

Formelsammlung für die schriftliche Prüfung Angewandte Mathematik in der Informatik

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1 Einführung

1.1 Funktionen $f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = d \quad (\text{konstante Funktion})$$

$$f(x) = kx + d \quad (\text{explizite lineare F.})$$

$$F(x, y) = \mathbf{n}\mathbf{x} + d = 0 \quad (\text{implizite lineare F.})$$

$$f(x) = ax^k \quad (\text{Potenzfunktion})$$

$$f(x) = \sqrt[k]{x} = x^{1/k} \quad (\text{Wurzelfunktion})$$

$$f(f^{-1}(x)) = x = f^{-1}(f(x)) \quad (\text{Umkehrfunktion})$$

1.2 Eigenschaften

$$f(x) = f(-x) \quad (\text{gerade Funktion})$$

$$f(x) = -f(-x) \quad (\text{ungerade Funktion})$$

1.3 Polynome

$$p(x) = \sum_{i=0}^n a_i x^i \quad (\text{Polynomfunktion})$$

$$r(x) = \frac{p_1(x)}{p_2(x)} \quad (\text{Rationale Funktion})$$

Faktorisierung

$$p(x) = (x - x_k)^{m_k} \dots (x - x_1)^{m_1} (x - x_0)^{m_0} \cdot q(x)$$

Nullstellen quadratisches Polynom

$$ax^2 + bx + c = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1.4 Trigonometrische Funktionen

Additionstheoreme

$$\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$$

$$\cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y$$

$$\tan(x \pm y) = \frac{\sin(x \pm y)}{\cos(x \pm y)} = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x) \tan(y)}$$

$$\cot(x \pm y) = \frac{\cos(x \pm y)}{\sin(x \pm y)} =$$

Hyperbolische Funktionen

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

1.5 Vektoroperationen

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \dots + v_n w_n \quad (\text{Inneres Produkt})$$

$$\|\mathbf{v}\|_2 = \sqrt{\mathbf{v} \cdot \mathbf{v}} \quad (\text{Betrag, Länge})$$

$$\hat{\mathbf{v}} = \text{nrm}(\mathbf{v}) = \frac{\mathbf{v}}{\|\mathbf{v}\|_2} \quad (\text{Einheitsvektor})$$

$$\varphi = \angle(\mathbf{v}, \mathbf{w}) \text{ mit } 0 \leq \varphi \leq \pi$$

$$\cos \varphi = \text{nrm}(\mathbf{v}) \cdot \text{nrm}(\mathbf{w}) \quad (\text{Winkel 1})$$

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|_2 \|\mathbf{w}\|_2 \cos \varphi \quad (\text{Winkel 2})$$

1.6 Matrixoperationen

$$\mathbf{A} = a_{ij}$$

$$\mathbf{A}^T = a_{ji} \quad (\text{Transponierte})$$

$$\mathbf{I} = \delta_{ij} = \text{diag}(1, 1, \dots, 1) \quad (\text{Einheitsmatrix})$$

$$\mathbf{A}\mathbf{I} = \mathbf{I}\mathbf{A} = \mathbf{A}$$

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \quad (\text{Inverse})$$

2 Differentialrechnung

2.1 Erste Ableitung

$$x(t)' = \frac{dx}{dt} = \frac{d}{dt}x = \dot{x} = \lim_{h \rightarrow 0} \frac{x(t+h) - x(t)}{h}$$

