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Formelsammlung für die schriftliche Prüfung Angewandte Mathematik in der Informatik

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1 Einführung

1.1 Funktionen $f : \mathbb{R} \to \mathbb{R}$

$$
\begin{aligned}
f(x) &= d && \text{(konstante Funktion)}\\
f(x) &= kx + d && \text{(explizite lineare F.)}\\
F(x, y) &= \mathbf{nx} + d = 0 && \text{(implizite lineare F.)}\\
f(x) &= ax^k && \text{(Potenzfunktion)}\\
f(x) &= \sqrt[k]{x} = x^{1/k} && \text{(Wurzelfunktion)}\\
f(f^{-1}(x)) &= x = f^{-1}(f(x)) && \text{(Umkehrfunktion)}
\end{aligned}
$$

1.2 Eigenschaften

$$
\begin{aligned}
f(x) &= f(-x) && \text{(gerade Funktion)}\\
f(x) &= -f(-x) && \text{(ungerade Funktion)}
\end{aligned}
$$

1.3 Polynome

$$
\begin{aligned}
p(x) &= \sum_{i=0}^{n} a_i x^i && \text{(Polynomfunktion)}\\
r(x) &= \frac{p_1(x)}{p_2(x)} && \text{(Rationale Funktion)}
\end{aligned}
$$

Faktorisierung

$$p(x) = (x - x_k)^{m_k} \; \ldots \; (x - x_1)^{m_1} (x - x_0)^{m_0} \cdot q(x)$$

Nullstellen quadratisches Polynom

$$
\begin{aligned}
a\,x^2 + b\,x + c &= 0\\
x_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

1.4 Trigonometrische Funktionen

Additionstheoreme

$$
\begin{aligned}
\sin(x \pm y) &= \sin x \cdot \cos y \pm \cos x \cdot \sin y\\
\cos(x \pm y) &= \cos x \cdot \cos y \mp \sin x \cdot \sin y\\
\tan(x \pm y) &= \frac{\sin(x \pm y)}{\cos(x \pm y)} = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}\\
\cot(x \pm y) &= \frac{\cos(x \pm y)}{\sin(x \pm y)} =
\end{aligned}
$$

Hyperbolische Funktionen

$$
\begin{aligned}
\sinh(x) &= \frac{e^x - e^{-x}}{2}\\
\cosh(x) &= \frac{e^x + e^{-x}}{2}\\
\tanh(x) &= \frac{\sinh(x)}{\cosh(x)}
\end{aligned}
$$

1.5 Vektoroperationen

$$
\begin{aligned}
&\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \cdots + v_n w_n && \text{(Inneres Produkt)}\\
&||\mathbf{v}||_2 = \sqrt{\mathbf{v} \cdot \mathbf{v}} && \text{(Betrag, Länge)}\\
&\hat{\mathbf{v}} = \mathrm{nrm}(\mathbf{v}) = \frac{\mathbf{v}}{||\mathbf{v}||_2} && \text{(Einheitsvektor)}\\
&\varphi = \angle(\mathbf{v}, \mathbf{w}) \text{ mit } 0 \le \varphi \le \pi &&\\
&\cos\varphi = \mathrm{nrm}(\mathbf{v}) \cdot \mathrm{nrm}(\mathbf{w}) && \text{(Winkel 1)}\\
&\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||_2 \, ||\mathbf{w}||_2 \, \cos\varphi && \text{(Winkel 2)}
\end{aligned}
$$

1.6 Matrixoperationen

$$
\begin{aligned}
&\mathbf{A} = a_{ij} &&\\
&\mathbf{A}^\top = a_{ji} && \text{(Transponierte)}\\
&\mathbf{I} = \delta_{ij} = \mathrm{diag}(1, 1, \ldots, 1) && \text{(Einheitsmatrix)}\\
&\mathbf{AI} = \mathbf{IA} = \mathbf{A} &&\\
&\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} && \text{(Inverse)}
\end{aligned}
$$

2 Differentialrechnung

2.1 Erste Ableitung

$$x(t)' = \frac{dx}{dt} = \frac{d}{dt}x = \dot{x} = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$

