General Information: This assignment contains written and/or programming tasks. Combine all the answers to the written tasks in a single PDF document, named `{lastname}-written.pdf`. You can also scan or take pictures of (readable) handwritten papers. JPEG/PNG image files are accepted in this case and they should be named `{exercisenumber}-{lastname}-written.{jpeg/png}`. Make sure that we can follow the manual calculations. Do not combine too many small steps into one. The programming tasks have to be solved in *Julia* and the source code files have to be submitted using the following naming scheme: `{exercisenumber}-{lastname}.jl`.

(1) (3 points) For $z \in \mathbb{C}$, we have:

$$z = a + \mathrm{i} \cdot b = r(\cos(\varphi) + \mathrm{i} \cdot \sin(\varphi)) = r \cdot e^{\mathrm{i} \cdot \varphi},$$

with $a, b \in \mathbb{R}$ and $|z| = r = \sqrt{a^2 + b^2}$.

a) (1.2 points) Given the complex numbers $x = (\cos(0) + \mathrm{i}\sin(0))$, $y = 5e^{\mathrm{i}0.64}$, and $z = -5 + 2\mathrm{i}$, state the form they are in and determine the missing forms. Think about the meaning of the components to find a formula for φ and make use of sketches to visualize and support your arguments.

Solution:

φ is the angle from the origin to r. We have several options to compute φ, using trigonometric functions, for example:

$$\varphi = \begin{cases} \arctan(\frac{b}{a}), & a > 0 \\ \arctan(\frac{b}{a}) + \pi, & a < 0 \\ \frac{1}{2}\pi, & a = 0,\ b > 0 \\ -\frac{1}{2}\pi, & a = 0,\ b < 0 \\ 0, & a = 0,\ b = 0. \end{cases}$$

For x, we have:

$$r = 1, \quad \varphi = 0$$

$$\begin{aligned} x_{\text{Polar}} &= (\cos(0) + \mathrm{i}\sin(0)) \\ x_{\text{Euler}} &= e^{\mathrm{i}0} \\ x_{\text{Cartesian}} &= (1 + \mathrm{i}0) = 1. \end{aligned}$$

For y, we have:

$$r = 5, \quad \varphi = 0.64$$

$$\begin{aligned} y_{\text{Euler}} &= 5e^{\mathrm{i}0.64} \\ y_{\text{Polar}} &= 5(\cos(0.64) + \mathrm{i}\sin(0.64)) \\ y_{\text{Cartesian}} &\approx 4 + \mathrm{i}3 \end{aligned}$$

For z, we have:

$$r = \sqrt{(-5)^2 + 3^2} \approx 5.39, \quad \varphi = \arctan(\frac{2}{-5}) + \pi \approx 2.76$$

$$
\begin{aligned}
z_{\text{Cartesian}} &= -5 + \mathrm{i}2 \\
z_{\text{Euler}} &= 5.39e^{\mathrm{i}2.76} \\
z_{\text{Polar}} &= 5.39(\cos(2.76) + \mathrm{i}\sin(2.76))
\end{aligned}
$$

b) (0.2 points) Compute the sums $b_1 = x + z$, $b_2 = y + z$.

Solution:

$$
\begin{aligned}
b_1 &= x + z = -4 + \mathrm{i}2 \\
b_2 &= y + z = -1 + \mathrm{i}5.
\end{aligned}
$$

c) (0.2 points) The products $c_1 = x \cdot z$, $c_2 = y \cdot z$.

Solution:

$$
\begin{aligned}
c_1 &= x \cdot z = z \\
c_2 &= y \cdot z = (4 \cdot (-5) - 3 \cdot 2) + \mathrm{i}(4 \cdot 2 + 3 \cdot (-5)) = -26 - \mathrm{i}7.
\end{aligned}
$$

Or using polar forms here:

$$
c_2 = 5e^{\mathrm{i}0.64} \cdot 5.39e^{\mathrm{i}2.76} = 5 \cdot 5.39e^{\mathrm{i}(0.64+2.76)}
$$

d) (0.8 points) Complex division $\frac{a}{b}$ can be achieved by taking the complex conjugate of the denominator and multiplying it with the numerator and denominator $\frac{a\bar{b}}{b\bar{b}}$. The complex conjugate of a complex number $\bar{b}$ is achieved by swapping the sign of the imaginary component. Compute:

$$
d_1 = \frac{y}{x},\ d_2 = \frac{x}{y},\ d_3 = \frac{z}{y},\ d_4 = \frac{y}{z}
$$

Solution: Complex conjugates:

$$
\begin{aligned}
\bar{x} &= 1 \\
\bar{y} &= 4 - \mathrm{i}3 \\
\bar{z} &= -5 - \mathrm{i}2.
\end{aligned}
$$

