int_{0}^{\pi} f(x)\sin(kx)\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} 1\sin(kx)\mathrm{d}x + \int_{0}^{\pi} (-1)\sin(kx)\mathrm{d}x\right] \\
&= -\frac{2}{\pi}\left[\int_{0}^{\pi} \sin(kx)\mathrm{d}x\right] \\
&= -\frac{2}{\pi}\left[-\frac{1}{k}\cos(kx)\right]_0^{\pi} \\
&= -\frac{2}{\pi}\left[-\frac{1}{k}\cos(k\pi) - (-\frac{1}{k}\cos(k\cdot 0))\right] \\
&= -\frac{2}{\pi}\left[-\frac{1}{k}\cos(k\pi) + \frac{1}{k}\right] \\
&= -\frac{2}{k\pi}\left[-\cos(k\pi) + 1)\right].
\end{aligned}$$

Plugging in each k gives: $b_1 = -\frac{4}{\pi}, b_3 = -\frac{4}{3\pi}, b_5 = -\frac{4}{5\pi}$.

Construct the series:

$$0 + 0\cdot\cos(x) + \frac{-4}{\pi}\sin(x) + 0\cdot\cos(3x) + \frac{-4}{3\pi}\sin(3x) + 0\cdot\cos(5x) + \frac{-4}{5\pi}\sin(5x).$$

Additionally, see `fourier_solution.jl`.

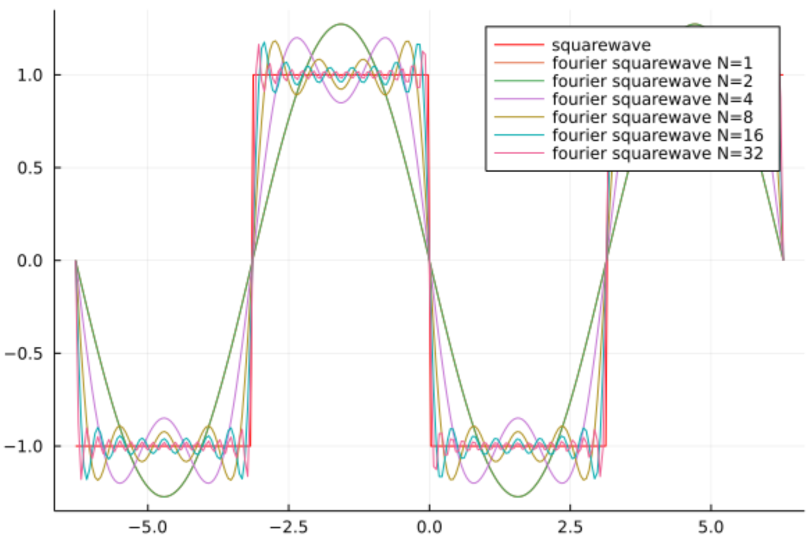

Figure 3: Plot of square wave function and (N-th) partial sums of the corresponding Fourier series.