2.2 Ableitungsregeln

$$(a)' = 0 \quad (\text{Konstante})$$

$$(x^k)' = kx^{k-1} \quad (\text{Potenz})$$

$$(af \pm bg)' = af' \pm bg' \quad (\text{Summenregel})$$

$$(f \cdot g)' = f' \cdot g + f \cdot g' \quad (\text{Produktregel})$$

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2} \quad (\text{Quotientenregel})$$

$$(f \circ g)' = f'(g) \cdot g' \quad (\text{Kettenregel})$$

$$(\sin x)' = \cos x \quad (\text{Kreisfunktionen 1})$$

$$(\cos x)' = -\sin x \quad (\text{Kreisfunktionen 2})$$

$$(\sinh x)' = \cosh(x) \quad (\text{Hyperbelfunktionen 1})$$

$$(\cosh x)' = \sinh(x) \quad (\text{Hyperbelfunktionen 2})$$

$$(a^x)' = \ln a \cdot a^x \quad (\text{Exponential 1})$$

$$(e^x)' = e^x \quad (\text{Exponential 2})$$

$$(e^{f(x)})' = e^{f(x)} f'(x) \quad (\text{Exponential 3})$$

$$(\ln(f(x)))' = \frac{f'(x)}{f(x)} \quad (\text{Logarithmus 1})$$

$$(\ln(x))' = \frac{1}{x} \quad (\text{Logarithmus 2})$$

2.3 Newton Verfahren (siehe 7.2)

2.4 Zweite Ableitung

$$x(t)'' = (x')' = \frac{d}{dt} \frac{d}{dt}(x) = \frac{d^2}{dt^2} x = \ddot{x} = x^{(2)}$$

2.5 Partielle Ableitung

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} f(x, y) = f_x = \\ &= \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} f(x, y) = f_y = \\ &= \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} \end{aligned}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2}{\partial x^2} f = \frac{\partial}{\partial x} \frac{\partial}{\partial x} f = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2}{\partial y^2} f = \frac{\partial}{\partial y} \frac{\partial}{\partial y} f = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y} f = \frac{\partial}{\partial x} \frac{\partial}{\partial y} f = f_{xy}$$

2.6 Gradient

$$\nabla f = \text{grad } f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)^T f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)^T$$

2.7 Sätze zu Funktionen

Stetigkeit

$$\begin{aligned} f : D \rightarrow \mathbb{R}, \quad D \subset \mathbb{R}, f \text{ ist stetig in } x_0 \\ \iff \lim_{x \rightarrow x_0} f(x) = f(x_0) \end{aligned}$$

Zwischenwertsatz

$$\begin{aligned} f : [a, b] \rightarrow \mathbb{R} \text{ stetig} \\ \wedge (f(a) < C < f(b) \vee f(a) > C > f(b)) \\ \Rightarrow \exists c \in]a, b[: f(c) = C \end{aligned}$$

Satz von Rolle

$$\begin{aligned} f : [a, b] \rightarrow \mathbb{R} \text{ stetig} \\ \wedge \text{ in }]a, b[\text{ differenzierbar} \wedge f(a) = f(b) \\ \Rightarrow \exists x_0 \in]a, b[: f'(x_0) = 0 \end{aligned}$$

Mittelwertsatz der Differentialrechnung

$$\begin{aligned} f : [a, b] \rightarrow \mathbb{R} \text{ stetig} \\ \wedge \text{ in }]a, b[\text{ differenzierbar} \\ \Rightarrow \exists x_0 \in]a, b[: f'(x_0) = \frac{f(b) - f(a)}{b - a} \end{aligned}$$

Monotonie

$$\begin{aligned} f : [a, b] \rightarrow \mathbb{R} \text{ stetig} \\ \wedge \text{ in }]a, b[\text{ differenzierbar} \rightarrow \\ \begin{cases} \forall x \in]a, b[: f'(x) \geq 0, & \text{monoton steigend} \\ \forall x \in]a, b[: f'(x) \leq 0, & \text{monoton fallend} \\ \forall x \in]a, b[: f'(x) = 0, & \text{konstant} \end{cases} \end{aligned}$$

3 Integralrechnung

3.1 Definition

$$\lim_{n \rightarrow \infty, \Delta x_k \rightarrow 0} \sum_{k=0}^{n-1} f(\xi_k) \Delta x_k = \int_a^b f(x) dx$$