2.2 Ableitungsregeln

$$
\begin{aligned}
(a)' &= 0 && \text{(Konstante)}\\
(x^k)' &= kx^{k-1} && \text{(Potenz)}\\
(af \pm bg)' &= af' \pm bg' && \text{(Summenregel)}\\
(f \cdot g)' &= f' \cdot g + f \cdot g' && \text{(Produktregel)}\\
\left(\frac{f}{g}\right)' &= \frac{f' \cdot g - f \cdot g'}{g^2} && \text{(Quotientenregel)}
\end{aligned}
$$

$$(f \circ g)' = f'(g) \cdot g' \qquad \text{(Kettenregel)}$$
$$(\sin x)' = \cos x \qquad \text{(Kreisfunktionen 1)}$$
$$(\cos x)' = -\sin x \qquad \text{(Kreisfunktionen 2)}$$
$$(\sinh x)' = \cosh(x) \qquad \text{(Hyperbelfunktionen 1)}$$
$$(\cosh x)' = \sinh(x) \qquad \text{(Hyperbelfunktionen 2)}$$
$$(a^x)' = \ln a \cdot a^x \qquad \text{(Exponential 1)}$$
$$(e^x)' = e^x \qquad \text{(Exponential 2)}$$
$$(e^{f(x)})' = e^{f(x)} f'(x) \qquad \text{(Exponential 3)}$$
$$(\ln(f(x)))' = \frac{f'(x)}{f(x)} \qquad \text{(Logarithmus 1)}$$
$$(\ln(x))' = \frac{1}{x} \qquad \text{(Logarithmus 2)}$$

2.3 Newton Verfahren (siehe 7.2)

2.4 Zweite Ableitung

$$x(t)'' = (x')' = \frac{d}{dt}\frac{d}{dt}(x) = \frac{d^2}{dt^2}x = \ddot{x} = x^{(2)}$$

2.5 Partielle Ableitung

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} f(x,y) = f_x = \\ &= \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} f(x,y) = f_y = \\ &= \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}\end{aligned}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2}{\partial x^2} f = \frac{\partial}{\partial x}\frac{\partial}{\partial x} f = f_{xx}$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2}{\partial y^2} f = \frac{\partial}{\partial y}\frac{\partial}{\partial y} f = f_{yy}$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y} f = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f = f_{xy}$$

2.6 Gradient

$$\nabla f = \operatorname{grad} f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^{\mathsf{T}} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)^{\mathsf{T}}$$

2.7 Sätze zu Funktionen

Stetigkeit

$$\begin{aligned}&f : D \to \mathbb{R},\ D \subset \mathbb{R}, f \text{ ist stetig in } x_0 \\ &\iff \lim_{x \to x_0} f(x) = f(x_0)\end{aligned}$$

Zwischenwertsatz

$$\begin{aligned}&f : [a,b] \to \mathbb{R} \text{ stetig} \\ &\wedge (f(a) < C < f(b) \vee f(a) > C > f(b)) \\ &\Rightarrow \exists\ c \in\ ]a,b[: f(c) = C\end{aligned}$$

Satz von Rolle

$$\begin{aligned}&f : [a,b] \to \mathbb{R} \text{ stetig} \\ &\wedge \text{ in }]a,b[\text{ differenzierbar} \wedge f(a) = f(b) \\ &\Rightarrow \exists\ x_0 \in\ ]a,b[: f'(x_0) = 0\end{aligned}$$

Mittelwertsatz der Differentialrechnung

$$\begin{aligned}&f : [a,b] \to \mathbb{R} \text{ stetig} \\ &\wedge \text{ in }]a,b[\text{ differenzierbar} \\ &\Rightarrow \exists\ x_0 \in\ ]a,b[: f'(x_0) = \frac{f(b) - f(a)}{b - a}\end{aligned}$$

Monotonie

$$\begin{aligned}&f : [a,b] \to \mathbb{R} \text{ stetig} \\ &\wedge \text{ in }]a,b[\text{ differenzierbar} \to \\ &\begin{cases} \forall x \in\ ]a,b[: f'(x) \geq 0, & \text{monoton steigend} \\ \forall x \in\ ]a,b[: f'(x) \leq 0, & \text{monoton fallend} \\ \forall x \in\ ]a,b[: f'(x) = 0, & \text{konstant} \end{cases}\end{aligned}$$