Division:

$$
\begin{aligned}
d_1 &= \frac{y}{x} = \frac{y}{1} = y \\
d_2 &= \frac{x}{y} = \frac{1\bar{y}}{y\bar{y}} = \frac{\bar{y}}{16 - \mathrm{i}12 + \mathrm{i}12 - \mathrm{i}^2 9} = \frac{4 - \mathrm{i}3}{25} \\
d_3 &= \frac{z}{y} = \frac{z\bar{y}}{y\bar{y}} = \frac{(-5 + \mathrm{i}2)(4 - \mathrm{i}3)}{16 - \mathrm{i}12 + \mathrm{i}12 - \mathrm{i}^2 9} = \frac{-14 + \mathrm{i}23}{25} \\
d_4 &= \frac{y}{z} = \frac{y\bar{z}}{z\bar{z}} = \frac{(4 + \mathrm{i}3)(-5 - \mathrm{i}2)}{(-5 + \mathrm{i}2)(-5 - \mathrm{i}2)} = \frac{-20 - \mathrm{i}8 - \mathrm{i}15 - \mathrm{i}^2 6}{29} = \frac{-14 - \mathrm{i}23}{29}
\end{aligned}
$$

Or using polar forms with complex numbers x and y:

$$
\frac{x}{y} = \frac{r_x}{r_y} e^{\mathrm{i}(\varphi_x - \varphi_y)}
$$

e) (0.2 points) The norm of a complex number $z = a + ib$ is defined as $|z| = \sqrt{a^2 + b^2}$. Compute the norms for y, and z.

Solution:

$$e_1 = |y| = \sqrt{4^2 + 3^2} = 5$$
$$e_2 = |z| = \sqrt{-5^2 + 2^2} = \sqrt{29}$$

f) (0.4 points) The reciprocal of a complex number is denoted as follows:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

Compute the reciprocal for y, and z.

Solution:

$$f_1 = \frac{1}{y} = \frac{\bar{y}}{|y|^2} = \frac{4 - \mathrm{i}3}{25}$$
$$f_2 = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{-5 - \mathrm{i}2}{29}$$

(2) (2 points) Use a Taylor Series to derive Euler's formula. Given the Taylor Series approximation for $x \in \mathbb{R}$:

$$e^x \approx \sum_{k=0}^{N} \frac{x^k}{k!},$$

can be extended as follows:

$$e^{\mathrm{i}x} \approx \sum_{k=0}^{N} \frac{(\mathrm{i}x)^k}{k!}.$$

Show that $e^{\mathrm{i}x} = \cos(x) + \mathrm{i} \cdot \sin(x)$.

Hint: You might want to have a look at the solutions on how to approximate $\sin(x)$ for the exercise sheet about series and sequences. Derive a respective approximation for $\cos(x)$.

Solution:

For $N = \infty$ the approximation becomes an exact solution:

$$e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!}$$

$$\begin{aligned}
e^{ix} &= \frac{(ix)^0}{0!} + \frac{(ix)^1}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + ... \\
&= \frac{1}{1} + \frac{ix}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + ... \\
&= 1 + ix + \frac{(-1)x^2}{2!} + \frac{(-i)x^3}{3!} + \frac{(1)x^4}{4!} + \frac{(i)x^5}{5!} + ... \\
&= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + ... \\
&= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...) \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... \\
e^{ix} &= \cos(x) + i\sin(x)
\end{aligned}$$

(3) (2 points) We are revisiting the Fourier series representation of a square wave function we constructed in the sequences and series exercise. Our goal is to convert the Real series into a Complex series. We had:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) = \sum_{k=-\infty}^{\infty} c_n e^{ikx}$$

with

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\mathrm{dx}, \quad k \geq 0 \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\mathrm{dx}, \quad k \geq 1,
\end{aligned}$$

as coefficients for the real version of the series and similarly we have

$$c_k = \begin{cases} a_0 & , k = 0 \\ \frac{(a_k - \mathrm{i}b_k)}{2} & , k > 0 \\ \frac{(a_{|k|} + \mathrm{i}b_{|k|})}{2} & , k < 0 \end{cases}$$

as coefficient for the complex version of the series.

a) (1 point) Compute c_k by plugging in a_k and b_k.