3.2 Eigenschaften

$$\begin{aligned} a, b \in \mathbb{R} \wedge a < b \quad f, g : [a, b] \rightarrow \mathbb{R} \text{ integrierbar} \\ \text{für } f(x) \geq 0 : \int_a^b f(x) dx \geq 0 \\ \text{für } f(x) \leq 0 : \int_a^b f(x) dx \leq 0 \\ \text{für } f(x) \leq g(x) : \int_a^b f(x) dx \leq \int_a^b g(x) dx \\ \int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx \\ \int_a^a f(x) dx = 0 \\ \int_a^b f(x) dx = - \int_b^a f(x) dx \end{aligned}$$

3.3 Mittelwertsatz der Integralrechnung

$$\begin{aligned} f : [a, b] \rightarrow \mathbb{R} \text{ stetig} \Rightarrow \exists \xi \in [a, b] \text{ sodass} \\ \int_a^b f(x) dx = f(\xi)(b - a) \end{aligned}$$

Mittelwert einer Funktion

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx$$

Quadratisches Mittel

$$\bar{f} = \sqrt{\frac{1}{b-a} \int_a^b (f(x))^2 dx}$$

3.4 Hauptsatz Different.-/Integralrechnung

$$\begin{aligned} f : [a, b] \rightarrow \mathbb{R} \text{ stetig} \wedge \lambda \in [a, b] \Rightarrow \\ F_\lambda(x) = \int_\lambda^x f(t) dt \quad (\text{Stammfunktion von } f) \\ F'_\lambda(x) = f(x) \end{aligned}$$

Bestimmtes Integral

$$\int_a^b f(x)dx = [F(x)]_a^b = F(x)|_a^b = F(b) - F(a)$$

Unbestimmtes Integral

$$\int f(x)dx = F(x) + C$$

3.5 Elementare Integrationsregeln

$a, b, \lambda_1, k, C \in \mathbb{R} \quad f, g : [a, b] \rightarrow \mathbb{R}$ integrierbar

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b \lambda_1 f(x)dx = \lambda_1 \int_a^b f(x)dx \quad (\text{Linearität})$$

$$\int k dx = kx + C \quad (\text{Konstante})$$

$$\int_a^b dx = \int_a^b 1 \cdot dx = x|_a^b = b - a \quad (\text{Bereich})$$

$$\int x^k dx = \frac{x^{k+1}}{k+1} + C \quad (\text{Potenz})$$

$$\int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C \quad (\text{Wurzel})$$

$$\int e^{kx} dx = \frac{1}{k} e^{kx} + C \quad (\text{e})$$

$$\int \frac{1}{x} dx = \ln |x| + C \quad (\text{Kehrwert})$$

$$\int \sin x dx = -\cos x + C \quad (\text{Sinus})$$

$$\int \cos x dx = \sin x + C \quad (\text{Cosinus})$$

$$\int \sinh x dx = \cosh x + C \quad (\text{Sinus Hyp.})$$

$$\int \cosh x dx = \sinh x + C \quad (\text{Cosinus Hyp.})$$

$$\int \ln x dx = x \ln x - x + C \quad (\ln)$$

3.6 Partielle Integration

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx + C$$

3.7 Integration mit Substitution

$$\int_a^b f(g(t))g'(t)dt = \int_{g(a)}^{g(b)} f(x)dx = F(x) \Big|_{g(a)}^{g(b)}$$

3.8 Integration 2D

$$V = \iint_D f(x, y)dD = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x, y) dx dy$$

Normalbereich

mit $D = \{(x, y) : a \leq x \leq b, g(x) \leq y \leq h(x)\}$

$$\iint_D f(x, y)dx dy = \int_{x=a}^b \left(\int_{y=g(x)}^{h(x)} f(x, y) dx \right) dy$$

3.9 Integration 3D

$$\iiint_{\Omega} f(x, y, z)dx dy dz = \int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x, y, z) dx dy dz$$