3 Integralrechnung

3.1 Definition

$$\lim_{n \to \infty, \Delta x_k \to 0} \sum_{k=0}^{n-1} f(\xi_k)\Delta_k = \int_a^b f(x)\ dx$$

3.2 Eigenschaften

$$a, b \in \mathbb{R} \wedge a < b \quad f, g : [a,b] \to \mathbb{R} \text{ integrierbar}$$
$$\text{für } f(x) \geq 0 : \int_a^b f(x)dx \geq 0$$
$$\text{für } f(x) \leq 0 : \int_a^b f(x)dx \leq 0$$
$$\text{für } f(x) \leq g(x) : \int_a^b f(x)dx \leq \int_a^b g(x)dx$$
$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$
$$\int_a^a f(x)dx = 0$$
$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

3.3 Mittelwertsatz der Integralrechnung

$$f : [a,b] \to \mathbb{R} \text{ stetig} \Rightarrow \exists\ \xi \in [a,b] \text{ sodass}$$
$$\int_a^b f(x)dx = f(\xi)(b-a)$$

Mittelwert einer Funktion

$$\bar{f} = \tfrac{1}{b-a} \int_a^b f(x)dx$$

Quadratisches Mittel

$$\bar{f} = \sqrt{\tfrac{1}{b-a} \int_a^b (f(x))^2 dx}$$

3.4 Hauptsatz Different.-/Integralrechnung

$$f : [a,b] \to \mathbb{R} \text{ stetig} \wedge \lambda \in [a,b] \Rightarrow$$
$$F_\lambda(x) = \int_\lambda^x f(t)dt \qquad \text{(Stammfunktion von } f\text{)}$$
$$F'_\lambda(x) = f(x)$$

Bestimmtes Integral

$$\int_a^b f(x)dx = \big[F(x)\big]_a^b = F(x)\Big|_a^b = F(b) - F(a)$$

Unbestimmtes Integral

$$\int f(x)dx = F(x) + C$$

3.5 Elementare Integrationsregeln

$$a,b,\lambda_1,k,C \in \mathbb{R} \qquad f,g: \ [a,b] \to \mathbb{R} \text{ integrierbar}$$
$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$\int_a^b \lambda_1 f(x)dx = \lambda_1 \int_a^b f(x)dx$	(Linearität)
$\int kdx = kx + C$	(Konstante)
$\int_a^b dx = \int_a^b 1 \cdot dx = x\Big\vert_a^b = b - a$	(Bereich)
$\int x^k dx = \frac{x^{k+1}}{k+1} + C$	(Potenz)
$\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{x^{3/2}}{3/2} + C$	(Wurzel)
$\int e^{kx}dx = \frac{1}{k}e^{kx} + C$	(e)
$\int \frac{1}{x}dx = \ln\vert x\vert + C$	(Kehrwert)
$\int \sin xdx = -\cos x + C$	(Sinus)
$\int \cos xdx = \sin x + C$	(Cosinus)
$\int \sinh xdx = \cosh x + C$	(Sinus Hyp.)
$\int \cosh xdx = \sinh x + C$	(Cosinus Hyp.)
$\int \ln xdx = x \ \ln x - x + C$	(ln)

3.6 Partielle Integration

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx + C$$

3.7 Integration mit Substitution

$$\int_a^b f\big(g(t)\big)g'(t)dt = \int_{g(a)}^{g(b)} f(x)dx = F(x)\Big|_{g(a)}^{g(b)}$$

3.8 Integration 2D

$$V = \iint_D f(x,y)dD = \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x,y) \ dxdy$$

Normalbereich

$$\text{mit } D = \{(x,y) : a \le x \le b, g(x) \le y \le h(x)\}$$

$$\iint_D f(x,y)dxdy = \int_{x=a}^{b}\Big(\int_{y=g(x)}^{h(x)} f(x,y) \ dx\Big)dy$$

