General Information: This assignment contains written and/or programming tasks. Combine all the answers to the written tasks in a single PDF document, named `{lastname}-written.pdf`. You can also scan or take pictures of (readable) handwritten papers. JPEG/PNG image files are accepted in this case and they should be named `{exercisenumber}-{lastname}-written.{jpeg/png}`. Make sure that we can follow the manual calculations. Do not combine too many small steps into one.
The programming tasks have to be solved in *Julia* and the source code files have to be submitted using the following naming scheme: `{exercisenumber}-{lastname}.jl`.

(1) (1 point) Determine the limit if there is one for each of the following sequences:

a) (0.25 points)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n^2 + 45n - 15}{\sqrt{36n^4 - 16n - 32}}$$

b) (0.25 points)

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n}\sin(n)$$

c) (0.25 points)

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \ln\left(2\frac{n^2}{4} \cdot \frac{1}{n^3} \cdot \sqrt{\frac{1}{n}}\right)$$

d) (0.25 points) **Hint**: L'Hôpital's rule states that for two differentiable functions f and g where $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ results in an indeterminate form, the following expression holds: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$.

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} n^5 e^{-n}$$

(2) (2 points) Do the following series converge?

a) (0.5 points) Use the ratio test to determine whether the following series converges:

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}.$$

Hint: The ratio test utilizes the limit:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Following cases are distinguished:

- if $L < 1$, the series converges,
- if $L > 1$, the series diverges,
- if $L = 1$, n/a.

b) (0.5 points) Use the integral test to determine whether the following series converges:

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)^2}.$$

Hint: According to the integral test, a series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the integral $\lim_{a\to\infty} \int_1^a f(n)\mathrm{d}n$ converges.

c) (0.5 points) Use the direct comparison test to determine whether the following series converges:

$$\sum_{n=1}^{\infty} \frac{n^2}{n^4 + n + 4}.$$

Hint: The direct comparison test states that if the infinite series $\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$, then the infinite series $\sum_{n=0}^{\infty} a_n$ converges as well.

d) (0.5 points) Use the alternating series test to determine whether the following series converges:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+3}.$$

Hint: The alternating series test states that a series which can be rewritten as $\sum_{n=1}^{\infty} (-1)^n a_n$ converges if the following conditions are met:

i. $|a_n|$ decreases monotonically (check if $|a_{n+1}| \leq |a_n|$)

ii. $\lim_{n\to\infty} a_n = 0$

(3) (2 points) A differentiable function $f(x)$ can be approximated at a point a using the *Taylor series*:

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots, = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

with $f^{(k)}(a)$ the k-th derivative of f at the point a.

a) (1.0 point) Write the sum of the first 8 terms of the Taylor series of the function $f(x) = \sin(x)$ at the point $a = 0$. Show that this series can be written with the following formula:

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}. \tag{1}$$

b) (0.5 points) Using Julia, implement the formula and plot the Taylor series for $n = 5$, $n = 10$, and $n = 15$ (see Figure below). Use the template `sine.jl`.

c) (0.5 points) For each value of n, plot the absolute error between the Taylor series approximation and the real function $\sin(x)$. Use the template `sine.jl`.

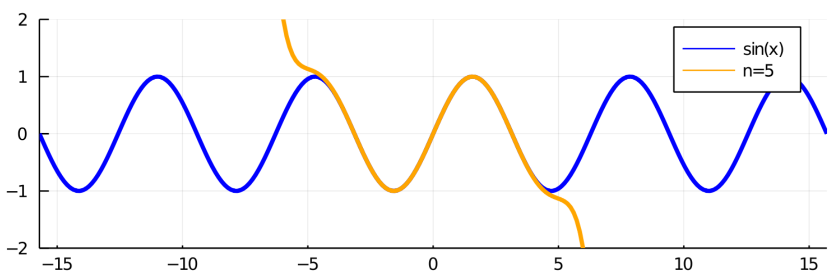

Figure 1: Plot of function $\sin(x)$ (blue) and its Taylor series approximation for $n = 5$.

(4) (2 points) A periodic piecewise continuous function f on the interval $[-\pi, \pi]$ has a *Fourier Series Representation*:

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k sin(kx),$$

with the following coefficients:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(kx)\mathrm{d}x, \quad k \geq 0$$
$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(kx)\mathrm{d}x, \quad k \geq 1.$$

Compute the terms for $k \in \{1, 3, 5\}$ of the Fourier Series representation for $f(x)$ with the period 2π:

$$f(x) = \begin{cases} 1 & -\pi < x < 0 \\ -1 & 0 < x < \pi \end{cases} .$$

Copy your resulting a_0, a_k, b_k into the respective place in the `fourier.jl` file and compare your result to Figure 2.

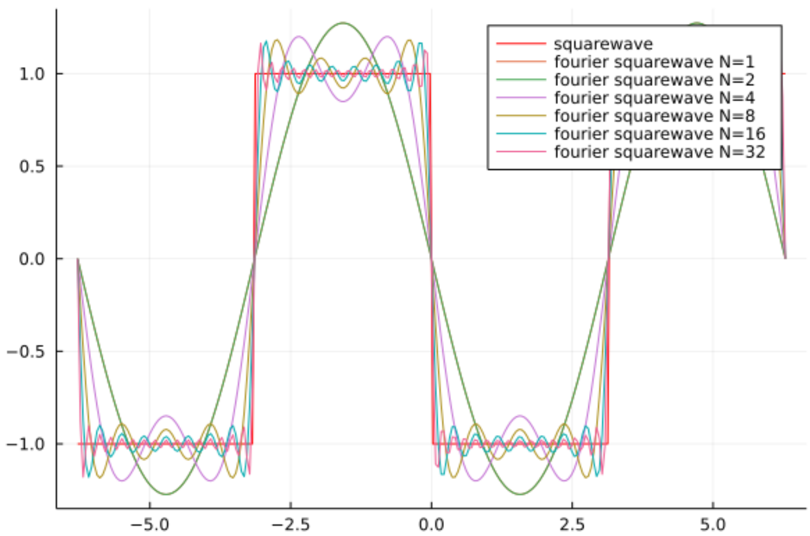

Figure 2: Plot of square wave function and (N-th) partial sums of the corresponding Fourier series.