Normalbereich

mit $\Omega = \{(x, y, z) : a_1 \leq x_1 \leq b_1,$

$$a_2(x_1) \leq x_2 \leq b_2(x),$$

$$a_3(x_1, x_2) \leq x_3 \leq b_3(x_1, x_2)\}$$

$$\iiint_{\Omega} f(x_1, x_2, x_3)dx_3 dx_2 dx_1 =$$

$$\int_{x_1=a_1}^{b_1} \int_{x_2=a_2(x_1)}^{b_2(x_1)} \int_{x_3=a_3(x_1, x_2)}^{b_3(x_1, x_2)} f(x_1, x_2, x_3) dx_3 dx_2 dx_1$$

3.10 Satz von Fubini

$$\int_{\Omega} f(\mathbf{x})d\mathbf{x} = \int_{\Omega} f(x_1, \dots, x_n)dx_1 \dots dx_n =$$

$$\int_{I_n} (\dots (\int_{I_2} (\dots (\int_{I_1} f(x_1, \dots, x_n)dx_1)dx_2) \dots dx_n =$$

$$\int_{I_1} (\dots (\int_{I_2} (\dots (\int_{I_n} f(x_1, \dots, x_n)dx_n)dx_{n-1}) \dots dx_1$$

3.11 Uneigentliche Integrale

$$\lim_{b \rightarrow \infty} \int_a^b f(x)dx = \int_a^{\infty} f(x)dx \quad (\text{unbegrenzt oben})$$

$$\lim_{a \rightarrow -\infty} \int_a^b f(x)dx = \int_{-\infty}^b f(x)dx \quad (\text{unbegrenzt unten})$$

$$\lim_{c \nearrow b} \int_a^c f(x)dx \quad (\text{Grenzwert } c \rightarrow b)$$

4 Differentialgleichungen

4.1 GDGL

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)}) \quad (\text{explizit})$$

$$F(x, y, y', y'', \dots, y^{(n)}) = 0 \quad (\text{implizit})$$

$$F(y, y', y'', \dots, y^{(n)}) = g(x) = 0 \quad (\text{homogen})$$

$$F(y, y', y'', \dots, y^{(n)}) = g(x) \neq 0 \quad (\text{inhomogen})$$

Linearität

$$a_n(\cdot)y_{(n)} + a_{n-1}(\cdot)y_{(n-1)} + \dots + a_0(\cdot)y + b(\cdot) = 0$$

mit differenzierbaren Funktionen $a_i(\cdot), b(\cdot)$

Ordnung

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)}) \rightarrow (n\text{-te Ordnung})$$

4.2 Elementar lösbare GDGL

$$y'(x) = a(x)y(x) + b(x) \quad (\text{expl. lin. 1. Ordn.})$$

$$y'(x) = a(x)y(x) + b(x)(y(x))^k \quad (\text{Bernoulli})$$

$$y'(x) = f(y(x)) \cdot g(x) \quad (\text{trennbare Variablen})$$

$$y'(x) = f(ax + by(x) + c) \quad (\text{Substitution})$$

4.3 Differentialoperatoren

Gradient (siehe 2.6)

Divergenz

$$\nabla \cdot \mathbf{v} = \operatorname{div} \mathbf{v} \stackrel{(\text{in 2D})}{=} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$$

Laplace

$$\Delta = \nabla \cdot \nabla = \nabla^2 \stackrel{(\text{in 2D})}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

4.4 PDGL 2. Ordnung

$$a \cdot u_{xx} + b \cdot u_{xy} + d \cdot u_x + e \cdot u_y + k \cdot u = g(x, y)$$
$$b^2 - 4ac > 0 \quad (\text{hyperbolisch})$$
$$b^2 - 4ac = 0 \quad (\text{parabolisch})$$
$$b^2 - 4ac < 0 \quad (\text{elliptisch})$$

4.5 Numerische Integration

Explizites Eulerverfahren (siehe 7.6)