3.9 Integration 3D

$$\iiint_\Omega f(x,y,z)dxdydz = \int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x,y,z) \ dxdydz$$

Normalbereich

$$\text{mit } \Omega = \{(x,y,z): \ a_1 \le x_1 \le b_1, \ a_2(x_1) \le x_2 \le b_2(x), \ a_3(x_1,x_2) \le x_3 \le b_3(x_1,x_2)\}$$

$$\iiint_\Omega f(x_1,x_2,x_3)dx_3dx_2dx_1 =$$

$$\int_{x_1=a_1}^{b_1}\int_{x_2=a_2(x_1)}^{b_2(x_1)}\int_{x_3=a_3(x_1,x_2)}^{b_3(x_1,x_2)} f(x_1,x_2,x_3) \ dx_3dx_2dx_1$$

3.10 Satz von Fubini

$$\int_\Omega f(\mathbf{x})d\mathbf{x} = \int_\Omega f(x_1,\dots,x_n)dx_1\dots dx_n =$$
$$\int_{I_n}(\dots(\int_{I_2}\dots(\int_{I_1} f(x_1,\dots,x_n)dx_1)dx_2)\dots dx_n =$$
$$\int_{I_1}(\dots(\int_{I_2}\dots(\int_{I_n} f(x_1,\dots,x_n)dx_n)dx_{n-1})\dots dx_1$$

3.11 Uneigentliche Integrale

$\lim\limits_{b\to\infty}\int_a^b f(x)dx = \int_a^\infty f(x)dx$	(unbegrenzt oben)
$\lim\limits_{a\to-\infty}\int_a^b f(x)dx = \int_{-\infty}^b f(x)dx$	(unbegrenzt unten)
$\lim\limits_{c\nearrow b}\int_a^c f(x)dx$	(Grenzwert c $\to$ b)

4 Differentialgleichungen

4.1 GDGL

$y^{(n)} = f(x,y,y',y'',\dots,y^{(n-1)})$	(explizit)
$F(x,y,y',y'',\dots,y^{(n)}) = 0$	(implizit)
$F(y,y',y'',\dots,y^{(n)}) = g(x) = 0$	(homogen)
$F(y,y',y'',\dots,y^{(n)}) = g(x) \neq 0$	(inhomogen)

Linearität

$$a_n(.)y_{(n)} + a_{n-1}(.)y_{(n-1)} + \dots + a_0(.)y + b(.) = 0$$
mit differenzierbaren Funktionen $a_i(.), b(.)$

Ordnung

$$y^{(n)} = f(x,y,y',y'',\dots,y^{(n-1)}) \quad \to (n\text{-te Ordnung})$$

4.2 Elementar lösbare GDGL

$y'(x) = a(x)y(x) + b(x)$	(expl. lin. 1. Ordn.)
$y'(x) = a(x)y(x) + b(x)(y(x))^k$	(Bernoulli)
$y'(x) = f(y(x)) \cdot g(x)$	(trennbare Variablen)
$y'(x) = f(ax + by(x) + c)$	(Substitution)

4.3 Differentialoperatoren

Gradient (siehe 2.6)

Divergenz

$$\nabla \cdot \mathbf{v} = \text{div } \mathbf{v} \overset{\text{(in 2D)}}{=} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$$

Laplace

$$\Delta = \nabla \cdot \nabla = \nabla^2 \overset{\text{(in 2D)}}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

4.4 PDGL 2. Ordnung

$$\begin{aligned} &a \cdot u_{xx} + b \cdot u_{xy} + d \cdot u_x + e \cdot u_y + \quad k \cdot u = g(x,y) \\ &b^2 - 4ac > 0 && \text{(hyperbolisch)} \\ &b^2 - 4ac = 0 && \text{(parabolisch)} \\ &b^2 - 4ac < 0 && \text{(elliptisch)} \end{aligned}$$

4.5 Numerische Integration

Explizites Eulerverfahren (siehe 7.6)

5 Weitere Analysis Themen

5.1 Kugelkoordinaten

$$x = r\cos\varphi\sin\theta, \quad y = r\sin\varphi\sin\theta, \quad z = r\cos\theta$$
$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \arccos\frac{z}{r}, \quad \varphi = \arctan\frac{y}{x}$$