Solution: $a_k = 0$ for any k and $b_k = -\frac{2}{k\pi}(-\cos(k\pi) + 1)$:

$$c_k = \begin{cases} 0 & , k = 0 \\ \frac{(0 - \mathrm{i}(-\frac{2}{k\pi}(-\cos(k\pi)+1)))}{2} & , k > 0 \\ \frac{(0 + \mathrm{i}(-\frac{2}{|k|\pi}(-\cos(|k|\pi)+1)))}{2} & , k < 0 \end{cases}$$

$b_k = -\frac{2}{k\pi}(-\cos(k\pi)+1)$ where $-\cos(k\pi)+1$ evaluates to 0 for even and 2 for odd k:

$$c_k = \begin{cases} 0 & , k \text{ is even} \\ \frac{-\mathrm{i}(-\frac{2}{k\pi}(2))}{2} & , k > 0,\ k \text{ is odd} \\ \frac{+\mathrm{i}(-\frac{2}{|k|\pi}(2))}{2} & , k < 0,\ k \text{ is odd} \end{cases}$$

after simplification we end up with:

$$c_k = \begin{cases} 0 & , k \text{ is even} \\ \mathrm{i}\frac{2}{k\pi} & , k > 0,\ k \text{ is odd} \\ \mathrm{i}\frac{-2}{|k|\pi} & , k < 0.\ k \text{ is odd.} \end{cases}$$

b) Implement the complex version of the square wave Fourier series representation, using `fourier.jl`.

Solution: see `fourier_solution.jl`.

You might want to have a look at how to operate complex numbers with Julia[1].

For the sake of completeness, you can check whether your result resembles the solution of the sequences and series exercise sheet just like Figure 1.

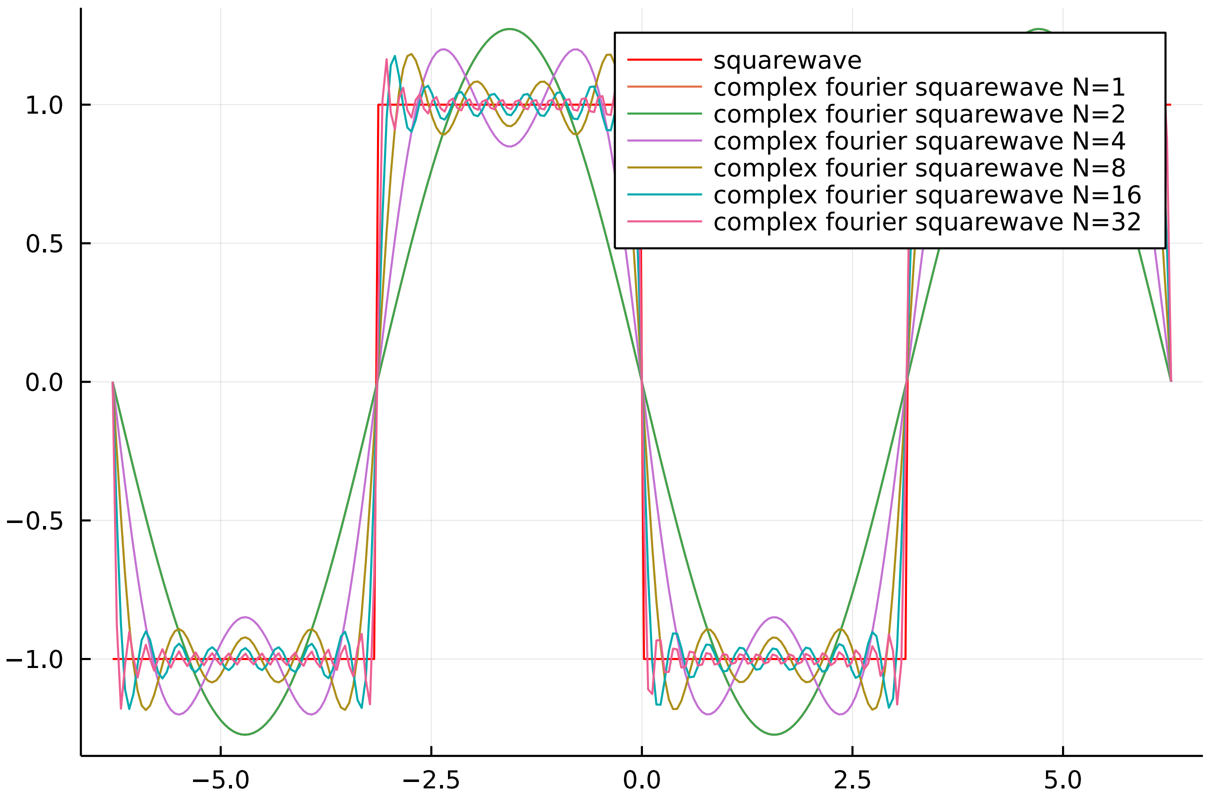

Figure 1: Complex Fourier series representation of a square wave function.

[1] https://docs.julialang.org/en/v1/manual/complex-and-rational-numbers/