5 Weitere Analysis Themen

5.1 Kugelkoordinaten

$$x = r \cos \varphi \sin \theta, \quad y = r \sin \varphi \sin \theta, \quad z = r \cos \theta$$
$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arccos \frac{z}{r}, \quad \varphi = \arctan \frac{y}{x}$$

5.2 Rotationsmatrix

$$\mathbf{p}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{R}(\theta) \mathbf{p}$$

5.3 Parametrische Abbildung

$$f(t_1, \dots, t_n) = (x_1(t_1, \dots, t_n), \dots, x_m(t_1, \dots, t_n))^T$$
$$f(t) = (x(t), y(t))^T \quad (2D \text{ Kurve})$$
$$f(t) = (x(t), y(t), z(t))^T \quad (3D \text{ Kurve})$$
$$f(s, t) = (x(s, t), y(s, t), z(s, t))^T \quad (3D \text{ Fläche})$$

5.4 Tangentialvektor

$$\dot{\gamma}(\tau) = (\gamma'_1(\tau), \dots, \gamma'_n(\tau))^T \neq \emptyset$$
$$\mathbf{t}(\tau) = \frac{\dot{\gamma}(\tau)}{\|\dot{\gamma}(\tau)\|_2}$$

5.5 Bogenlänge einer Kurve

$$L = \int_a^b \|\dot{\gamma}(t)\|_2 \, dt = \int_a^b \left(\sum_{i=1}^n (\gamma'_i(t))^2 \right)^{\frac{1}{2}} dt$$

Spezialfall \mathbb{R}^2

$$L = \int_a^b \sqrt{1 + (f'(t))^2} \, dt \quad \text{mit} \quad \gamma(t) = (t, f(t))^T$$

5.6 Weitere Funktionen

$$\gamma(\theta) = \begin{pmatrix} a \cdot \theta \cdot \cos \theta \\ a \cdot \theta \cdot \sin \theta \end{pmatrix} \quad (\text{Archim. Spirale})$$

$$\gamma(\theta) = \begin{pmatrix} a \cdot \cos \theta \\ a \cdot \sin \theta \\ b \cdot \theta \end{pmatrix} \quad (\text{Helix})$$

5.7 Multivariate Extrema

Tangentialhyperebene

$$g(\mathbf{x}_0 + \Delta \mathbf{x}) = f(\mathbf{x}_0) + \frac{\partial f}{\partial x_1}(\mathbf{x}_0) \Delta x_1 + \dots$$
$$\dots + \frac{\partial f}{\partial x_n}(\mathbf{x}_0) \Delta x_n$$

mit $\Delta x_i = x_i - x_{0,i}$ und $\Delta \mathbf{x} = (\Delta x_1, \dots, \Delta x_n)$

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \Delta x_i \quad (\text{Totales Differential})$$

Hessematrix \mathbf{H}_f

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \dots & \dots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

5.8 Jakobi Matrix \mathbf{J}_f

$$\left(\frac{\partial f_i}{\partial x_j} \right)_{i=1, \dots, m, j=1, \dots, n} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

5.9 Jakobi Verfahren (multivariat)

$$\mathbf{x}_{n+1} = \mathbf{x}_n - (\mathbf{J}_f(\mathbf{x}_n))^{-1} f(\mathbf{x}_n)$$

5.10 Kettenregel

$f, g: \mathbb{R} \rightarrow \mathbb{R}$ (Erinnerung)

$$(f \circ g)' = f'(g(x)) \cdot g'(x)$$

$\gamma: [a, b] \rightarrow \mathbb{R}^n, t \rightarrow (x_1(t), \dots, x_n(t))^T, f: \mathbb{R}^n \rightarrow \mathbb{R},$
 $\mathbf{x} = (x_1, \dots, x_n) \rightarrow f(x_1, \dots, x_n) = f(\mathbf{x})$

$$\frac{d(f \circ g)}{dt}(\mathbf{x}(t)) = \frac{\partial f}{\partial x_1}(\mathbf{x}(t)) \frac{dx_1}{dt}(t) + \dots$$
$$+ \frac{\partial f}{\partial x_n}(\mathbf{x}(t)) \frac{dx_n}{dt}(t) = \nabla f \cdot \frac{d\mathbf{x}}{dt}$$

für $n = 1, 2$

$$\frac{df}{dt} = \frac{df}{dg} \frac{dg}{dt}, \quad \frac{df}{dt} = \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt}$$