5.2 Rotationsmatrix

$$\mathbf{p}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{R}(\theta)\mathbf{p}$$

5.3 Parametrische Abbildung

$$f(t_1,\dots,t_n) = \big(x_1(t_1,\dots,t_n),\dots,x_m(t_1,\dots,t_n)\big)^\mathsf{T}$$
$$\begin{aligned} f(t) &= \big(x(t), y(t)\big)^\mathsf{T} && \text{(2D Kurve)} \\ f(t) &= \big(x(t), y(t), z(t)\big)^\mathsf{T} && \text{(3D Kurve)} \\ f(s,t) &= \big(x(s,t), y(s,t), z(s,t)\big)^\mathsf{T} && \text{(3D Fläche)} \end{aligned}$$

5.4 Tangentialvektor

$$\dot\gamma(\tau) = \big(\gamma_1'(\tau),\dots,\gamma_n'(\tau)\big)^\mathsf{T} \neq \emptyset$$
$$\mathbf{t}(\tau) = \frac{\dot\gamma(\tau)}{||\dot\gamma(\tau)||_2}$$

5.5 Bogenlänge einer Kurve

$$L = \int_a^b ||\dot\gamma(t)||_2 \; dt = \int_a^b \left(\sum_{i=1}^n \big(\gamma_1'(t)\big)^2\right)^{\frac{1}{2}}$$

Spezialfall $\mathbb{R}^2$

$$L = \int_a^b \sqrt{1 + \big(f'(t)\big)^2} \quad \text{mit} \quad \gamma(t) = \big(t, f(t)\big)^\mathsf{T}$$

5.6 Weitere Funktionen

$$\gamma(\theta) = \begin{pmatrix} a \cdot \theta \cdot \cos\theta \\ a \cdot \theta \cdot \sin\theta \end{pmatrix} \qquad \text{(Archim. Spirale)}$$

$$\gamma(\theta) = \begin{pmatrix} a \cdot \cos\theta \\ a \cdot \sin\theta \\ b \cdot \theta \end{pmatrix} \qquad \text{(Helix)}$$

5.7 Multivariate Extrema

Tangentialhyperebene

$$\begin{aligned} g(\mathbf{x}_0 + \Delta\mathbf{x}) =& f(\mathbf{x}_0) + \frac{\partial f}{\partial x_1}(\mathbf{x}_0)\Delta x_1 + \dots \\ & \dots + \frac{\partial f}{\partial x_n}(\mathbf{x}_0)\Delta x_n \end{aligned}$$

mit $\Delta x_i = x_i - x_{0,i}$ und $\Delta\mathbf{x} = (\Delta x_1, \dots, \Delta x_n)$

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Delta x_i \qquad \text{(Totales Differential)}$$

Hessematrix $\mathbf{H}_f$

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

5.8 Jakobi Matrix $\mathbf{J}_f$

$$\left(\frac{\partial f_i}{\partial x_j}\right)_{i=1,\dots,m, j=1,\dots,n} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \\ \frac{\partial f_m}{\partial x_1} & \cdots & & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

5.9 Jakobi Verfahren (multivariat)

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \big(\mathbf{J}_f(\mathbf{x}_n)\big)^{-1} f(\mathbf{x}_n)$$

5.10 Kettenregel

$f, g : \mathbb{R} \to \mathbb{R}$ (Erinnerung)

$$\big(f \circ g\big)' = f'\big(g(x)\big) \cdot g'(x)$$

$\gamma : [a,b] \to \mathbb{R}^n$, $t \to (x_1(t), \dots, x_n(t))^\mathsf{T}$, $f : \mathbb{R}^n \to \mathbb{R}$,
$\mathbf{x} = (x_1, \dots, x_n) \to f(x_1, \dots, x_n) = f(\mathbf{x})$

$$\begin{aligned} \frac{d(f \circ g)}{dt}\big(\mathbf{x}(t)\big) =& \frac{\partial f}{\partial x_1}\big(\mathbf{x}(t)\big)\frac{dx_1}{dt}(t) + \dots \\ & + \frac{\partial f}{\partial x_n}\big(\mathbf{x}(t)\big)\frac{dx_n}{dt}(t) = \nabla f \cdot \frac{d\mathbf{x}}{dt} \end{aligned}$$

für $n = 1, 2$

$$\frac{df}{dt} = \frac{df}{dg}\frac{dg}{dt}, \qquad \frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt}$$

5.11 Kurvenintegral 1. Art

$$\int_a^b f(\gamma(t))\ ||\gamma'(t)||_2\ dt = \int_C f ds$$

5.12 Kurvenintegral 2. Art

$$\int_a^b \mathbf{f}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{f}\ ds$$

6 Folgen und Reihen

6.1 Konvergenz

$$\lim_{n\to\infty} a_n = a \iff \forall \varepsilon > 0\ \exists\ n_0 \in \mathbb{N}\ \forall n > n_0 : \\ |a_n - a| < \varepsilon$$

Geometrische Reihe

$$\lim_{m\to\infty} \sum_{k=0}^{m-1} q^k = \lim_{m\to\infty} \frac{1-q^m}{1-q} = \frac{1}{1-q}$$

Quotientenkriterium

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \begin{cases} < 1 & \text{konvergent} \\ > 1 & \text{divergent} \end{cases}$$

6.2 Gamma Funktion

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0$$

6.3 Binomischer Lehrsatz

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k!)}$$

6.4 Exponentialfunktion

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

6.5 Potenzreihe

$$\sum_{k=0}^\infty c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

6.6 Taylorreihe

$$f(x) = \sum_{k=0}^\infty \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$

$$T_{f,n}(x) = T_{f,n}(x_0 + h) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (h)^k \\ = f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + \ldots$$

6.7 Fourierreihe

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^\infty (a_k \cos(kx) + b_x \sin(kx))$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx, \quad k \geq 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx, \quad k \geq 1$$

7 Numerik

7.1 Maschinengenauigkeit

$\xi = m \cdot 2^e \in \mathbb{R},\ 0 \leq m < 1,\ x_{\text{li}} < \xi < x_{\text{re}}$

mit $x_{\text{li}}, x_{\text{re}}\ldots$ nächste normalisierte Maschinenzahlen mit k Bit Mantisse

$$\frac{\Delta\xi}{\xi} \leq \frac{2^{e-k-1}}{m \cdot 2^e} = \frac{1}{2} 2^{-k} = \varepsilon$$

7.2 Newton Verfahren

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

7.3 Konvergenzgeschwindigkeit

$$|x_{k+1} - x^*| \leq c|x_k - x^*|^p$$

7.4 Finite Differenzen

f' Vorwärts

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} + O(h)$$

f' Rückwärts

$$f'(x_0) = \frac{f(x_0) - f(x_0-h)}{h} + O(h)$$

f' Zentral

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} + O(h^2)$$

f" Zentral

$$f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} + O(h^2)$$

Multivariat, $f : \mathbb{R}^2 \to \mathbb{R}$

$$f_x(x,y) \approx \big(f(x+h_x, y) - f(x,y)\big)/h_x$$

$$f_y(x,y) \approx \big(f(x, y+h_y) - f(x,y)\big)/h_y$$

$$f_{xx}(x,y) \approx \frac{f(x+h_x,y) - 2f(x,y) + f(x-h_x,y)}{h_x^2}$$

$$f_{yy}(x,y) \approx \frac{f(x,y+h_y) - 2f(x,y) + f(x,y-h_y)}{h_y^2}$$

$$f_{xy}(x,y) \approx \big[f(x+h_x, y+h_y) - f(x+h_x, y-h_y) - \\ f(x-h_x, y+h_y) + f(x-h_x, y-h_y)\big] \cdot \\ \cdot \frac{1}{4h_x h_y}$$

7.5 Quadratur

$$I \approx Q_n[f] = \sum_{i=0}^{n} w_i f(x_i)$$

Untersumme

$$Q_n[f] = \frac{b-a}{n} \sum_{k=0}^{n-1} f(x_k)$$

Trapezregel

$$Q_n[f] = \frac{b-a}{n} \sum_{k=0}^{n-1} \frac{f(x_k)+f(x_{k+1})}{2}$$

Simpsonregel

$$\frac{b-a}{n \cdot 6}\Big(f(x_0)+f(x_n)+4\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2} + 2\sum_{k=0}^{n-2} f(x_{k+1})\Big)$$