5.11 Kurvenintegral 1. Art

$$\int_a^b f(\gamma(t)) \|\gamma'(t)\|_2 dt = \int_C f ds$$

5.12 Kurvenintegral 2. Art

$$\int_a^b \mathbf{f}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{f} ds$$

6 Folgen und Reihen

6.1 Konvergenz

$$\lim_{n \rightarrow \infty} a_n = a \iff \forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall n > n_0 : |a_n - a| < \varepsilon$$

Geometrische Reihe

$$\lim_{m \rightarrow \infty} \sum_{k=0}^{m-1} q^k = \lim_{m \rightarrow \infty} \frac{1 - q^m}{1 - q} = \frac{1}{1 - q}$$

Quotientenkriterium

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \begin{cases} < 1 & \text{konvergent} \\ > 1 & \text{divergent} \end{cases}$$

6.2 Gamma Funktion

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0$$

6.3 Binomischer Lehrsatz

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

6.4 Exponentialfunktion

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

6.5 Potenzreihe

$$\sum_{k=0}^{\infty} c_k (x - a)^k = c_0 + c_1(x - a) + c_2(x - a)^2 + \dots$$

6.6 Taylorreihe

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$
$$T_{f,n}(x) = T_{f,n}(x_0 + h) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (h)^k$$
$$= f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + \dots$$

6.7 Fourierreihe

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx))$$
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx, \quad k \geq 0$$
$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx, \quad k \geq 1$$

7 Numerik

7.1 Maschinengenauigkeit

$\xi = m \cdot 2^e \in \mathbb{R}$, $0 \leq m < 1$, $x_{\text{li}} < \xi < x_{\text{re}}$
mit $x_{\text{li}}, x_{\text{re}} \dots$ nächste normalisierte Maschinenzahlen
mit k Bit Mantisse

$$\frac{\Delta \xi}{\xi} \leq \frac{2^{e-k-1}}{m \cdot 2^e} = \frac{1}{2} 2^{-k} = \varepsilon$$

7.2 Newton Verfahren

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

7.3 Konvergenzgeschwindigkeit

$$|x_{k+1} - x^*| \leq c |x_k - x^*|^p$$

7.4 Finite Differenzen

f' Vorwärts

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} + O(h)$$

f' Rückwärts

$$f'(x_0) = \frac{f(x_0) - f(x_0 - h)}{h} + O(h)$$

f' Zentral

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h} + O(h^2)$$

f'' Zentral

$$f''(x_0) = \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h))}{h^2} + O(h^2)$$

Multivariat, $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f_x(x, y) \approx (f(x + h_x, y) - f(x, y)) / h_x$$

$$f_y(x, y) \approx (f(x, y + h_y) - f(x, y)) / h_y$$

$$f_{xx}(x, y) \approx \frac{f(x + h_x, y) - 2f(x, y) + f(x - h_x, y))}{h_x^2}$$

$$f_{yy}(x, y) \approx \frac{f(x, y + h_y) - 2f(x, y) + f(x, y - h_y))}{h_y^2}$$

$$f_{xy}(x, y) \approx [f(x + h_x, y + h_y) - f(x + h_x, y - h_y) - f(x - h_x, y + h_y) + f(x - h_x, y - h_y)] \cdot \frac{1}{4h_x h_y}$$

7.5 Quadratur

$$I \approx Q_n[f] = \sum_{i=0}^n w_i f(x_i)$$

Untersumme

$$Q_n[f] = \frac{b-a}{n} \sum_{k=0}^{n-1} f(x_k)$$

Trapezregel

$$Q_n[f] = \frac{b-a}{n} \sum_{k=0}^{n-1} \frac{f(x_k) + f(x_{k+1})}{2}$$

Simpsonregel

$$\frac{b-a}{n \cdot 6} \left(f(x_0) + f(x_n) + 4 \sum_{k=0}^{n-1} \frac{f(x_k) + f(x_{k+1})}{2} + 2 \sum_{k=0}^{n-2} f(x_{k+1}) \right)$$

7.6 Explizites Eulerverfahren

$$y_{n+1} = y_n + h \cdot f(t_n, y_n), \quad \text{mit } f(t_n, y_n) = y'_n$$

8 Lösen linearer Gleichungssys.

8.1 LU Zerlegung

$$\mathbf{L}_{(0)}[j] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & & \\ & & 1 & \vdots \\ \vdots & & l_{j+1,j} & \ddots \\ 0 & & l_{n,j} & 0 & 1 \end{bmatrix}, \quad l_{i,j} = -\frac{a_{ij}}{a_{jj}}$$

$$\begin{aligned} \mathbf{A} &= \mathbf{L}_{(0)}^{-1} \mathbf{L}_{(0)} \mathbf{A} = \mathbf{L}_{(0)}^{-1} \mathbf{A}_{(1)} = \mathbf{L}_{(0)}^{-1} \mathbf{L}_{(1)}^{-1} \mathbf{L}_{(1)} \mathbf{A}_{(1)} \\ &= \mathbf{L}_{(0)}^{-1} \mathbf{L}_{(1)}^{-1} \mathbf{A}_{(2)} = \dots = \mathbf{L}_{(0)}^{-1} \mathbf{L}_{(1)}^{-1} \dots \mathbf{L}_{(n-2)}^{-1} \mathbf{A}_{(n-1)} \\ &= \mathbf{L} \cdot \mathbf{U} \end{aligned}$$

8.2 Cholesky Zerlegung

$$\mathbf{A} = \mathbf{L} \mathbf{L}^\top$$

$$l_{jj} = \pm \sqrt{a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2}, \quad l_{ij} = \frac{1}{l_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk} \right)$$

8.3 Jacobi Verfahren

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1, j \neq i}^n a_{ij} x_j^{(k)} \right)$$

8.4 Gauß-Seidel-Verfahren

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^n a_{ij} x_j^{(k)} \right)$$

8.5 SOR-Verfahren

$$x_i^{(k+1)}(\text{SOR}) = (1 - \omega) x_i^{(k)} + \omega x_i^{(k+1)}(\text{Gauß})$$

8.6 Verfahren des steilsten Abstiegs

$$\begin{aligned} \mathbf{x}^{(k+1)} &= \mathbf{x}^{(k)} - \alpha^{(k)} \nabla f(\mathbf{x}^{(k)}) \\ -\nabla f(\mathbf{x}^{(k)}) &= \mathbf{b} - \mathbf{A} \mathbf{x}^{(k)} = \mathbf{r}^{(k)} \\ \alpha^{(k)} &= \frac{(\mathbf{r}^{(k)})^\top \mathbf{r}^{(k)}}{(\mathbf{r}^{(k)})^\top \mathbf{A} \mathbf{r}^{(k)}} \end{aligned}$$

9 Interpolation

9.1 Vandermonde-Matrix

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

9.2 Lagrange Polynom

$$\begin{aligned} \ell_i^{[n]} &= \prod_{j=0, j \neq i}^n \frac{x - x_j}{x_i - x_j} \\ p(x) &= \sum_{i=0}^n y_i \cdot \ell_i^{[n]}(x) \end{aligned}$$

9.3 Spline

$$\begin{aligned} f(x) &\approx \sum_i f(x_i) \cdot N_i(x) \\ N_i(x_i) &= 1 \\ N_i(x_{j, i \neq j}) &= 0 \end{aligned}$$

9.4 Bilinear

$$f(x, y) = \begin{bmatrix} 1 & x \end{bmatrix}^\top \begin{bmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{bmatrix} \begin{bmatrix} 1 & y \end{bmatrix}^\top$$