7.6 Explizites Eulerverfahren

$$y_{n+1} = y_n + h \cdot f(t_n, y_n)\,, \quad \text{mit } f(t_n, y_n) = y_n'$$

8 Lösen linearer Gleichungssys.

8.1 LU Zerlegung

$$\mathbf{L}_{(0)}[j] = \begin{bmatrix} 1 & 0 & \dots & 0 & \\ 0 & \ddots & & & \\ & & 1 & \vdots & \\ \vdots & & l_{j+1,j} & & \\ & & \vdots & \ddots & 0 \\ 0 & & l_{n,j} & 0 & 1 \end{bmatrix}, \quad l_{i,j} = -\frac{a_{ij}}{a_{jj}}$$

$$\begin{aligned}\mathbf{A} &= \mathbf{L}_{(0)}^{-1}\mathbf{L}_{(0)}\mathbf{A} = \mathbf{L}_{(0)}^{-1}\mathbf{A}_{(1)} = \mathbf{L}_{(0)}^{-1}\mathbf{L}_{(1)}^{-1}\mathbf{L}_{(1)}\mathbf{A}_{(1)} \\ &= \mathbf{L}_{(0)}^{-1}\mathbf{L}_{(1)}^{-1}\mathbf{A}_{(2)} = \dots = \mathbf{L}_{(0)}^{-1}\mathbf{L}_{(1)}^{-1}\dots\mathbf{L}_{(n-2)}^{-1}\mathbf{A}_{(n-1)} \\ &= \mathbf{L}\cdot\mathbf{U}\end{aligned}$$

8.2 Cholesky Zerlegung

$$\mathbf{A} = \mathbf{L}\ \mathbf{L}^{\mathsf{T}}$$

$$l_{jj} = \pm\sqrt{a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2}, \quad l_{ij} = \frac{1}{l_{jj}}\left(a_{ij} - \sum_{k=1}^{j-1} l_{ik}\ l_{jk}\right)$$

8.3 Jacobi Verfahren

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1, j\neq i}^{n} a_{ij}\ x_j^{(k)}\right)$$

8.4 Gauß-Seidel-Verfahren

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}\ x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}\ x_j^{(k)}\right)$$

8.5 SOR-Verfahren

$$x_{i\ (\text{sor})}^{(k+1)} = (1-\omega)\, x_i^{(k)} + \omega\, x_{i\ (\text{gauß})}^{(k+1)}$$

8.6 Residuum

$$\mathbf{r}^{(k)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(k)}$$

8.7 Verfahren des steilsten Abstiegs

$$\begin{aligned}\mathbf{x}^{(k+1)} &= \mathbf{x}^{(k)} - \alpha^{(k)}\nabla f(\mathbf{x}^{(k)}) \\ -\nabla f(\mathbf{x}^{(k)}) &= \mathbf{b} - \mathbf{A}\mathbf{x}^{(k)} = \mathbf{r}^{(k)} \\ \alpha^{(k)} &= \frac{\left(\mathbf{r}^{(k)}\right)^{\mathsf{T}}\mathbf{r}^{(k)}}{\left(\mathbf{r}^{(k)}\right)^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{r}^{(k)}}\end{aligned}$$

9 Interpolation

9.1 Vandermonde-Matrix

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

9.2 Lagrange Polynom

$$\ell_i^{[n]} = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

$$p(x) = \sum_{i=0}^{n} y_i \cdot \ell_i^{[n]}(x)$$

9.3 Spline

$$\begin{aligned} f(x) &\approx \sum_i f(x_i)\cdot N_i(x) \\ N_i(x_i) &= 1 \\ N_i(x_{j,i\neq j}) &= 0 \end{aligned}$$

9.4 Bilinear

$$f(x,y) = \begin{bmatrix} 1-x \\ x \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{bmatrix} \begin{bmatrix} 1-y \\ y \end{bmatrix}^{\mathsf{T}}$$

9.5 Normalisierte baryzentrische Koordinaten

$$\mathbf{q} = \lambda_0\mathbf{p}_0 + \lambda_1\mathbf{p}_1 + \dots + \lambda_{n-1}\mathbf{p}_{n-1}$$

$$\sum_{i=0}^{n-1} \lambda_i = 1$$

9.6 Polynomapproximation

$$y_i = a_0 + a_1 x_i + a_2 x_i^2 + \dots + a_n x_i^n \quad \text{(Polynom)}$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^n \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{bmatrix}$$

$$\mathbf{a} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{y}$$

Annahme Grad $n = 2$

$$a_1 = \frac{\sum_{i=0}^{m} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=0}^{m} (x_i - \bar{x})}, \qquad a_0 = \bar{y} - c_1 \bar{x}$$

10 Zufallszahlen

10.1 Monte Carlo Integration

$$\int_0^1 f(x)dx \approx \frac{1}{N} \sum_{i=1}^{N} f(\xi_i) \quad \text{mit} \quad \xi_i \in [0,1]$$

10.2 Linearer Kongruenzgenerator

$$z_{i+1} = (a \cdot z_i + d) \bmod m$$
$$r_i = z_i / m$$

10.3 van-der-Corput Sequenz

$$g_b(n) = \sum_{k=0}^{L-1} d_k(n) b^{-(1+k)}$$

10.4 Wahrscheinlichkeit innerhalb Intervall

$$P(a \leq X \leq b) = \int_a^b p(x)dx$$
$$\int_{-\infty}^{\infty} p(x)dx = 1$$

10.5 Kumulierte Wahrscheinlichkeitsdichtef.

$$P(X) = \int_{-\infty}^{X} p(x)dx$$
$$P(X,Y) = \int_{-\infty}^{Y} \int_{-\infty}^{X} p(x,y)dxdy$$

Randdichten

$$p(x) = \int p(x,y)dy$$
$$p(y) = \int p(x,y)dx$$

Bedingte WDF

$$p(y|x) = \frac{p(x,y)}{p(x)}$$

11 Komplexe Zahlen

$$z = a + ib \qquad \text{(kartesisch)}$$
$$i^2 = -1$$
$$i = \sqrt{-1} \qquad \text{(imaginäre Einheit)}$$

11.1 Operationen kartesisch

Addition / Multiplikation

$$(a+ib) + (c+id) = ((a+c) + i(b+d))$$
$$(a+ib) \cdot (c+id) = ((ac-bd) + i(ad+bc))$$

Division

$$|z| = \sqrt{a^2 + b^2} \qquad \text{(Betrag)}$$
$$\bar{z} = a - ib \qquad \text{(konjugiert Komplex)}$$
$$z = \frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_1}{\bar{z}_2} = \frac{z_1 \cdot \bar{z}_2}{|z_2|^2}$$

11.2 Euler'sche Formel

$$e^{ix} = \cos x + i \sin x$$
$$z = r \cdot e^{i\varphi}$$
$$e^z = e^{a+ib} = e^a(\cos b + i \sin b)$$

Multiplikation

$$z = z_1 \cdot z_2 = r_1 \cdot r_2 \cdot e^{i(\varphi_1 + \varphi_2)}$$

Trigonometrischer Zusammenhang

$$\cos x = \mathrm{Re}(e^{ix}) = \frac{e^{ix} + e^{-ix}}{2}$$
$$\sin x = \mathrm{Im}(e^{ix}) = \frac{e^{ix} - e^{-ix}}{2i}$$

11.3 Fourier Transformation

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi u \cdot x} dx$$
$$= \int_{-\infty}^{\infty} f(x)(\cos(2\pi u \cdot x) - i \sin(2\pi u \cdot x))dx$$
$$f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi x \cdot u} du$$

Diskret

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i\frac{2\pi}{N}kn}$$
$$= \sum_{n=0}^{N-1} x_n \cdot \big(\cos(\frac{2\pi}{N}kn) - i \sin(\frac{2\pi}{N}kn)\big)$$