9.5 Normalisierte baryzentrische Koordinaten

$$\mathbf{q} = \lambda_0 \mathbf{p}_0 + \lambda_1 \mathbf{p}_1 + \dots + \lambda_{n-1} \mathbf{p}_{n-1}$$

$$\sum_{i=0}^{n-1} \lambda_i = 1$$

9.6 Polynomapproximation

$$y_i = a_0 + a_1 x_i + a_2 x_i^2 + \dots + a_n x_i^n \quad (\text{Polynom})$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^n \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{bmatrix}$$

$$\mathbf{a} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{y}$$

Annahme Grad $n = 2$

$$a_1 = \frac{\sum_{i=0}^m (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=0}^m (x_i - \bar{x})}, \quad a_0 = \bar{y} - c_1 \bar{x}$$

10 Zufallszahlen

10.1 Monte Carlo Integration

$$\int_0^1 f(x) dx \approx \frac{1}{N} \sum_{i=1}^N f(\xi_i) \quad \text{mit} \quad \xi_i \in [0, 1]$$

10.2 Linearer Kongruenzgenerator

$$z_{i+1} = (a \cdot z_i + d) \bmod m$$

10.3 van-der-Corput Sequenz

$$g_b(n) = \sum_{k=0}^{L-1} d_k(n) b^{-(1+k)}$$

10.4 Wahrscheinlichkeit innerhalb Intervall

$$P(a \leq X \leq b) = \int_a^b p(x) dx$$
$$\int_{-\infty}^{\infty} p(x) dx = 1$$

10.5 Kumulierte Wahrscheinlichkeitsdichtef.

$$P(X) = \int_{-\infty}^X p(x) dx$$
$$P(X, Y) = \int_{-\infty}^Y \int_{-\infty}^X p(x, y) dx dy$$

Randdichten

$$p(x) = \int p(x, y) dy$$
$$p(y) = \int p(x, y) dx$$

Bedingte WDF

$$p(y|x) = \frac{p(x, y)}{p(x)}$$

11 Komplexe Zahlen

$$z = a + ib \quad (\text{kartesisch})$$

$$i^2 = -1$$

$$i = \sqrt{-1} \quad (\text{imaginäre Einheit})$$

11.1 Operationen kartesisch

Addition / Multiplikation

$$(a + ib) + (c + id) = ((a + c) + i(b + d))$$

$$(a + ib) \cdot (c + id) = ((ac - bd) + i(ad + bc))$$

Division

$$|z| = \sqrt{a^2 + b^2} \quad (\text{Betrag})$$

$$\bar{z} = a - ib \quad (\text{konjugiert Komplex})$$

$$z = \frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1 \cdot \bar{z}_2}{|z_2|^2}$$

11.2 Euler'sche Formel

$$e^{ix} = \cos x + i \sin x$$

$$z = r \cdot e^{i\varphi}$$

$$e^z = e^{a+ib} = e^a (\cos b + i \sin b)$$

Multiplikation

$$z = z_1 \cdot z_2 = r_1 \cdot r_2 \cdot e^{i(\varphi_1 + \varphi_2)}$$

Trigonometrischer Zusammenhang

$$\cos x = \operatorname{Re}(e^{ix}) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \operatorname{Im}(e^{ix}) = \frac{e^{ix} - e^{-ix}}{2i}$$

11.3 Fourier Transformation

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi u \cdot x} dx$$
$$= \int_{-\infty}^{\infty} f(x) (\cos(2\pi u \cdot x) - i \sin(2\pi u \cdot x)) dx$$
$$f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi x \cdot u} du$$

Diskret

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i \frac{2\pi}{N} kn}$$
$$= \sum_{n=0}^{N-1} x_n \cdot \left(\cos\left(\frac{2\pi}{N} kn\right) - i \sin\left(\frac{2\pi}{N} kn\right) \right)$$